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Generalized inverses, ideals, and projectors in rings

Patricia Mariela Morillas[a]

[a] *Instituto de Matemática Aplicada San Luis (UNSL-CONICET), Ejército de los Andes 950, CP 5700, San Luis, Argentina.*

Abstract. The theory of generalized inverses of matrices and operators is closely connected with projections, i.e., idempotent (bounded) linear transformations. We show that a similar situation occurs in any associative ring $\mathcal{R}$ with a unit $1 \neq 0$. We prove that generalized inverses in $\mathcal{R}$ are related to idempotent group endomorphisms $\rho : \mathcal{R} \to \mathcal{R}$, called projectors. We use these relations to give characterizations and existence conditions for $\{1\}$, $\{2\}$, and $\{1,2\}$-inverses with any given principal/annihilator ideals. As a consequence, we obtain sufficient conditions for any right/left ideal of $\mathcal{R}$ to be a principal or an annihilator ideal of an idempotent element of $\mathcal{R}$. We also study some particular generalized inverses: Drazin and (b,c) inverses, and (e,f) Moore-Penrose, e-core, f-dual core, w-core, dual v-core, right w-core, left dual v-core, and (p,q) inverses in rings with involution.

1. Introduction

For non-invertible operators and matrices, and more generally, for non-invertible elements of semigroups and rings, several generalized inverses were defined and studied. Each generalized inverse is used to study specific types of problems. They are useful for solving matrix and operator equations (including integral and differential equations), in probability theory, in the study of algebras, rings, and semigroups, among others. See, e.g., [3, 6, 8, 10, 36, 41] and references therein.

Throughout this paper, $\mathcal{R}$ will be an associative ring with a unit $1 \neq 0$. An involution $*$ of $\mathcal{R}$ is an involutory anti-automorphism $a \mapsto a^*$, i.e., $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{R}$. For $a \in \mathcal{R}$, consider the *principal right (resp. left) ideal of* $\mathcal{R}$ *with generator* a, $a\mathcal{R} = \{ar : r \in \mathcal{R}\}$ (resp. $\mathcal{R}a = \{ra : r \in \mathcal{R}\}$), and the *right* (resp. *left*) *annihilator of* a, $\mathrm{rann}(a) = \{r \in \mathcal{R} : ar = 0\}$ (resp. $\mathrm{lann}(a) = \{r \in \mathcal{R} : ra = 0\}$). For $a, x \in \mathcal{R}$, consider the following equalities:

$$axa = a, \quad (1) \qquad xax = x, \quad (2) \qquad (xa)^* = xa, \quad (3) \qquad (ax)^* = ax, \quad (4)$$

$$ax = xa, \quad (5) \qquad xa^2 = a, \quad (6) \qquad ax^2 = x, \quad (7) \qquad a^2x = a, \quad (8) \qquad x^2a = x, \quad (9)$$

$$xa^{k+1} = a^k \text{ for some } k \in \{1, 2, \ldots\}, \quad (1^k) \qquad a^{k+1}x = a^k \text{ for some } k \in \{1, 2, \ldots\}, \quad ({}^k1)$$

where (3) and (4) require $\mathcal{R}$ to be a $*$-ring.

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Email address: morillas.unsl@gmail.com (Patricia Mariela Morillas)

Definition 1.1. *For any $a \in \mathcal{R}$, let $a\{i, j, \dots, l\}$ denote the set of elements $x \in \mathcal{R}$ which satisfy equations (i), (j), . . . , (l) from among equations (1)–(9),(1^k) and ($^k1$). An element $x \in a\{i, j, \dots, l\}$ is called an $\{i, j, \dots, l\}$-inverse of a, and denoted by $a^{(i,j,\dots,l)}$.*

The relation of generalized inverses of matrices and operators to oblique and orthogonal projections is one of the most important properties for their study and applications. Before describing the approach and the results for generalized inverses in rings that we present in this paper, we now recall some of these relations.

Let $\mathbb{C}^{m\times n}$ denote the set of matrices of order $m \times n$. Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$. Then

$$X \in A\{1\} \Leftrightarrow AX = P_{\mathrm{R}(A),S} \text{ and } XA = P_{T,\mathrm{N}(A)},$$

where S is some subspace of $\mathbb{C}^m$ complementary to the range $\mathrm{R}(A)$ of A, $P_{\mathrm{R}(A),S}$ is the oblique projection onto $\mathrm{R}(A)$ along S, T is some subspace of $\mathbb{C}^n$ complementary to the null space $\mathrm{N}(A)$ of A, and $P_{T,\mathrm{N}(A)}$ is the oblique projection onto T along $\mathrm{N}(A)$. We also have

$$X \in A\{1,3\} \Leftrightarrow AX = P_{\mathrm{R}(A)}$$

and

$$X \in A\{1,4\} \Leftrightarrow XA = P_{\mathrm{R}(A^*)},$$

where A^* is the conjugate transpose of A, and $P_{\mathrm{R}(A)}$ and $P_{\mathrm{R}(A^*)}$ denote the orthogonal projections onto the $\mathrm{R}(A)$ and $\mathrm{R}(A^*)$, respectively. See, e.g., [4] for more details.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $\mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ with closed range. If $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$, then the Moore-Penrose inverse $A^\dagger = A^{(1,2,3,4)}$ exists and

$$AA^\dagger = P_{\mathrm{R}(A)} \text{ and } A^\dagger A = P_{\mathrm{R}(A^*)},$$

where A^* is the adjoint of A (see, e.g., [32, Theorem 1]).

Let now $\mathcal{V}$ be a complex Banach space. If $A \in \mathcal{B}(\mathcal{V})$ has finite index k, then the Drazin inverse $A^D = A^{(1^k,2,5)}$ of A exists and satisfies

$$AA^D = A^D A = P_{\mathrm{R}(A^k),\mathrm{N}(A^k)},$$

where $P_{\mathrm{R}(A^k),\mathrm{N}(A^k)}$ is the oblique projection onto $\mathrm{R}(A^k)$ along $\mathrm{N}(A^k)$ (see, e.g., [20, Theorem 4 and its proof]).

The core $A^{\#\!\!\!\bigcirc} = A^{(1,2,3,6,7)}$ and the dual core $A_{\#\!\!\!\bigcirc} = A^{(1,2,4,8,9)}$ inverses of $A \in \mathbb{C}^{n\times n}$ are defined by the conditions

$$AA^{\#\!\!\!\bigcirc} = P_{\mathrm{R}(A)} \text{ and } \mathrm{R}(A^{\#\!\!\!\bigcirc}) \subseteq \mathrm{R}(A) \tag{10}$$

and

$$A_{\#\!\!\!\bigcirc}A = P_{\mathrm{R}(A^*)} \text{ and } \mathrm{R}(A_{\#\!\!\!\bigcirc}) \subseteq \mathrm{R}(A^*), \tag{11}$$

respectively (see [2, Definition 1, (i) and (ii) on page 693]).

Since the theory of generalized inverses of matrices and operators is closely connected with projections, i.e., idempotent (bounded) linear transformations, it is natural to think about a similar situation when working in any ring. In this paper, we show that we can similarly relate generalized inverses in a ring $\mathcal{R}$ to idempotent group endomorphisms $\rho : \mathcal{R} \to \mathcal{R}$, called projectors, which are linked to direct sum decompositions of $\mathcal{R}$, and use these relations in their study.

The organization of the paper is as follows:

In Section 2, we present some properties of generalized inverses, elements, direct sums and projectors in rings that we use throughout the article.

In Section 3, we relate {1}, {2}, {1, 2}, {1, 5}, and Drazin inverses to projectors.

Numerous particular generalized inverses are defined or studied by means of their associated principal or annihilator ideals. In Sections 4, 5, and 6, we address this topic with generality using projectors. Let $a \in \mathcal{R}$, $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$, and $\mathcal{S}', \mathcal{T}'$ be left ideals of $\mathcal{R}$. We study {1}, {2}, and {1, 2}-inverses x of a such that one of the following conditions holds: $xa\mathcal{R} = \mathcal{S}$ and $\text{rann}(xa) = \mathcal{T}$; $\mathcal{R}xa = \mathcal{S}'$ and $\text{lann}(xa) = \mathcal{T}'$; $xa\mathcal{R} = \mathcal{S}$ and $\mathcal{R}xa = \mathcal{S}'$; or $\text{rann}(xa) = \mathcal{T}$ and $\text{lann}(xa) = \mathcal{T}'$. We also consider {1} and {1, 2}-inverses such that one of the following conditions holds: $xa\mathcal{R} = \mathcal{S}$; $\text{rann}(xa) = \mathcal{T}$; $\mathcal{R}xa = \mathcal{S}'$; or $\text{lann}(xa) = \mathcal{T}'$. We give several characterizations and existence conditions for these generalized inverses. We establish the connection of the Mitsch partial order [27] with the {2}-inverses considered in Section 5. Since in these sections, we do not make any additional a priori assumption for the ideals $\mathcal{S}, \mathcal{T}, \mathcal{S}'$, and $\mathcal{T}'$, we can consider a large variety of particular cases (as we do in Section 7). Moreover, as a consequence, we obtain sufficient conditions for any right/left ideal of $\mathcal{R}$ to be a principal or an annihilator ideal of an idempotent element of $\mathcal{R}$.

In Section 7, we apply results of the previous sections to study some classes of {1}, {2}, and {1, 2}-inverses. First, we consider {1, 3}, {1, 4}, {1, 3, 4}, {1, 3, 6}, {1, 4, 8}, {1, 3, 7}, and {1, 4, 9}-inverses. Then, we analyze the (e, f) Moore-Penrose inverse [30], the e-core and f-dual core inverses [29], the w-core and the dual v-core inverses [47], and the right w-core and left dual v-core inverses [48]. Finally, we obtain some properties of two types of {2}-inverses: the (b, c) inverses [13] and the (p, q) inverse [11]. We end Section 7 by giving an illustrative example with a matrix over a field.

2. Preliminaries

This section presents notations, definitions and results about generalized inverses, elements, direct sums and projectors in rings, that will be used later.

2.1. Generalized inverses

Let $a \in \mathcal{R}$. If $a\{1\} \neq \emptyset$, then a is called *regular* (in the sense of von Neumann) and an $x \in a\{1\}$ is called an *inner inverse* of a. If $a\{2\} \neq \emptyset$, then a is called *anti-regular* and an $x \in a\{2\}$ is called an *outer inverse* of a.

Note that if $x \in a\{5\}$, then (1^k) is equivalent to $({}^k1)$. If there exists $k \in \{1, 2, \ldots\}$ such that $a\{2, 5, 1^k\} \neq \emptyset$, then a is called *Drazin invertible*. The smallest of these positive integers k is called the index of a. The set $a\{2, 5, 1^k\}$ has a unique element called the *Drazin inverse* of a and denoted by a^D. In particular, if $a\{1, 2, 5\} \neq \emptyset$, then a is called *group invertible* and the *group inverse* of a is denoted by $a^\#$. For more details about these inverses in rings see, e.g., [12, 23, 31].

Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}$. If $a\{1, 2, 3, 4\}$ is not empty, then a is called *Moore-Penrose invertible*. In this case, $a\{1, 2, 3, 4\}$ has a unique element called the *Moore-Penrose inverse* of a and denoted by $a^\dagger$. See, e.g., [16, 21, 22, 42] for properties of the Moore-Penrose inverse in $*$-rings.

Baksalary and Trenkler [2] introduced two generalized inverses for complex matrices (see (10) and (11)). Later, Rakić, Dinčić, and Djordjević [35] generalized these notions to an arbitrary $*$-ring. Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then x is a *core* (resp. *dual core*) *inverse* of a if $x \in a\{1\}$ and $x\mathcal{R} = x^*\mathcal{R} = a\mathcal{R}$ (resp. $x\mathcal{R} = x^*\mathcal{R} = a^*\mathcal{R}$). (see [35, Definitions 2.3 and 2.4]). If they exist, the core and the dual core inverses of an element $a \in \mathcal{R}$ are unique and are denoted by $a^{\#\!\!\!\!\bigcirc}$ and $a_{\#\!\!\!\!\bigcirc}$, respectively. By [35, Lemmas 2.1 and 2.2], the conditions of [35, Definitions 2.3 and 2.4] for the core and dual core inverses are equivalent to the conditions (10) and (11) for finite complex matrices. In [35], it is proved that $a^{\#\!\!\!\!\bigcirc} = a^{(1,2,3,6,7)}$ and $a_{\#\!\!\!\!\bigcirc} = a^{(1,2,4,8,9)}$. In [43], it is proved that $a^{\#\!\!\!\!\bigcirc} = a^{(3,6,7)}$, moreover, by the proof of [43, Theorem 3.1], $a\{6, 7\} \subseteq a\{1, 2\}$. Similarly, $a_{\#\!\!\!\!\bigcirc} = a^{(4,8,9)}$ and $a\{8, 9\} \subseteq a\{1, 2\}$. Properties of the core and dual core inverse in $*$-rings can be found in, e.g., [24, 35, 43].

Let us denote the sets of all Drazin invertible, group invertible, Moore-Penrose invertible, core invertible and dual core invertible elements in $\mathcal{R}$ by $\mathcal{R}^D$, $\mathcal{R}^\#$, $\mathcal{R}^\dagger$, $\mathcal{R}^{\#\!\!\!\!\bigcirc}$ and $\mathcal{R}_{\#\!\!\!\!\bigcirc}$, respectively. We will use the following well-known equalities:

1. Let $a \in \mathcal{R}$ and $a^{(1)} \in a\{1\}$. Then $aa^{(1)}\mathcal{R} = a\mathcal{R}$, $\text{rann}(a^{(1)}a) = \text{rann}(a)$, $\text{lann}(aa^{(1)}) = \text{lann}(a)$, and $\mathcal{R}a^{(1)}a = \mathcal{R}a$.
2. Let $a \in \mathcal{R}$ and $a^{(1)} \in a\{2\}$. Then $\text{rann}(aa^{(2)}) = \text{rann}(a^{(2)})$, $a^{(2)}a\mathcal{R} = a^{(2)}\mathcal{R}$, $\mathcal{R}aa^{(2)} = \mathcal{R}a^{(2)}$, and $\text{lann}(a^{(2)}a) = \text{lann}(a^{(2)})$.
3. Let $a \in \mathcal{R}^D$ with index k and $l \geq k$. Then $a^Da\mathcal{R} = aa^D\mathcal{R} = a^D\mathcal{R} = a^l\mathcal{R}$, $\text{rann}(aa^D) = \text{rann}(a^Da) = \text{rann}(a^D) = \text{rann}(a^l)$, $\mathcal{R}aa^D = \mathcal{R}a^Da = \mathcal{R}a^D = \mathcal{R}a^l$ and $\text{lann}(aa^D) = \text{lann}(a^Da) = \text{lann}(a^D) = \text{lann}(a^l)$.

2.2. Elements in rings

An element $p \in \mathcal{R}$ is an *idempotent* if $p = p^2$. If $\mathcal{R}$ is a $*$-ring and $p = p^*$, then p is said to be *symmetric*. If $p = p^2 = p^*$, then p is called a *projection*. The set of invertible and idempotent elements in $\mathcal{R}$ are denoted with $\mathcal{R}^{-1}$ and $\mathcal{R}^\bullet$, respectively. If $\mathcal{R}$ is a $*$-ring, the set of symmetric elements in $\mathcal{R}$ is denoted with $\mathcal{R}^{\mathrm{sym}}$.

For $a \in \mathcal{R}$, we consider the group endomorphisms $\varphi_a : \mathcal{R} \to \mathcal{R}$ given by $\varphi_a(x) = ax$ and ${}_a\varphi : \mathcal{R} \to \mathcal{R}$ given by ${}_a\varphi(x) = xa$. We have, $\mathrm{im}(\varphi_a) = a\mathcal{R}$, $\ker(\varphi_a) = \mathrm{rann}(a)$, $\mathrm{im}({}_a\varphi) = \mathcal{R}a$, and $\ker({}_a\varphi) = \mathrm{lann}(a)$.

Lemma 2.1. *Let $a \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $a \in \mathcal{R}^{-1}$.
2. $a\mathcal{R} = \mathcal{R}$ *and* $\mathrm{rann}(a) = \{0\}$.
3. $\mathcal{R}a = \mathcal{R}$ *and* $\mathrm{lann}(a) = \{0\}$.

Proof. (1) $\Rightarrow$ (2)(3): It is immediate.

(2) $\Rightarrow$ (1): Assume that $a\mathcal{R} = \mathcal{R}$ and $\mathrm{rann}(a) = \{0\}$. Then φ_a is a group automorphism. Let ψ be the group automorphism such that $\psi = \varphi_a^{-1}$. Since $\varphi_a(1) = a$, we have $\psi(a) = 1$. For each $s \in \mathcal{R}$ there exists a unique $r \in \mathcal{R}$ such that $ar = s$. In particular, there exists a unique $b \in \mathcal{R}$ such that $ab = 1$. Then, $\varphi_a(r) = s \Leftrightarrow ar = s \Leftrightarrow ar = abs \Leftrightarrow r = bs$. Hence, $\psi(s) = r = bs$. From here, $\psi = \varphi_b$ and $ba = \varphi_b(a) = \psi(a) = 1$. This shows that $b = a^{-1}$. Therefore, $a \in \mathcal{R}^{-1}$.

(3) $\Rightarrow$ (1): It is similar to the proof of (2) $\Rightarrow$ (1). □

Lemma 2.2. *Let $p, q \in \mathcal{R}^\bullet$. Then:*

1. $p\mathcal{R} = \mathrm{rann}(1-p)$, $\mathcal{R}p = \mathrm{lann}(1-p)$.
2. $p\mathcal{R} \subseteq q\mathcal{R} \Leftrightarrow \mathrm{lann}(q) \subseteq \mathrm{lann}(p)$, $\mathcal{R}p \subseteq \mathcal{R}q \Leftrightarrow \mathrm{rann}(q) \subseteq \mathrm{rann}(p)$.
3. $q = p \Leftrightarrow \{q\mathcal{R} \subseteq p\mathcal{R}$ *and* $\mathrm{rann}(q) \subseteq \mathrm{rann}(p)\}$.

Lemma 2.3. *[35, Lemmas 2.5 and 2.6] Let $a, b \in \mathcal{R}$. Then:*

1. *If* $a\mathcal{R} \subseteq b\mathcal{R}$, *then* $\mathrm{lann}(b) \subseteq \mathrm{lann}(a)$.
2. *If* $\mathrm{lann}(b) \subseteq \mathrm{lann}(a)$ *and* $b\{1\} \neq \emptyset$, *then* $a\mathcal{R} \subseteq b\mathcal{R}$.
3. *If* $\mathcal{R}a \subseteq \mathcal{R}b$, *then* $\mathrm{rann}(b) \subseteq \mathrm{rann}(a)$.
4. *If* $\mathrm{rann}(b) \subseteq \mathrm{rann}(a)$ *and* $b\{1\} \neq \emptyset$, *then* $\mathcal{R}a \subseteq \mathcal{R}b$.

Lemma 2.4. *Let $a, b \in \mathcal{R}$ be such that $b\{1\} \neq \emptyset$. Then:*

1. *If* $\mathrm{rann}(b) \subseteq \mathrm{rann}(a)$ *and* $\mathcal{R}b \subseteq \mathcal{R}a$, *then* $a\{1\} \neq \emptyset$.
2. *If* $\mathrm{lann}(b) \subseteq \mathrm{lann}(a)$ *and* $b\mathcal{R} \subseteq a\mathcal{R}$, *then* $a\{1\} \neq \emptyset$.

Proof. (1): Let $x \in b\{1\}$. Since $\mathrm{rann}(b) \subseteq \mathrm{rann}(a)$ and $1 - xb \in \mathrm{rann}(b)$, we have $a(1 - xb) = 0$. From here, $a = axb \in aRb \subseteq aRa$. Therefore, $a\{1\} \neq \emptyset$.

(2): The proof is similar to the proof of (1). □

As a consequence of Lemmas 2.3(2)(4) and 2.4 we get the following result.

Lemma 2.5. *Let $a, b \in \mathcal{R}$ be such that $b\{1\} \neq \emptyset$.*

1. *If* $\mathrm{rann}(a) = \mathrm{rann}(b)$, *then* $\mathcal{R}b \subseteq \mathcal{R}a$ *if and only if* $a\{1\} \neq \emptyset$.
2. *If* $\mathrm{lann}(a) = \mathrm{lann}(b)$, *then* $b\mathcal{R} \subseteq a\mathcal{R}$ *if and only if* $a\{1\} \neq \emptyset$.

If $\mathcal{R}$ is a $*$-ring, then $a\mathcal{R} \subseteq b\mathcal{R} \Leftrightarrow \mathcal{R}a^* \subseteq \mathcal{R}b^*$ and $\mathrm{rann}(a) \subseteq \mathrm{rann}(b) \Leftrightarrow \mathrm{lann}(a^*) \subseteq \mathrm{lann}(b^*)$.

Definition 2.6. *Let $\mathcal{R}$ be a $*$-ring.*

1. *Let $a, b \in \mathcal{R}$. Then a and b are* right orthogonal *(resp.* left orthogonal*), written $a \perp_r b$ or $b \perp_r a$ (resp. $a \perp_l b$ or $b \perp_l a$), if $a^*b = 0$ (resp. $ab^* = 0$).*

2. *Let $\mathcal{S},\mathcal{T} \subseteq \mathcal{R}$. Then $\mathcal{S}$ and $\mathcal{T}$ are* right orthogonal *(resp.* left orthogonal*), written $\mathcal{S} \perp_r \mathcal{T}$ or $\mathcal{T} \perp_r \mathcal{S}$ (resp. $\mathcal{S} \perp_l \mathcal{T}$ or $\mathcal{T} \perp_l \mathcal{S}$), if $a \perp_r b$ (resp. $a \perp_l b$) for each $a \in \mathcal{S}$ and $b \in \mathcal{T}$.*

Lemma 2.7. *Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}$. Then:*

1. $a\mathcal{R} \perp_r \operatorname{rann}(a^*)$.
2. *If $a \in \mathcal{R}^{\text{sym}}$ then $a\mathcal{R} \perp_r \operatorname{rann}(a)$.*
3. *If $a \in \mathcal{R}^\bullet$ and $a\mathcal{R} \perp_r \operatorname{rann}(a)$, then $a \in \mathcal{R}^{\text{sym}}$.*

Proof. (1): Let $x \in a\mathcal{R}$ and $y \in \operatorname{rann}(a^*)$. Hence, there exists $r \in \mathcal{R}$ such that $x = ar$ and $a^*y = 0$. Then, $x^*y = r^*a^*y = 0$. This proves that $a\mathcal{R} \perp_r \operatorname{rann}(a^*)$.

(2): It follows from (1).

(3): Assume that $a \in \mathcal{R}^\bullet$ and $a\mathcal{R} \perp_r \operatorname{rann}(a)$. Since $a \in \mathcal{R}^\bullet$, it follows that $1 - a \in \operatorname{rann}(a)$. Then, since $a\mathcal{R} \perp_r \operatorname{rann}(a)$, we have $a^*(1 - a) = 0$. This last equality is equivalent to $a^* = a^*a$. Then, $a^* = a$. □

Analogously to Lemma 2.7, we have:

Lemma 2.8. *Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}$. Then:*

1. $\mathcal{R}a \perp_l \operatorname{lann}(a^*)$.
2. *If $a \in \mathcal{R}^{\text{sym}}$ then $\mathcal{R}a \perp_l \operatorname{lann}(a)$.*
3. *If $a \in \mathcal{R}^\bullet$ and $\mathcal{R}a \perp_l \operatorname{lann}(a)$, then $a \in \mathcal{R}^{\text{sym}}$.*

2.3. Projectors

In vector spaces and modules, idempotent linear transformations are well known as (oblique) projections or projectors (see, e.g., [1, 37]). In rings, we consider idempotent group endomorphisms, called projectors, to use them to study generalized inverses. Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$. Let

$$\mathcal{S} + \mathcal{T} = \{s + t : s \in \mathcal{S} \text{ and } t \in \mathcal{T}\}.$$

Definition 2.9. *Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$. Then $\mathcal{R}$ is the (*internal*)* direct sum *of $\mathcal{S}$ and $\mathcal{T}$, written $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$, if $\mathcal{R} = \mathcal{S} + \mathcal{T}$ and $\mathcal{S} \cap \mathcal{T} = \{0\}$. In this case, $\mathcal{S}$ and $\mathcal{T}$ are called* direct summands *of $\mathcal{R}$, and $\mathcal{T}$ (resp. $\mathcal{S}$) is called a* complement *of $\mathcal{S}$ (resp. $\mathcal{T}$) in $\mathcal{R}$.*

Associated with a direct sum decomposition of $\mathcal{R}$ we have a group endomorphism:

Definition 2.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$. The group endomorphism $\rho_{\mathcal{S},\mathcal{T}} : \mathcal{R} \to \mathcal{R}$ defined by $\rho_{\mathcal{S},\mathcal{T}}(s + t) = s$, where $s \in \mathcal{S}$ and $t \in \mathcal{T}$, is called the* oblique projector onto $\mathcal{S}$ along $\mathcal{T}$. *If $\mathcal{R}$ is a $*$-ring and $\mathcal{S} \perp_r \mathcal{T}$ (resp. $\mathcal{S} \perp_l \mathcal{T}$), we say that $\rho_{\mathcal{S},\mathcal{T}}$ is a* right (resp. left) orthogonal projector.

We usually say projector and orthogonal projector instead of oblique projector and right (left) orthogonal projector, respectively. From Definition 2.10 we obtain:

Lemma 2.11. *Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$. Then:*

1. $\rho_{\mathcal{S},\mathcal{T}} + \rho_{\mathcal{T},\mathcal{S}} = \operatorname{id}_{\mathcal{R}}$.
2. $\operatorname{im}(\rho_{\mathcal{S},\mathcal{T}}) = \mathcal{S}$ *and* $\ker(\rho_{\mathcal{S},\mathcal{T}}) = \mathcal{T}$.
3. $r \in \operatorname{im}(\rho_{\mathcal{S},\mathcal{T}}) \Leftrightarrow \rho_{\mathcal{S},\mathcal{T}}(r) = r$.

Another property of projectors derived easily from Definition 2.10 is the following:

Lemma 2.12. *If $\varphi : \mathcal{R} \to \mathcal{R}$ is a group endomorphism such that $\mathcal{R} = \operatorname{im}(\varphi) \oplus \ker(\varphi)$ and $\varphi_{|\operatorname{im}(\varphi)} = \operatorname{id}_{\operatorname{im}(\varphi)}$, then $\varphi = \rho_{\operatorname{im}(\varphi),\ker(\varphi)}$.*

The next lemma asserts that projectors are precisely idempotent group endomorphisms.

Lemma 2.13. *A group endomorphism $\varphi : \mathcal{R} \to \mathcal{R}$ is a projector if and only if φ is idempotent, and in this case, $\mathcal{R} = \operatorname{im}(\varphi) \oplus \ker(\varphi)$ and $\varphi = \rho_{\operatorname{im}(\varphi),\ker(\varphi)}$.*

Proof. Assume that there exist $\mathcal{S}$ and $\mathcal{T}$ subgroups of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$ and $\varphi = \rho_{\mathcal{S},\mathcal{T}}$. Let $r \in \mathcal{R}$. There exist $s \in \mathcal{S}$ and $t \in \mathcal{T}$ such that $r = s + t$. Then $\varphi^2(r) = \rho_{\mathcal{S},\mathcal{T}}(\rho_{\mathcal{S},\mathcal{T}}(s + t)) = \rho_{\mathcal{S},\mathcal{T}}(s) = s = \rho_{\mathcal{S},\mathcal{T}}(r) = \varphi(r)$. Hence, $\varphi^2 = \varphi$.

Conversely, suppose that φ is idempotent. Let $r \in \mathcal{R}$, $s = \varphi(r)$ and $t = r - s$. Then $\varphi(t) = \varphi(r - s) = \varphi(r)-\varphi(s) = \varphi(r)-\varphi^2(r) = 0$. Thus, $r = s+t$ with $s \in \mathrm{im}(\varphi)$ and $t \in \ker(\varphi)$. This shows that $\mathcal{R} = \mathrm{im}(\varphi)+\ker(\varphi)$. If $r \in \mathrm{im}(\varphi)$, then there exists $s \in \mathcal{R}$ such that

$$r = \varphi(s) = \varphi^2(s) = \varphi(r). \tag{12}$$

Hence, $\varphi_{|\mathrm{im}(\varphi)} = \mathrm{id}_{\mathrm{im}(\varphi)}$. By (12), if $r \in \mathrm{im}(\varphi) \cap \ker(\varphi)$, then $r = 0$. Thus, $\mathrm{im}(\varphi) \cap \ker(\varphi) = \{0\}$. Therefore, $\mathcal{R} = \mathrm{im}(\varphi) \oplus \ker(\varphi)$. Applying now Lemma 2.12, we get $\varphi = \rho_{\mathrm{im}(\varphi),\ker(\varphi)}$. □

So far, we only required $\mathcal{S}$ and $\mathcal{T}$ to be subgroups of $\mathcal{R}$. The next two lemmas give properties of the projectors when $\mathcal{S}$ and $\mathcal{T}$ are right (left) ideals of $\mathcal{R}$.

Lemma 2.14. *Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$. Then:*

1. *$\mathcal{S}$ and $\mathcal{T}$ are right ideals of $\mathcal{R}$ if and only if $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = \rho_{\mathcal{S},\mathcal{T}}(r_1)r_2$ for each $r_1, r_2 \in \mathcal{R}$.*
2. *$\mathcal{S}$ and $\mathcal{T}$ are left ideals of $\mathcal{R}$ if and only if $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = r_1\rho_{\mathcal{S},\mathcal{T}}(r_2)$ for each $r_1, r_2 \in \mathcal{R}$.*

Proof. We prove (1). The proof of (2) is similar.

Assume that $\mathcal{S}$ and $\mathcal{T}$ are right ideals of $\mathcal{R}$. Let $r_1, r_2 \in \mathcal{R}$. Since $\rho_{\mathcal{S},\mathcal{T}}(r_1)r_2 \in \mathcal{S}$ and $\rho_{\mathcal{T},\mathcal{S}}(r_1)r_2 \in \mathcal{T}$, we have $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = \rho_{\mathcal{S},\mathcal{T}}(\rho_{\mathcal{S},\mathcal{T}}(r_1)r_2 + \rho_{\mathcal{T},\mathcal{S}}(r_1)r_2) = \rho_{\mathcal{S},\mathcal{T}}(r_1)r_2$.

Conversely, assume that $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = \rho_{\mathcal{S},\mathcal{T}}(r_1)r_2$ for each $r_1, r_2 \in \mathcal{R}$. Take $r_1 \in \mathcal{S}$ and $r_2 \in \mathcal{R}$. Then $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = r_1r_2$ and by Lemma 2.11(2), $r_1r_2 \in \mathcal{S}$. This shows that $\mathcal{S}$ is a right ideal of $\mathcal{R}$. Take now $r_1 \in \mathcal{T}$ and $r_2 \in \mathcal{R}$. Then $\rho_{\mathcal{S},\mathcal{T}}(r_1r_2) = 0$ and by Lemma 2.11(2), $r_1r_2 \in \mathcal{T}$. This shows that $\mathcal{T}$ is a right ideal of $\mathcal{R}$. □

As a consequence of Lemmas 2.11 and 2.14 we get:

Lemma 2.15. *Let $\mathcal{S}$ and $\mathcal{T}$ be subgroups of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$ and $a \in \mathcal{R}$. Then the following assertions hold:*

1. *If $\mathcal{S}$ and $\mathcal{T}$ are right ideals of $\mathcal{R}$, then $\rho_{\mathcal{S},\mathcal{T}}(1)a = a \Leftrightarrow a\mathcal{R} \subseteq \mathcal{S}$ and $a\rho_{\mathcal{S},\mathcal{T}}(1) = a \Leftrightarrow \mathcal{T} \subseteq \mathrm{rann}(a)$.*
2. *If $\mathcal{S}$ and $\mathcal{T}$ are left ideals of $\mathcal{R}$, then $a\rho_{\mathcal{S},\mathcal{T}}(1) = a \Leftrightarrow \mathcal{R}a \subseteq \mathcal{S}$ and $\rho_{\mathcal{S},\mathcal{T}}(1)a = a \Leftrightarrow \mathcal{T} \subseteq \mathrm{lann}(a)$.*

Proof. Assume that $\mathcal{S}$ and $\mathcal{T}$ are right ideals of $\mathcal{R}$. Then $\rho_{\mathcal{S},\mathcal{T}}(1)a = a \Leftrightarrow \rho_{\mathcal{S},\mathcal{T}}(a) = a \Leftrightarrow a \in \mathcal{S} \Leftrightarrow a\mathcal{R} \subseteq \mathcal{S}$ and $a\rho_{\mathcal{S},\mathcal{T}}(1) = a \Leftrightarrow a\rho_{\mathcal{S},\mathcal{T}}(1) = a(\rho_{\mathcal{S},\mathcal{T}}(1) + \rho_{\mathcal{T},\mathcal{S}}(1)) \Leftrightarrow a\rho_{\mathcal{T},\mathcal{S}}(1) = 0 \Leftrightarrow \forall r \in \mathcal{R} : a\rho_{\mathcal{T},\mathcal{S}}(1)r = 0 \Leftrightarrow \forall r \in \mathcal{R} : a\rho_{\mathcal{T},\mathcal{S}}(r) = 0 \Leftrightarrow \mathcal{T} \subseteq \mathrm{rann}(a)$. This proves (1). The proof of (2) is similar. □

In what follows, whenever we write $\rho_{\mathcal{S},\mathcal{T}}$, we are implicity asserting that $\mathcal{R} = \mathcal{S}\oplus\mathcal{T}$. Part (1) of the following corollary is a consequence of Lemma 2.13 whereas part (2) follows from Lemma 2.14.

Corollary 2.16. *The following assertions hold:*

1. *If $a \in \mathcal{R}^\bullet$, then $\varphi_a = \rho_{a\mathcal{R},\mathrm{rann}(a)}$ and ${}_a\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$.*
2. *Let $\mathcal{S}$, $\mathcal{T}$ be right (resp. left) ideals of $\mathcal{R}$ and $a = \rho_{\mathcal{S},\mathcal{T}}(1)$, then $a \in \mathcal{R}^\bullet$ and $\varphi_a = \rho_{\mathcal{S},\mathcal{T}}$ (resp. ${}_a\varphi = \rho_{\mathcal{S},\mathcal{T}}$).*

3. Relations of {1}, {2}, {1, 2}, {1, 5}, and Drazin inverses to projectors

Let $a\{1\} \neq \emptyset$ and $a^{(1)} \in a\{1\}$. Since $aa^{(1)}, a^{(1)}a \in \mathcal{R}^\bullet$, Corollary 2.16(1) yields the next theorem that relates {1}-inverses to projectors.

Theorem 3.1. *Let $a \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(ax)}$.
3. $\varphi_{xa} = \rho_{xa\mathcal{R},\mathrm{rann}(a)}$.
4. ${}_{ax}\varphi = \rho_{\mathcal{R}ax,\mathrm{lann}(a)}$.

5. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(xa)}$.

Since $a^{(2)} \in a\{2\}$ if and only if $a \in a^{(2)}\{1\}$, Theorem 3.1 has an analogous version for {2}-inverses:

Theorem 3.2. *Let $a \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{2\}$.
2. $\varphi_{ax} = \rho_{ax\mathcal{R},\mathrm{rann}(x)}$.
3. $\varphi_{xa} = \rho_{x\mathcal{R},\mathrm{rann}(xa)}$.
4. ${}_{ax}\varphi = \rho_{\mathcal{R}x,\mathrm{lann}(ax)}$.
5. ${}_{xa}\varphi = \rho_{\mathcal{R}xa,\mathrm{lann}(x)}$.

The following lemma will be used later.

Lemma 3.3. *Let $a, b \in \mathcal{R}$ be such that $(ab)\{1\} \neq \emptyset$. Let $(ab)^{(1)} \in (ab)\{1\}$. Then:*

1. $ab(ab)^{(1)}a = a \Leftrightarrow ab\mathcal{R} = a\mathcal{R} \Leftrightarrow \mathrm{lann}(ab) = \mathrm{lann}(a)$.
2. $b(ab)^{(1)}ab = b \Leftrightarrow \mathrm{rann}(ab) = \mathrm{rann}(b) \Leftrightarrow \mathcal{R}ab = \mathcal{R}b$.

Proof. We always have $ab\mathcal{R} \subseteq a\mathcal{R}$, $\mathrm{lann}(a) \subseteq \mathrm{lann}(ab)$, $\mathrm{rann}(b) \subseteq \mathrm{rann}(ab)$ and $\mathcal{R}ab \subseteq \mathcal{R}b$. Using Theorem 3.1 and Lemma 2.15 we get:

$$ab(ab)^{(1)}a = a \Leftrightarrow \rho_{ab\mathcal{R},\mathrm{rann}(ab(ab)^{(1)})}(1)a = a \Leftrightarrow a\mathcal{R} \subseteq ab\mathcal{R},$$

$$ab(ab)^{(1)}a = a \Leftrightarrow \rho_{\mathcal{R}ab(ab)^{(1)},\mathrm{lann}(ab)}(1)a = a \Leftrightarrow \mathrm{lann}(ab) \subseteq \mathrm{lann}(a),$$

$$b(ab)^{(1)}ab = b \Leftrightarrow b\rho_{(ab)^{(1)}ab\mathcal{R},\mathrm{rann}(ab)}(1) = b \Leftrightarrow \mathrm{rann}(ab) \subseteq \mathrm{rann}(b)$$

and

$$b(ab)^{(1)}ab = b \Leftrightarrow b\rho_{\mathcal{R}ab,\mathrm{lann}((ab)^{(1)}ab)}(1) \Leftrightarrow \mathcal{R}b \subseteq \mathcal{R}ab.$$

This proves (1) and (2). □

The next theorem is a consequence of Theorems 3.1 and 3.2. It relates {1, 2}-inverses to projectors.

Theorem 3.4. *Let $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 2\}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(x)}$.
3. $\varphi_{xa} = \rho_{x\mathcal{R},\mathrm{rann}(a)}$.
4. ${}_{ax}\varphi = \rho_{\mathcal{R}x,\mathrm{lann}(a)}$.
5. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(x)}$.

We note that Theorem 3.1(1)(2)(3) is related to [3, Lemma 1.1(f) and (2.28)], whereas Lemma 3.3 and Theorem 3.4 generalize [3, Lemma 1.2], [3, Ex. 2.21] and [3, Corollary 2.7] (see also [41, Corollary 1.3.2]), respectively.

Remark 3.5. *Let $a \in \mathcal{R}$ and $x \in a\{1, 2\}$. Using an argument similar to the one used in the proof of the* if *part of [3, Theorem 1.2] we get:*

1. *If $a \in \mathcal{R}$, $x \in a\{1\}$ and $x\mathcal{R} = xa\mathcal{R}$ (or $\mathcal{R}x = \mathcal{R}ax$), then $x \in a\{1, 2\}$.*
2. *If $a \in \mathcal{R}$, $x \in a\{2\}$ and $a\mathcal{R} = ax\mathcal{R}$ (or $\mathcal{R}a = \mathcal{R}xa$), then $x \in a\{1, 2\}$.*

The characterizations of Moore-Penrose, core, and dual core inverses using φ_{ax} (resp. ${}_{ax}\varphi$) and φ_{xa} (resp. ${}_{xa}\varphi$) appear in Section 7.2 as a consequence of some general results. Now, we present an example to show that only these group endomorphisms are not sufficient to characterize x as the Drazin, the Moore-Penrose, the core, or the dual core inverse.

Example 3.6. *Consider the real matrices* $X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ *and* $A = \begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$. *Then* $A^{\#} = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 0 \end{pmatrix}$, $A^{\dagger} = \begin{pmatrix} 1/4 & 0 \\ -1/4 & 0 \end{pmatrix}$, $A^{\textcircled{\#}} = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$, *and* $A_{\textcircled{\#}} = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}$. *We have:*

1. $AX = XA = AA^{\#} = A^{\#}A$ *if and only if* $x_{1,1} = 1/2$, $x_{2,1} = 0$ *and* $x_{2,2} = x_{1,2} + 1/2$ *if and only if* $X \in A\{1, 5\}$.
2. $AX = AA^{\dagger}$ *and* $XA = A^{\dagger}A$ *if and only if* $x_{1,1} = -x_{2,1} = 1/4$, *and* $x_{1,2} = x_{2,2}$ *if and only if* $X \in A\{1, 3, 4\}$.
3. $AX = AA^{\textcircled{\#}}$ *and* $XA = A^{\textcircled{\#}}A$ *if and only if* $x_{1,1} = 1/2$, $x_{2,1} = 0$ *and* $x_{1,2} = x_{2,2}$ *if and only* $X \in A\{3, 6\}$.
4. $AX = AA_{\textcircled{\#}}$ *and* $XA = A_{\textcircled{\#}}A$ *if and only if* $x_{1,1} = -x_{2,1} = 1/4$ *and* $x_{2,2} = x_{1,2} + 1/2$ *if and only* $X \in A\{4, 8\}$.

We have the following immediate result.

Theorem 3.7. *Let* $a \in \mathcal{R}$. *Then the following assertions are equivalent:*

1. $x \in a\{1, 5\}$.
2. $\varphi_{ax} = \varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$.
3. ${}_{ax}\varphi = {}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$.

The next theorem relates the Drazin inverse to projectors.

Theorem 3.8. *Let* $a, x \in \mathcal{R}$. *Then the following assertions are equivalent:*

1. $a \in \mathcal{R}^D$ *with index* $k \leq l$ *and* $x = a^D$.
2. $\varphi_{xa} = \varphi_{ax} = \rho_{a^l\mathcal{R},\mathrm{rann}(a^l)}$ *and* $x\mathcal{R} \subseteq a^l\mathcal{R}$.
3. $\varphi_{xa} = \varphi_{ax} = \rho_{a^l\mathcal{R},\mathrm{rann}(a^l)}$ *and* $\mathrm{rann}(a^l) \subseteq \mathrm{rann}(x)$.
4. ${}_{xa}\varphi = {}_{xa}\varphi = \rho_{\mathcal{R}a^l,\mathrm{lann}(a^l)}$ *and* $\mathcal{R}x \subseteq \mathcal{R}a^l$.
5. ${}_{xa}\varphi = {}_{xa}\varphi = \rho_{\mathcal{R}a^l,\mathrm{lann}(a^l)}$ *and* $\mathrm{lann}(a^l) \subseteq \mathrm{lann}(x)$.

Proof. The implications (1) ⇒ (2)–(5) follow from the definition of the Drazin inverse and Corollary 2.16.

If $\varphi_{xa} = \varphi_{ax} = \rho_{a^l\mathcal{R},\mathrm{rann}(a^l)}$, then $ax = xa$ and $xa^{l+1} = \rho_{a^l\mathcal{R},\mathrm{rann}(a^l)}(a^l) = a^l$. Thus, $x \in a\{5, 1^l\}$. Similarly, if ${}_{xa}\varphi = {}_{xa}\varphi = \rho_{\mathcal{R}a^l,\mathrm{lann}(a^l)}$, then $x \in a\{5, 1^l\}$.

(2) ⇒ (1): Assume that (2) holds. Then $x \in a\{5, 1^l\}$ and $xax = \rho_{a^l\mathcal{R},\mathrm{rann}(a^l)}(x) = x$. So, $a \in \mathcal{R}^D$ with index $k \leq l$ and $x = a^D$.

The proofs of the other implications are similar. □

4. {1}-inverses with prescribed principal and annihilator ideals

As a consequence of Theorem 4.1 below, we can assert that if we choose arbitrary right ideals $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{R}$ complementary to $\mathrm{rann}(a)$ and $a\mathcal{R}$, respectively, then there exists $x \in a\{1\}$ such that $xa\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(ax) = \mathcal{T}$. Similar considerations can be made for Theorems 4.2-4.8. Before we enunciate these theorems, we observe that if $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$, then there exists $z \in \mathcal{R}$ such that $az = \rho_{a\mathcal{R},\mathcal{T}}(1)$ and $aza = \rho_{a\mathcal{R},\mathcal{T}}(a) = a$. Hence, $a\{1\} \neq \emptyset$. Analogously, if $\mathcal{T}'$ is a left ideal of $\mathcal{R}$ and $\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}'$, then $a\{1\} \neq \emptyset$.

Theorem 4.1. *Let* $a, x \in \mathcal{R}$ *and* $\mathcal{S}, \mathcal{T}$ *be right ideals of* $\mathcal{R}$. *Then the following assertions are equivalent:*

1. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, *and* $\mathrm{rann}(ax) = \mathcal{T}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$ *and* $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$.
3. $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

Proof. (1) ⇒ (2): It follows from Theorem 3.1.

(2) ⇒ (3): Assume that (2) holds. Then, $x \in a\{1\}$ and

$$\rho_{\mathcal{S},\mathrm{rann}(a)}(x\rho_{a\mathcal{R},\mathcal{T}}(1)) + (1 - xa)x(1 - ax) = xa(x(ax)) + x - xax - xax + xaxax = xax + x - xax - xax + xax = x.$$

Thus, (3) holds with $a^{(1)} = y = x$.

(3) ⇒ (1): Assume that (3) holds. Using Lemma 2.15(1) we obtain, $ax = aa^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$, $xa = \rho_{\mathcal{S},\mathrm{rann}(a)}(a^{(1)}a)$ and $axa = a$. Hence, $x \in a\{1\}$, and by Theorem 3.1, $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(ax)}$ and $\varphi_{xa} = \rho_{xa\mathcal{R},\mathrm{rann}(a)}$.

Let $r \in \mathcal{R}$. Then

$$\begin{aligned} axr = 0 \Leftrightarrow aa^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(r) = 0 &\Leftrightarrow \{\exists s \in \mathcal{R},\ \rho_{a\mathcal{R},\mathcal{T}}(r) = as \text{ and } aa^{(1)}as = 0\} \\ &\Leftrightarrow \{\exists s \in \mathcal{R},\ \rho_{a\mathcal{R},\mathcal{T}}(r) = as \text{ and } as = 0\} \Leftrightarrow r \in \mathcal{T}. \end{aligned}$$

Thus, $\mathrm{rann}(ax) = \mathcal{T}$. By Theorem 3.1, $\varphi_{1-a^{(1)}a} = \rho_{\mathrm{rann}(a),a^{(1)}a\mathcal{R}}$. Then, $\rho_{\mathcal{S},\mathrm{rann}(a)}((1 - a^{(1)}a)\mathcal{R}) = \{0\}$ and

$$\rho_{\mathcal{S},\mathrm{rann}(a)}(r) = \rho_{\mathcal{S},\mathrm{rann}(a)}(a^{(1)}ar) + \rho_{\mathcal{S},\mathrm{rann}(a)}((1 - a^{(1)}a)r) = \rho_{\mathcal{S},\mathrm{rann}(a)}(a^{(1)}ar) = xar.$$

From here, $xa\mathcal{R} = \mathcal{S}$. Consequently, (1) holds. □

Theorem 4.1 generalizes [3, Theorem 2.12(a)(b)]. Using Theorem 3.1 and Lemma 2.15(2), we analogously prove the next result about {1}-inverses with given left principal and annihilator ideals.

Theorem 4.2. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, *and* $\mathrm{lann}(xa) = \mathcal{T}$.
2. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$.
3. $x = \rho_{\mathcal{R}a,\mathcal{T}}(1)a^{(1)}\rho_{\mathcal{S},\mathrm{lann}(a)}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

With similar proofs, we obtain the next six theorems. The first is about {1}-inverses with given right and left principal ideals.

Theorem 4.3. *Let $a, x \in \mathcal{R}$, $\mathcal{S}$ be a right ideal of $\mathcal{R}$, and $\mathcal{S}'$ be a left ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, *and* $\mathcal{R}ax = \mathcal{S}'$.
2. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$ *and* ${}_{ax}\varphi = \rho_{\mathcal{S}',\mathrm{lann}(a)}$.
3. $a\{1\} \neq \emptyset$ *and* $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{\mathcal{S}',\mathrm{lann}(a)}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

Now we give a theorem for {1}-inverses with given right and left annihilator ideals.

Theorem 4.4. *Let $a, x \in \mathcal{R}$, $\mathcal{T}$ be a right ideal of $\mathcal{R}$, and $\mathcal{T}'$ be a left ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$, $\mathrm{rann}(ax) = \mathcal{T}$, *and* $\mathrm{lann}(xa) = \mathcal{T}'$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}'}$.
3. $x = \rho_{\mathcal{R}a,\mathcal{T}'}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

The following two theorems are about {1}-inverses for which only a right principal or annihilator ideal is prefixed.

Theorem 4.5. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}$ be a right ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$.
2. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$.
3. $a\{1\} \neq \emptyset$ *and* $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)} + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

Theorem 4.6. *Let $a, x \in \mathcal{R}$ and $\mathcal{T}$ be a right ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$ *and* $\mathrm{rann}(ax) = \mathcal{T}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$.
3. $x = a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

In the next two theorems, we consider the case of {1}-inverses for which only a left principal or annihilator ideal is prefixed.

Theorem 4.7. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}$ be a left ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$ *and* $\mathcal{R}ax = \mathcal{S}$.
2. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$.
3. $a\{1\} \neq \emptyset$ *and* $x = a^{(1)}\rho_{\mathcal{S},\mathrm{lann}(a)}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

Theorem 4.8. *Let $a, x \in \mathcal{R}$ and $\mathcal{T}$ be left a ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1\}$ *and* $\mathrm{lann}(xa) = \mathcal{T}$.
2. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$.
3. $x = \rho_{\mathcal{R}a,\mathcal{T}}(1)a^{(1)} + (1 - a^{(1)}a)y(1 - aa^{(1)})$ *where* $a^{(1)} \in a\{1\}$ *and* $y \in \mathcal{R}$.

In the previous results, we can replace the hypothesis $a\{1\} \neq \emptyset$ by the condition $\rho_{\mathcal{S},\mathrm{rann}(a)}(1) \in a\mathcal{R}$ (resp. $\rho_{\mathcal{S},\mathrm{lann}(a)}(1) \in \mathcal{R}a$).

Part (1) of Proposition 4.9 generalizes [3, Corollary 2.9].

Proposition 4.9. *Let $a \in \mathcal{R}$ and $a^{(1)} \in a\{1\}$. Let $\mathcal{A} = \{a^{(1)} + (1 - a^{(1)}a)y(1 - aa^{(1)}) : y \in \mathcal{R}\}$. The following assertions hold:*

1. *Let $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$ such that $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$. If $a^{(1)}a\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(aa^{(1)}) = \mathcal{T}$, then $\mathcal{A} = \{x \in a\{1\} : xa\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(ax) = \mathcal{T}\}$.*
2. *Let $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{S} \oplus \mathrm{lann}(a)$. If $\mathcal{R}aa^{(1)} = \mathcal{S}$ and $\mathrm{lann}(a^{(1)}a) = \mathcal{T}$, then $\mathcal{A} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{S}$ and $\mathrm{lann}(xa) = \mathcal{T}\}$.*
3. *Let $\mathcal{S}$ be a right ideal of $\mathcal{R}$ and $\mathcal{S}'$ be a left ideal of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$ and $\mathcal{R} = \mathcal{S}' \oplus \mathrm{lann}(a)$. If $a^{(1)}a\mathcal{R} = \mathcal{S}$ and $\mathcal{R}aa^{(1)} = \mathcal{S}'$, then $\mathcal{A} = \{x \in a\{1\} : xa\mathcal{R} = \mathcal{S}$ and $\mathcal{R}ax = \mathcal{S}'\}$.*
4. *Let $\mathcal{T}$ be a right ideal of $\mathcal{R}$ and $\mathcal{T}'$ be a left ideal of $\mathcal{R}$ such that $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}'$. If $\mathrm{rann}(aa^{(1)}) = \mathcal{T}$ and $\mathrm{lann}(a^{(1)}a) = \mathcal{T}'$, then $\mathcal{A} = \{x \in a\{1\} : \mathrm{rann}(ax) = \mathcal{T}$ and $\mathrm{lann}(xa) = \mathcal{T}'\}$.*
5. *Let $\mathcal{S}$ be a right ideal of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$. If $a^{(1)}a\mathcal{R} = \mathcal{S}$, then $\mathcal{A} = \{x \in a\{1\} : xa\mathcal{R} = \mathcal{S}\}$.*
6. *Let $\mathcal{S}$ be a left ideal of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathrm{lann}(a)$. If $\mathcal{R}aa^{(1)} = \mathcal{S}$, then $\mathcal{A} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{S}\}$.*
7. *Let $\mathcal{T}$ be a right ideal of $\mathcal{R}$ such that $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$. If $\mathrm{rann}(aa^{(1)}) = \mathcal{T}$, then $\mathcal{A} = \{x \in a\{1\} : \mathrm{rann}(ax) = \mathcal{T}\}$.*
8. *Let $\mathcal{T}'$ be a left ideal of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}'$. If $\mathrm{lann}(a^{(1)}a) = \mathcal{T}'$, then $\mathcal{A} = \{x \in a\{1\} : \mathrm{lann}(xa) = \mathcal{T}'\}$.*

Proof. (1): By Theorem 4.1, $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, and $\mathrm{rann}(ax) = \mathcal{T}$ if and only if $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$ and $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$. Then $ax = aa^{(1)}$ and $xa = a^{(1)}a$. Consequently, $x = a^{(1)} + (1 - a^{(1)}a)y(1 - aa^{(1)})$ with $y = x - a^{(1)}$. Conversely if there exists $y \in \mathcal{R}$ such that $x = a^{(1)} + (1 - a^{(1)}a)y(1 - aa^{(1)})$, then $ax = aa^{(1)}$ and $xa = a^{(1)}a$. This implies that $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, and $\mathrm{rann}(ax) = \mathcal{T}$. This shows that (1) holds.

Using Theorems 4.2-4.8, parts (2)-(8) can be similarly proved. □

5. {2}-inverses with prescribed principal and annihilator ideals

Taking into account that $x \in a\{2\}$ if and only if $a \in x\{1\}$, applying the results of Section 4 we obtain corresponding results for {2}-inverses x of a such that $ax\mathcal{R}$, $\mathrm{rann}(xa)$, $\mathcal{R}xa$, and/or $\mathrm{lann}(ax)$ are given. In this section, we further study {2}-inverses with prescribed principal and/or annihilator ideals. We begin with the following two results about uniqueness and relations to projectors.

Theorem 5.1. *Let $a \in \mathcal{R}$, let $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$. If a has a {2}-inverse x such that $x\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(x) = \mathcal{T}$, then $\varphi_{ax} = \rho_{\varphi_a(\mathcal{S}),\mathcal{T}}$ and $\varphi_{xa} = \rho_{\mathcal{S},\varphi_a^{-1}(\mathcal{T})}$. If there exists, this {2}-inverse x is unique and will be denoted by $a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$.*

Proof. Assume that $x \in a\{2\}$, $x\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(x) = \mathcal{T}$. We have $ax\mathcal{R} = \varphi_a(\mathcal{S})$ and $\mathrm{rann}(xa) = \varphi_a^{-1}(\mathcal{T})$. Thus, by Theorem 3.2, $\varphi_{ax} = \rho_{\varphi_a(\mathcal{S}),\mathcal{T}}$ and $\varphi_{xa} = \rho_{\mathcal{S},\varphi_a^{-1}(\mathcal{T})}$. Assume that $x_1, x_2 \in a\{2\}$, $x_1\mathcal{R} = x_2\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(x_1) = \mathrm{rann}(x_2) = \mathcal{T}$. Using Lemma 2.15(1) we get, $x_1 = \rho_{\mathcal{S},\varphi_a^{-1}(\mathcal{T})}(x_1) = x_2ax_1 = x_2\rho_{\varphi_a(\mathcal{S}),\mathcal{T}}(1) = x_2$. This shows that x is unique. □

Using Theorem 3.2 and Lemma 2.15(2), we analogously obtain:

Theorem 5.2. *Let $a \in \mathcal{R}$ and let $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$. If a has a $\{2\}$-inverse x such that $\mathcal{R}x = \mathcal{S}$, and* $\mathrm{lann}(x) = \mathcal{T}$*, then ${}_{xa}\varphi = \rho_{{}_a\varphi(\mathcal{S}),\mathcal{T}}$ and ${}_{ax}\varphi = \rho_{\mathcal{S},{}_a\varphi^{-1}(\mathcal{T})}$. If there exists, this $\{2\}$-inverse x is unique and will be denoted by $a^{(2)}_{\mathrm{lprin}=\mathcal{S},\mathrm{lann}=\mathcal{T}}$.*

The next theorem gives necessary and sufficient conditions for the existence of $a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$. The equivalence (1) ⇔ (3) is a generalization of [9, Theorem 2.8] in complex Banach algebras and of [45, Theorem 2.4] in rings. In [9, Theorem 2.8], $\mathcal{S} = p\mathcal{R}$ and $\mathcal{T} = q\mathcal{R}$ with $p, q \in \mathcal{R}^\bullet$. In [45, Theorem 2.4], $\mathcal{S} = b\mathcal{R}$ and $\mathcal{T} = \mathrm{rann}(c)$ with $b, c \in \mathcal{R}$.

Theorem 5.3. *Let $a \in \mathcal{R}$ and let $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$. Then the following statements are equivalent:*

1. *$a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ exists.*
2. *There exists $x \in \mathcal{S}$ such that $\varphi_{ax} = \rho_{a\mathcal{S},\mathcal{T}}$ and* $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$.
3. *$\mathcal{R} = a\mathcal{S} \oplus \mathcal{T}$ and* $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$.
4. *There exists $x \in \mathcal{S}$ such that $xas = s$ for all $s \in \mathcal{S}$, $1 - ax \in \mathcal{T}$, and $xt = 0$ for all $t \in \mathcal{T}$.*

Proof. (2) ⇒ (3) and (1) ⇒ (4) are immediate.

(1) ⇒ (2): Suppose that $a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ exists and $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$. Then $x \in \mathcal{S}$ and, by Theorem 5.1, $\varphi_{ax} = \rho_{a\mathcal{S},\mathcal{T}}$ and $\varphi_{xa} = \rho_{\mathcal{S},\varphi_a^{-1}(\mathcal{T})}$. In particular, $xas = s$ for all $s \in \mathcal{S}$. From this last property, $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$.

(2) ⇒ (1): Assume that there exists $x \in \mathcal{S}$ such that $\varphi_{ax} = \rho_{a\mathcal{S},\mathcal{T}}$ and $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$. Then, $x\mathcal{R} \subseteq \mathcal{S}$, $\mathrm{rann}(ax) = \mathcal{T}$ and $axax = ax$. Since $x \in \mathcal{S}$, from the last equality we obtain, $xax - x \in \mathrm{rann}(a) \cap \mathcal{S}$. Thus, $xax = x$.

By Theorem 5.1, $\varphi_{xa} = \rho_{x\mathcal{R},\varphi_a^{-1}(\mathcal{T})}$. Let $s \in \mathcal{S}$ and $r, y \in \mathcal{R}$ be such that $s = xr + y$ is the unique decomposition of s as a sum of an element in $x\mathcal{R}$ and an element in $\varphi_a^{-1}(\mathcal{T})$. Then $0 = a(xr - s) + ay$ where $a(xr - s) \in a\mathcal{S}$ and $ay \in \mathcal{T}$. Since $a\mathcal{S} \cap \mathcal{T} = \{0\}$, it follows that $a(xr - s) = 0$. From this last equality, using that $xr - s \in \mathcal{S}$ and $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$, we obtain $s = xr$. Thus, $\mathcal{S} \subseteq x\mathcal{R}$.

We have proved that $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$.

(3) ⇒ (2): Assume that $\mathcal{R} = a\mathcal{S} \oplus \mathcal{T}$ and $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$. Then, there exists a unique $x \in \mathcal{S}$ such that $ax = \rho_{a\mathcal{S},\mathcal{T}}(1)$. Hence, $\varphi_{ax} = \rho_{a\mathcal{S},\mathcal{T}}$.

(4) ⇒ (1): Suppose that there exists $x \in \mathcal{S}$ such that $xas = s$ for all $s \in \mathcal{S}$, $1 - ax \in \mathcal{T}$, and $xt = 0$ for all $t \in \mathcal{T}$. Clearly, $x \in a\{2\}$. We also have, $x \in \mathcal{S} \Rightarrow x\mathcal{R} \subseteq \mathcal{S}$, $\{xas = s \text{ for all } s \in \mathcal{S}\} \Rightarrow \mathcal{S} \subseteq x\mathcal{R}$, $1 - ax \in \mathcal{T} \Rightarrow (1 - ax)\mathcal{R} \subseteq \mathcal{T}$, and $\{xt = 0 \text{ for all } t \in \mathcal{T}\} \Rightarrow \mathcal{T} \subseteq \mathrm{rann}(x)$. Since $\mathrm{rann}(x) = \mathrm{rann}(ax) = (1 - ax)\mathcal{R}$, we conclude that (1) holds. □

Using Theorem 5.2, we obtain the next theorem which is analogous to Theorem 5.3.

Theorem 5.4. *Let $a \in \mathcal{R}$ and let $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$. Then the following statements are equivalent:*

1. *$a^{(2)}_{\mathrm{lprin}=\mathcal{S},\mathrm{lann}=\mathcal{T}}$ exists.*
2. *There exists $x \in \mathcal{S}$ such that ${}_{xa}\varphi = \rho_{\mathcal{S}a,\mathcal{T}}$ and* $\mathrm{lann}(a) \cap \mathcal{S} = \{0\}$.
3. *$\mathcal{R} = \mathcal{S}a \oplus \mathcal{T}$ and* $\mathrm{lann}(a) \cap \mathcal{S} = \{0\}$.
4. *There exists $x \in \mathcal{S}$ such that $sax = s$ for all $s \in \mathcal{S}$, $1 - xa \in \mathcal{T}$, and $tx = 0$ for all $t \in \mathcal{T}$.*

The next theorem is about the elements $x \in a\{2\}$ such that $x\mathcal{R}$ and $\mathcal{R}x$ are given. If $\mathcal{A} \subseteq \mathcal{R}$, we consider $\mathrm{rann}(\mathcal{A}) = \{r \in \mathcal{R} : ar = 0 \text{ for all } a \in \mathcal{A}\}$ and $\mathrm{lann}(\mathcal{A}) = \{r \in \mathcal{R} : ra = 0 \text{ for all } a \in \mathcal{A}\}$.

Theorem 5.5. *Let $a \in \mathcal{R}$, $\mathcal{S}$ be a right ideal of $\mathcal{R}$ and $\mathcal{S}'$ be a left ideal of $\mathcal{R}$. Then the following statements are equivalent:*

1. *There exists $x \in a\{2\}$ such that $x\mathcal{R} = \mathcal{S}$ and $\mathcal{R}x = \mathcal{S}'$.*
2. *$\mathcal{R} = a\mathcal{S} \oplus \mathrm{rann}(\mathcal{S}')$ and $\mathcal{R} = \mathcal{S}'a \oplus \mathrm{lann}(\mathcal{S})$.*
3. *There exists $x \in \mathcal{S} \cap \mathcal{S}'$ such that $xas = s$ for all $s \in \mathcal{S}$ and $sax = s$ for all $s \in \mathcal{S}'$.*
4. *There exists $x \in \mathcal{S} \cap \mathcal{S}'$ such that $\varphi_{ax} = \rho_{a\mathcal{S},\mathrm{rann}(\mathcal{S}')}$ and ${}_{xa}\varphi = \rho_{\mathcal{S}'a,\mathrm{lann}(\mathcal{S})}$.*

If the above x exists, then it is unique. This generalized inverse is equal to $a^{(2)}_{\text{rprin}=\mathcal{S},\text{rann}=\text{rann}(\mathcal{S}')}$ *(resp.* $a^{(2)}_{\text{lprin}=\mathcal{S}',\text{lann}=\text{lann}(\mathcal{S})}$*) and will be denoted by* $a^{(2)}_{\text{rprin}=\mathcal{S},\text{lprin}=\mathcal{S}'}$.

Proof. (1) ⇒ (2)(4): Since $\text{rann}(x) = \text{rann}(\mathcal{R}x)$ and $\text{lann}(x) = \text{lann}(x\mathcal{R})$, the implications follow from Theorems 5.3 and 5.4.

(2) ⇒ (1): Suppose that (2) holds. Assume that $s \in \mathcal{S}$ and $as = 0$. Let $z \in \mathcal{S}'$ such that ${}_{za}\varphi = \rho_{\mathcal{S}'a,\text{lann}(\mathcal{S})}$. Then $s = 1s = \rho_{\mathcal{S}'a,\text{lann}(\mathcal{S})}(1)s = zas = 0$. This shows that $\text{rann}(a) \cap \mathcal{S} = \{0\}$. Similarly, $\text{lann}(a) \cap \mathcal{S}' = \{0\}$. By Theorems 5.3 and 5.4, $a^{(2)}_{\text{rprin}=\mathcal{S},\text{rann}=\text{rann}(\mathcal{S}')}$ and $a^{(2)}_{\text{lprin}=\mathcal{S}',\text{lann}=\text{lann}(\mathcal{S})}$ exist. Denote them by x and y, respectively. Since $\text{rann}(x) = \text{rann}(\mathcal{S}') = \text{rann}(\mathcal{R}y) = \text{rann}(y)$ and $\text{lann}(y) = \text{lann}(\mathcal{S}) = \text{lann}(x\mathcal{R}) = \text{lann}(x)$, it follows that $y(ax) = y\rho_{a\mathcal{S},\text{rann}(\mathcal{S}')}(1) = y$, and $(ya)x = \rho_{\mathcal{S}'a,\text{lann}(\mathcal{S})}(1)x = x$. Hence, $x = y = a^{(2)}_{\text{rprin}=\mathcal{S},\text{rann}=\text{rann}(\mathcal{S}')} = a^{(2)}_{\text{lprin}=\mathcal{S}',\text{lann}=\text{lann}(\mathcal{S})}$. In particular, $x \in a\{2\}$, $x\mathcal{R} = \mathcal{S}$, $\mathcal{R}x = \mathcal{S}'$, and by Theorem 5.1 (or Theorem 5.2), this generalized inverse is unique.

(1) ⇔ (3): It is immediate.

(4) ⇒ (1): Since $x \in \mathcal{S} \cap \mathcal{S}'$, we have $x\mathcal{R} \subseteq \mathcal{S}$, $\mathcal{R}x \subseteq \mathcal{S}'$, $\text{lann}(\mathcal{S}) \subseteq \text{lann}(x)$, $\text{rann}(\mathcal{S}') \subseteq \text{rann}(x)$, $xax = x\rho_{a\mathcal{S},\text{rann}(\mathcal{S}')}(1) = \rho_{\mathcal{S}'a,\text{lann}(\mathcal{S})}(1)x = x$. If $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$, then $xas = \rho_{\mathcal{S}'a,\text{lann}(\mathcal{S})}(1)s = s$ and $s'ax = s'\rho_{a\mathcal{S},\text{rann}(\mathcal{S}')}(1) = s'$. Thus, $\mathcal{S} \subseteq x\mathcal{R}$, $\mathcal{S}' \subseteq \mathcal{R}x$. We conclude that (1) holds. □

Assume that there exist $b, c \in \mathcal{R}$ such that $\mathcal{S} = b\mathcal{R}$ and $\mathcal{S}' = \mathcal{R}c$. In this case, Theorem 5.5(2) coincides with [13, Proposition 2.7(ii)], whereas Theorem 5.5(3) coincides with the definition of the (b, c) inverse (see [13, Definition 1.3] and [33, page 103]).

The following theorem is about {2}-inverses x such that $\text{rann}(x)$ and $\text{lann}(x)$ are given.

Theorem 5.6. *Let* $a, x \in \mathcal{R}$*,* $\mathcal{T}$ *be a right ideal of* $\mathcal{R}$*, and* $\mathcal{T}'$ *be a left ideal of* $\mathcal{R}$*. Then the following statements are equivalent:*

1. $x \in a\{2\}$, $\text{rann}(x) = \mathcal{T}$, *and* $\text{lann}(x) = \mathcal{T}'$.
2. $\varphi_{ax} = \rho_{ax\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}xa,\mathcal{T}'}$, *and* $\text{rann}(a) \cap x\mathcal{R} = \{0\}$.
3. $\varphi_{ax} = \rho_{ax\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}xa,\mathcal{T}'}$, *and* $\text{lann}(a) \cap \mathcal{R}x = \{0\}$.
4. $\varphi_{ax} = \rho_{ax\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}xa,\mathcal{T}'}$, *and* $\mathcal{T} \subseteq \text{rann}(x)$.
5. $\varphi_{ax} = \rho_{ax\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}xa,\mathcal{T}'}$, *and* $\mathcal{T}' \subseteq \text{lann}(x)$.
6. $1 - ax \in \mathcal{T}$, $xt = 0$ *for all* $t \in \mathcal{T}$, $1 - xa \in \mathcal{T}'$, *and* $tx = 0$ *for all* $t \in \mathcal{T}'$.

If the above x exists, then it is unique and will be denoted by $a^{(2)}_{\text{lann}=\mathcal{T}',\text{rann}=\mathcal{T}}$.

Proof. The implications (1) ⇒ (2)(3) follow from Theorems 5.3 and 5.4, the equivalences (1) ⇔ (4) and (1) ⇔ (5) follow from Theorem 3.2, and the implication (1) ⇒ (6) is immediate.

Assume that (2) holds. Then $xax - x \in \text{rann}(a) \cap x\mathcal{R}$. Taking into account that $\text{rann}(a) \cap x\mathcal{R} = \{0\}$, we get $x \in a\{2\}$. Applying Theorem 3.2, we conclude that (1) holds. The proof of (3) ⇒ (1) is similar.

The proof of (6) ⇒ (1) is similar to the proof of (3) ⇒ (1) in Theorems 5.3 and 5.4.

Let $x, y \in a\{2\}$ be such that $\text{rann}(x) = \text{rann}(y) = \mathcal{T}$ and $\text{lann}(x) = \text{lann}(y) = \mathcal{T}'$. Since $x\{1\} \neq \emptyset$ and $y\{1\} \neq \emptyset$, from Lemma 2.3, $x\mathcal{R} = y\mathcal{R}$ and $x\mathcal{R} = \mathcal{R}y$. By Theorem 5.1 (or Theorem 5.2), $x = y$. This proves the uniqueness. □

We note that [45, Proposition 3.1], in which $\mathcal{T} = \text{rann}(c)$ and $\mathcal{T}' = \text{lann}(b)$ for some $b, c \in \mathcal{R}$, follows from Theorem 5.6(1)(6).

In [13], the connection of (b, c) inverses with the Mitsch $\mathcal{M}$ partial order [27] in a semigroup is established. Here, we consider $\mathcal{M}$ in relation to the {2}-inverses considered in this section obtaining results similar to [13, Lemmas 4.2 and 6.5, Theorems 4.3 and 6.6]. We recall that if $y, z \in \mathcal{R}$, then $y\mathcal{M}z$ if there exists $v, w \in \mathcal{R}$ such that $vz = vy = y = yw = zw$. Let $\mathcal{S}, \mathcal{T}$ (resp. $\mathcal{S}', \mathcal{T}'$) be right (resp. left) ideals of $\mathcal{R}$. Consider the pair of sets

$$Y_{a,\mathcal{S},\mathcal{S}'} = \{y \in \mathcal{R} : y \in a\{2\}, y \in \mathcal{S} \cap \mathcal{S}'\},$$
$$Z_{a,\mathcal{S},\mathcal{S}'} = \{z \in \mathcal{R} : z \in a\{2\}, \mathcal{S} \subseteq z\mathcal{R}, \mathcal{S}' \subseteq \mathcal{R}z\},$$

$$Y_{a,\mathcal{S},\mathcal{T}} = \{y \in \mathcal{R} : y \in a\{2\}, y \in \mathcal{S}, \mathcal{T} \subseteq \mathrm{rann}(y)\},$$
$$Z_{a,\mathcal{S},\mathcal{T}} = \{z \in \mathcal{R} : z \in a\{2\}, \mathcal{S} \subseteq z\mathcal{R}, \mathrm{rann}(z) \subseteq \mathcal{T}\},$$
$$Y_{a,\mathcal{S}',\mathcal{T}'} = \{y \in \mathcal{R} : y \in a\{2\}, y \in \mathcal{S}', \mathcal{T}' \subseteq \mathrm{lann}(y)\},$$
$$Z_{a,\mathcal{S}',\mathcal{T}'} = \{z \in \mathcal{R} : z \in a\{2\}, \mathcal{S}' \subseteq \mathcal{R}z, \mathrm{lann}(z) \subseteq \mathcal{T}'\},$$
$$Y_{a,\mathcal{T},\mathcal{T}'} = \{y \in \mathcal{R} : y \in a\{2\}, \mathcal{T} \subseteq \mathrm{rann}(y), \mathcal{T}' \subseteq \mathrm{lann}(y)\},$$
$$Z_{a,\mathcal{T},\mathcal{T}'} = \{z \in \mathcal{R} : z \in a\{2\}, \mathrm{rann}(z) \subseteq \mathcal{T}, \mathrm{lann}(z) \subseteq \mathcal{T}'\}.$$

Lemma 5.7. *Let $a \in \mathcal{R}$, and $\mathcal{S}$, $\mathcal{T}$ (resp. $\mathcal{S}'$, $\mathcal{T}'$) be right (resp. left) ideals of $\mathcal{R}$. Then:*

1. *$y\mathcal{M}z$ for each $y \in Y_{a,\mathcal{S},\mathcal{S}'}$ and $z \in Z_{a,\mathcal{S},\mathcal{S}'}$.*
2. *$y\mathcal{M}z$ for each $y \in Y_{a,\mathcal{S},\mathcal{T}}$ and $z \in Z_{a,\mathcal{S},\mathcal{T}}$.*
3. *$y\mathcal{M}z$ for each $y \in Y_{a,\mathcal{S}',\mathcal{T}'}$ and $z \in Z_{a,\mathcal{S}',\mathcal{T}'}$.*
4. *$y\mathcal{M}z$ for each $y \in Y_{a,\mathcal{T},\mathcal{T}'}$ and $z \in Z_{a,\mathcal{T},\mathcal{T}'}$.*

Proof. (1): Let $y \in Y_{a,\mathcal{S},\mathcal{S}'}$ and $z \in Z_{a,\mathcal{S},\mathcal{S}'}$. Then $y, z \in \{2\}$, $\varphi_{za} = \rho_{z\mathcal{R},\mathrm{rann}(za)}$, ${}_{az}\varphi = \rho_{\mathcal{R}z,\mathrm{lann}(az)}$, $y \in \mathcal{S} \subseteq z\mathcal{R} = za\mathcal{R}$, and $y \in \mathcal{S}' \subseteq \mathcal{R}z = \mathcal{R}az$. Therefore, $zay = \rho_{z\mathcal{R},\mathrm{rann}(za)}(y) = y$ and $yaz = \rho_{\mathcal{R}z,\mathrm{lann}(az)}(y) = y$. Now, as in the proof of [13, Lemma 4.2], $(ya)z = (ya)y = y = y(ay) = z(ay)$.

The proofs of (2), (3), and (4) are similar to the proof of (1). □

Theorem 5.8. *Let $a \in \mathcal{R}$, and $\mathcal{S}$, $\mathcal{T}$ (resp. $\mathcal{S}, \mathcal{T}'$) be right (resp. left) ideals of $\mathcal{R}$. Then:*

1. $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'} \Leftrightarrow x \in Y_{a,\mathcal{S},\mathcal{S}'} \cap Z_{a,\mathcal{S},\mathcal{S}'} \Leftrightarrow x = \max_{\mathcal{M}} Y_{a,\mathcal{S},\mathcal{S}'} = \min_{\mathcal{M}} Z_{a,\mathcal{S},\mathcal{S}'}$.
2. $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}} \Leftrightarrow x \in Y_{a,\mathcal{S},\mathcal{T}} \cap Z_{a,\mathcal{S},\mathcal{T}} \Leftrightarrow x = \max_{\mathcal{M}} Y_{a,\mathcal{S},\mathcal{T}} = \min_{\mathcal{M}} Z_{a,\mathcal{S},\mathcal{T}}$.
3. $x = a^{(2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'} \Leftrightarrow x \in Y_{a,\mathcal{S}',\mathcal{T}'} \cap Z_{a,\mathcal{S}',\mathcal{T}'} \Leftrightarrow x = \max_{\mathcal{M}} Y_{a,\mathcal{S}',\mathcal{T}'} = \min_{\mathcal{M}} Z_{a,\mathcal{S}',\mathcal{T}'}$.
4. $x = a^{(2)}_{\mathrm{rann}=\mathcal{T},\mathrm{lann}=\mathcal{T}'} \Leftrightarrow x \in Y_{a,\mathcal{T},\mathcal{T}'} \cap Z_{a,\mathcal{T},\mathcal{T}'} \Leftrightarrow x = \max_{\mathcal{M}} Y_{a,\mathcal{T},\mathcal{T}'} = \min_{\mathcal{M}} Z_{a,\mathcal{T},\mathcal{T}'}$.

Proof. (1): The equivalence $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'} \Leftrightarrow x \in Y_{a,\mathcal{S},\mathcal{S}'} \cap Z_{a,\mathcal{S},\mathcal{S}'}$ and the implication $x = \max_{\mathcal{M}} Y_{a,\mathcal{S},\mathcal{S}'} = \min_{\mathcal{M}} Z_{a,\mathcal{S},\mathcal{S}'} \Rightarrow x \in Y_{a,\mathcal{S},\mathcal{S}'} \cap Z_{a,\mathcal{S},\mathcal{S}'}$ are immediate. The implication $x \in Y_{a,\mathcal{S},\mathcal{S}'} \cap Z_{a,\mathcal{S},\mathcal{S}'} \Rightarrow x = \max_{\mathcal{M}} Y_{a,\mathcal{S},\mathcal{S}'} = \min_{\mathcal{M}} Z_{a,\mathcal{S},\mathcal{S}'}$ follows from Lemma 5.7(1).

The proofs of (2), (3), and (4) are similar to the proof of (1). □

6. {1, 2}-inverses with prescribed principal and annihilator ideals

Let $\mathcal{S}$, $\mathcal{T}$ be right ideals of $\mathcal{R}$. If $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}} \in a\{1\}$, then we write $x = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$. Similar meaning will have $a^{(1,2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'}$, $a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'}$, and $a^{(1,2)}_{\mathrm{lann}=\mathcal{T}',\mathrm{rann}=\mathcal{T}}$, where $\mathcal{S}'$ and $\mathcal{T}'$ are left ideals of $\mathcal{R}$.

Theorem 6.1 below characterizes $a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ for right ideals $\mathcal{S}$, $\mathcal{T}$ of $\mathcal{R}$. The equivalences (1) ⇔ (4) ⇔ (8) are a generalization of [3, Theorem 2.12(c) and Ex. 2.37] for finite complex matrices.

Theorem 6.1. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}$, $\mathcal{T}$ be right ideals of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$, $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, *and* $x \in \mathcal{S}$.
3. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$, $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, *and* $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
4. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$, $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, *and* $\mathcal{T} \subseteq \mathrm{rann}(x)$.
5. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathrm{rann}(ax) = \mathcal{T}$, *and* $x \in \mathcal{S}$.
6. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathrm{rann}(ax) = \mathcal{T}$, *and* $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
7. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathrm{rann}(ax) = \mathcal{T}$, *and* $\mathcal{T} \subseteq \mathrm{rann}(x)$.
8. $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$ *where* $a^{(1)} \in a\{1\}$.

Proof. The equivalences (1) ⇔ (2) and (1) ⇔ (4) follow from Theorem 3.4. Note that if $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, then $1 - xa \in \mathrm{lann}(\mathcal{S})$. Hence, by Theorem 3.4, (1) ⇔ (3).

The implications (1) ⇒ (5)(6)(7) are immediate. The implications (5) ⇒ (2), (6) ⇒ (3), and (7) ⇒ (4) follow from Theorem 3.1.

(1) ⇒ (8): By Theorem 4.1, $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1) + (1 - a^{(1)}a)y(1 - aa^{(1)})$ where $a^{(1)} \in a\{1\}$ and $y \in \mathcal{R}$. Then $ax = aa^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$ and $xa = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}a$. Let $r \in \mathcal{R}$ be such that $\rho_{a\mathcal{R},\mathcal{T}}(1) = ar$. Since $x \in a\{2\}$, we get $x = x(ax) = (xa)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1) = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}aa^{(1)}ar = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}ar = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$. This shows that (8) holds.

(8) ⇒ (1): By Theorem 4.1, $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(ax) = \mathcal{T}$. As in the proof of (1) ⇒ (8), $xax = \rho_{\mathcal{S},\mathrm{rann}(a)}(a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)) = x$. Hence, (1) holds. □

Using Theorems 3.1, 3.4, and 4.2, we analogously obtain:

Theorem 6.2. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x = a^{(1,2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'}$.
2. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$, ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$ *and* $x \in \mathcal{S}$.
3. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$, ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$ *and* $\mathrm{rann}(\mathcal{S}) \subseteq \mathrm{rann}(x)$.
4. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$, ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$ *and* $\mathcal{T} \subseteq \mathrm{lann}(x)$.
5. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, $\mathrm{lann}(ax) = \mathcal{T}$, $x \in \mathcal{S}$.
6. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, $\mathrm{lann}(ax) = \mathcal{T}$, $\mathrm{rann}(\mathcal{S}) \subseteq \mathrm{rann}(x)$.
7. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, $\mathrm{lann}(ax) = \mathcal{T}$, $\mathcal{T} \subseteq \mathrm{lann}(x)$.
8. $x = \rho_{\mathcal{R}a,\mathcal{T}}(1)a^{(1)}\rho_{\mathcal{S},\mathrm{lann}(a)}(1)$ *for some* $a^{(1)} \in a\{1\}$.

Analogously we have the following six theorems with proofs similar to the proof of Theorem 6.1. As in Section 4, in the next results, we can replace the hypothesis $a\{1\} \neq \emptyset$ by the condition $\rho_{\mathcal{S},\mathrm{rann}(a)}(1) \in a\mathcal{R}$ (resp. $\rho_{\mathcal{S},\mathrm{lann}(a)}(1) \in \mathcal{R}a$).

Theorem 6.3. *Let $a, x \in \mathcal{R}$, $\mathcal{S}$ be a right ideal of $\mathcal{R}$, and $\mathcal{S}'$ be left a ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'}$.
2. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, ${}_{ax}\varphi = \rho_{\mathcal{S}',\mathrm{lann}(a)}$ *and* $x \in \mathcal{S} \cup \mathcal{S}'$.
3. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, ${}_{ax}\varphi = \rho_{\mathcal{S}',\mathrm{lann}(a)}$ *and* $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
4. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$, ${}_{ax}\varphi = \rho_{\mathcal{S}',\mathrm{lann}(a)}$ *and* $\mathrm{rann}(\mathcal{S}') \subseteq \mathrm{rann}(x)$.
5. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathcal{R}ax = \mathcal{S}'$, $x \in \mathcal{S} \cup \mathcal{S}'$.
6. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathcal{R}ax = \mathcal{S}'$, $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
7. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, $\mathcal{R}ax = \mathcal{S}'$, $\mathrm{rann}(\mathcal{S}') \subseteq \mathrm{rann}(x)$.
8. $a\{1\} \neq \emptyset$ *and* $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}\rho_{\mathcal{S}',\mathrm{lann}(a)}(1)$ *where* $a^{(1)} \in a\{1\}$.

Theorem 6.4. *Let $a, x \in \mathcal{R}$, $\mathcal{T}$ be a right ideal of $\mathcal{R}$, and $\mathcal{T}'$ be left a ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x = a^{(1,2)}_{\mathrm{lann}=\mathcal{T}',\mathrm{rann}=\mathcal{T}}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}'}$ *and* $\mathcal{T} \subseteq \mathrm{rann}(x)$.
3. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$, ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}'}$ *and* $\mathcal{T}' \subseteq \mathrm{lann}(x)$.
4. $x \in a\{1\}$, $\mathrm{rann}(ax) = \mathcal{T}$, $\mathrm{lann}(ax) = \mathcal{T}'$, $\mathcal{T} \subseteq \mathrm{rann}(x)$.
5. $x \in a\{1\}$, $\mathrm{rann}(ax) = \mathcal{T}$, $\mathrm{lann}(ax) = \mathcal{T}'$, $\mathcal{T}' \subseteq \mathrm{lann}(x)$.
6. $x = \rho_{\mathcal{R}a,\mathcal{T}'}(1)a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$ *where* $a^{(1)} \in a\{1\}$.

Theorem 6.5. *Let $a, x \in \mathcal{R}$ and $\mathcal{S}$ be a right ideal of $\mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 2\}$ *and* $x\mathcal{R} = \mathcal{S}$.

2. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$ *and* $x \in \mathcal{S}$.
3. $\varphi_{xa} = \rho_{\mathcal{S},\mathrm{rann}(a)}$ *and* $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
4. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, *and* $x \in \mathcal{S}$.
5. $x \in a\{1\}$, $xa\mathcal{R} = \mathcal{S}$, *and* $\mathrm{lann}(\mathcal{S}) \subseteq \mathrm{lann}(x)$.
6. $a\{1\} \neq \emptyset$ *and* $x = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)a^{(1)}$ *where* $a^{(1)} \in a\{1\}$.

Theorem 6.6. *Let* $a, x \in \mathcal{R}$ *and* $\mathcal{T}$ *be a right ideal of* $\mathcal{R}$. *Then the following assertions are equivalent:*

1. $x \in a\{1, 2\}$ *and* $\mathrm{rann}(x) = \mathcal{T}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathcal{T}}$ *and* $\mathcal{T} \subseteq \mathrm{rann}(x)$.
3. $x \in a\{1\}$, $\mathrm{rann}(ax) = \mathcal{T}$, *and* $\mathcal{T} \subseteq \mathrm{rann}(x)$.
4. $x = a^{(1)}\rho_{a\mathcal{R},\mathcal{T}}(1)$ *where* $a^{(1)} \in a\{1\}$.

Theorem 6.7. *Let* $a, x \in \mathcal{R}$ *and* $\mathcal{S}$ *be a left ideal of* $\mathcal{R}$. *Then the following assertions are equivalent:*

1. $x \in a\{1, 2\}$ *and* $\mathcal{R}x = \mathcal{S}$.
2. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$ *and* $x \in \mathcal{S}$.
3. ${}_{ax}\varphi = \rho_{\mathcal{S},\mathrm{lann}(a)}$ *and* $\mathrm{rann}(\mathcal{S}) \subseteq \mathrm{rann}(x)$.
4. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, $x \in \mathcal{S}$.
5. $x \in a\{1\}$, $\mathcal{R}ax = \mathcal{S}$, *and* $\mathrm{rann}(\mathcal{S}) \subseteq \mathrm{rann}(x)$.
6. $a\{1\} \neq \emptyset$ *and* $x = a^{(1)}\rho_{\mathcal{S},\mathrm{lann}(a)}(1)$ *for some* $a^{(1)} \in a\{1\}$.

Theorem 6.8. *Let* $a, x \in \mathcal{R}$ *and* $\mathcal{T}$ *be left a ideal of* $\mathcal{R}$. *Then the following assertions are equivalent:*

1. $x \in a\{1, 2\}$ *and* $\mathrm{lann}(x) = \mathcal{T}$.
2. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathcal{T}}$ *and* $\mathcal{T} \subseteq \mathrm{lann}(x)$.
3. $x \in a\{1\}$, $\mathrm{lann}(ax) = \mathcal{T}$, $\mathcal{T} \subseteq \mathrm{lann}(x)$.
4. $x = \rho_{\mathcal{R}a,\mathcal{T}}(1)a^{(1)}$ *for some* $a^{(1)} \in a\{1\}$.

If the $\{1, 2\}$-inverses characterized in Theorems 6.5-6.8 exist, then they are not necessarily unique. To see this note that if $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger} \cap \mathcal{R}^{\textcircled{\#}} \cap \mathcal{R}_{\textcircled{\#}}$, then $a^{\#}, a^{\dagger}, a^{\textcircled{\#}}, a_{\textcircled{\#}} \in a\{1, 2\}$, $a^{\#}\mathcal{R} = a^{\textcircled{\#}}\mathcal{R} = a\mathcal{R}$, $a^{\dagger}\mathcal{R} = a_{\textcircled{\#}}\mathcal{R} = a^*\mathcal{R}$, $\mathrm{rann}(a^{\#}) = \mathrm{rann}(a_{\textcircled{\#}}) = \mathrm{rann}(a)$, $\mathrm{rann}(a^{\dagger}) = \mathrm{rann}(a^{\textcircled{\#}}) = \mathrm{rann}(a^*)$, $\mathcal{R}a^{\#} = \mathcal{R}a_{\textcircled{\#}} = \mathcal{R}a$, $\mathcal{R}a^{\dagger} = \mathcal{R}a^{\textcircled{\#}} = \mathcal{R}a^*$, $\mathrm{lann}(a^{\#}) = \mathrm{lann}(a^{\textcircled{\#}}) = \mathrm{lann}(a)$, and $\mathrm{lann}(a^{\dagger}) = \mathrm{lann}(a_{\textcircled{\#}}) = \mathrm{lann}(a^*)$.

Theorems 6.9 and 6.10 give other characterizations of $a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ and $a^{(1,2)}_{\mathrm{lprin}=\mathcal{S},\mathrm{lann}=\mathcal{T}}$, respectively.

Theorem 6.9. *Let* $a \in \mathcal{R}$. *Let* $\mathcal{S}$, $\mathcal{T}$ *be right ideals of* $\mathcal{R}$ *such that* $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$ *and* $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$. *Let* $(\varphi_a)_{|\mathcal{S}}$ *be the group isomorphism obtained by the restriction of* φ_a *from* $\mathcal{S}$ *to* $a\mathcal{R}$. *Let* $\psi : \mathcal{R} \to \mathcal{R}$ *be the group endomorphism given by* $\psi(r) = ((\varphi_a)_{|\mathcal{S}})^{-1}(\rho_{a\mathcal{R},\mathcal{T}}(r))$ *for each* $r \in \mathcal{R}$. *Then the following assertions are equivalent:*

1. *b is the unique element in $\mathcal{S}$ such that* $ab = \rho_{a\mathcal{R},\mathcal{T}}(1)$.
2. $\psi = \varphi_b$.
3. $b = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$.

Proof. Note that $\ker(\psi) = \mathcal{T}$ and $\mathrm{im}(\psi) = \mathcal{S}$. Since $\varphi_a(\rho_{\mathcal{S},\mathrm{rann}(a)}(1)) = \varphi_a(1) = \rho_{a\mathcal{R},\mathcal{T}}(a)$, we obtain $\psi(a) = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)$.

(1) ⇒ (2): Let $r, s \in \mathcal{R}$ be such that $\psi(r) = s$. We have $\psi(r) = s$ if and only if $s \in \mathcal{S}$ and $(\varphi_a)_{|\mathcal{S}}(s) = \rho_{a\mathcal{R},\mathcal{T}}(r)$. Thus, $as = (\varphi_a)_{|\mathcal{S}}(s) = \rho_{a\mathcal{R},\mathcal{T}}(r) = \rho_{a\mathcal{R},\mathcal{T}}(1)r = abr$. Then $s = br$. From here, $\psi = \varphi_b$.

(2) ⇒ (3): Suppose that $\psi = \varphi_b$. Then $b\mathcal{R} = \mathrm{im}(\psi) = \mathcal{S}$, $\mathrm{rann}(b) = \ker(\psi) = \mathcal{T}$, $ba = \varphi_b(a) = \psi(a) = \rho_{\mathcal{S},\mathrm{rann}(a)}(1)$, $aba = a\rho_{\mathcal{S},\mathrm{rann}(a)}(1) = a$, and $bab = \rho_{\mathcal{S},\mathrm{rann}(a)}(b) = b$. Therefore, (3) holds.

(3) ⇒ (1): Assume that (3) holds. Then, $b \in b\mathcal{R} = \mathcal{S}$ and, by Theorem 3.4, $\varphi_{ab} = \rho_{a\mathcal{R},\mathcal{T}}$. In particular, $ab = \rho_{a\mathcal{R},\mathcal{T}}(1)$. Since $\mathcal{S} \cap \mathrm{rann}(a) = \{0\}$, there is a unique element that satisfies (1). □

Theorem 6.9 can be viewed as a generalization of [8, Theorem 6.2.1] about the equivalence of three definitions of $\{1, 2\}$-inverses of linear transformations with prescribed range and null spaces. Analogously to Theorem 6.9, we get:

Theorem 6.10. *Let $a \in \mathcal{R}$. Let $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{S} \oplus \mathrm{lann}(a)$ and $\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}$. Let $({}_a\varphi)_{|\mathcal{S}}$ be the group isomorphism obtained as the restriction of ${}_a\varphi$ from $\mathcal{S}$ to $\mathcal{R}a$. Let $\psi : \mathcal{R} \to \mathcal{R}$ be the group endomorphism given by $\psi(r) = (({}_a\varphi)_{|\mathcal{S}})^{-1}(\rho_{\mathcal{R}a,\mathcal{T}}(r))$ for each $r \in \mathcal{R}$. Then the following assertions are equivalent:*

1. *b is the unique element in $\mathcal{S}$ such that $ba = \rho_{\mathcal{R}a,\mathcal{T}}(1)$.*
2. *$\psi = {}_b\varphi$.*
3. *$b = a^{(1,2)}_{\mathrm{lprin}=\mathcal{S},\mathrm{lann}=\mathcal{T}}$.*

The next theorem generalizes [44, Theorem 3.1] for finite matrices over an associative ring and [9, Theorem 3.3] in complex Banach algebras with $\mathcal{S} = p\mathcal{R}$, $\mathcal{T} = q\mathcal{R}$, and $p, q \in \mathcal{R}^\bullet$. The proof of (2) ⇒ (3) ⇒ (1) is similar to the one presented for Theorem 3.3 in [9]. Here we use Theorem 5.3 to prove (3) ⇒ (1).

Theorem 6.11. *Let $a \in \mathcal{R}$ and let $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$. Then the following statements are equivalent:*

1. *$a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ exists.*
2. *$\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$.*
3. *$\mathcal{R} = a\mathcal{S} \oplus \mathcal{T}$, $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$, and $a\mathcal{R} \cap \mathcal{T} = \{0\}$.*
4. *There exists $x \in \mathcal{S}$ such that $\varphi_{ax} = \rho_{a\mathcal{S},\mathcal{T}}$, $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$, and $a\mathcal{R} \cap \mathcal{T} = \{0\}$.*

Proof. (1) ⇒ (2): It follows from the definition of $a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ and Theorem 3.4.

(2) ⇒ (3): Suppose that $\mathcal{R} = a\mathcal{R} \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$. Then $a\mathcal{R} \cap \mathcal{T} = \{0\}$ and $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$. Clearly, $a\mathcal{S} \subseteq a\mathcal{R}$. Let $t \in a\mathcal{R}$. Then there exists $s \in \mathcal{R}$ such that $t = as = a\rho_{\mathcal{S},\mathrm{rann}(a)}(s)$. Hence, $a\mathcal{R} \subseteq a\mathcal{S}$.

(3) ⇒ (1): Assume that $\mathcal{R} = a\mathcal{S} \oplus \mathcal{T}$, $\mathrm{rann}(a) \cap \mathcal{S} = \{0\}$, and $a\mathcal{R} \cap \mathcal{T} = \{0\}$. By Theorem 5.3, a has a $\{2\}$-inverse x such that $x\mathcal{R} = \mathcal{S}$ and $\mathrm{rann}(x) = \mathcal{T}$. Then $axa - a \in a\mathcal{R} \cap \mathcal{T}$, and consequently $axa = a$. Hence, $x = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$.

(3) ⇔ (4): It follows from Theorem 5.3. □

Using Theorems 3.4 and 5.4, we obtain the following result.

Theorem 6.12. *Let $a \in \mathcal{R}$ and let $\mathcal{S}, \mathcal{T}$ be left ideals of $\mathcal{R}$. Then the following statements are equivalent:*

1. *$a^{(1,2)}_{\mathrm{lprin}=\mathcal{S},\mathrm{lann}=\mathcal{T}}$ exists.*
2. *$\mathcal{R} = \mathcal{R}a \oplus \mathcal{T}$ and $\mathcal{R} = \mathcal{S} \oplus \mathrm{lann}(a)$.*
3. *$\mathcal{R} = \mathcal{S}a \oplus \mathcal{T}$, $\mathrm{lann}(a) \cap \mathcal{S} = \{0\}$, and $\mathcal{R}a \cap \mathcal{T} = \{0\}$.*
4. *There exists $x \in \mathcal{S}$ such that ${}_{xa}\varphi = \rho_{\mathcal{S}a,\mathcal{T}}$, $\mathrm{lann}(a) \cap \mathcal{S} = \{0\}$, and $\mathcal{R}a \cap \mathcal{T} = \{0\}$.*

We note that (2) ⇒ (1) in Theorem 6.11 (resp. Theorem 6.12) follows also from Theorem 6.9 (resp. Theorem 6.10). Now we obtain a theorem that gives necessary and sufficient conditions for the existence of $\{1, 2\}$-inverses with given right and left principal ideals.

Theorem 6.13. *Let $a \in \mathcal{R}$, let $\mathcal{S}$ be a right ideal of $\mathcal{R}$, and $\mathcal{S}'$ be a left ideal of $\mathcal{R}$. Then the following statements are equivalent:*

1. *$a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'}$ exists.*
2. *$\mathcal{R} = a\mathcal{R} \oplus \mathrm{rann}(\mathcal{S}')$, $\mathcal{R} = \mathcal{S} \oplus \mathrm{rann}(a)$, $\mathcal{R} = \mathcal{R}a \oplus \mathrm{lann}(\mathcal{S})$, and $\mathcal{R} = \mathcal{S}' \oplus \mathrm{lann}(a)$.*
3. *$\mathcal{R} = a\mathcal{S} \oplus \mathrm{rann}(\mathcal{S}')$, $a\mathcal{R} \cap \mathrm{rann}(\mathcal{S}') = \{0\}$, $\mathcal{R} = \mathcal{S}a \oplus \mathrm{lann}(\mathcal{S})$, and $\mathcal{R}a \cap \mathrm{lann}(\mathcal{S}) = \{0\}$.*
4. *There exists $x \in \mathcal{S} \cap \mathcal{S}'$ such that $\varphi_{ax} = \rho_{a\mathcal{S},\mathrm{rann}(\mathcal{S}')}$, $a\mathcal{R} \cap \mathrm{rann}(\mathcal{S}') = \{0\}$, ${}_{xa}\varphi = \rho_{\mathcal{S}'a,\mathrm{lann}(\mathcal{S})}$, and $\mathcal{R}a \cap \mathrm{lann}(\mathcal{S}) = \{0\}$.*

Proof. From Theorems 6.11 and 6.12, we get (1) ⇒ (2) ⇒ (3). The implication (1) ⇒ (4) follows from Theorem 3.4. The proof of (3) ⇒ (1) is similar to the proof of (2) ⇒ (1) in Theorem 5.5 and is based on Theorems 6.11 and 6.12. The implication (4) ⇒ (3) is immediate. □

From previous results, we derive the next sufficient conditions for right/left ideal of $\mathcal{R}$ to be principal/annhililator ideals of idempotent elements of $\mathcal{R}$.

Corollary 6.14. *Let $a \in \mathcal{R}$, $\mathcal{S}, \mathcal{T}$ be right ideals of $\mathcal{R}$ and $\mathcal{S}', \mathcal{T}'$ be left ideals of $\mathcal{R}$. The following assertions hold:*

1. *If any of the equivalent statements (1)-(4) of Theorem 5.3 (or Theorem 6.11) holds, then there exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{S} = p\mathcal{R}$ and $\mathcal{T} = \mathrm{rann}(q)$.*
2. *If any of the equivalent statements (1)-(4) of Theorem 5.4 (or Theorem 6.12) holds, then there exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{S} = \mathcal{R}p$ and $\mathcal{T} = \mathrm{lann}(q)$.*
3. *If any of the equivalent statements (1)-(4) of Theorem 5.5 (or Theorem 6.13) holds, then there exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{S} = p\mathcal{R}$ and $\mathcal{S}' = \mathcal{R}q$.*
4. *If any of the equivalent statements (1)-(6) of Theorem 5.6 holds, then there exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{T} = \mathrm{rann}(p)$ and $\mathcal{T}' = \mathrm{lann}(q)$.*

Proof. (1): By Theorem 5.3 (resp. Theorem 6.11), $x = a^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$ (resp. $x = a^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}}$) exists and the conclusion follows taking $p = xa$ and $q = ax$. The rest of the proof is similar. □

7. Particular classes of {1}, {2}, and {1, 2}-inverses

In this section, we apply previous results to study particular classes of {1}, {2}, and {1, 2}-inverses. We also give an illustrative example with a matrix over a field.

7.1. {1, 3}, {1, 4}, {1, 3, 4}, {1, 3, 6}, {1, 4, 8}, {1, 3, 7}, and {1, 4, 9}-inverses

For {1, 3}-inverses we have:

Theorem 7.1. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 3\}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$.
3. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$.

Proof. (1) ⇒ (2): It follows from Theorem 3.1 and the equality $\mathrm{rann}(ax) = \mathrm{rann}(a^*)$.

(2) ⇒ (1): If $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$, then $x \in a\{1\}$, $ax \in R^\bullet$, $ax\mathcal{R} = a\mathcal{R}$, and $\mathrm{rann}(ax) = \mathrm{rann}(a^*)$. By Lemma 2.7, $ax \in R^{\mathrm{sym}}$, i.e, $x \in a\{3\}$.

(1) ⇔ (3): It is analogous to the proof of (1) ⇔ (2). □

Similar to Theorem 7.1, we obtain:

Theorem 7.2. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 4\}$.
2. $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$.
3. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$.

As a consequence of Theorems 7.1 and 7.2, we obtain the next theorem.

Theorem 7.3. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 3, 4\}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$ *and* $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$.
3. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$.
4. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$ *and* $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$.
5. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$.

In the next two theorems, we give the projectors associated with {1, 3, 6} and {1, 4, 8}-inverses.

Theorem 7.4. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Each of the assertions*

1. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$ *and* $\varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$,

2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$,
3. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$ *and* $\varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$,
4. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$,

implies $x \in a\{1, 3, 6\}$.

Proof. Suppose that (1) holds. By Theorem 7.1, $x \in a\{1, 3\}$. Since $xa^2 = \rho_{a\mathcal{R},\mathrm{rann}(a)}(a) = a$, we have $x \in a\{6\}$. The remainder of the implications can be similarly proved. □

We analogously have:

Theorem 7.5. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Each of the assertions*

1. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$ *and* $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$,
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$,
3. ${}_{ax}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$ *and* $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$,
4. ${}_{ax}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}$ *and* ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$,

implies $x \in a\{1, 4, 8\}$.

From Theorems 7.1 and 7.2, we obtain the next two theorems.

Theorem 7.6. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 3, 7\}$.
2. $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$ *and* $x \in a\mathcal{R}$.
3. ${}_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)}$ *and* $\mathrm{lann}(a) \subseteq \mathrm{lann}(x)$.

We note that the elements of $a\{1, 3, 7\}$ are the *right core inverses* of a which are a particular case of right (b, c) inverse of a (see [14, 39, 40], in particular, [39, Theorem 5.1]).

Theorem 7.7. *Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$. Then the following assertions are equivalent:*

1. $x \in a\{1, 4, 9\}$.
2. $\varphi_{xa} = \rho_{a^*\mathcal{R},\mathrm{rann}(a)}$ *and* $\mathrm{rann}(a) \subseteq \mathrm{rann}(x)$.
3. ${}_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)}$ *and* $x \in \mathcal{R}a$.

In Section 4, we studied {1}-inverses with given principal and annihilator ideals. Now, we consider some examples of sets of these {1}-inverses. Let $a \in \mathcal{R}$. From Theorems 3.1 and 3.7,

$$\begin{aligned} a\{1, 5\} &= \{x \in a\{1\} : xa\mathcal{R} = a\mathcal{R} \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a)\} \\ &= \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a \text{ and } \mathrm{lann}(xa) = \mathrm{lann}(a)\}. \end{aligned}$$

Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}$. By Theorems 4.6, 4.7 and 7.1,

$$a\{1, 3\} = \{x \in a\{1\} : \mathrm{rann}(ax) = \mathrm{rann}(a^*)\} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a^*\},$$

and by Theorems 4.5, 4.8 and 7.2,

$$a\{1, 4\} = \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R}\} = \{x \in a\{1\} : \mathrm{lann}(xa) = \mathrm{lann}(a^*)\}.$$

From the above equalities, we get

$$\begin{aligned} a\{1, 3, 4\} &= \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R} \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a^*)\} \\ &= \{x \in a\{1\} : \mathrm{lann}(xa) = \mathrm{lann}(a^*) \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a^*)\} \\ &= \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R} \text{ and } \mathcal{R}ax = \mathcal{R}a^*\} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a^* \text{ and } \mathrm{lann}(xa) = \mathrm{lann}(a^*)\}. \end{aligned}$$

From Theorems 4.1 and 7.4,

$$\begin{aligned} a\{1,3,6\} &\supseteq \{x \in a\{1\} : xa\mathcal{R} = a\mathcal{R} \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a^*)\} \\ &= \{x \in a\{1\} : \mathrm{lann}(xa) = \mathrm{lann}(a) \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a^*)\} \\ &= \{x \in a\{1\} : xa\mathcal{R} = a\mathcal{R} \text{ and } \mathcal{R}ax = \mathcal{R}a^*\} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a^* \text{ and } \mathrm{lann}(xa) = \mathrm{lann}(a))\} \end{aligned} \tag{13}$$

and from Theorems 4.2 and 7.5,

$$\begin{aligned} a\{1,4,8\} &\supseteq \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R} \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a)\} \\ &= \{x \in a\{1\} : \mathrm{lann}(xa) = \mathrm{lann}(a^*) \text{ and } \mathrm{rann}(ax) = \mathrm{rann}(a)\} \\ &= \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R} \text{ and } \mathcal{R}ax = \mathcal{R}a\} = \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a \text{ and } \mathrm{lann}(xa) = \mathrm{lann}(a^*)\}. \end{aligned}$$

We comment on the previous inclusions. Let $\mathcal{R}$ be a $*$-ring and $a, x \in \mathcal{R}$ be such that $x \in a\{1,3,6\}$. Then $a\mathcal{R} \subseteq xa\mathcal{R}$ and $\mathrm{rann}(ax) = \mathrm{rann}(x^*a^*) = \mathrm{rann}(a^*)$. Depending on the ring $\mathcal{R}$, we can always have $a\mathcal{R} = xa\mathcal{R}$ or not; consequently, the equality in (13) is always satisfied or not. We next consider two examples. Let $\mathcal{R} = \mathbb{C}^{n\times n}$ and $A, X \in \mathbb{C}^{n\times n}$ be such that $X \in A\{1,3,6\}$. Then $\mathrm{R}(A) \subseteq \mathrm{R}(XA)$ and $\dim(\mathrm{R}(XA)) = n - \dim(\mathrm{N}(XA)) = n - \dim(\mathrm{N}(A)) = \dim(\mathrm{R}(A))$. Thus, $\mathrm{R}(XA) = \mathrm{R}(A)$ and the equality in (13) holds for A. Let now $\ell^2(\mathbb{N})$ be the Hilbert space of the complex sequences $x = (x_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$ with inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}$. Let $\mathcal{R} = \mathcal{B}(\ell^2(\mathbb{N}))$ be the ring of all bounded linear operators from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$. Let $A, X \in \mathcal{B}(\ell^2(\mathbb{N}))$ defined by $A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and $X(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. These operators were considered in [34, Remark 3.1] and satisfy $X \in A\{1,3,6\}$ and $\mathrm{R}(A) \subsetneq \mathrm{R}(XA) = \ell^2(\mathbb{N})$. Therefore, the strict inclusion in (13) holds for A. Similar considerations are valid for the other inclusion.

By Theorems 4.6, 4.7 and 7.6,

$$\begin{aligned} a\{1,3,7\} &= \{x \in a\{1\} : \mathrm{rann}(ax) = \mathrm{rann}(a^*) \text{ and } x \in a\mathcal{R}\} \\ &= \{x \in a\{1\} : \mathcal{R}ax = \mathcal{R}a^* \text{ and } \mathrm{lann}(a) \subseteq \mathrm{lann}(x)\} \end{aligned}$$

and by Theorems 4.5, 4.8 and 7.7,

$$\begin{aligned} a\{1,4,9\} &= \{x \in a\{1\} : xa\mathcal{R} = a^*\mathcal{R} \text{ and } \mathrm{rann}(a) \subseteq \mathrm{rann}(x)\} \\ &= \{x \in a\{1\} : \mathrm{lann}(xa) = \mathrm{lann}(a^*) \text{ and } x \in \mathcal{R}a\}. \end{aligned}$$

7.2. Generalizations of Moore-Penrose, core, and dual core inverses

In this section, we consider some generalizations of Moore-Penrose, core, and dual core inverses. As in Section 7.1, the focus is on their relation to projectors.

7.2.1. The (e, f) Moore-Penrose inverse

Let $\mathcal{R}$ be a $*$-ring. Let $a \in \mathcal{R}$ and $e, f \in \mathcal{R}^{-1} \cap \mathcal{R}^{\mathrm{sym}}$. In [30], if $x \in a\{1,2\}$, $(eax)^* = eax$ and $(fxa)^* = fxa$, then $x \in \mathcal{R}$ is called the *(weighted) (e, f) Moore-Penrose inverse* of a. This generalized inverse is denoted by $a^\dagger_{e,f}$ and $a^\dagger = a^\dagger_{1,1}$. If $x = a^\dagger_{e,f}$, then $xe^{-1} \in (ea)\{1\}$ and $fx \in (af^{-1})\{1\}$. More details can be found in, e.g., [30, 38, 46]. If $a^\dagger_{e,f}$ exists, then

$$a^\dagger_{e,f} = a^{(1,2)}_{\mathrm{rprin}=f^{-1}a^*\mathcal{R},\mathrm{rann}=\mathrm{rann}(a^*e)} = a^{(1,2)}_{\mathrm{lprin}=\mathcal{R}a^*e,\mathrm{lann}=\mathrm{lann}(f^{-1}a^*)} = a^{(1,2)}_{\mathrm{rprin}=f^{-1}a^*\mathcal{R},\mathrm{lprin}=\mathcal{R}a^*e} = a^{(1,2)}_{\mathrm{lann}=\mathrm{lann}(f^{-1}a^*),\mathrm{rann}=\mathrm{rann}(a^*e)}.$$

Consider the conditions

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*e)},\ \varphi_{xa} = \rho_{f^{-1}a^*\mathcal{R},\mathrm{rann}(a)}. \tag{14a}$$

$$_{ax}\varphi = \rho_{\mathcal{R}a^*e,\mathrm{lann}(a)},\ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(f^{-1}a^*)}. \tag{14b}$$

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*e)},\ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(f^{-1}a^*)}. \tag{14c}$$

$$_{ax}\varphi = \rho_{\mathcal{R}a^*e,\mathrm{lann}(a)},\ \varphi_{xa} = \rho_{f^{-1}a^*\mathcal{R},\mathrm{rann}(a)}. \tag{14d}$$

$$x\mathcal{R} \subseteq f^{-1}a^*\mathcal{R}. \tag{14e}$$

$$\mathrm{lann}(f^{-1}a^*) \subseteq \mathrm{lann}(x). \tag{14f}$$

$$\mathcal{R}x \subseteq \mathcal{R}a^*e. \tag{14g}$$

$$\mathrm{rann}(a^*e) \subseteq \mathrm{rann}(x). \tag{14h}$$

From Theorems 6.1-6.4, $a^{\dagger}_{e,f}$ exists and $x = a^{\dagger}_{e,f}$ if and only if one of the conditions (14a)-(14d) holds and one of the conditions (14e)-(14h) holds. Let now $e = f = 1$. For Hilbert space operators with closed range, conditions (14a) and (14e) coincide with the conditions (iii) of [32, Theorem 1]. Conditions (14a) and (14h) are a generalization of the conditions of [3, Ex. 2.58]. The relations of generalized inverses to projectors can be used to extend to any ring results of matrices and operators. In particular, the characterizations of the Moore-Penrose inverse using orthogonal projectors given in Section 7.2.1 can be used to generalize results of [28, Section 2] to ∗-rings.

7.2.2. The e-core and the f-dual core inverses

Let $\mathcal{R}$ be a ∗-ring. Let $a, x \in \mathcal{R}$ and $e, f \in \mathcal{R}^{-1} \cap \mathcal{R}^{\mathrm{sym}}$. Then x is the *e-core inverse* of a if $x \in a\{1\}$, $x\mathcal{R} = a\mathcal{R}$, and $\mathcal{R}x = \mathcal{R}a^*e$, whereas x is the *f-dual core inverse* of a if $x \in a\{1\}$, $x\mathcal{R} = f^{-1}a^*\mathcal{R}$, and $\mathcal{R}x = \mathcal{R}a$. These generalized inverses were defined and studied in [29] (see also, e.g., [46] for more properties). If they exist, then they are unique, and we denote them with $a^{\text{\textcircled{\#}},e}$ and $a_{\text{\textcircled{\#}},f}$, respectively. We have $a^{\text{\textcircled{\#}},1} = a^{\text{\textcircled{\#}}}$ and $a_{\text{\textcircled{\#}},1} = a_{\text{\textcircled{\#}}}$. By [29, Theorems 2.1 and 2.2], $a^{\text{\textcircled{\#}},e}, a_{\text{\textcircled{\#}},f} \in a\{1, 2\}$.

If $a^{\text{\textcircled{\#}},e}$ exists, then

$$a^{\text{\textcircled{\#}},e} = a^{(1,2)}_{\mathrm{rprin}=a\mathcal{R},\mathrm{lprin}=\mathcal{R}a^*e} = a^{(1,2)}_{\mathrm{rprin}=a\mathcal{R},\mathrm{rann}=\mathrm{rann}(a^*e)} = a^{(1,2)}_{\mathrm{lprin}=\mathcal{R}a^*e,\mathrm{lann}=\mathrm{lann}(a)} = a^{(1,2)}_{\mathrm{lann}=\mathrm{lann}(a),\mathrm{rann}=\mathrm{rann}(a^*e)}.$$

If $a_{\text{\textcircled{\#}},f}$ exists, then

$$a_{\text{\textcircled{\#}},f} = a^{(1,2)}_{\mathrm{rprin}=f^{-1}a^*\mathcal{R},\mathrm{lprin}=\mathcal{R}a} = a^{(1,2)}_{\mathrm{rprin}=f^{-1}a^*\mathcal{R},\mathrm{rann}=\mathrm{rann}(a)} = a^{(1,2)}_{\mathrm{lprin}=\mathcal{R}a,\mathrm{lann}=\mathrm{lann}(f^{-1}a^*)} = a^{(1,2)}_{\mathrm{lann}=\mathrm{lann}(f^{-1}a^*),\mathrm{rann}=\mathrm{rann}(a)}.$$

As a consequence of Theorems 6.1-6.4, $a^{\text{\textcircled{\#}},e}$ exists and $x = a^{\text{\textcircled{\#}},e}$ if and only if one of the conditions (15a)-(15d) holds and one of the conditions (15e)-(15h) holds, where the conditions are

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*e)}, \varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}. \quad (15a) \qquad x\mathcal{R} \subseteq a\mathcal{R}. \quad (15e)$$

$$_{ax}\varphi = \rho_{\mathcal{R}a^*e,\mathrm{lann}(a)}, \ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}. \quad (15b) \qquad \mathrm{lann}(a) \subseteq \mathrm{lann}(x). \quad (15f)$$

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*e)}, \ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}. \quad (15c) \qquad \mathcal{R}x \subseteq \mathcal{R}a^*e. \quad (15g)$$

$$_{ax}\varphi = \rho_{\mathcal{R}a^*e,\mathrm{lann}(a)}, \varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}. \quad (15d) \qquad \mathrm{rann}(a^*e) \subseteq \mathrm{rann}(x). \quad (15h)$$

Similarly, $a_{\text{\textcircled{\#}},f}$ exists and $x = a_{\text{\textcircled{\#}},f}$ if and only if one of the conditions (16a)-(16d) holds and one of the conditions (16e)-(16h) holds, where the conditions are

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a)}, \varphi_{xa} = \rho_{f^{-1}a^*\mathcal{R},\mathrm{rann}(a)}, \quad (16a) \qquad x\mathcal{R} \subseteq f^{-1}a^*\mathcal{R}, \quad (16e)$$

$$_{ax}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}, \ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(f^{-1}a^*)}, \quad (16b) \qquad \mathrm{lann}(f^{-1}a^*) \subseteq \mathrm{lann}(x), \quad (16f)$$

$$\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a)}, \ _{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(f^{-1}a^*)}, \quad (16c) \qquad \mathcal{R}x \subseteq \mathcal{R}a, \quad (16g)$$

$$_{ax}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a)}, \varphi_{xa} = \rho_{f^{-1}a^*\mathcal{R},\mathrm{rann}(a)}, \quad (16d) \qquad \mathrm{rann}(a) \subseteq \mathrm{rann}(x). \quad (16h)$$

7.2.3. The w-core and the dual v-core inverses

Let $\mathcal{R}$ be a ∗-ring and $a, x, w, v \in \mathcal{R}$. Then x is the *w-core inverse* of a if $(awx)^* = awx$, $xawa = a$, and $awx^2 = x$. Similarly, x is the *dual v-core inverse* of a if $(xva)^* = xva$, $avax = a$, and $x^2va = x$. The w-core and the dual v-core inverses of a are unique if they exist, and are denoted by $a^{\text{\textcircled{\#}},w}$ and $a_{\text{\textcircled{\#}},v}$, respectively. We have $a^{\text{\textcircled{\#}}}_1 = a^{\text{\textcircled{\#}}}$ and $a_{\text{\textcircled{\#}},1} = a_{\text{\textcircled{\#}}}$. We refer the reader to, e.g. [17, 47] for more details. We note that the equivalence (1) ⇔ (2) of the next proposition was given in [47, Theorem 2.10]. Here, we present a proof based on Theorem 3.4.

Proposition 7.8. *Let $\mathcal{R}$ be a ∗-ring and $a, w, x \in \mathcal{R}$. The following assertions are equivalent:*

1. $x = a^{\text{\textcircled{\#}},w}$.
2. $x = (aw)^{\text{\textcircled{\#}}}$ *and* $a\mathcal{R} \subseteq aw\mathcal{R}$.
3. $x = (aw)^{\text{\textcircled{\#}}}$ *and* $\mathrm{lann}(aw) \subseteq \mathrm{lann}(a)$.

Proof. (1) ⇒ (2): If $x = a^{\textcircled{\#},w}$, then $x = (aw)^{\textcircled{\#}}$ and, by Theorem 3.4, $a = xawa = \rho_{aw\mathcal{R},\mathrm{rann}(aw)}(a)$. This last equality implies that $a\mathcal{R} \subseteq aw\mathcal{R}$.

(2) ⇒ (3) is immediate.

(3) ⇒ (1): If $x = (aw)^{\textcircled{\#}}$ and $\mathrm{lann}(aw) \subseteq \mathrm{lann}(a)$, then $(awx)^* = awx$, $awx^2 = x$ and, by Theorem 3.4, $xawa = \rho_{\mathcal{R}aw,\mathrm{lann}(aw)}(1)a = a$. □

For the dual v-core inverse we analogously have:

Proposition 7.9. *Let $\mathcal{R}$ be a $*$-ring and $a, v, x \in \mathcal{R}$. The following assertions are equivalent:*

1. $x = a_{\textcircled{\#},v}$.
2. $x = (va)_{\textcircled{\#}}$ *and* $\mathcal{R}a \subseteq \mathcal{R}va$.
3. $x = (va)_{\textcircled{\#}}$ *and* $\mathrm{rann}(va) \subseteq \mathrm{rann}(a)$.

By Proposition 7.8, if $a^{\textcircled{\#},w}$ exists, then

$$a^{\textcircled{\#},w} = a^{(1,2)}_{\mathrm{rprin}=aw\mathcal{R},\mathrm{rann}=\mathrm{rann}((aw)^*)} = a^{(1,2)}_{\mathrm{lprin}=\mathcal{R}(aw)^*,\mathrm{lann}=\mathrm{lann}(aw)} = a^{(1,2)}_{\mathrm{rprin}=aw\mathcal{R},\mathrm{lprin}=\mathcal{R}(aw)^*} = a^{(1,2)}_{\mathrm{lann}=\mathrm{lann}(aw),\mathrm{rann}=\mathrm{rann}((aw)^*)}.$$

If $a_{\textcircled{\#},v}$ exists, then

$$a_{\textcircled{\#},v} = a^{(1,2)}_{\mathrm{rprin}=(va)^*\mathcal{R},\mathrm{rann}=\mathrm{rann}(va)} = a^{(1,2)}_{\mathrm{lprin}=\mathcal{R}va,\mathrm{lann}=\mathrm{lann}((va)^*)} = a^{(1,2)}_{\mathrm{rprin}=(va)^*\mathcal{R},\mathrm{lprin}=\mathcal{R}va} = a^{(1,2)}_{\mathrm{lann}=\mathrm{lann}((va)^*),\mathrm{rann}=\mathrm{rann}(va)}.$$

Let $b = aw$ and $c = va$. By Theorems 6.1-6.4 and Proposition 7.8 (resp. Proposition 7.9), $a^{\textcircled{\#},w}$ exists and $x = a^{\textcircled{\#},w}$ if and only if one of the conditions (17a)-(17d) holds, one of the conditions (17e)-(17h) holds, and one of the conditions (17i)-(17j) holds, where the conditions are

$$\varphi_{bx} = \rho_{b\mathcal{R},\mathrm{rann}(b^*)}, \varphi_{xa} = \rho_{b\mathcal{R},\mathrm{rann}(b)}, \quad (17a) \qquad x\mathcal{R} \subseteq b\mathcal{R}, \quad (17e)$$

$$_{bx}\varphi = \rho_{\mathcal{R}b^*,\mathrm{lann}(b)}, {}_{xa}\varphi = \rho_{\mathcal{R}b,\mathrm{lann}(b)}, \quad (17b) \qquad \mathrm{lann}(b) \subseteq \mathrm{lann}(x), \quad (17f) \qquad a\mathcal{R} \subseteq b\mathcal{R}, \quad (17i)$$

$$\varphi_{bx} = \rho_{b\mathcal{R},\mathrm{rann}(b^*)}, {}_{xa}\varphi = \rho_{\mathcal{R}b,\mathrm{lann}(b)}, \quad (17c) \qquad \mathcal{R}x \subseteq \mathcal{R}b^*, \quad (17g) \qquad \mathrm{lann}(b) \subseteq \mathrm{lann}(a). \quad (17j)$$

$$_{bx}\varphi = \rho_{\mathcal{R}b^*,\mathrm{lann}(b)}, \varphi_{xa} = \rho_{b\mathcal{R},\mathrm{rann}(b)}, \quad (17d) \qquad \mathrm{rann}(b^*) \subseteq \mathrm{rann}(x), \quad (17h)$$

We also have $a_{\textcircled{\#},v}$ exists and $x = a_{\textcircled{\#},v}$ if and only if one of the conditions (18a)-(18d) holds, one of the conditions (18e)-(18h) holds, and one of the conditions (18i)-(18j) holds, where the conditions are

$$\varphi_{cx} = \rho_{c\mathcal{R},\mathrm{rann}(c)}, \varphi_{xc} = \rho_{c^*\mathcal{R},\mathrm{rann}(c)}, \quad (18a) \qquad x\mathcal{R} \subseteq c^*\mathcal{R}, \quad (18e)$$

$$_{cx}\varphi = \rho_{\mathcal{R}c,\mathrm{lann}(c)}, {}_{xc}\varphi = \rho_{\mathcal{R}c,\mathrm{lann}(c^*)}, \quad (18b) \qquad \mathrm{lann}(c^*) \subseteq \mathrm{lann}(x), \quad (18f) \qquad \mathcal{R}a \subseteq \mathcal{R}c, \quad (18i)$$

$$\varphi_{cx} = \rho_{c\mathcal{R},\mathrm{rann}(c)}, {}_{xc}\varphi = \rho_{\mathcal{R}c,\mathrm{lann}(c^*)}, \quad (18c) \qquad \mathcal{R}x \subseteq \mathcal{R}c, \quad (18g) \qquad \mathrm{rann}(c) \subseteq \mathrm{rann}(a). \quad (18j)$$

$$_{cx}\varphi = \rho_{\mathcal{R}c,\mathrm{lann}(c)}, \varphi_{xc} = \rho_{c^*\mathcal{R},\mathrm{rann}(c)}, \quad (18d) \qquad \mathrm{rann}(c) \subseteq \mathrm{rann}(x), \quad (18h)$$

Let $w = 1$. The conditions $\varphi_{ax} = \rho_{a\mathcal{R},\mathrm{rann}(a^*)}$, $\varphi_{xa} = \rho_{a\mathcal{R},\mathrm{rann}(a)}$ and $x\mathcal{R} \subseteq a\mathcal{R}$ are stronger than the conditions (10) of the definition of the core inverse for finite complex matrices. Let $\mathcal{H}$ be an arbitrary Hilbert space and $L(\mathcal{H})$ be the ring of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. In [34, Remark 3.1], it is shown that the conditions $A, X \in L(\mathcal{H})$, $AX = P_{\mathrm{R}(A)}$ and $\mathrm{R}(X) \subseteq \mathrm{R}(A)$, do not imply that $X = A^{\textcircled{\#}}$. Similar considerations can be made for the dual core inverse.

7.2.4. The right w-core and left dual v-core inverses

The *right w-core inverse* x of a is defined by the equations $awxa = a$, $(awx)^* = awx$, and $awx^2 = x$. We analogously have the *left dual v-core inverse* x of a defined by the equations $axva = a$, $(xva)^* = xva$, and $x^2va = x$. These generalized inverses were defined in [48]. From Theorem 7.6 we obtain:

Proposition 7.10. *Let $\mathcal{R}$ be a $*$-ring and $a, w, x \in \mathcal{R}$. The following assertions are equivalent:*

1. *x is a right w-core inverse of a.*
2. $x \in (aw)\{1, 3, 7\}$ *and* $a\mathcal{R} \subseteq aw\mathcal{R}$.

3. $x \in (aw)\{1,3,7\}$ *and* $\mathrm{lann}(aw) \subseteq \mathrm{lann}(a)$.

We note that the equivalence (1) ⇔ (2) of Proposition 7.10 appears in [48, Theorem 2.14]. Analogously, using Theorem 7.7 we get:

Proposition 7.11. *Let $\mathcal{R}$ be a $*$-ring and $a, v, x \in \mathcal{R}$. The following assertions are equivalent:*

1. *x is a left dual v-core inverse of a.*
2. $x \in (va)\{1,4,9\}$ *and* $\mathcal{R}a \subseteq \mathcal{R}va$.
3. $x \in (va)\{1,4,9\}$ *and* $\mathrm{rann}(va) \subseteq \mathrm{rann}(a)$.

From Proposition 7.10 and Theorem 7.6, x is a right w-core inverse of a if and only if one of the conditions (19a)-(19b) holds and one of the conditions (19c)-(19d) holds, where the conditions are

$$\varphi_{awx} = \rho_{aw\mathcal{R},\mathrm{rann}((aw)^*)},\ x \in aw\mathcal{R}, \quad (19a) \qquad a\mathcal{R} \subseteq aw\mathcal{R}, \quad (19c)$$
$$_{ax}\varphi = \rho_{\mathcal{R}a^*,\mathrm{lann}(a)},\ \mathrm{lann}(aw) \subseteq \mathrm{lann}(x), \quad (19b) \qquad \mathrm{lann}(aw) \subseteq \mathrm{lann}(a). \quad (19d)$$

By Proposition 7.11 and Theorem 7.7, x is a left dual v-core inverse of a if and only if one of the one of the conditions (20a)-(20b) holds and one of the conditions (20c)-(20d) holds, where the conditions are

$$\varphi_{xva} = \rho_{(va)^*\mathcal{R},\mathrm{rann}(va)},\ x \in \mathcal{R}va, \quad (20a) \qquad \mathcal{R}a \subseteq \mathcal{R}va, \quad (20c)$$
$$_{xa}\varphi = \rho_{\mathcal{R}a,\mathrm{lann}(a^*)},\ \mathrm{rann}(va) \subseteq \mathrm{rann}(x), \quad (20b) \qquad \mathrm{rann}(va) \subseteq \mathrm{rann}(a). \quad (20d)$$

7.3. (b, c)-inverses

Let $a, x, b, c, d \in \mathcal{R}$. In this section, we consider the (b, c) inverses defined by Drazin in [13]: $a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathcal{R}c}$ is the (b, c) *inverse* of a, $a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{rann}=\mathrm{rann}(c)}$ is the *right hybrid* (b, c) *inverse* of a, $a^{(2)}_{\mathrm{rprin}=\mathcal{R}b,\mathrm{rann}=\mathrm{lann}(c)}$ is the *left hybrid* (b, c) *inverse* of a, and $a^{(2)}_{\mathrm{lann}=\mathrm{lann}(b),\mathrm{rann}=\mathrm{rann}(c)}$ is the *annihilator* (b, c) *inverse* of a. See, e.g., [5, 13, 15, 18, 19, 25, 26, 33, 45]. Indeed in [13], Drazin calls x the (b, c)-inverse of a if $x \in b\mathcal{R}x \cap x\mathcal{R}c$, $xab = b$, and $cax = c$. In [13, page 1922], Drazin shows that this definition is equivalent to the given here for the (b, c)-inverse. The (d, d)-inverse of a is the *inverse of a along d* defined by Mary in [25]. Conversely, each (b, c)-inverse is a (d, d)-inverse [26]. In [15], the previous inverses were generalized as follows. Let $v, w \in \mathcal{R}$. Then, $(vaw)^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathcal{R}c}$ is the (w, v)-*weighted* (b, c)-*inverse* of a, $(vaw)^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{rann}=\mathrm{rann}(c)}$ is the *right hybrid* (w, v)-*weighted* (b, c)-*inverse* of a, $(vaw)^{(2)}_{\mathrm{rprin}=\mathcal{R}b,\mathrm{rann}=\mathrm{lann}(c)}$ is the *left hybrid* (w, v)-*weighted* (b, c)-*inverse* of a, and $(vaw)^{(2)}_{\mathrm{lann}=\mathrm{lann}(b),\mathrm{rann}=\mathrm{rann}(c)}$ is the *annihilator* (w, v)-*weighted* (b, c)-*inverse* of a.

Theorem 7.12(1)-(3) below can be seen as a generalization of [3, Theorem 2.13].

Theorem 7.12. *Let $a, b, c \in \mathcal{R}$ be such that $(cab)\{1\} \neq \emptyset$. Let $(cab)^{(1)} \in (cab)\{1\}$ and $x = b(cab)^{(1)}c$. Then:*

1. $x \in a\{1\} \Leftrightarrow \{ab\mathcal{R} = a\mathcal{R}$ *and* $\mathrm{rann}(cab) = \mathrm{rann}(ab)\} \Leftrightarrow \{ab\mathcal{R} = a\mathcal{R}$ *and* $\mathcal{R}cab = \mathcal{R}ab\}$.
2. $\{x \in a\{2\}$ *and* $x\mathcal{R} = b\mathcal{R}\} \Leftrightarrow \mathrm{rann}(cab) = \mathrm{rann}(b) \Leftrightarrow \mathcal{R}cab = \mathcal{R}b$.
3. $\{x \in a\{2\}$ *and* $\mathrm{rann}(x) = \mathrm{rann}(c)\} \Leftrightarrow cab\mathcal{R} = c\mathcal{R} \Leftrightarrow \mathrm{lann}(cab) = \mathrm{lann}(c)$.
4. $\{x \in a\{2\}$ *and* $\mathcal{R}x = \mathcal{R}c\} \Leftrightarrow \mathrm{lann}(cab) = \mathrm{lann}(c) \Leftrightarrow cab\mathcal{R} = c\mathcal{R}$.
5. $\{x \in a\{2\}$ *and* $\mathrm{lann}(x) = \mathrm{lann}(b)\} \Leftrightarrow \mathcal{R}cab = \mathcal{R}b \Leftrightarrow \mathrm{rann}(cab) = \mathrm{rann}(b)$.

Proof. We first observe that by Lemma 3.3, $cab\mathcal{R} = ca\mathcal{R}$ (resp. $cab\mathcal{R} = c\mathcal{R}$) if and only if $\mathrm{lann}(cab) = \mathrm{lann}(ca)$ (resp. $\mathrm{lann}(cab) = \mathrm{lann}(c)$), and $\mathcal{R}cab = \mathcal{R}ab$ (resp. $\mathcal{R}cab = \mathcal{R}b$) if and only if $\mathrm{rann}(cab) = \mathrm{rann}(ab)$ (resp. $\mathrm{rann}(cab) = \mathrm{rann}(b)$).

(1): If $x \in a\{1\}$, then $a = ab(cab)^{(1)}ca$ and $ab = ab(cab)^{(1)}(cab)$. By the last equalities and Lemma 3.3(2), $ab\mathcal{R} = a\mathcal{R}$ and $\mathrm{rann}(cab) = \mathrm{rann}(ab)$. Conversely, if $ab\mathcal{R} = a\mathcal{R}$ and $\mathrm{rann}(cab) = \mathrm{rann}(ab)$, then there exists $r \in \mathcal{R}$ such that $a = abr$ and, using again Lemma 3.3(2), $axa = a(b(cab)^{(1)}c)a = (ab)(cab)^{(1)}(cab)r = abr = a$. Hence, $x \in a\{1\}$.

(2): We have, $x \in a\{2\}$ and $x\mathcal{R} = b\mathcal{R}$ if and only if $x = xax$ and there exists $r \in \mathcal{R}$ such that $b = xr$. By the expression of x, we obtain $b = xr = xaxr = b(cab)^{(1)}cab$. By Lemma 3.3(2), this last equality is equivalent to $\mathrm{rann}(cab) = \mathrm{rann}(b)$. Conversely, by Lemma 3.3(2), if $\mathrm{rann}(cab) = \mathrm{rann}(b)$, then $xab = b(cab)^{(1)}cab = b$. From the last equality, $x = b(cab)^{(1)}c = xab(cab)^{(1)}c = xax$ and $x\mathcal{R} = b\mathcal{R}$.

(3): By Theorem 3.2, if $x \in a\{2\}$ and $\mathrm{rann}(x) = \mathrm{rann}(c)$, then $\mathcal{R} = ab(cab)^{(1)}c\mathcal{R} \oplus \mathrm{rann}(c)$. Therefore, $c\mathcal{R} \subseteq cab(cab)^{(1)}c\mathcal{R} \subseteq cab\mathcal{R}$. This implies that $c\mathcal{R} = cab\mathcal{R}$. Conversely, using Lemma 3.3(1), we obtain that if $cab\mathcal{R} = c\mathcal{R}$, then $cax = cab(cab)^{(1)}c = c$. From here, $\mathrm{rann}(x) = \mathrm{rann}(c)$ and $xax = b(cab)^{(1)}cax = b(cab)^{(1)}c = x$.

The proofs of (4) and (5) are similar to the proofs of (2) and (3), respectively. □

We get the next result for right hybrid (b, c) inverses.

Theorem 7.13. *Let $a, b, c \in \mathcal{R}$. If any of the conditions*

1. $\mathrm{rann}(ab) = \{0\}$, *$c\mathcal{R} = \mathcal{R}$ and $\mathcal{R} = ab\mathcal{R} \oplus \mathrm{rann}(c)$, or*
2. $\mathrm{rann}(b) = \{0\}$, *$ca\mathcal{R} = \mathcal{R}$ and $\mathcal{R} = b\mathcal{R} \oplus \varphi_a^{-1}(\mathrm{rann}(c))$*

holds, then $cab \in \mathcal{R}^{-1}$ and $b(cab)^{-1}c = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{rann}=\mathrm{rann}(c)}$.

Proof. We only prove (1) since the proof of (2) is similar. Assume that $\mathrm{rann}(ab) = \{0\}$, $c\mathcal{R} = \mathcal{R}$ and $\mathcal{R} = ab\mathcal{R} \oplus \mathrm{rann}(c)$. Then $cab\mathcal{R} = c\mathcal{R} = \mathcal{R}$. Let $r \in \mathcal{R}$. If $cabr = 0$, then $abr \in ab\mathcal{R} \cap \mathrm{rann}(c)$. Hence, $abr = 0$. This shows that $\mathrm{rann}(cab) = \mathrm{rann}(ab) = \mathrm{rann}(b) = \{0\}$. By Lemma 2.1, $cab \in \mathcal{R}^{-1}$, and by Theorem 7.12(2)(3), if $x = b(cab)^{-1}c$, then $x \in a\{2\}$, $x\mathcal{R} = b\mathcal{R}$ and $\mathrm{rann}(x) = \mathrm{rann}(c)$. □

Theorems 5.1 and 7.13 can be seen as generalizations of [3, Theorem 2.14]. Let $\mathbb{C}_r^{m\times n}$ denote the class of complex matrices of rank r. Let $A \in \mathbb{C}_r^{m\times n}$, $s \le r$, $U \in \mathbb{C}_s^{n\times s}$ and $V \in \mathbb{C}^{s\times n}$. Then, $\mathrm{rank}(V) = s \Leftrightarrow \mathrm{R}(V) = \mathbb{C}^s$ and equality [3, (2.62)] is equivalent to $\mathrm{N}(AU) = \{0\}$. This shows that the hypotheses $\mathrm{rann}(ab) = \{0\}$ and $c\mathcal{R} = \mathcal{R}$ in Theorem 7.13 are natural generalizations of the hypotheses of [3, Theorem 2.14].

Using Lemma 2.1 and Theorem 7.12(4)(5), we analogously obtain:

Theorem 7.14. *Let $a, b, c \in \mathcal{R}$. If any of the conditions*

1. $\mathrm{lann}(ca) = \{0\}$, *$\mathcal{R}b = \mathcal{R}$ and $\mathcal{R} = \mathcal{R}ca \oplus \mathrm{lann}(b)$, or*
2. $\mathrm{lann}(c) = \{0\}$, *$\mathcal{R}ab = \mathcal{R}$ and $\mathcal{R} = \mathcal{R}c \oplus {}_a\varphi^{-1}(\mathrm{lann}(b))$*

holds, then $cab \in \mathcal{R}^{-1}$ and $b(cab)^{-1}c = a^{(2)}_{\mathrm{lprin}=\mathcal{R}c,\mathrm{lann}=\mathrm{lann}(b)}$.

The next theorem gives relations between the different types of (b, c)-inverses.

Theorem 7.15. *Let $a, b, c, x \in \mathcal{R}$. The following assertions are equivalent:*

1. $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{rann}=\mathrm{rann}(c)}$ *and $x \in \mathcal{R}c$ (or $c\{1\} \neq \emptyset$).*
2. $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{rann}=\mathrm{rann}(c)}$ *and $(cab)\{1\} \neq \emptyset$.*
3. $x = a^{(2)}_{\mathrm{lprin}=\mathcal{R}c,\mathrm{lann}=\mathrm{lann}(b)}$ *and $x \in b\mathcal{R}$ (or $b\{1\} \neq \emptyset$).*
4. $x = a^{(2)}_{\mathrm{lprin}=\mathcal{R}c,\mathrm{lann}=\mathrm{lann}(b)}$ *and $(cab)\{1\} \neq \emptyset$.*
5. $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathcal{R}c}$.
6. $x = a^{(2)}_{\mathrm{lann}=\mathrm{lann}(b),\mathrm{rann}=\mathrm{rann}(c)}$ *and $x \in b\mathcal{R}$ (or $b\{1\} \neq \emptyset$) and $x \in \mathcal{R}c$ (or $c\{1\} \neq \emptyset$).*
7. $x = a^{(2)}_{\mathrm{lann}=\mathrm{lann}(b),\mathrm{rann}=\mathrm{rann}(c)}$ *and $(cab)\{1\} \neq \emptyset$.*
8. $(cab)\{1\} \neq \emptyset$, *$\mathcal{R}cab = \mathcal{R}b$ (or $\mathrm{rann}(cab) = \mathrm{rann}(b)$), $cab\mathcal{R} = c\mathcal{R}$ (or $\mathrm{lann}(cab) = \mathrm{lann}(c)$), and $x = b(cab)^{(1)}c$ for any $(cab)^{(1)} \in (cab)\{1\}$.*

Proof. If $x \in \{2\}$, then $a \in x\{1\}$. Hence, from Lemma 2.5, in (1) and (6): $x \in \mathcal{R}c \Leftrightarrow c\{1\} \neq \emptyset$, and in (3) and (6): $x \in b\mathcal{R} \Leftrightarrow b\{1\} \neq \emptyset$.

As a consequence of [45, Theorem 2.4] (or Theorem 5.3) and [45, Corollary 2.6], we get (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5). From Theorem 5.4 and [45, Corollary 2.6], we obtain (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(5) $\Rightarrow$ (6): It is immediate.

(6) $\Rightarrow$ (5): Suppose that $x = a^{(2)}_{\mathrm{lann}=\mathrm{lann}(b),\mathrm{rann}=\mathrm{rann}(c)}$, $x \in b\mathcal{R}$, and $x \in \mathcal{R}c$. By Theorem 5.6(6), $c = cax$ and $b = xab$. Hence, $\mathcal{R}c \subseteq \mathcal{R}x$ and $b\mathcal{R} \subseteq x\mathcal{R}$. We conclude that $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathcal{R}c}$.

(5) $\Rightarrow$ (8): Suppose that $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathcal{R}c}$. Then $x = a^{(2)}_{\mathrm{rprin}=b\mathcal{R},\mathrm{lprin}=\mathrm{rann}(c)}$. Thus, (8) follows from [45, Proposition 2.5] and Theorem 7.12(2)(4).

(8) $\Rightarrow$ (5): It follows from Theorem 7.12(2)(4).

(7) $\Leftrightarrow$ (8): It follows from Theorem 7.12(3)(5). $\square$

From Theorems 5.3(3), 5.4(3) and 7.15(5)(1)(3), we get [19, Proposition 2.7].

7.4. The (p, q) inverse

We now consider $\{2\}$-inverses defined using prefixed idempotent elements in $\mathcal{R}$. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. Then $a^{(2)}_{\mathrm{rprin}=p\mathcal{R},\mathrm{rann}=q\mathcal{R}}$ is the *image-kernel* (p, q) *inverse* of a (see [18, Definition 3.1]). The image-kernel (p, q) inverse is the Cao-Xue (p, q, l) inverse (see [9, Definition 2.10]).

Let $a \in \mathcal{R}$ and $p \in \mathcal{R}^{\bullet}$. Then a is called *Bott-Duffin invertible* if $1 - p + ap \in \mathcal{R}^{-1}$, and in this case, the *Bott-Duffin p inverse* of a is $p(1-p+ap)^{-1}$ (see [7, Definitions (c)]). If $x \in \mathcal{R}$ is such that $x = px = xq$, $xap = p$, and $qax = q$, then x is called the *Bott-Duffin* (p, q) *inverse* of a (see [13, Definition 3.2]). The Bott-Duffin (p, p) inverse of a is the Bott-Duffin p inverse of a (see [13, Proposition 3.1]). By [18, Proposition 3.4], $x = a^{(2)}_{\mathrm{rprin}=p\mathcal{R},\mathrm{rann}=q\mathcal{R}}$ if and only if x is the Bott-Duffin $(p, 1 - q)$ inverse of a.

Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. Then $x \in \mathcal{R}$ is called the *Djordjević-Wei* (p, q) *inverse* of a if $x \in a\{2\}$, $xa = p$ and $ax = 1 - q$ (see [11, Definition 2.1]). We note that $a^{(2)}_{p,q} = a^{(2)}_{\mathrm{rprin}=p\mathcal{R},\mathrm{rann}=q\mathcal{R}}$ with $\mathrm{rann}(a^{(2)}_{p,q}a) = \mathrm{rann}(p)$ and $aa^{(2)}_{p,q}\mathcal{R} = \mathrm{rann}(q)$. We also have, $a^{(2)}_{p,q} = a^{(2)}_{\mathrm{lprin}=\mathrm{lann}(q),\mathrm{lann}=\mathrm{lann}(p)}$ with $\mathcal{R}a^{(2)}_{p,q}a = \mathcal{R}p$ and $\mathrm{lann}(aa^{(2)}_{p,q}) = \mathcal{R}q$. Since $(1-q)a = (1-q)ap \Leftrightarrow a(1-p) = qa(1-p) \Rightarrow a(1-p)\mathcal{R} \subseteq q\mathcal{R}$, the conditions of part (2) of the following theorem are weaker than the conditions of [11, Theorem 2.1(2)] that include the equality $px = x$. The conditions of part (4) are with inclusions instead of with equalities as in [9, Theorem 2.4(2)] in a complex Banach algebra.

Theorem 7.16. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. Then the following statements are equivalent:*

1. *$x \in \mathcal{R}$ is the Djordjević-Wei (p, q) inverse of a.*
2. *$a(1 - p)\mathcal{R} \subseteq q\mathcal{R}$, $xap = p$, $1 - q = ax$, and $xq = 0$.*
3. *$\mathcal{R}qa \subseteq \mathcal{R}(1 - p)$, $p = xa$, $px = x$, and $(1 - q)ax = 1 - q$.*
4. *$x \in a\{2\}$, $xa\mathcal{R} \subseteq p\mathcal{R}$, $\mathrm{rann}(xa) \subseteq \mathrm{rann}(p)$ (or $p\mathcal{R} \subseteq xa\mathcal{R}$, $\mathrm{rann}(p) \subseteq \mathrm{rann}(xa)$) and $ax\mathcal{R} \subseteq \mathrm{rann}(q)$, $\mathrm{rann}(ax) \subseteq q\mathcal{R}$ (or $\mathrm{rann}(q) \subseteq ax\mathcal{R}$, $q\mathcal{R} \subseteq \mathrm{rann}(ax)$).*

If $a^{(2)}_{p,q}$ exists, then $\mathrm{rann}(p) = \varphi_a^{-1}(q\mathcal{R})$ and $\mathcal{R}q = {}_a\varphi^{-1}(\mathrm{lann}(p))$.

Proof. (1) $\Rightarrow$ (2)(3): It follows from the definition of $a^{(2)}_{p,q}$ that $xap = p$, $1 - q = ax$, $xq = 0$, $p = xa$, $px = x$, and $(1 - q)ax = 1 - q$.

By Theorems 5.1 and 5.2, $\mathrm{rann}(p) = (1 - p)\mathcal{R} = \mathrm{rann}(xa) = \varphi_a^{-1}(q\mathcal{R})$ and $\mathcal{R}q = \mathrm{lann}(1 - q) = {}_a\varphi^{-1}(\mathcal{R}(1 - p))$. Then $a(1 - p)\mathcal{R} \subseteq q\mathcal{R}$ and $\mathcal{R}qa \subseteq \mathcal{R}(1 - p)$.

(2) $\Rightarrow$ (1): Assume that $a(1 - p)\mathcal{R} \subseteq q\mathcal{R}$ and there exists $x \in \mathcal{R}$ such that $xap = p$, $1 - q = ax$, and $xq = 0$. Then $xax = x$ and $xa(1 - p) = 0$. Since $xap = p$ and $xa(1 - p) = 0$, we have $xa = p$.

The proof of (3) $\Rightarrow$ (1) is similar to the proof of (2) $\Rightarrow$ (1) and the proof of (1) $\Rightarrow$ (4) is immediate.

If x is a $\{2\}$-inverse of a, then $ax, xa \in \mathcal{R}^{\bullet}$. Hence, (4) $\Rightarrow$ (1) follows from Lemma 2.2(3). $\square$

In [18], it is noted that if $x = a^{(2)}_{\mathrm{rprin}=p\mathcal{R},\mathrm{rann}=q\mathcal{R}}$, then x is the Djordjević-Wei $(pxa, q(1 - ax))$ inverse of a. Using Theorems 5.1 and 5.2, we obtain the following proposition.

Proposition 7.17. *Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. Then the following statements are equivalent:*

1. *x is the Djordjević-Wei (p, q) inverse of a.*
2. *$x = a^{(2)}_{\text{rprin}=p\mathcal{R},\text{rann}=q\mathcal{R}}$, $ap\mathcal{R} = (1-q)\mathcal{R}$, and $\varphi_a^{-1}(q\mathcal{R}) = (1-p)\mathcal{R}$.*
3. *$x = a^{(2)}_{\text{lprin}=\text{lann}(q),\text{lann}=\text{lann}(p)}$, $\mathcal{R}(1-q)a = \mathcal{R}p$, and ${}_a\varphi^{-1}(\mathcal{R}(1-p)) = \mathcal{R}q$.*

Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$. If $a^{(2)}_{p,q}$ exists and $a^{(2)}_{p,q} \in a\{1\}$, then $a^{(2)}_{p,q}$ is the *Djordjević-Wei (p, q)-reflexive generalized inverse* of a and it is denoted by $a^{(1,2)}_{p,q}$ (see [11, page 3054]).

Example 7.18. *If $a \in \mathcal{R}^D$ with $\text{ind}(a) \leq l$, then $a^D = a^{(2)}_{aa^D,1-aa^D}$. Let $\mathcal{R}$ be a $*$-ring. If $a \in \mathcal{R}^\dagger$, then $a^\dagger = a^{(2)}_{a^\dagger a,1-aa^\dagger} = a^{(1,2)}_{a^\dagger a,1-aa^\dagger}$. If $a \in \mathcal{R}^{\textcircled{\dagger}}$, then $a^{\textcircled{\dagger}} = a^{(2)}_{a^{\textcircled{\dagger}}a,1-aa^{\textcircled{\dagger}}} = a^{(1,2)}_{a^{\textcircled{\dagger}}a,1-aa^{\textcircled{\dagger}}}$. If $a \in \mathcal{R}_{\textcircled{\dagger}}$, then $a_{\textcircled{\dagger}} = a^{(2)}_{a_{\textcircled{\dagger}}a,1-aa_{\textcircled{\dagger}}} = a^{(1,2)}_{a_{\textcircled{\dagger}}a,1-aa_{\textcircled{\dagger}}}$.*

The equivalence (1) $\Leftrightarrow$ (2) of the following proposition coincides with [9, Proposition 3.1] for complex Banach algebras.

Proposition 7.19. *Let $a \in \mathcal{R}$. Then the following statements are equivalent:*

1. *$a\{1, 2\} \neq \emptyset$.*
2. *There exist $p, q \in \mathcal{R}^\bullet$ such that $\text{rann}(a) = \text{rann}(p)$ and $a\mathcal{R} = q\mathcal{R}$.*
3. *There exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{R}a = \mathcal{R}p$ and $\text{lann}(a) = \text{lann}(q)$.*
4. *There exist $p, q \in \mathcal{R}^\bullet$ such that $\mathcal{R}a = \mathcal{R}p$ and $a\mathcal{R} = q\mathcal{R}$.*

Proof. (1) $\Rightarrow$ (2)(3): Let $x \in a\{1, 2\}$. Setting $p = xa$ and $q = ax$, these implications follow from Theorem 3.4.

(2) $\Rightarrow$ (1): From the hypotheses, $\mathcal{R} = p\mathcal{R} \oplus \text{rann}(p) = p\mathcal{R} \oplus \text{rann}(a)$ and $\mathcal{R} = q\mathcal{R} \oplus \text{rann}(q) = q\mathcal{R} \oplus a\mathcal{R}$. By Theorem 6.11, $a^{(1,2)}_{\text{rprin}=p\mathcal{R},\text{rann}=q\mathcal{R}}$ exists. Thus, $a\{1, 2\} \neq \emptyset$.

Using Theorems 6.12 and 6.13, the proofs of the remainder implications are similar to the proof of (2) $\Rightarrow$ (1). □

7.5. An example

Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$ and $E_{i,j} = e_i e_j^t \in \mathbb{F}^{2\times 2}$ for each $i, j \in \{1, 2\}$ where $e_1 = (1, 0)^t$ and $e_2 = (0, 1)^t$. Let $A = E_{1,2}$. Then $A^2 = 0$, $A\mathbb{F}^{2\times 2} = \text{rann}(A)$, $\mathbb{F}^{2\times 2}A = \text{lann}(A)$, $A\mathbb{F}^{2\times 2} = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{2,1} = x_{2,2} = 0\}$, and $\mathbb{F}^{2\times 2}A = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{1,1} = x_{2,1} = 0\}$.

By a direct computation, we get $A\{1\} = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{2,1} = 1\}$ and

$$A\{2\} = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{1,1} = x_{1,1}x_{2,1},\ x_{1,2} = x_{1,1}x_{2,2},\ x_{2,1} = x_{2,1}x_{2,1},\ x_{2,2} = x_{2,1}x_{2,2}\}.$$

Hence, $A\{1, 2\} = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{2,1} = 1 \text{ and } x_{1,2} = x_{1,1}x_{2,2}\}$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be the right and the left ideals of $\mathbb{F}^{2\times 2}$ such that $\mathbb{F}^{2\times 2} = A\mathbb{F}^{2\times 2} \oplus \mathcal{S}$ and $\mathbb{F}^{2\times 2} = \mathbb{F}^{2\times 2}A \oplus \mathcal{S}'$. Then $\mathcal{S} = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{1,1} = x_{1,2} = 0\}$ and $\mathcal{S}' = \{(x_{i,j}) \in \mathbb{F}^{2\times 2} : x_{1,2} = x_{2,2} = 0\}$. We set $\mathcal{T} = \mathcal{S}$ and $\mathcal{T}' = \mathcal{S}'$. We have $\rho_{A\mathbb{F}^{2\times 2},\mathcal{T}}(I) = \rho_{\mathcal{S}',\text{lann}(A)}(I) = E_{1,1}$ and $\rho_{\mathcal{S},\text{rann}(A)}(I) = \rho_{\mathbb{F}^{2\times 2}A,\mathcal{T}'}(I) = E_{2,2}$.

Let $Z \in A\{1\}$ and $Y = (y_{i,j}) \in \mathbb{F}^{2\times 2}$. Then $\rho_{\mathcal{S},\text{rann}(A)}(I)Z = \rho_{\mathbb{F}^{2\times 2}A,\mathcal{T}'}(I)Z = E_{2,1} + z_{2,2}E_{2,2}$, $Z\rho_{A\mathbb{F}^{2\times 2},\mathcal{T}}(I) = Z\rho_{\mathcal{S}',\text{lann}(A)}(I) = z_{1,1}E_{1,1} + E_{2,1}$, $\rho_{\mathcal{S},\text{rann}(A)}(I)Z\rho_{A\mathbb{F}^{2\times 2},\mathcal{T}}(I) = \rho_{\mathbb{F}^{2\times 2}A,\mathcal{T}'}(I)Z\rho_{\mathcal{S}',\text{lann}(A)}(I) = \rho_{\mathcal{S},\text{rann}(A)}(I)Z\rho_{\mathcal{S}',\text{lann}(A)}(I) = E_{2,1}$, and $(I - ZA)Y(I - AZ) = (y_{1,2} - z_{1,1}y_{2,2} - z_{2,2}y_{1,1} - z_{1,1}z_{2,2}y_{2,1})E_{1,2}$.

By Theorems 4.1(3)-4.4(3),

$$\begin{aligned}
\{aE_{1,2} + E_{2,1} : a \in \mathbb{F}\} &= \{X \in A\{1\} : XA\mathbb{F}^{2\times 2} = \mathcal{S} \text{ and } \text{rann}(AX) = \mathcal{T}\} \\
&= \{X \in A\{1\} : \mathbb{F}^{2\times 2}AX = \mathcal{S}' \text{ and } \text{lann}(XA) = \mathcal{T}'\} \\
&= \{X \in A\{1\} : \text{lann}(XA) = \mathcal{T}' \text{ and } \text{rann}(AX) = \mathcal{T}\} \\
&= \{X \in A\{1\} : XA\mathbb{F}^{2\times 2} = \mathcal{S} \text{ and } \mathbb{F}^{2\times 2}AX = \mathcal{S}'\}.
\end{aligned}$$

As a consequence of Theorems 4.5(3) and 4.8(3), we obtain

$$\{aE_{2,2} + E_{2,1} + bE_{1,2} : a, b \in \mathbb{F}\} = \{X \in A\{1\} : XA\mathbb{F}^{2\times 2} = \mathcal{S}\} = \{X \in A\{1\} : \text{lann}(XA) = \mathcal{T}'\},$$

and from Theorems 4.6(3) and 4.7(3) we get,

$$\{aE_{1,2} + E_{2,1} + bE_{1,1} : a, b \in \mathbb{F}\} = \{X \in A\{1\} : \mathrm{rann}(AX) = \mathcal{T}\} = \{X \in A\{1\} : \mathbb{F}^{2\times 2}AX = \mathcal{S}'\}.$$

Let $\mathcal{Z}_r = \{Z \in \mathbb{F}^{2\times 2} : AZ = \rho_{A\mathbb{F}^{2\times 2},\mathcal{T}}(I)\}$ and $\mathcal{Z}_l = \{Z \in \mathbb{F}^{2\times 2} : ZA = \rho_{\mathbb{F}^{2\times 2}A,\mathcal{T}'}(I)\}$. Then $\mathcal{Z}_r \cap \mathcal{Z}_l \subseteq A\{1\}$, $\mathcal{Z}_r = \{(z_{i,j}) \in \mathbb{F}^{2\times 2} : z_{2,1} = 1 \text{ and } z_{2,2} = 0\}$ and $\mathcal{Z}_l = \{(z_{i,j}) \in \mathbb{F}^{2\times 2} : z_{1,1} = 0 \text{ and } z_{2,1} = 1\}$. Since $\mathbb{F}^{2\times 2} = A\mathbb{F}^{2\times 2} \oplus \mathcal{S}$ and $A^2 = 0$, we have $A\mathbb{F}^{2\times 2} = A\mathcal{S}$. Similarly, $\mathbb{F}^{2\times 2}A = \mathcal{S}'A$. We also have $\mathrm{rann}(\mathcal{S}') = \mathcal{T}$ and $\mathrm{lann}(\mathcal{S}) = \mathcal{T}'$. Hence, by Theorem 5.3(2), $A^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}} \in \mathcal{S} \cap \mathcal{Z}_r$, by Theorem 5.4(2), $A^{(2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'} \in \mathcal{S}' \cap \mathcal{Z}_l$, and by Theorem 5.5(4), $A^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'} \in \mathcal{S} \cap \mathcal{S}' \cap \mathcal{Z}_r \cap \mathcal{Z}_l$. From here,

$$E_{2,1} = A^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}} = A^{(2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'} = A^{(2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'}.$$

Applying Theorem 5.6(6), we obtain $E_{2,1} = A^{(2)}_{\mathrm{lann}=\mathcal{T}',\mathrm{rann}=\mathcal{T}}$.

By parts (8) of Theorems 6.1-6.4,

$$E_{2,1} = A^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{rann}=\mathcal{T}} = A^{(1,2)}_{\mathrm{lprin}=\mathcal{S}',\mathrm{lann}=\mathcal{T}'} = A^{(1,2)}_{\mathrm{rprin}=\mathcal{S},\mathrm{lprin}=\mathcal{S}'} = A^{(1,2)}_{\mathrm{lann}=\mathcal{T}',\mathrm{rann}=\mathcal{T}}.$$

By Theorems 6.5(6) and 6.8(4),

$$\{E_{2,1} + aE_{2,2} : a \in \mathbb{F}\} = \{X \in A\{1,2\} : XA\mathbb{F}^{2\times 2} = \mathcal{S}\} = \{X \in A\{1,2\} : \mathrm{lann}(XA) = \mathcal{T}'\}.$$

By Theorems 6.6(6) and 6.7(4),

$$\{aE_{1,1} + E_{2,1} : a \in \mathbb{F}\} = \{X \in A\{1,2\} : \mathrm{rann}(AX) = \mathcal{T}\} = \{X \in A\{1,2\} : \mathbb{F}^{2\times 2}AX = \mathcal{S}'\}.$$

It is easy to see that

$$\{B \in \mathbb{F}^{2\times 2} : \mathcal{S} = B\mathbb{F}^{2\times 2}\} = \{B \in \mathbb{F}^{2\times 2} : \mathcal{T}' = \mathrm{lann}(B)\} = \{(b_{i,j}) \in \mathbb{F}^{2\times 2} : b_{1,1} = b_{1,2} = 0 \text{ and } (b_{2,1}, b_{2,2}) \neq 0\}$$

and

$$\{C \in \mathbb{F}^{2\times 2} : \mathcal{T} = \mathrm{rann}(C)\} = \{C \in \mathbb{F}^{2\times 2} : \mathcal{S}' = \mathbb{F}^{2\times 2}C\} = \{(c_{i,j}) \in \mathbb{F}^{2\times 2} : c_{1,2} = c_{2,2} = 0 \text{ and } (c_{1,1}, c_{2,1}) \neq 0\}.$$

We note that we have obtained the unique {2}-inverse corresponding to the right ideals $\mathcal{S}$ and $\mathcal{T}$ such that $\mathbb{F}^{2\times 2} = A\mathbb{F}^{2\times 2} \oplus \mathcal{T}$ and $\mathbb{F}^{2\times 2} = \mathcal{S} \oplus \mathrm{rann}(A)$ (resp. left ideals $\mathcal{S}'$ and $\mathcal{T}'$ such that $\mathbb{F}^{2\times 2} = \mathbb{F}^{2\times 2}A \oplus \mathcal{T}'$ and $\mathbb{F}^{2\times 2} = \mathcal{S}' \oplus \mathrm{lann}(A)$). There are other {2}-inverses with other principal/annihilator ideals. For example, as in [13, Example 2.5], we can consider $B = (\lambda, 1)^t(\alpha, \beta)$ and $C = (\gamma, \delta)^t(1, \mu)$ with $(\alpha, \beta), (\gamma, \delta) \in \mathbb{F}^2 \setminus \{0\}$ and $\lambda, \mu \in \mathbb{F}$. The conditions that any pair of ideals must satisfy are given in Theorems 5.3-5.6.

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