Filomat 38:19 (2024), 6703–6713
https://doi.org/10.2298/FIL2419703C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New characterizations and representations of the weak group inverse

Yang Chen[a], **Li Yuan**[a,*], **Xiangyu Zhang**[b]

[a]*Department of Mathematics and Computer Science, Hanjiang Normal University, Shiyan, China*
[b]*School of Mathematics and Statistics, Hubei Normal University, Huangshi, China*

Abstract. The weak group inverse is characterized from an algebraic point of view. Some equivalent conditions for a matrix to be the weak group inverse are established using range, null space and several matrix equations. Based on the group inverse, Bott-Duffin inverse and certain projections, some representations of the weak group inverse are given. In addition, splitting method for computing the weak group inverse is presented.

1. Introduction

The sets of all natural number, complex number, n dimensional column vectors and $m \times n$ complex matrices will be denoted by $\mathbb{N}$, $\mathbb{C}$, $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$, respectively. The identity matrix in $\mathbb{C}^{n\times n}$ and the null matrix in $\mathbb{C}^{m\times n}$ are denoted by I_n and O. For $A \in \mathbb{C}^{m\times n}$, let A^*, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the conjugate transpose, the rank, the range and the null space of A, respectively. For $A \in \mathbb{C}^{n\times n}$, the index of A, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$. The symbol $\mathbb{C}_k^{n\times n}$ stands for the set of all $n \times n$ complex matrices with index k.

The definitions of several helpful generalized inverses are stated now. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies $XAX = X$ is called an outer inverse of $A \in \mathbb{C}^{m\times n}$ and denoted by $A^{(2)}$. The outer inverse of $A \in \mathbb{C}^{m\times n}$ with the range $\mathcal{T}$ and null space $\mathcal{S}$ is the unique matrix $A^{(2)}_{\mathcal{T},\mathcal{S}} = X \in \mathbb{C}^{n\times m}$ satisfying $XAX = X$, $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$, where A has rank r and the two subspaces $\mathcal{T}$ and $\mathcal{S}$ of $\mathbb{C}^n$ and $\mathbb{C}^m$ are of dimensions $s \le r$ and $m - s$, respectively. We know that such X exists if and only if $A\mathcal{T} \oplus \mathcal{S} = \mathbb{C}^m$. For main properties please see [1, 4, 5, 18].

The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ [1, 5, 13] such that $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, $(A^\dagger A)^* = A^\dagger A$.

The Drazin inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix A^D [1, 5, 6] satisfying $A^DAA^D = A^D$, $AA^D = A^DA$, $A^DA^{k+1} = A^k$. In a particular case that $\mathrm{ind}(A) = 1$, the Drazin inverse becomes the group inverse $A^D = A^\#$.

The core-EP inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix $A^{\dagger\!\!\!\bigcirc}$ [14] such that $A^{\dagger\!\!\!\bigcirc}AA^{\dagger\!\!\!\bigcirc} = A^{\dagger\!\!\!\bigcirc}$ and $\mathcal{R}(A^{\dagger\!\!\!\bigcirc}) = \mathcal{R}((A^{\dagger\!\!\!\bigcirc})^*) = \mathcal{R}(A^k)$. It is known that [8], $A^{\dagger\!\!\!\bigcirc} = A(A^{\dagger\!\!\!\bigcirc})^2 = A^DA^k(A^k)^\dagger$.

2020 *Mathematics Subject Classification*. 15A09.
Keywords. weak group inverse, core-EP inverse, core-EP decomposition.
Received: 18 October 2023;Revised: 29 January 2024; Accepted: 20 February 2024
Communicated by Dragana Cvetković-Ilić
This research is supported by the Science and Technology Research Program of the Hanjiang Normal University under Grant No.2023A03, Outstanding Young and Middle-aged Science and Technology Innovation Team Project of Universities in Hubei Province under Grant No.T2022035.
* Corresponding author: Li Yuan
Email addresses: 275477236@qq.com (Yang Chen), yuanli813@126.com (Li Yuan), xiangyuz021@163.com (Xiangyu Zhang)

For square matrices with arbitrary index, the weak group inverse was defined in [16] as a generalization of the group inverse. Precisely, the weak group inverse of $A \in \mathbb{C}_k^{n\times n}$ is the unique matrix $A^{ⓦ} = X \in \mathbb{C}^{n\times n}$ satisfying the system of equations

$$AX^2 = X, \;\; AX = A^{\textcircled{\dagger}}A.$$

Notice that, by [16], $A^{ⓦ} = (A^{\textcircled{\dagger}})^2 A$. Some interesting properties of the weak group inverse were established in [7, 12, 16, 17, 19].

Inspired by recent research about weak group inverse, our aim is to present new characterizations and representations for the weak group inverse. In Section 2, new characterizations of weak group inverse are given by equations and subspaces. In Section 3, some representations for the weak group inverse are given based on the Bott-Duffin inverse and certain projections. Section 4 gives splitting method for computing the weak group inverse.

2. New characterizations of weak group inverse by equations and subspaces

We begin with several lemmas which will be used in later.

Lemma 2.1. [1] *Let $\mathcal{L}$ and $\mathcal{M}$ be complementary subspaces of $\mathbb{C}^n$, $P_{\mathcal{L},\mathcal{M}}$ be a projection onto $\mathcal{L}$ along $\mathcal{M}$ and $A \in \mathbb{C}^{n\times n}$. Then*
(a) $P_{\mathcal{L},\mathcal{M}}A = A$ if and only if $\mathcal{R}(A) \subseteq \mathcal{L}$;
(b) $AP_{\mathcal{L},\mathcal{M}} = A$ if and only if $\mathcal{M} \subseteq \mathcal{N}(A)$.

The Lemma 2.2 can be got by [12, Lemma 2.4].

Lemma 2.2. [12] *Let $A \in \mathbb{C}_k^{n\times n}$. Then*
(a) $A^{ⓦ} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)}$;
(b) $AA^{ⓦ} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$;
(c) $A^{ⓦ}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$.

Lemma 2.3. [12, Theorem 2.1] *Let $A \in \mathbb{C}_k^{n\times n}$ and $k \le l \in \mathbb{N}$. Then $A^{ⓦ} = A^l(A^{l+2})^\dagger A$.*

The core-EP decomposition of a square matrix was given in [15] and the corresponding formula of the weak group inverse was verified in [16].

Lemma 2.4. [15, 16] *Let $A \in \mathbb{C}_k^{n\times n}$ and $r(A^k) = t$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = A_1 + A_2 = U\begin{bmatrix} T & S \\ O & N \end{bmatrix}U^*, \tag{1}$$

$$A_1 = U\begin{bmatrix} T & S \\ O & O \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} O & O \\ O & N \end{bmatrix}U^*,$$

where N is nilpotent with index k, T is $t\times t$ invertible matrix. The representation is called the core-EP decomposition of A, while $A_1 = AA^{\textcircled{\dagger}}A$ and A_2 are the core part and nilpotent part of A, respectively. In addition,

$$A^{ⓦ} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ O & O \end{bmatrix}U^*. \tag{2}$$

In the following theorem, we will show that the condition $\mathcal{R}(X) = \mathcal{R}(A^k)$ in [19, Theorem 3.1 $(d)-(f)$] can be relaxed as the condition $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$.

Theorem 2.5. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $A = A_1 + A_2$ is the core-EP decomposition of A, where A_1 and A_2 are the core part and nilpotent part of the core-EP decomposition of A. Then the following are equivalent:*
(a) $X = A^{ⓦ}$;
(b) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $AX = A^{ⓣ}A$;
(c) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $A^2X = P_{\mathcal{R}(A^k)}A$;
(d) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $(A^k)^*A^2X = (A^k)^*A$;
(e) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $A_1X = A^{ⓣ}A$.

Proof. (a) $\Rightarrow$ (b). It can be obtained directly from Lemma 2.2 and the definition of the weak group inverse.
(b) $\Rightarrow$ (c). Follows by $AA^{ⓣ} = P_{\mathcal{R}(A^k)}$.
(c) $\Rightarrow$ (d). Pre-multiplying by $(A^k)^*$ on $A^2X = P_{\mathcal{R}(A^k)}A$ gives $(A^k)^*A^2X = (A^k)^*A$ by $(A^k)^*P_{\mathcal{R}(A^k)} = (A^k)^*$.
(d) $\Rightarrow$ (a). We have $r(X) \le r(A^k)$ by $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, then $r(A^k) = r((A^k)^*A) = r((A^k)^*A^2X) \le r(X) \le r(A^k)$ implies $r(X) = r(A^k)$. So $\mathcal{R}(X) = \mathcal{R}(A^k)$. The proof is finished by [19, Theorem 3.1 (f)].
(b) $\Rightarrow$ (e). Follows directly from $A^{ⓣ} = A(A^{ⓣ})^2$.
(e) $\Rightarrow$ (a). Firstly note that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$ implies $A^{ⓣ}AX = X$. Also, by $A_1X = A^{ⓣ}A$ we obtain $X = A^{ⓣ}AX = A^{ⓣ}AA^{ⓣ}AX = A^{ⓣ}A^{ⓣ}A = A^{ⓦ}$. $\square$

Using the result of Lemma 2.2 that $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(A^{ⓦ})$, we will give several different characterizations of the weak group inverse of a matrix A.

Theorem 2.6. *Let $A \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*
(a) $X = A^{ⓦ}$;
(b) $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$;
(c) $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $XAA^{ⓣ} = A^{ⓣ}$;
(d) $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $XA^D = (A^D)^2$;
(e) $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $XA^{ⓣ} = (A^{ⓣ})^2$.

Proof. (a) $\Rightarrow$ (b). By Lemma 2.3 and Lemma 2.1 (b), we have $XA^{k+1} = A^k(A^{k+2})^\dagger AA^{k+1} = A^k$. The condition $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$ holds by Lemma 2.2.
(b) $\Rightarrow$ (c). Multiplying $XA^{k+1} = A^k$ from the right side by $(A^{k+1})^\dagger$, we get $XAA^{ⓣ} = A^{ⓣ}$ by $A^{ⓣ} = A^k(A^{k+1})^\dagger$.
(c) $\Rightarrow$ (e). Post-multiplying by $A^{ⓣ}$ on $XAA^{ⓣ} = A^{ⓣ}$ gives $XA^{ⓣ} = (A^{ⓣ})^2$ by $A(A^{ⓣ})^2 = A^{ⓣ}$.
(a) $\Rightarrow$ (d). By Lemma 2.2, we have $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$ and $XA^D = XA(A^D)^2 = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}(A^D)^2 = (A^D)^2$ by Lemma 2.1 (a).
(d) $\Rightarrow$ (e). By $XA^D = (A^D)^2$ and $A^{ⓣ} = A^DA^k(A^k)^\dagger$, we get that $XA^{ⓣ} = (A^D)^2A^k(A^k)^\dagger = A^DA^{ⓣ} = A^DA(A^{ⓣ})^2 = P_{\mathcal{R}(A^k)}(A^{ⓣ})^2$ which by Lemma 2.1 (a) yields $XA^{ⓣ} = (A^{ⓣ})^2$.
(e) $\Rightarrow$ (a). By $XA^{ⓣ} = (A^{ⓣ})^2$, we get $\mathcal{R}(A^k) = \mathcal{R}((A^{ⓣ})^2) \subseteq \mathcal{R}(X)$ and $r(A^k) = r((A^{ⓣ})^2) = r(XA^{ⓣ}) \le r(X)$, and by $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, we have that $r(X) \le r((A^k)^*A)$ which together give $r(X) = r((A^k)^*A) = r(A^k)$. Then we have $\mathcal{N}((A^k)^*A) = \mathcal{N}(X)$ and $\mathcal{R}(X) = \mathcal{R}(A^k) = \mathcal{R}((A^{ⓣ})^2)$, which implies that $X = (A^{ⓣ})^2L$ for some $L \in \mathbb{C}^{n\times n}$. Post-multiplying on $XA^{ⓣ} = (A^{ⓣ})^2$ by L, we obtain $XAX = X$. Hence, we have $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)} = A^{ⓦ}$ by Lemma 2.2 (a). $\square$

The conditions $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}((A^k)^*A) = \mathcal{N}(X)$ in [19, Theorem 3.2 (b)] can be relaxed. Therefore, we have the following Theorem 2.7.

Theorem 2.7. *Let $A \in \mathbb{C}_k^{n\times n}$ and$X \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*
(a) $X = A^{ⓦ}$;
(b) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $\mathcal{N}(X) \subseteq \mathcal{N}((A^k)^*A)$, $XAX = X$;
(c) $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$, $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $XAX = X$;
(d) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $AXA = A^{ⓣ}A^2$;
(e) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $\mathcal{N}((A^k)^*A) \subseteq \mathcal{N}(X)$, $AXA^{ⓣ} = A^{ⓣ}$.

Proof. That (a) implies all other items $(b)-(e)$ can be checked directly using the definition of the weak group inverse and Lemma 2.2.

$(b) \Rightarrow (a)$. By $XAX = X$, we have $X(AX - I_n) = O$ and by $\mathcal{N}(X) \subseteq \mathcal{N}((A^k)^*A)$, we have $(A^k)^*A(AX - I_n) = O$, that is $(A^k)^*A^2X = (A^k)^*A$. Hence, by Theorem 2.5 $(d) \Rightarrow (a)$ it follows that $X = A^{\textcircled{w}}$.

$(c) \Rightarrow (a)$. By $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$, we have $A^k = XL$ for some $L \in \mathbb{C}^{n\times n}$. Then $A^k = XL = XAXL = XA^{k+1}$. Thus $X = A^{\textcircled{w}}$ by Theorem 2.6 $(b) \Rightarrow (a)$.

$(d) \Rightarrow (e)$. From $A^{\textcircled{\dagger}} = A(A^{\textcircled{\dagger}})^2$ and $AXA = A^{\textcircled{\dagger}}A^2$, we obtain $AXA^{\textcircled{\dagger}} = AXA(A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}A^2(A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.

$(e) \Rightarrow (a)$. Since $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, we get $A^{\textcircled{\dagger}}AX = X$. Pre-multiplying on $AXA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ by $A^{\textcircled{\dagger}}$, we get $XA^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^2$. Hence, $X = A^{\textcircled{w}}$ by Theorem 2.6 $(e) \Rightarrow (a)$. □

In the following theorems, we present some necessary and sufficient conditions for a matrix X to be $A^{\textcircled{w}}$ based on the conditions $A_1X^2 = X$, $XAX = X$ and $XA_1X = X$, where A_1 is the core part of the core-EP decomposition of A.

Theorem 2.8. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $A = A_1 + A_2$ is the core-EP decomposition of A, where A_1 and A_2 are the core part and nilpotent part of the core-EP decomposition of A. Then the following are equivalent:*
(a) $X = A^{\textcircled{w}}$;
(b) $A_1X^2 = X$, $AX = A^{\textcircled{\dagger}}A$;
(c) $A_1X^2 = X$, $A_1X = A^{\textcircled{\dagger}}A$;
(d) $A_1X^2 = X$, $A_1^2X = A_1$;
(e) $A_1X^2 = X$, $A_1X = A_1^{\#}A_1$;
(f) $AX^2 = X$, $(A^k)^*A^2X = (A^k)^*A$.

Proof. $(a) \Rightarrow (b)$. By the definition of the weak group inverse, we have

$$A_1X^2 = AA^{\textcircled{\dagger}}AXX = AA^{\textcircled{\dagger}}X = AA^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = X.$$

$(b) \Rightarrow (c)$. From $AX = A^{\textcircled{\dagger}}A$ and $A(A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}$, we get $A_1X = AA^{\textcircled{\dagger}}AX = AA^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$.

$(c) \Rightarrow (d)$. It follows by $A_1^2X = A_1(A_1X) = A_1A^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = A_1$.

$(d) \Rightarrow (a)$. By $A_1X^2 = X$, we get $\mathcal{R}(X) \subseteq \mathcal{R}(A_1) \subseteq \mathcal{R}(AA^{\textcircled{\dagger}}) = \mathcal{R}(A^k)$. Pre-multiplying by $A^{\textcircled{\dagger}}$ on $A_1^2X = A_1$ gives $A^{\textcircled{\dagger}}AA_1X = A^{\textcircled{\dagger}}A$. Hence, we get $A_1X = A^{\textcircled{\dagger}}A$ by Lemma 2.1 (a). The rest follows by Theorem 2.5 $(e) \Rightarrow (a)$.

$(a) \Rightarrow (e)$. From [16, Theorem 3.2], we know $A^{\textcircled{w}} = A_1^{\#}$. Hence, it is obvious that $A_1X = A_1A_1^{\#} = A_1^{\#}A_1$.

$(e) \Rightarrow (a)$. Consequently by $X = A_1X^2 = (A_1X)X = A_1^{\#}(A_1X) = A_1^{\#}A_1^{\#}A_1 = A_1^{\#} = A^{\textcircled{w}}$.

$(a) \Rightarrow (f)$. This implication is obvious.

$(f) \Rightarrow (a)$. Since $AX^2 = X$ implies that $X = A^kX^{k+1}$ it follows that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Hence, $X = A^{\textcircled{w}}$ by Theorem 2.5 $(d) \Rightarrow (a)$. □

Theorem 2.9. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $A = A_1 + A_2$ is the core-EP decomposition of A, where A_1 and A_2 are the core part and nilpotent part of the core-EP decomposition of A. Then the following are equivalent:*
(a) $X = A^{\textcircled{w}}$;
(b) $XAX = X$, $XA^{k+1} = A^k$, $AX = A^{\textcircled{\dagger}}A$;
(c) $XAX = X$, $XA = (A^{\textcircled{\dagger}})^2A^2$, $(A^k)^*A^2X = (A^k)^*A$;
(d) $XA_1X = X$, $AX = A^{\textcircled{\dagger}}A$, $XA_1 = A^{\textcircled{\dagger}}A$;
(e) $XA_1X = X$, $A_1X = A^{\textcircled{\dagger}}A$, $XA_1 = A^{\textcircled{\dagger}}A$;
(f) $XA_1X = X$, $A_1X = A^{\textcircled{\dagger}}A$, $XAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$;
(g) $XA_1X = X$, $A_1X = A^{\textcircled{\dagger}}A$, $XA^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^2$;
(h) $XA_1X = X$, $A_1X = A^{\textcircled{\dagger}}A$, $XA_1^2 = A_1$;
(i) $XA_1X = X$, $A_1X = XA_1$, $XA_1^2 = A_1$;
(j) $XA_1X = X$, $A_1X = XA_1$, $A_1XA_1 = A_1$.

Proof. $(a) \Rightarrow (b), (c)$. It is obvious by the definition of the weak group inverse and Lemma 2.2.

$(b) \Rightarrow (a)$. By $XA^{k+1} = A^k$, we get $\mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{\dagger}}) \subseteq \mathcal{R}(XA)$. By the conditions $XAX = X$, $AX = A^{\textcircled{\dagger}}A$ and Lemma 2.1 (a), we have

$$X = X(AX) = XA^{\textcircled{\dagger}}A = XAA^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = P_{\mathcal{R}(XA),\mathcal{N}(XA)}A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{w}}.$$

$(c) \Rightarrow (a)$. Since $(A^k)^*A^2X = (A^k)^*A$, we get $\mathcal{N}(AX) \subseteq \mathcal{N}((A^k)^*A) = \mathcal{N}(A^{\textcircled{\dagger}}A)$. Then

$$X = (XA)X = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}AAX = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}AP_{\mathcal{R}(AX),\mathcal{N}(AX)} = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{w}}.$$

Notice that we use Lemma 2.1 (b) in the previous equality.

$(a) \Rightarrow (d)$. Using Lemmas 2.1 and 2.2, we verify that $XA_1 = XAA^{\textcircled{\dagger}}A = P_{\mathcal{R}(XA),\mathcal{N}(XA)}A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$ and $XA_1X = A^{\textcircled{\dagger}}AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}X = X$.

$(d) \Rightarrow (e)$. This follows similarly as Theorem 2.8 $(b) \Rightarrow (c)$.

$(e) \Rightarrow (f)$. Post-multiplying by $A^{\textcircled{\dagger}}$ on $XA_1 = A^{\textcircled{\dagger}}A$ gives $XAA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ by $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.

$(f) \Rightarrow (g)$. It follows by $A(A^{\textcircled{\dagger}})^2 = A^{\textcircled{\dagger}}$.

$(g) \Rightarrow (a)$. Notice that $X = X(A_1X) = XA^{\textcircled{\dagger}}A = (A^{\textcircled{\dagger}})^2A = A^{\textcircled{w}}$.

$(e) \Rightarrow (h)$. By $XA_1 = A^{\textcircled{\dagger}}A$ and Lemma 2.1 (a), we have $XA_1^2 = (XA_1)A_1 = A^{\textcircled{\dagger}}AA_1 = A_1$.

$(h) \Rightarrow (i)$. This implication is clear by $A_1X = XA_1(A_1X) = XA_1A^{\textcircled{\dagger}}A = XAA^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A = XA_1$.

$(i) \Rightarrow (j)$. Consequently by $A_1XA_1 = XA_1(A_1X)A_1 = (XA_1X)A_1A_1 = XA_1A_1 = A_1$.

$(j) \Rightarrow (a)$. By $XA_1X = X$ and $A_1X = XA_1$, we have $X = A_1X^2$. From $A_1XA_1 = A_1$ and $A_1X = XA_1$, we get $A_1 = A_1^2X$. Hence, $X = A^{\textcircled{w}}$ by Theorem 2.8 $(d) \Rightarrow (a)$. □

3. Representations of the weak group inverse

Using Lemma 2.2 (a) and the representation of $A^{(2)}_{\mathcal{T},\mathcal{S}}$ inverse from [18, Theorem 2.1], we get new representations for the weak group inverse.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}_k$. Then*

$$A^{\textcircled{w}} = A^k(A^k)^*A(A^{k+1}(A^k)^*A)^{\#} = (A^k(A^k)^*A^2)^{\#}A^k(A^k)^*A.$$

Mary [11] introduced the inverse along an element, the Lemma 2.2 (a) shows that the weak group is the inverse along $A^k(A^k)^*A$. Thus, the Theorem 3.1 also can be got by [11, Theorem 7].

Bott and Duffin [2] defined the Bott-Duffin inverse of $A \in \mathbb{C}^{n\times n}$ by $A^{(-1)}_{\mathcal{L}} = P_{\mathcal{L}}(AP_{\mathcal{L}} + I_n - P_{\mathcal{L}})^{-1}$ when $AP_{\mathcal{L}} + I_n - P_{\mathcal{L}}$ is nonsingular, where $\mathcal{L}$ is a subspace of $\mathbb{C}^n$. In [19], the authors showed the weak group inverse by a special Bott-Duffin inverse. Inspired by that, the following expressions for weak group inverse are given using different Bott-Duffin inverse.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}_k$. Then*

$$\begin{aligned} A^{\textcircled{w}} &= (A^*P_{\mathcal{R}(A^*A^k)}A)^{(-1)}_{\mathcal{R}(A^k)}A^*P_{\mathcal{R}(A^*A^k)} \\ &= P_{\mathcal{R}(A^k)}(A^*P_{\mathcal{R}(A^*A^k)}AP_{\mathcal{R}(A^k)} + I_n - P_{\mathcal{R}(A^k)})^{-1}A^*P_{\mathcal{R}(A^*A^k)}. \end{aligned}$$

Proof. Assume that A is given by (1) and $\triangle = T^k(T^k)^* + \widetilde{T}(\widetilde{T})^*$, $L = TT^* + SS^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$.

By (1), we get $A^k = U\begin{bmatrix} T^k & \widetilde{T} \\ O & O \end{bmatrix}U^*$, then $P_{\mathcal{R}(A^k)} = A^k(A^k)^{\dagger} = U\begin{bmatrix} I_t & O \\ O & O \end{bmatrix}U^*$ and

$$A^*A^k = U\begin{bmatrix} T^*T^k & T^*\widetilde{T} \\ S^*T^k & S^*\widetilde{T} \end{bmatrix}U^*. \tag{3}$$

Applying [4, Ch.3 Corollary 2.3] to (3), we get

$$(A^*A^k)^\dagger = U\begin{bmatrix} (T^k)^*\triangle^{-1}L^{-1}T & (T^k)^*\triangle^{-1}L^{-1}S \\ (\widetilde{T})^*\triangle^{-1}L^{-1}T & (\widetilde{T})^*\triangle^{-1}L^{-1}S \end{bmatrix}U^*,$$

which yields

$$P_{\mathcal{R}(A^*A^k)} = U\begin{bmatrix} T^*L^{-1}T & T^*L^{-1}S \\ S^*L^{-1}T & S^*L^{-1}S \end{bmatrix}U^*. \tag{4}$$

Let $M = P_{\mathcal{R}(A^k)}(A^*P_{\mathcal{R}(A^*A^k)}AP_{\mathcal{R}(A^k)} + I_n - P_{\mathcal{R}(A^k)})^{-1}A^*P_{\mathcal{R}(A^*A^k)}$. A straightforward calculation gives that

$$\begin{aligned} M &= U\begin{bmatrix} I_t & O \\ O & O \end{bmatrix}\begin{bmatrix} (T^*)^2L^{-1}T^2 & O \\ (S^*T^* + N^*S^*)L^{-1}T^2 & I_{n-t} \end{bmatrix}^{-1}\begin{bmatrix} (T^*)^2L^{-1}T & (T^*)^2L^{-1}S \\ (S^*T^* + N^*S^*)L^{-1}T & (S^*T^* + N^*S^*)L^{-1}S \end{bmatrix}U^* \\ &= U\begin{bmatrix} I_t & O \\ O & O \end{bmatrix}\begin{bmatrix} T^{-2}L(T^*)^{-2} & O \\ -(S^*T^* + N^*S^*)(T^*)^{-2} & I_{n-t} \end{bmatrix}\begin{bmatrix} (T^*)^2L^{-1}T & (T^*)^2L^{-1}S \\ (S^*T^* + N^*S^*)L^{-1}T & (S^*T^* + N^*S^*)L^{-1}S \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}S \\ O & O \end{bmatrix}U^* \\ &= A^{Ⓦ}. \end{aligned}$$

□

Theorem 3.3. *Let $A \in \mathbb{C}_k^{n\times n}$. Then $A^{Ⓦ} = P_{\mathcal{R}(A^k)}A^*(AP_{\mathcal{R}(A^k)}A^* + I_n - P_{\mathcal{R}(A^*A^k)})^{-1}$.*

Proof. Assume that A is given by (1) and $L = TT^* + SS^*$, $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$. By (1) and (4), we get

$$\begin{aligned} M_1 &= P_{\mathcal{R}(A^k)}A^*(AP_{\mathcal{R}(A^k)}A^* + I_n - P_{\mathcal{R}(A^*A^k)})^{-1} \\ &= U\begin{bmatrix} T^* & O \\ O & O \end{bmatrix}U^*U\begin{bmatrix} TT^* + I_t - T^*L^{-1}T & -T^*L^{-1}S \\ -S^*L^{-1}T & I_{n-t} - S^*L^{-1}S \end{bmatrix}^{-1}U^* \\ &= U\begin{bmatrix} T^* & O \\ O & O \end{bmatrix}\begin{bmatrix} (T^*)^{-1}T^{-1} & (T^*)^{-1}T^{-2}S \\ S^*(T^*)^{-2}T^{-1} & I_n + S^*((T^*)^{-1}T^{-1} + (T^*)^{-2}T^{-2})S \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}S \\ O & O \end{bmatrix}U^* \\ &= A^{Ⓦ}. \end{aligned}$$

□

Example 3.4. *Let*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -a & 0 & 1 \\ a & 0 & 0 \end{bmatrix}$$

with $\operatorname{ind}(A) = 2$, *where a is a real number. By Lemma* 2.3, *the weak group inverse of A is given by*

$$A^{Ⓦ} = A^2(A^4)^\dagger A = \begin{bmatrix} \frac{3a^2+8}{5a^2+16} & 0 & \frac{-a}{5a^2+16} \\ \frac{-3a^3-8a}{20a^2+64} & 0 & \frac{-a}{5a^2+16} \\ \frac{3a^3+8a}{10a^2+32} & 0 & \frac{-a^2}{10a^2+32} \end{bmatrix}.$$

By calculation, we get that

$$P_{\mathcal{R}(A^2)} = \begin{bmatrix} \frac{16}{5a^2+16} & \frac{-4a}{5a^2+16} & \frac{8a}{5a^2+16} \\ \frac{-4a}{5a^2+16} & \frac{a^2}{5a^2+16} & \frac{-2a^2}{5a^2+16} \\ \frac{8a}{5a^2+16} & \frac{-2a^2}{5a^2+16} & \frac{4a^2}{5a^2+16} \end{bmatrix}, \ P_{\mathcal{R}(A^*A^2)} = \begin{bmatrix} \frac{9a^4+48a^2+64}{9a^4+49a^2+64} & 0 & \frac{-3a^3-8a}{9a^4+49a^2+64} \\ 0 & 0 & 0 \\ \frac{-3a^3-8a}{9a^4+49a^2+64} & 0 & \frac{a^2}{9a^4+49a^2+64} \end{bmatrix},$$

$$(A^*P_{\mathcal{R}(A^*A^2)}AP_{\mathcal{R}(A^2)} + I_3 - P_{\mathcal{R}(A^2)})^{-1} = \begin{bmatrix} \frac{9a^4+48a^2+64}{25a^4+160a^2+256} & \frac{16a^5+111a^3+192a}{100a^4+640a^2+1024} & \frac{-16a^5-111a^3-192a}{50a^4+320a^2+512} \\ \frac{-9a^5-49a^3-64a}{100a^4+640a^2+1024} & \frac{9a^6+449a^4+2624a^2+4096}{400a^4+2560a^2+4096} & \frac{-9a^6-49a^4-64a^2}{200a^4+1280a^2+2048} \\ \frac{9a^5+49a^3+64a}{50a^4+320a^2+512} & \frac{-9a^6-49a^4-64a^2}{200a^4+1280a^2+2048} & \frac{9a^6+149a^4+704a^2+1024}{100a^4+640a^2+1024} \end{bmatrix},$$

$$(AP_{\mathcal{R}(A^2)}A^* + I_3 - P_{\mathcal{R}(A^*A^2)})^{-1} = \begin{bmatrix} \frac{45a^4+128a^2+64}{80a^2+256} & \frac{3a^3+8a}{10a^2+32} & \frac{-63a^3-136a}{80a^2+256} \\ \frac{3a^3+8a}{10a^2+32} & 1 & \frac{-a^2}{10a^2+32} \\ \frac{-63a^3-136a}{80a^2+256} & \frac{-a^2}{10a^2+32} & \frac{117a^2+256}{80a^2+256} \end{bmatrix}.$$

Then it can be verified that $(A^*P_{\mathcal{R}(A^*A^2)}A)^{(-1)}_{\mathcal{R}(A^2)}A^*P_{\mathcal{R}(A^*A^2)} = A^{ⓦ}$, $P_{\mathcal{R}(A^2)}A^*(AP_{\mathcal{R}(A^2)}A^* + I_3 - P_{\mathcal{R}(A^*A^2)})^{-1} = A^{ⓦ}$.

The following theorem provides new formulae for the weak group inverse $A^{ⓦ}$ based on projections $X = P_{\mathcal{N}((A^k)^*A^2),\mathcal{R}(A^k)}$ and $Y = P_{\mathcal{N}((A^k)^*A),\mathcal{R}(A^k)}$.

Theorem 3.5. *Let* $A \in \mathbb{C}^{n\times n}_k$, $X = P_{\mathcal{N}((A^k)^*A^2),\mathcal{R}(A^k)}$ *and* $Y = P_{\mathcal{N}((A^k)^*A),\mathcal{R}(A^k)}$. *Then for any* $a, b \in \mathbb{C} \setminus \{0\}$, *we have*

$$\begin{aligned} A^{ⓦ} &= (A^k(A^k)^*A^2 + aX)^{-1}A^k(A^k)^*A(I_n - Y) \\ &= (I_n - X)A^k(A^k)^*A(A^{k+1}(A^k)^*A + bY)^{-1}. \end{aligned}$$

Proof. By Lemma 2.1 and Lemma 2.2, it is not difficult to conclude that

$$(A^k(A^k)^*A^2 + aX)A^{ⓦ} = A^k(A^k)^*A(I_n - Y).$$

Now we only need to show the invertibility of $A^k(A^k)^*A^2 + aX$. Let $(A^k(A^k)^*A^2 + aX)\xi = 0$ for some $\xi \in \mathbb{C}^n$. Then $A^k(A^k)^*A^2\xi = -aX\xi$. By Lemma 2.2, we have

$$A^k(A^k)^*A^2\xi = -aX\xi \in \mathcal{R}(A^k(A^k)^*A^2) \cap \mathcal{R}(X) = \mathcal{R}(A^k(A^k)^*A^2) \cap \mathcal{N}((A^k)^*A^2) \subseteq \mathcal{R}(A^k) \cap \mathcal{N}((A^k)^*A^2) = \{0\},$$

which gives $A^k(A^k)^*A^2\xi = -aX\xi = 0$. Hence,

$$\xi \in \mathcal{N}(A^k(A^k)^*A^2) \cap \mathcal{N}(X) = \mathcal{N}(A^k(A^k)^*A^2) \cap \mathcal{R}(A^k) \subseteq \mathcal{N}((A^k)^*A^2) \cap \mathcal{R}(A^k) = \{0\}.$$

Thus, $\xi = 0$ and $A^k(A^k)^*A^2 + aX$ is invertible.

Analogously, it can be verified that $A^{k+1}(A^k)^*A+bY$ is nonsingular and $A^{ⓦ} = (I_n-X)A^k(A^k)^*A(A^{k+1}(A^k)^*A+bY)^{-1}$. □

Example 3.6. *In order to illustrate the representations of Theorem* 3.5, *let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -i & 0 & i \\ 2 & 0 & 0 \end{bmatrix}$$

with $\mathrm{ind}(A) = 2$, $a = -\frac{1}{5}$ *and* $b = 2i$, *where* i *stands for the imaginary unit. According to Lemma* 2.3, *exact calculation in Mathematica gives*

$$A^{ⓦ} = A^2(A^4)^\dagger A = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{6} \\ \frac{2}{3}i & 0 & \frac{1}{6}i \\ \frac{4}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Simple calculation gives

$$X = P_{\mathcal{N}((A^2)^*A^2),\mathcal{R}(A^2)} = \begin{bmatrix} 0 & 0 & 0 \\ -i & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \ Y = P_{\mathcal{N}((A^2)^*A),\, \mathcal{R}(A^2)} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{2}{3} & 1 & -\frac{1}{6}i \\ -\frac{4}{3} & 0 & \frac{2}{3} \end{bmatrix},$$

$$A^2(A^2)^*A = \begin{bmatrix} 4 & 0 & 1 \\ 4i & 0 & i \\ 8 & 0 & 2 \end{bmatrix}, \ (A^2(A^2)^*A^2 - \frac{1}{5}X)^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{31}{6}i & -5 & 0 \\ \frac{31}{3} & 0 & -5 \end{bmatrix},$$

$$(A^3A^2(A^2)^* + 2iY)^{-1} = \begin{bmatrix} \frac{1}{9} - \frac{1}{6}i & 0 & \frac{1}{36} + \frac{1}{12}i \\ -\frac{1}{3} + \frac{1}{9}i & -\frac{1}{2}i & -\frac{1}{12} + \frac{1}{12}i \\ \frac{2}{9} + \frac{2}{3}i & 0 & \frac{1}{18} - \frac{1}{3}i \end{bmatrix}.$$

Further, it can be verified that Theorem 3.5 *is valid in this example.*

Some representations for generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ of matrices were given in [3]. For the weak group inverse we have the following results.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}_k$, $a, b, c, d \in \mathbb{C} \setminus \{0\}$. Assume that F and E^* are full column rank matrices, which satisfy $\mathcal{R}(A^k) = \mathcal{R}(F)$ and $\mathcal{N}((A^k)^*A) = \mathcal{N}(E)$. Then*

$$A^{\textcircled{w}} = b(aP_{\mathcal{N}((A^k)^*)} + bFEA)^{-1}FE \tag{5}$$

$$= dFE(cP_{\mathcal{N}((A^k)^*A)} + dAFE)^{-1}. \tag{6}$$

Proof. In order to show that $aP_{\mathcal{N}((A^k)^*)} + bFEA$ is nonsingular, let $(aP_{\mathcal{N}((A^k)^*)} + bFEA)x = 0$ for some $x \in \mathbb{C}^n$. Then $aP_{\mathcal{N}((A^k)^*)}x = -bFEAx$, we have $-FEAx \in \mathcal{R}(FEA) \subseteq \mathcal{R}(F) = \mathcal{R}(A^k)$ and $aP_{\mathcal{N}((A^k)^*)}x \in \mathcal{N}((A^k)^*)$, i.e.,

$$aP_{\mathcal{N}((A^k)^*)}x = -bFEAx \in \mathcal{N}((A^k)^*) \cap \mathcal{R}(A^k) = \{0\}.$$

Hence $P_{\mathcal{N}((A^k)^*)}x = 0$ and $FEAx = 0$. It follows that $x \in \mathcal{N}((A^k)^*)^\perp = \mathcal{R}(A^k)$. Since F is full column rank matrix, we get $EAx = 0$, which gives

$$x \in \mathcal{N}(EA) = \mathcal{R}(A^*E^*)^\perp = (A^*\mathcal{R}(E^*))^\perp = \mathcal{R}(A^*A^*A^k)^\perp = \mathcal{N}((A^k)^*A^2).$$

Hence $x \in \mathcal{R}(A^k) \cap \mathcal{N}((A^k)^*A^2) = \{0\}$, so $x = 0$ and $aP_{\mathcal{N}((A^k)^*)} + bFEA$ is nonsingular. By Lemma 2.1(b) and Lemma 2.2, we obtain $P_{\mathcal{N}((A^k)^*)}A^{\textcircled{w}} = O$ and $EAA^{\textcircled{w}} = E$ which together give (5).

Similarly, (6) can be verified. □

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}_k$, $a, b, c, d \in \mathbb{C} \setminus \{0\}$. Suppose that B and C^* are full column rank matrices which satisfy $\mathcal{N}((A^k)^*A) = \mathcal{R}(B)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$. Let $E_B = I_n - BB^\dagger$, $F_C = I_n - C^\dagger C$. Then,*

$$A^{\textcircled{w}} = a(aA^*E_BA + bC^*C)^{-1}A^*E_B \tag{7}$$

$$= cF_CA^*(cAF_CA^* + dBB^*)^{-1}. \tag{8}$$

Proof. We show that $aA^*E_BA + bC^*C$ is nonsingular. Assume that $(aA^*E_BA + bC^*C)x = 0$ for some $x \in \mathbb{C}^n$. Then, we have $bC^*Cx = -aA^*E_BAx$,

$$x \in \mathcal{R}(C^*C) \cap \mathcal{R}(A^*E_BA) = \mathcal{R}(C^*) \cap \mathcal{R}(A^*E_B) = \mathcal{R}(A^k)^\perp \cap \mathcal{N}((A^k)^*A^2)^\perp = \{0\},$$

which implies $C^*Cx = 0$ and $A^*E_BAx = 0$. Hence $Cx = 0$, $E_BAx = 0$ yield

$$x \in \mathcal{N}(C) \cap \mathcal{N}(E_BA) = \mathcal{R}(A^k) \cap \mathcal{N}((A^k)^*A^2) = \{0\}.$$

Thus $x = 0$ and $aA^*E_BA + bC^*C$ is nonsingular. Hence, since $\mathcal{R}(A^{\textcircled{w}}) = \mathcal{R}(A^k) = \mathcal{N}(C)$, we get $CA^{\textcircled{w}} = O$. By $\mathcal{N}(E_B) = \mathcal{R}(B) = \mathcal{N}((A^k)^*A)$, Lemmas 2.1 and 2.2, we obtain $E_BAA^{\textcircled{w}} = E_B$. Therefore, $A^{\textcircled{w}} = a(aA^*E_BA + bC^*C)^{-1}A^*E_B$.

Similarly, (8) can be verified. □

As we know, $A^{\textcircled{w}}$ is an outer inverse of A with rang $\mathcal{R}(A^k)$ and null space $\mathcal{N}((A^k)^*A)$. The results of Theorems 2.2 and 2.4 in [3] are applicable to the weak group inverse.

Corollary 3.9. *Let $A \in \mathbb{C}_k^{n\times n}$. Let B and C^* be of full column rank matrices and satisfy*

$$\mathcal{N}((A^k)^*A) = \mathcal{R}(B), \quad \mathcal{R}(A^k) = \mathcal{N}(C).$$

Let $E_B = I_n - BB^\dagger$, $F_C = I_n - C^\dagger C$. Then,

$$\begin{bmatrix} A^{\textcircled{w}} \\ O \end{bmatrix} = \begin{bmatrix} A^*E_BA & C^* \\ C & O \end{bmatrix}^{-1} \begin{bmatrix} A^*E_B \\ O \end{bmatrix},$$

$$\begin{bmatrix} A^{\textcircled{w}} & O \end{bmatrix} = \begin{bmatrix} F_CA^* & O \end{bmatrix} \begin{bmatrix} AF_CA^* & B \\ B^* & O \end{bmatrix}^{-1}.$$

Corollary 3.10. *Let $A \in \mathbb{C}_k^{n\times n}$.*

(*a*) *Let E and C be of full row rank matrices and satisfy $\mathcal{N}((A^k)^*A) = \mathcal{N}(E)$, $\mathcal{R}(A^k) = \mathcal{N}(C)$. Then*

$$A^{\textcircled{w}} = \begin{bmatrix} EA \\ C \end{bmatrix}^{-1} \begin{bmatrix} E \\ O \end{bmatrix}; \tag{9}$$

(*b*) *Let F and B be of full column rank matrices and satisfy $\mathcal{N}((A^k)^*A) = \mathcal{R}(B)$, $\mathcal{R}(A^k) = \mathcal{R}(F)$. Then*

$$A^{\textcircled{w}} = \begin{bmatrix} F & O \end{bmatrix} \begin{bmatrix} AF & B \end{bmatrix}^{-1}. \tag{10}$$

Example 3.11. *Let*

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\mathrm{ind}(A) = 2$. *Using Lemma* 2.3, *the weak group inverse of A is given by*

$$A^{\textcircled{w}} = A^2(A^4)^\dagger A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

Let

$$B = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let $a = \frac{1}{5}$, $b = 3 + 2i$, $c = -3i$ and $d = 4$.

In order to verify the representations (5) *and* (6), *it is necessary to compute*

$$(\frac{1}{5}P_{\mathcal{N}((A^2)^*)} + (3+2i)FEA)^{-1} = \begin{bmatrix} \frac{3}{52} - \frac{1}{26}i & 0 & -\frac{5}{2} & -\frac{25}{4} \\ 0 & \frac{3}{26} - \frac{1}{13}i & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

$$(-3iP_{\mathcal{N}((A^2)^*A)} + 4AFE)^{-1} = \begin{bmatrix} \frac{1}{24} & 0 & \frac{1}{48} - \frac{1}{6}i & \frac{1}{48} - \frac{1}{6}i \\ 0 & -\frac{1}{8} & 0 & 0 \\ \frac{1}{48} & 0 & \frac{1}{96} + \frac{1}{3}i & \frac{1}{96} \\ \frac{1}{48} & 0 & \frac{1}{96} & \frac{1}{96} + \frac{1}{3}i \end{bmatrix}.$$

Further calculation gives $(3+2i)(\frac{1}{5}P_{\mathcal{N}((A^2)^*)}+(3+2i)FEA)^{-1}FE = A^{ⓦ}$ *and* $4FE(-3iP_{\mathcal{N}((A^2)^*A)}+4AFE)^{-1} = A^{ⓦ}$.

According to the representations (7) *and* (8), *it is required to compute*

$$E_B = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \ F_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(\frac{1}{5}A^*E_BA+(3+2i)C^*C)^{-1} = \begin{bmatrix} \frac{477}{208}-\frac{29}{104}i & 0 & -\frac{3}{26}+\frac{1}{13}i & -\frac{15}{52}+\frac{5}{26}i \\ 0 & \frac{5}{4} & 0 & 0 \\ -\frac{3}{26}+\frac{1}{13}i & 0 & \frac{3}{13}-\frac{2}{13}i & 0 \\ -\frac{15}{52}+\frac{5}{26}i & 0 & 0 & \frac{3}{13}-\frac{2}{13}i \end{bmatrix},$$

$$(-3iAF_CA^*+4BB^*)^{-1} = \begin{bmatrix} \frac{1}{12}i & 0 & \frac{1}{24}i & \frac{1}{24}i \\ 0 & \frac{1}{12}i & 0 & 0 \\ \frac{1}{24}i & 0 & \frac{1}{16}+\frac{1}{48}i & \frac{1}{48}i \\ \frac{1}{24}i & 0 & \frac{1}{48}i & \frac{1}{16}+\frac{1}{48}i \end{bmatrix}.$$

It can be verified that both expressions $\frac{1}{5}(\frac{1}{5}A^*E_BA+(3+2i)C^*C)^{-1}A^*E_B$ *and* $-3iF_CA^*(-3iAF_CA^*+4BB^*)^{-1}$ *are equal to* $A^{ⓦ}$ *as* (11).

On the other hand, according to the representations (9) *and* (10), *we compute*

$$\begin{bmatrix} EA \\ C \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{5}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \begin{bmatrix} AF & B \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Simple calculation verifies that the identities in (9) *and* (10) *coincide with* $A^{ⓦ}$.

4. Splitting method for computing the weak group inverse

Many characterizations of several generalized inverses were investigated in terms of splitting methods [9, 10, 12]. Corresponding splitting method for finding the weak group inverse is verified in this section.

Theorem 4.1. *Let* $A \in \mathbb{C}^{n\times n}_k$. *Suppose that* $A^k(A^k)^*A^2 = H-K$, $\mathcal{R}(A^k) = \mathcal{R}(H)$ *and* $\mathcal{N}((A^k)^*A) = \mathcal{N}(H)$. *Then*
(*a*) $H^\#$ *exists;*
(*b*) $I_n - H^\#K$ *is invertible;*
(*c*) $A^{ⓦ} = (I_n - H^\#K)^{-1}H^\#A^k(A^k)^*A$.

Proof. (*a*). Notice that $\mathrm{ind}(H) = 1$ by $\mathbb{C}^n = \mathcal{R}(H)\oplus\mathcal{N}(H) = \mathcal{R}(A^k)\oplus\mathcal{N}((A^k)^*A)$.

(*b*). In order to check that $I_n - H^\#K$ is nonsingular, let $(I_n - H^\#K)x = 0$ for some $x \in \mathbb{C}^n$. Then

$$x = H^\#Kx \in \mathcal{R}(H^\#) = \mathcal{R}(H) = \mathcal{R}(A^k)$$

and

$$x = H^\#Kx = H^\#(H - A^k(A^k)^*A^2)x = H^\#Hx - H^\#A^k(A^k)^*A^2x = x - H^\#A^k(A^k)^*A^2x.$$

Hence, we get $H^\#A^k(A^k)^*A^2x = 0$. Pre-multiply both sides by H and apply Lemma 2.1 to yield $A^k(A^k)^*A^2x = 0$. Thus, $x \in \mathcal{R}(A^k)\cap\mathcal{N}(A^k(A^k)^*A^2) = \{0\}$. Therefore, $I_n - H^\#K$ is invertible.

(*c*). By Lemma 2.1, we have

$$(I_n - H^\#K)A^{ⓦ} = A^{ⓦ} - H^\#HA^{ⓦ} + H^\#A^k(A^k)^*A^2A^{ⓦ} = H^\#A^k(A^k)^*A.$$

□

Example 4.2. *Let A and $A^{ⓦ}$ as in Example 3.11. To verify Theorem 4.1, let*

$$H = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We calculate the matrices

$$H^{\#} = H(H^3)^{\dagger}H = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K = \begin{bmatrix} -179 & 0 & -\frac{179}{2} & -\frac{449}{2} \\ 0 & -63 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ (I_4 - H^{\#}K)^{-1} = \begin{bmatrix} \frac{1}{180} & 0 & -\frac{179}{360} & -\frac{449}{360} \\ 0 & \frac{1}{64} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Further verification confirms that the expression $(I_4 - H^{\#}K)^{-1}H^{\#}A^2(A^2)^*A$ coincides with $A^{ⓦ}$.

5. Conclusion

The aim of this paper is provide characterizations and representations of the weak group inverse. Several different characterizations of the weak group inverse are presented based on its range and null space as well some algebraic ones. Representations using the Bott-Duffin inverse and projectors are given for the weak group inverse. Splitting method for calculating the weak group inverse is obtained. Some numerical examples are provided to illustrate the results obtained.

We believe that investigation related to the weak group inverse will attract attention, and we describe perspectives for further research:

(1) The reverse order law of the weak group inverse.

(2) Extending the weak group inverse inverse to finite potent endomorphisms on arbitrary vector spaces.

References

[1] A.Ben-Israel, T.N.E.Greville, Generalized Inverses: Theory and Applications, 2nd ed., New York, Springer-Verlag (2003).

[2] R.Bott, R.J.Duffin, On the algebra of networks, Trans. Amer. Math. Soc. 74 (1953) 99–109.

[3] Y.L.Chen, Expressions and determinantal formulas for the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and their applications, J. Natural Science Nanjing Normal University. 16 (1993) 3–16.

[4] Y.L.Chen, The theory and method of generalized inverse matrix. Nanjing Normal University press, Nanjing (2005) (in Chinese).

[5] D.S.Cvetković-Ilić, Y.M.Wei, Algebraic properties of generalized inverses, Developments in Mathematics, vol 52, Singapore, Springer (2017).

[6] M.P.Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Mon. 65 (1958) 506–514.

[7] D.E.Ferreyra, V.Orquera, N.Thome, A weak group inverse for rectangular matrices, RACSAM. 113 (2019) 3727–3740.

[8] Y.F.Gao, J.L.Chen, Pseudo core inverses in rings with involution, Comm. Algebra. 46 (2018) 38–50.

[9] X.J.Liu, S.W.Huang, Proper splitting for the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and its application on Banach spaces, Abstr. Appl. Anal. 2012 (2012) 1–9.

[10] H.F.Ma, P.S.Stanimirović, Characterizations, approximation and perturbations of the core-EP inverse, Appl. Math. Comput. 359 (2019) 404–417.

[11] X.Mary, On generalized inverse and Green's relations, Linear Algebra Appl. 434 (2011) 1836–1844.

[12] D.Mosić, P.Stanimirović, Representations for the weak group inverse Commun. Math. Res. 397 (2021) 125957.

[13] R.Penrose, A generalized inverse for matrices, Math. Proc. Camb. Philos. Soc. 51 (1955) 406–413.

[14] K.M.Prasad, K.S.Mohana, Core-EP inverse, Linear Multilinear Algebra. 62 (2014) 792–802.

[15] H.X.Wang, Core-EP decomposition and its applications, Linear Algebra Appl. 508 (2016) 289–300.

[16] H.X.Wang, J.L.Chen, Weak group inverse, Open Math. 16 (2018) 1218–1232.

[17] H.X.Wang, X.J.Liu, The weak group matrix, Aequationes math. 93 (2019) 1261–1273.

[18] Y.M.Wei, A characterization and representation of the generalized inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and its applications, Linear Algebra Appl. 280 (1998) 87–96.

[19] H.Yan, H.X.Wang, K.Z.Zuo, Y.Chen, Further characterizations of the weak group inverse of matrices and the weak group matrix, AIMS Math. 6 (2021) 9322–9341.