Filomat 38:19 (2024), 6743–6751
https://doi.org/10.2298/FIL2419743A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multiplicative order compact operators between vector lattices and Riesz algebras

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Abstract. In this paper, we present and examine the concept of multiplicative order compact operators from vector lattices to Riesz algebras. Specifically, a linear operator T from a vector lattice X to an Riesz algebra E is deemed $\mathbb{omo}$-compact, if every net x_α in an $\mathbb{o}$-bounded subset of X possesses a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{\mathbb{mo}} y$ for some $y \in E$. Moreover, we introduce and investigate $\mathbb{omo}$-M- and $\mathbb{omo}$-L-weakly compact operators.

1. Introduction

Compact operators are important in the operator theory and its applications. It has been demonstrated in [12, Thm. 2], [10, Thm. 5], and [11, Thm. 2.2] that distinct types of classical convergence, such as order convergence and relatively uniform convergence, lack topological features in vector lattices. It is worth noting, however, that even in the absence of topology, several natural categories of compact operators can be investigated (see for example [7]). In this paper, we introduce and investigate operators with $\mathbb{omo}$-compactness, ranging from vector lattices to Riesz algebras. We assume throughout this work that all vector lattices are real and Archimedean, and all operators are linear. Vector lattices are denoted by the letters X and Y, whereas Riesz algebras by E and F.

A net x_α in X:

- $\mathbb{o}$-*converges* to $x \in X$ (shortly, $x_\alpha \xrightarrow{\mathbb{o}} x$), if there exists a net $y_\beta \downarrow 0$ such that, for any β, there exists α_β satisfying $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$;

- $\mathbb{r}$-*converges* to $x \in X$ (shortly, $x_\alpha \xrightarrow{\mathbb{r}} x$) if, for some $u \in X_+$, there exists a sequence α_n of indexes such that $|x_\alpha - x| \le \frac{1}{n}u$ for all $\alpha \ge \alpha_n$ (see, e.g. [14, 1.3.4, p.20]).

An operator $T : X \to Y$ is called:

- $\mathbb{o}$-*bounded*, if T takes order bounded sets to order bounded ones;

2020 *Mathematics Subject Classification*. 46A40, 46B42, 46J40, 47B65
Keywords. Vector lattice, Riesz algebra, Multiplicative order compact operator, $\mathbb{omo}$-M-Weakly compact operator, $\mathbb{omo}$-L-Weakly compact operator
Received: 17 October 2023; Revised: 14 December 2023; Accepted: 21 December 2023
Communicated by Dijana Mosić
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- *regular*, if $T = T_1 - T_2$ with $T_1, T_2 \geq 0$;
- $\mathbb{o}$-*continuous*, if $Tx_\alpha \xrightarrow{\mathbb{o}} 0$ whenever $x_\alpha \xrightarrow{\mathbb{o}} 0$;
- $\mathbb{r}$-*continuous*, if $Tx_\alpha \xrightarrow{\mathbb{r}} 0$ whenever $x_\alpha \xrightarrow{\mathbb{r}} 0$.

The collection $\mathcal{L}_b(X,Y)$, which comprises all order bounded operators from X to Y, establishes itself as a vector space. It should be noted that every regular is order bounded. If we shift our focus to regular operators, the set $\mathcal{L}_r(X,Y)$, which constitutes all regular operators from X to Y, is an ordered vector space under the following ordering: $T \geq 0$ if $Tx \geq 0$ for each $x \in X_+$. In notation, $\mathcal{L}_r(X) := \mathcal{L}_r(X,X)$, $\mathcal{L}_b(X) = \mathcal{L}_b(X,X)$, and so on. When Y is Dedekind complete, $\mathcal{L}_b(X,Y)$ coincides with $\mathcal{L}_r(X,Y)$ and is a Dedekind complete vector lattice (cf., [1, Thm. 1.67]), whereas the collection $\mathcal{L}_n(X,Y)$ of all order continuous operators from X to Y is a band in $\mathcal{L}_r(X,Y)$ (cf., [1, Thm. 1.73]). Evidently, each positive operator, and as a result, each regular operator is $\mathbb{r}$-continuous.

Consider the case when the vector lattices X and Y have their own linear convergences, $\mathbb{c}_1$ and $\mathbb{c}_2$, respectively. An operator $T : X \to Y$ is called $\mathbb{c}_1\mathbb{c}_2$-*continuous* (cf. [7, Def. 1.4]) if $x_\alpha \xrightarrow{\mathbb{c}_1} 0$ in X implies $Tx_\alpha \xrightarrow{\mathbb{c}_2} 0$ in Y. In the case when $\mathbb{c}_1 = \mathbb{c}_2$, we say that T is $\mathbb{c}_1$-*continuous*. The collection of all $\mathbb{c}_1\mathbb{c}_2$-continuous operators from X to Y is denoted by $\mathcal{L}_{\mathbb{c}_1\mathbb{c}_2}(X,Y)$, and if $\mathbb{c}_1 = \mathbb{c}_2$, we denote $\mathcal{L}_{\mathbb{c}_1\mathbb{c}_2}(X,Y)$ by $\mathcal{L}_{\mathbb{c}_1}(X,Y)$, and $\mathcal{L}_{\mathbb{c}_1}(X,X)$ by $\mathcal{L}_{\mathbb{c}_1}(X)$.

If a vector lattice X that is an associative algebra satisfying $x \cdot y \in X_+$ for each $x, y \in X_+$, it is called *Riesz algebra* (or *l-algebra*). An *l*-algebra E is known as:

- *d-algebra*, if $u \cdot (x \wedge y) = (u \cdot x) \wedge (u \cdot y)$ and $(x \wedge y) \cdot u = (x \cdot u) \wedge (y \cdot u)$ for all $x, y \in E$ and $u \in E_+$;
- *f-algebra* if $x \wedge y = 0$ implies $(u \cdot x) \wedge y = (x \cdot u) \wedge y = 0$ for all $u \in E_+$;
- *semiprime* whenever the only nilpotent element in E is 0;
- *unital* if E has a positive multiplicative unit.
- *right straight l-algebra* (resp., *left straight l-algebra*) whenever $x \cdot u \geq 0$ (resp., $u \cdot x \geq 0$) for all $u \in E_+$ implies $x \geq 0$. If an *l*-algebra E is both left and right straight *l*-algebra, we say that E is a *straight l-algebra*

Each vector lattice X is a commutative f-algebra with respect to the *trivial algebra multiplication* given by $x \cdot y = 0$ for all $x, y \in X$. Every unital *l*-algebra is straight. In an *l*-algebra E, $x \geq y$ implies $x \cdot u \geq y \cdot u$ for all $u \in E_+$. But, in general, the inequality $x \cdot u \geq 0$ for all $u \in E_+$ does not imply $x \geq 0$. An algebra in [6, Ex. 2.8] is an example of d-algebra which is not a straight *l*-algebra.

Consider a linear convergence $\mathbb{c}$ on E (see [7, Def. 1.6]). The algebra multiplication in E is known as:

- *right* $\mathbb{c}$-*continuous* (*resp., left* $\mathbb{c}$-*continuous*) if $x_\alpha \xrightarrow{\mathbb{c}} x$ implies $x_\alpha \cdot y \xrightarrow{\mathbb{c}} x \cdot y$ (resp., $y \cdot x_\alpha \xrightarrow{\mathbb{c}} y \cdot x$) for every $y \in E$ (cf. [7, Def. 5.3]).
- The right $\mathbb{c}$-continuous algebra multiplication will be referred to as $\mathbb{c}$-*continuous multiplication*.

Example 1.1. Consider $T, T_k \in \mathcal{L}_r(\ell^\infty)$ defined as follows: $Tx := l(x) \cdot \mathbb{1}_{\mathbb{N}}$ and $T_k x = x \cdot \mathbb{1}_{\{m \in \mathbb{N}: m \geq k\}}$ for all $x \in \ell^\infty$ and $k \in \mathbb{N}$, where l is a positive extension to ℓ^∞ of the functional $l(x) = \lim_{n\to\infty} x_n$ on the space c of all convergent real sequences. Clearly, $T_k \downarrow \geq 0$. If $T_k \geq S \geq 0$ in $\mathcal{L}_r(\ell^\infty)$ for all $k \in \mathbb{N}$ then, for every $p \in \mathbb{N}$

$$T_k e_p \geq S e_p \geq 0 \quad (\forall k \in \mathbb{N}),$$

where $e_p = \mathbb{1}_{\{p\}} \in \ell^\infty$. Since $T_k e_p = 0$ for all $k > p$ then $Se_p = 0$ for all $p \in \mathbb{N}$. As $\ell^\infty = \ker(l) \oplus \mathbb{R} \cdot \mathbb{1}_{\mathbb{N}}$, $S = s \cdot T$ for some $s \in \mathbb{R}_+$, and hence

$$T_2 \mathbb{1}_{\mathbb{N}} = \mathbb{1}_{\{m \in \mathbb{N}: m \geq 2\}} \geq s \cdot T \mathbb{1}_{\mathbb{N}} = s \cdot \mathbb{1}_{\mathbb{N}},$$

which implies $s = 0$, and hence $S = 0$. Thus, $T_k \downarrow 0$. However, the sequence $T \circ T_k = T$ does not $\mathbb{o}$-converge to 0, showing that the algebra multiplication in $\mathcal{L}_r(\ell^\infty)$ is not left $\mathbb{o}$-continuous. This also shows that, in unital *l*-algebras, $\mathbb{o}$-convergence can be properly weaker than $\mathbb{mo}$-convergence.

A net x_α in E $\mathbb{m}_r\mathbb{c}$-*converges* ($\mathbb{m}_l\mathbb{c}$-*converges*) to x iff

$$|x_\alpha - x| \cdot u \xrightarrow{\mathbb{c}} 0 \quad (\text{respectively} \quad u \cdot |x_\alpha - x| \xrightarrow{\mathbb{c}} 0) \quad (\forall u \in E_+),$$

shortly $x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{c}} x$ and $x_\alpha \xrightarrow{\mathbb{m}_l\mathbb{c}} x$. In commutative algebras, $\mathbb{m}_l\mathbb{c}$ is equivalent to $\mathbb{m}_r\mathbb{c}$. Replacing the algebra multiplication in E by "$\hat{\cdot}$" defined as $x \hat{\cdot} y := y \cdot x$, we restrict ourselves to $\mathbb{m}_r\mathbb{c}$-convergence and denote it by $\mathbb{m}\mathbb{c}$-*convergence* (cf. [4, 5, 7]).

Suppose X is Dedekind complete. Then $\mathcal{L}_r(X)$ is a unital Dedekind complete l-algebra under the operator multiplication, containing $\mathcal{L}_n(X)$ as an l-subalgebra. The algebra multiplication is right $\mathbb{m}\mathbb{o}$-continuous in $\mathcal{L}_r(X)$ and is both left and right $\mathbb{m}\mathbb{o}$-continuous in $\mathcal{L}_n(X)$ [6, Thm. 2.1].

Example 1.2. (cf. [3, Ex. 3.1]). Let E be an f-algebra of all bounded real functions on $[0, 1]$ which differ from a constant on at most countable set of $[0, 1]$. Let $T : E \to E$ be an operator that assigns to each $f \in E$ the constant function Tf on $[0, 1]$ such that the set $\{x \in [0, 1] : f(x) \neq (Tf)(x)\}$ is at most countable. Then T is a rank one continuous in $\|.\|_\infty$-norm positive operator. Consider the following net indexed by finite subsets of $[0, 1]$:

$$f_\alpha(x) = \begin{cases} 1 & \text{if} \quad x \notin \alpha \\ 0 & \text{if} \quad x \in \alpha. \end{cases}$$

Then $f_\alpha \downarrow 0$ in E, yet $\|f_\alpha\|_\infty = 1$ for all α. Thus, T is neither $\mathbb{o}\mathbb{m}\mathbb{o}$- nor $\mathbb{m}\mathbb{o}$-continuous. However, T is $\mathbb{r}$-continuous and, since E is unital, T is $\mathbb{m}\mathbb{r}$-continuous.

The structure of the paper is as follows. In Section 2, we introduce $\mathbb{o}\mathbb{m}\mathbb{c}$-compact operators from a vector lattice to an l-algebra and investigate their general properties with an emphasis on $\mathbb{o}\mathbb{m}\mathbb{o}$- and $\mathbb{o}\mathbb{m}\mathbb{r}$-cases. In Section 3, we investigate the domination problem for $\mathbb{o}\mathbb{m}\mathbb{c}$-compact operators; we define and study $\mathbb{o}\mathbb{m}\mathbb{o}$-$M$- and $\mathbb{o}\mathbb{m}\mathbb{o}$-$L$-weakly compact operators. For further unexplained terminology and notations, we refer to [1, 2, 6–8, 13–17].

2. The properties of $\mathbb{o}\mathbb{m}\mathbb{o}$-compact operators

We begin with the following two definitions (cf. [6, Def. 2.12]).

Definition 2.1. A subset A of an l-algebra E is called $\mathbb{m}_r\mathbb{o}$-*bounded* (resp., $\mathbb{m}_l\mathbb{o}$-*bounded*) if the set $A \cdot u$ (resp., $u \cdot A$) is order bounded for every $u \in E_+$.

Definition 2.2. An operator T from a vector lattice X to an l-algebra E is called $\mathbb{m}_r\mathbb{o}$-*bounded* (resp., $\mathbb{m}_l\mathbb{o}$-*bounded*) if T maps order bounded subsets of X to $\mathbb{m}_r\mathbb{o}$-bounded (resp., $\mathbb{m}_l\mathbb{o}$-bounded) subsets of E.

As usual, we restrict our attention to $\mathbb{m}_r\mathbb{o}$-bounded subsets and operators, and refer to them as $\mathbb{m}\mathbb{o}$-bounded. In any l-algebra E with trivial multiplication, $x * y = 0$ for all $x, y \in E$, each subset A of E is $\mathbb{m}\mathbb{o}$-bounded and as result, every operator from any X to such an l-algebra E is $\mathbb{m}\mathbb{o}$-bounded. For elementary properties of $\mathbb{m}\mathbb{o}$-bounded operators in l-algebras, we refer the reader to the paper [6].

Example 2.3. (cf. [7, Ex. 6]). Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then a sequence λ_n of reals *converges along* $\mathcal{U}$ to λ whenever $\{k \in \mathbb{N} : |\lambda_k - \lambda| \leq \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. Hence, for any $x := (x_n)_{n=1}^\infty \in \ell^\infty$, the sequence x_n converges along $\mathcal{U}$ to $x_\mathcal{U} := \lim_\mathcal{U} x_n$. In that case, an l-algebra multiplication $*$ in ℓ^∞ can be defined as $x * y := (\lim_\mathcal{U} x_n) \cdot (\lim_\mathcal{U} y_n) \cdot \mathbb{1}$, where $\mathbb{1}$ is a sequence of reals that all equal 1. It is easy to see that $(\ell^\infty, *)$ is a d-algebra. Then the set $A = \{ke_k : k \in \mathbb{N}\}$ is $\gtrdot\rtimes$-bounded yet not $\rtimes$-bounded.

Remark 2.4. *Let T be an operator from X to an l-algebra E. The following hold.*

(i) If T is $\mathbb{o}$-bounded (in particular if T is regular) then T is $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-bounded.

(ii) If T is $\mathbb{m}_l\mathbb{o}$- or $\mathbb{m}_r\mathbb{o}$-bounded and E is unital l-algebra then T is $\mathbb{o}$-bounded.

(iii) By [6, Thm. 2.6], every $\mathbb{r}$-continuous operator T from an Archimedean vector lattice to an Archimedean l-algebra is $\mathbb{rmo}$-continuous and then, by [6, Thm. 2.15], T is $\mathbb{mo}$-bounded.

(iv) It follows from [1, Lem. 1.4] that every order continuous operator is $\mathbb{o}$-bounded and hence $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-bounded.

(v) Every $\mathbb{mo}$-, $\mathbb{omo}$-, or $\mathbb{rmo}$-continuous operator is $\mathbb{m}_l\mathbb{o}$-bounded and $\mathbb{m}_r\mathbb{o}$-bounded. Moreover, every $\mathbb{m}_l\mathbb{o}$-, $\mathbb{om}_l\mathbb{o}$-, or $\mathbb{rm}_l\mathbb{o}$-continuous (resp., $\mathbb{m}_r\mathbb{o}$-, $\mathbb{om}_r\mathbb{o}$-, or $\mathbb{rm}_r\mathbb{o}$–continuous) operator is $\mathbb{m}_l\mathbb{o}$-bounded (resp., $\mathbb{m}_r\mathbb{o}$-bounded) [6, Thm. 2.14].

The converse of Remark 2.4 (i) is not true in general. Indeed, in any l-algebra with trivial multiplication, every operator is $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-bounded. A more interesting example is given below.

Example 2.5. Consider an operator T from the f-algebra c to the f-algebra c_0, defined by

$$T(x_1, x_2, x_3, \cdots) = (x, x - x_1, x - x_2, x - x_3, \cdots),$$

where $x = \lim\limits_{n\to\infty} x_n$. Then T is an $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-bounded operator. However, it follows from $T(0, \cdots, 0, 1, 1, \cdots) = (1, \cdots, 1, 0, 0 \cdots)$ that $T([0, \mathbb{1}])$ is not $\mathbb{o}$-bounded in c_0, and so, T is not $\mathbb{o}$-bounded.

The converse of Remark 2.4 (iv) is not true in general. To see this, we include the following example.

Example 2.6. (cf. [6, Ex. 2.8]). Let $(\ell^\infty, *)$ be as in Example 2.3. Now, the identity operator $I : (\ell^\infty, *) \to (\ell^\infty, *)$ is $\mathbb{o}$-bounded, but not $\mathbb{omo}$-continuous. Indeed, take the characteristic functions $h_n = \mathbb{1}_{\{k\in\mathbb{N}:k\geq n\}} \in \ell^\infty$. Then $h_n \xrightarrow{\mathbb{o}} 0$ in ℓ^∞ yet the sequence $|I(h_n) - I(0)| * \mathbb{1} = h_n * \mathbb{1} = \mathbb{1}$ is not $\mathbb{o}$-null. Thus, the sequence $I(h_n)$ is not $\mathbb{mo}$-null, and hence I is not $\mathbb{omo}$-continuous.

Remind that an operator between normed spaces is called *compact* if it maps the closed unit ball to a relatively compact set. Equivalently, the operator is compact if, for each norm bounded sequence, there exists a subsequence such that the image of it is convergent. Motivated by this, we introduce the following notion.

Definition 2.7. An operator T from X to an l-algebra E is called

(a) $\mathbb{om}_r\mathbb{o}$-*compact* (resp., $\mathbb{om}_l\mathbb{o}$-*compact*) if, for every $\mathbb{o}$-bounded set $B \subseteq X$ and every net x_α in B, there exist a subnet x_{α_β} and $y \in E$ such that $Tx_{\alpha_\beta} \xrightarrow{\mathbb{m}_r\mathbb{o}} y$ (resp., $Tx_{\alpha_\beta} \xrightarrow{\mathbb{m}_l\mathbb{o}} y$);

(b) $\mathbb{omo}$-*compact* if T is both $\mathbb{om}_r\mathbb{o}$- and $\mathbb{om}_l\mathbb{o}$-compact;

(c) *sequentially* $\mathbb{om}_r\mathbb{o}$-*compact* (resp., $\mathbb{om}_l\mathbb{o}$-*compact*) if, for every $\mathbb{o}$-bounded set $B \subseteq X$ and every sequence x_n in B, there exist a subsequence x_{n_k} and $y \in E$ such that $Tx_{n_k} \xrightarrow{\mathbb{m}_r\mathbb{o}} y$ (resp., $Tx_{n_k} \xrightarrow{\mathbb{m}_l\mathbb{o}} y$);

(d) *sequentially* $\mathbb{omo}$-*compact* if T is both sequentially $\mathbb{om}_r\mathbb{o}$- and $\mathbb{om}_l\mathbb{o}$-compact.

Example 2.8. Define an operator $T : c_0 \to c_0$ by

$$T\left(\sum_{k=1}^{\infty} a_k e_k\right) = \sum_{k=1}^{\infty} \frac{a_k}{k} e_k,$$

where $e_k = \mathbb{1}_{\{k\}}$ and $\mathbb{R} \ni a_k \to 0$. Then T is compact on the f-algebra $(c_0, \|.\|_\infty)$, and is $\mathbb{omo}$-compact.

Example 2.9. The identity operator on the l-algebra $L_\infty[0, 1]$ with pointwise multiplication is neither $\mathbb{omo}$-compact nor sequentially $\mathbb{omo}$-compact. Indeed, take the sequence of the Rademacher functions $r_n(t) = sgn(\sin(2^n \pi t))$ on $[0, 1]$. Clearly, r_n is $\mathbb{o}$-bounded. Now, assume that r_n has a $\mathbb{mo}$-convergent subnet r_α, say $r_\alpha \xrightarrow{\mathbb{mo}} f$ for some $f \in L_\infty[0, 1]$. Then $r_\alpha \xrightarrow{\mathbb{o}} f$ and hence $r_\alpha(t) \to f(t)$ almost everywhere violating that $r_n(t)$ diverges on $[0, 1]$ except countably many points of form $\frac{k}{m}$ for $k, m \in \mathbb{N}$.

An $\mathbb{omo}$-compact operator need not be sequentially $\mathbb{omo}$-compact, as the next example ([9, Ex. 7]) shows.

Example 2.10. Consider the set $E := \mathbb{R}^X$ equipped with the product topology, where X is the set of all strictly increasing maps from $\mathbb{N}$ to $\mathbb{N}$. It follows from [15, Ex. 3.10 (i)] that E is a unital Dedekind complete f-algebra with respect to the pointwise operations and ordering.

(i) The identity map $\mathcal{I}$ on E is an $\mathbb{om}_r\mathbb{o}$-compact operator. Indeed, assume that f_α is a net in an $\mathbb{o}$-bounded subset of E. It follows from [9, Ex. 7 (1)] that there exists a subnet f_{α_β} such that $f_{\alpha_\beta} \xrightarrow{\mathbb{o}} f$ for some $f \in E$. Since every f-algebra has $\mathbb{o}$-continuous algebra multiplication, it follows from [7, Lm. 5.5] that $f_{\alpha_\beta} \xrightarrow{\mathbb{m}_r\mathbb{o}} f$. Therefore, $\mathcal{I}$ is $\mathbb{om}_r\mathbb{o}$-compact.

(ii) The operator $\mathcal{I}$ is not sequentially $\mathbb{om}_r\mathbb{o}$-compact. Consider a sequence f_n in $\{-1, 1\}^X$ as in [9, Ex. 7 (2)]. Then f_n is order bounded yet has no $\mathbb{o}$-convergent subsequence. Thus, every subsequence of f_n does not $\mathbb{mo}$-converge E has a unit element.

Remark 2.11. It is known that any compact operator is norm continuous, but in general there are $\mathbb{omo}$-compact operators that are not $\mathbb{mo}$-continuous. Indeed, denote by $\mathcal{B}$ the Boolean algebra of the Borel subsets of $[0, 1]$ equal up to measure null. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$. Then it can be shown that the linear operator $\varphi_{\mathcal{U}} : L_\infty[0, 1] \to \mathbb{R}$ defined by

$$\varphi_{\mathcal{U}}(f) := \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A f d\mu$$

is $\mathbb{omo}$-compact (see [7, Lem. 5.5]) because the algebra multiplication in $\mathbb{R}$ is order continuous (cf. [13, 15]). However, it is not $\mathbb{mo}$-continuous.

The following result is an $\mathbb{omo}$-version of [9, Thm.2].

Theorem 2.12. *Every $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact) operator T from a vector lattice X to an l-algebra E is $\mathbb{m}_r\mathbb{o}$-bounded (resp., $\mathbb{m}_l\mathbb{o}$-bounded).*

Proof. Let $T : X \to E$ be $\mathbb{om}_r\mathbb{o}$-compact. Suppose in contrary that T is not $\mathbb{m}_r\mathbb{o}$-bounded. Then, there exist $b \in X_+$ and $u \in E_+$ such that $(T[0, b]) \cdot u$ is not order bounded in E. For every $a \in E_+$ choose an $x_a \in [0, b]$ satisfying

$$|Tx_a| \cdot u \nleq a. \tag{1}$$

Since the net $(x_a)_{a \in E_+}$ is order bounded and T is $\mathbb{om}_r\mathbb{o}$-compact, there exist a subnet $(x_{a_\gamma})_{\gamma \in \Gamma}$ and $z \in E$ with $Tx_{a_\gamma} \xrightarrow{\mathbb{m}_r\mathbb{o}} z$, that is

$$|Tx_{a_\gamma} - z| \cdot v \xrightarrow{\mathbb{o}} 0 \qquad (\forall v \in E_+).$$

In particular, $|Tx_{a_\gamma} - z| \cdot u \xrightarrow{\mathbb{o}} 0$, and hence the net $(|Tx_{a_\gamma} - z| \cdot u)_{\gamma \in \Gamma}$ has an order bounded tail. Then there are $\gamma_0 \in \Gamma$ and $g \in E_+$ with $|Tx_{a_\gamma} - z| \cdot u \leq g$ for $\gamma \geq \gamma_0$. The inequality $|Tx_{a_\gamma}| \leq |Tx_{a_\gamma} - z| + |z|$ implies

$$|Tx_{a_\gamma}| \cdot u \leq |Tx_{a_\gamma} - z| \cdot u + |z| \cdot u \leq g + |z| \cdot u \quad (\forall \gamma \geq \gamma_0).$$

Now, let γ_1 be such that $\gamma_1 \geq \gamma_0$ and $a_{\gamma_1} \geq g + |z| \cdot u \in E_+$. Then

$$|Tx_{a_{\gamma_1}}| \cdot u \leq g + |z| \cdot u \leq a_{\gamma_1},$$

which contradicts (1). Therefore, T is not $\mathbb{m}_r\mathbb{o}$-bounded.

The case of $\mathbb{om}_l\mathbb{o}$-compact operator is similar. □

The following example shows that sequentially $\mathbb{omo}$-compact operators need not to be order bounded.

Example 2.13. Let $\ell^\infty_\omega(\mathbb{R})$ be the l-algebra of countably supported bounded real-valued functions on $\mathbb{R}$. Let $E = \text{span}\{\mathbb{1}, \ell^\infty_\omega\} \subset \ell^\infty(\mathbb{R})$, where $\mathbb{1}$ denotes the constant function on $\mathbb{R}$ taking the value 1. Consider a projection T of E onto $\ell^\infty_\omega(\mathbb{R})$ whose kernel is $\text{span}\{\mathbb{1}\}$. Take an order bounded sequence $f_n = \beta_n\mathbb{1} + g_n$ in E, where $\beta_n \in \mathbb{R}$ and $g_n \in \ell^\infty_\omega(\mathbb{R})$. As the set $\{g_n(t) : n \in \mathbb{N}\}$ of functions on $\mathbb{R}$ is countably supported, there exists a subsequence g_{n_k} such that $g_{n_k}(t) \to z(t)$ for all $t \in \mathbb{R}$. Since g_{n_k} is order bounded, then $Tf_{n_k} = g_{n_k} \xrightarrow{\mathbb{o}} z \in \ell^\infty_\omega(\mathbb{R})$. By [16, Thm. VIII.2.3], $Tf_{n_k} \xrightarrow{\mathbb{mo}} z \in \ell^\infty_\omega(\mathbb{R})$, and hence T is a sequentially $\mathbb{omo}$-compact. Since the set $\{T\mathbb{1}_x : x \in \mathbb{R}\}$ is not order bounded in $\ell^\infty_\omega(\mathbb{R})$, the operator T is not order bounded.

Proposition 2.14. *Let R, T, and S be operators on an l-algebra E.*

(i) *If T is (sequentially) $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact) and S is (sequentially) $\mathbb{m}_r\mathbb{o}$-continuous (resp., $\mathbb{m}_l\mathbb{o}$-continuous) then the operator $S \circ T$ is (sequentially) $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact).*

(ii) *If T is (sequentially) $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact) and R is $\mathbb{o}$-bounded, then $T \circ R$ is (sequentially) $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact).*

(iii) *Let T be an $\mathbb{m}_r\mathbb{om}_r\mathbb{o}$-continuous (resp., $\mathbb{m}_l\mathbb{om}_l\mathbb{o}$-continuous) operator, and let S be an $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact) operator. Then the operator $T \circ S$ is $\mathbb{om}_r\mathbb{o}$-compact (resp., $\mathbb{om}_l\mathbb{o}$-compact).*

Proof. (i) Let x_α be a net in an $\mathbb{o}$-bounded subset of E. Since T is $\mathbb{om}_r\mathbb{o}$-compact, there exist a subnet x_{α_β} and $x \in E$ such that $Tx_{\alpha_\beta} \xrightarrow{\mathbb{m_r o}} x$. It follows from the $\mathbb{m}_r\mathbb{o}$-continuity of S that $S(Tx_{\alpha_\beta}) \xrightarrow{\mathbb{m_r o}} S(x)$. Therefore, $S \circ T$ is $\mathbb{om}_r\mathbb{o}$-compact.

(ii) Let x_α to be net in an $\mathbb{o}$-bounded subset B of E. Since R is $\mathbb{o}$-bounded, the set $R(B)$ is $\mathbb{o}$-bounded. Now, the $\mathbb{om}_r\mathbb{o}$-compactness of T implies the existence of a subnet x_{α_β} in x_α and of some $z \in E$ such that $TRx_{\alpha_\beta} \xrightarrow{\mathbb{m_r o}} z$. Therefore, $T \circ R$ is $\mathbb{om}_r\mathbb{o}$-compact.

(iii) Let x_α to be net in an $\mathbb{o}$-bounded subset B of E. The $\mathbb{om}_r\mathbb{o}$-compactness of S implies existence of a subnet x_{α_β} in x_α and of some $z \in E$ such that $Sx_{\alpha_\beta} \xrightarrow{\mathbb{m_r o}} z$. That is, for every $u \in E$, $|Sx_{\alpha_\beta} - z| \cdot u \xrightarrow{\mathbb{o}} 0$. Since T is $\mathbb{m}_r\mathbb{om}_r\mathbb{o}$-continuous, $TSx_{\alpha_\beta} \xrightarrow{\mathbb{m_r o}} Tz$, and hence $T \circ S$ is $\mathbb{om}_r\mathbb{o}$-compact.

The sequential and $\mathbb{om}_l\mathbb{o}$-compact cases are analogous. □

Proposition 2.15. *Every $\mathbb{o}$-continuous finite rank operator on an l-algebra E with $\mathbb{o}$-continuous multiplication is $\mathbb{omo}$-compact.*

Proof. Let $T : E \to E$ be $\mathbb{o}$-continuous and $\dim(TE) < \infty$. Then

$$T = \sum_{k=1}^{m} x_k \otimes f_k \text{ for } x_1, \ldots, x_m \in E \text{ and } f_1, \ldots, f_m \in E'_n.$$

WLOG, we may assume $T = x_1 \otimes f_1$. Since E'_n is Dedekind complete, f_1 is regular, and T is also regular. WLOG, suppose $x_1 \geq 0$ and $f_1 \geq 0$. Let z_α be a net in an $\mathbb{o}$-bounded subset of E. Then $Tz_\alpha = (x_1 \otimes f_1)(z_\alpha) = f_1(z_\alpha)x_1$ is $\mathbb{o}$-bounded since every $\mathbb{o}$-continuous functional is $\mathbb{o}$-bounded. Since $\dim(TE) = 1$, there exists a subnet z_{α_β} such that $Tz_{\alpha_\beta} \xrightarrow{\mathbb{o}} y \in T(E)$. Using $\dim(TE) = 1$ again, we obtain $Tz_{\alpha_\beta} \xrightarrow{\mathbb{mo}} y$. Therefore T is $\mathbb{omo}$-compact. □

The following result is an extension of Example 2.8.

Proposition 2.16. *Let E be an l-algebra with $\mathbb{o}$-continuous algebra multiplication. Then the algebra $\mathcal{L}_{rc}(E)$ of regular order compact operators is a subspace of $\mathcal{L}_{romo}(E)$, which is itself a right algebra ideal of $\mathcal{L}_r(E)$.*

Proof. Suppose that T is a regular $\mathbb{o}$-compact operator on a right $\mathbb{o}$-continuous l-algebra E, and x_α is a net in an $\mathbb{o}$-bounded subset B of E. Then there exist a subnet x_{α_β} and $y \in E$ such that $Tx_{\alpha_\beta} \xrightarrow{\mathbb{o}} y$. It follows from [7, Lm. 5.5] that $Tx_{\alpha_\beta} \xrightarrow{\mathbb{m_r o}} y$. Thus, we obtain that T is $\mathbb{om}_r\mathbb{o}$-compact. As the proof of $\mathbb{om}_l\mathbb{o}$-compactness is analogous, $\mathcal{L}_{rc}(E)$ is subspace of $\mathcal{L}_{romo}(E)$. On the other hand, it is well known that $\mathcal{L}_r(E)$ is a subspace of $\mathcal{L}_b(E)$. It follows from Theorem 2.14 (ii) that $\mathcal{L}_{romo}(E)$ is a right algebra ideal of $\mathcal{L}_r(E)$. □

3. Domination problem for compact operators

In this section, we study the domination problem for $\mathbb{omo}$-compact operators, and introduce $\mathbb{omo}$-M- and $\mathbb{omo}$-L-weakly compact operators. Now, consider the domination problem for positive $\mathbb{mo}$($\mathbb{omo}$)-continuous and $\mathbb{omo}$-compact operators. We have a positive answer for $\mathbb{mo}$($\mathbb{omo}$)-continuous operators in the next lemma.

Lemma 3.1. *Let E and F be l-algebras and let operators $T, S : E \to F$ satisfy $0 \le S \le T$. If T is $\mathbb{m}_r\mathbb{o}$-continuous (resp., $\mathbb{m}_l\mathbb{o}$-, $\mathbb{om}_r\mathbb{o}$-, or $\mathbb{om}_l\mathbb{o}$-continuous), then S has the same property.*

Proof. Suppose T to be $\mathbb{m}_r\mathbb{o}$-continuous and $x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} x \in E$ for some $x \in E$. Then we have $Tx_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} Tx$ in F. Since

$$0 \le |Sx_\alpha - Sx| \le S(|x_\alpha - x|) \le T(|x_\alpha - x|) \quad (\forall \alpha),$$

we get

$$|Sx_\alpha - Sx| \cdot u \le T(|x_\alpha - x|) \cdot u \quad (\forall u \in F_+). \tag{2}$$

On the other hand, it follows from [4, Prop. 2.4] that $x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} x$ implies $|x_\alpha - x| \xrightarrow{\mathbb{m}_r\mathbb{o}} 0$, and so, we obtain $T(|x_\alpha - x|) \xrightarrow{\mathbb{m}_r\mathbb{o}} 0$ by the $\mathbb{m}_r\mathbb{o}$-continuity of T, i.e., $T(|x_\alpha - x|) \cdot u \xrightarrow{\mathbb{o}} 0$ for all $u \in F_+$. Hence, the desired result raises from the inequality (2), $Sx_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} Sx$ in F. The proof for the cases of $\mathbb{m}_l\mathbb{o}$-, $\mathbb{om}_r\mathbb{o}$- and $\mathbb{om}_l\mathbb{o}$-continuity are similar. □

Recall that a net $(x_\alpha)_{\alpha \in A}$ in an l-algebra is called $\mathbb{mo}$-Cauchy if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha') \in A \times A}$ is $\mathbb{mo}$-convergent to 0. Moreover, an l-algebra is called $\mathbb{mo}$-complete if every $\mathbb{mo}$-Cauchy net is $\mathbb{mo}$-convergent; see [4, Def. 2.11].

Theorem 3.2. *Let X be a vector lattice and E be a Dedekind and sequentially $\mathbb{mo}$-complete l-algebra with $\mathbb{o}$-continuous algebra multiplication. If $T_m : X \to E$ is a sequence of sequential $\mathbb{omo}$-compact operators and $T_m \xrightarrow{\mathbb{o}} T$ in $\mathcal{L}_b(X, E)$ then T is sequentially $\mathbb{omo}$-compact.*

Proof. Let x_n be a order bounded sequence in X, T_m be a sequence of sequential $\mathbb{om}_r\mathbb{o}$-compact operators and E be sequentially $\mathbb{m}_r\mathbb{o}$-complete. Then there is $w \in X_+$ such that $|x_n| \le w$ for all $n \in N$. Also, by a standard diagonal argument, there exists a subsequence x_{n_k} such that for any $m \in N$, $T_m x_{n_k} \xrightarrow{\mathbb{m}_r\mathbb{o}} y_m$ for some $y_m \in E$. Let's show that y_m is a $\mathbb{mo}$-Cauchy sequence in E. Fix an arbitrary $u \in E_+$. Then we have

$$|y_m - y_j| \cdot u \le |y_m - T_m x_{n_k}| \cdot u + |T_m x_{n_k} - T_j x_{n_k}| \cdot u + |T_j x_{n_k} - y_j| \cdot u.$$

Then the first and third terms in the last inequality both order converge to zero as $m \to \infty$ and $j \to \infty$, respectively. Since $T_m \xrightarrow{\mathbb{o}} T$ in vector lattice $\mathcal{L}_b(X, E)$, we have $|T_m - T_j| \xrightarrow{\mathbb{o}} 0$, and so, it follows from [16, Thm. VIII.2.3] that $|T_m - T_j|(x) \xrightarrow{\mathbb{o}} 0$ for all $x \in X$. Then, by using [1, Thm. 1.67(a)], we obtain the inequality

$$|T_m x_{n_k} - T_j x_{n_k}| \cdot u \le |T_m - T_j|(|x_{n_k}|) \cdot u \le |T_m - T_j|(w) \cdot u.$$

Since E has $\mathbb{o}$-continuous algebra multiplication, it follows from [7, *Lem.* 5.5] that $|T_m - T_j|(x) \xrightarrow{\mathbb{o}} 0$ implies $|T_m - T_j|(w) \cdot u \xrightarrow{\mathbb{o}} 0$. Hence, we obtain that $|T_m x_{n_k} - T_j x_{n_k}| \cdot u \xrightarrow{\mathbb{o}} 0$. Therefore, y_m is $\mathbb{mo}$-Cauchy. Now, by sequentially $\mathbb{m}_r\mathbb{o}$-completeness of E, there is $y \in E$ such that $y_m \xrightarrow{\mathbb{m}_r\mathbb{o}} y$ in E as $m \to \infty$. Hence,

$$\begin{aligned} |Tx_{n_k} - y| \cdot u &\le |Tx_{n_k} - T_m x_{n_k}| \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u \\ &\le |T_m - T|(|x_{n_k}|) \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u \\ &\le |T_m - T|(w) \cdot u + |T_m x_{n_k} - y_m| \cdot u + |y_m - y| \cdot u. \end{aligned}$$

Now, for fixed $m \in N$, and as $k \to \infty$, we have

$$\limsup_{k \to \infty} |Tx_{n_k} - y| \cdot u \le |T_m - T|(w) \cdot u + |y_m - y| \cdot u.$$

But $m \in N$ is arbitrary, so $\limsup\limits_{k\to\infty} |Tx_{n_k} - y| \cdot u = 0$. Thus, $|Tx_{n_k} - y| \cdot u \xrightarrow{\mathbb{o}} 0$, i.e., $Tx_{n_k} \xrightarrow{\mathbb{m}_r\mathbb{o}} y$. Therefore, T is sequentially $\mathbb{o}\mathbb{m}_r\mathbb{o}$-compact.

The sequentially $\mathbb{o}\mathbb{m}_l\mathbb{o}$-compact case is analogous. □

In the rest of the section, we discuss $\mathbb{omo}$-M- and $\mathbb{omo}$-L-weakly compact operators. Remind that a norm bounded operator T from a normed lattice X into a normed space Y is called M-weakly compact if $Tx_n \xrightarrow{\|\cdot\|} 0$ holds for every norm bounded disjoint sequence x_n in X. Also, a norm bounded operator T from a normed space Y into a normed lattice X is called L-weakly compact whenever $\lim \|x_n\| = 0$ holds for every disjoint sequence x_n in the solid hull $\mathrm{sol}(T(B_Y)) := \{x \in X : \exists y \in T(B_Y) \text{ with } |x| \leq |y|\}$ of $T(B_Y)$, where B_Y is the closed unit ball of Y. Similarly we have the following notion.

Definition 3.3. Let $T : X \to E$ be a sequentially $\mathbb{mo}$-continuous operator.

(1) If $Tx_n \xrightarrow{\mathbb{mo}} 0$ for every order bounded disjoint sequence x_n in X then T is said to be $\mathbb{omo}$-*M-weakly compact*.

(2) If $y_n \xrightarrow{\mathbb{m}_r\mathbb{o}} 0$ for every disjoint sequence y_n in $\mathrm{sol}(T(A))$, where A is any order bounded subset of X, then T is said to be $\mathbb{omo}$-*L-weakly compact*.

Proposition 3.4. *Let T be an order bounded σ-order continuous operator from a normed lattice X to an l-algebra E with $\mathbb{o}$-continuous algebra multiplication. Then T is $\mathbb{omo}$-M- and $\mathbb{omo}$-L-weakly compact.*

Proof. Clearly, T is sequentially $\mathbb{mo}$-continuous operator, because E has $\mathbb{o}$-continuous algebra multiplication; see [6, Lem. 5.5]. Let x_n be an $\mathbb{o}$-bounded disjoint sequence in X. Then by [8, Rem. 10] we get $x_n \xrightarrow{\mathbb{o}} 0$. Thus, we have $Tx_n \xrightarrow{\mathbb{mo}} 0$. Therefore, T is $\mathbb{omo}$-M-weakly compact.

Now, we show that T is $\mathbb{omo}$-L-weakly compact. Let A be an order bounded set in X. Thus, $T(A)$ is order bounded, and so, $\mathrm{sol}(T(A))$ is an order bounded set in E. Take an arbitrary disjoint sequence y_n in $\mathrm{sol}(T(A))$. Then, using [8, Rem. 10], we have $y_n \xrightarrow{\mathbb{o}} 0$, and so, $y_n \xrightarrow{\mathbb{mo}} o$ since E has $\mathbb{o}$-continuous algebra multiplication; see [6, Lem. 5.5]. Thus, T is $\mathbb{omo}$-L-weakly compact. □

Similarly to [3, Cor. 2.3], we obtain the following result.

Theorem 3.5. *Let $T, S : X \to E$ be two linear operators from a normed lattice X to an l-algebra E such that $0 \leq S \leq T$. If T is $\mathbb{omo}$-M- or $\mathbb{omo}$-L-weakly compact then S has the same property.*

Proof. Suppose that T is an $\mathbb{omo}$-M-weakly compact operator. Thus, it follows from Lemma 3.1 that S is an $\mathbb{mo}$-continuous operator. Let x_α be an order bounded disjoint net in X. So, $|x_n|$ is also order bounded and disjoint. Since T is $\mathbb{omo}$-M-weakly compact, $T(|x_n|) \xrightarrow{\mathbb{mo}} 0$ in E. Following from the inequality

$$0 \leq |Sx_n| \cdot u \leq S(|x_n|) \cdot u \leq T(|x_n|) \cdot u \tag{3}$$

for all $n \in \mathbb{N}$ and for every $u \in E_+$ (cf. [2, Lem. 1.6]), we get $Sx_n \xrightarrow{\mathbb{mo}} 0$ in E. Thus, S is $\mathbb{omo}$-M-weakly compact.

Next, we show that S is $\mathbb{omo}$-L-weakly compact. Let A be an order bounded subset of X. Put $|A| = \{|a| : a \in A\}$. Clearly, $\mathrm{sol}(S(A)) \subseteq \mathrm{sol}(S(|A|))$ and since $0 \leq S \leq T$, we have $\mathrm{sol}(S(|A|)) \subseteq \mathrm{sol}(T(|A|))$. Let y_n be a disjoint sequence in $\mathrm{sol}(S(A))$ then y_n is in $\mathrm{sol}(T(|A|))$ and, since T is $\mathbb{omo}$-L-weakly compact then $T(|x_n|) \xrightarrow{\mathbb{mo}} 0$ in E. Therefore, by inequality (3), S is $\mathbb{omo}$-L-weakly compact. □

Proposition 3.6. *If $T : X \to E$ is an $\mathbb{omo}$-L-weakly compact lattice homomorphism then T is $\mathbb{omo}$-M-weakly compact.*

Proof. Take an order bounded disjoint sequence x_n in X. Since T is lattice homomorphism, we have that Tx_n is disjoint in E. Clearly $Tx_n \in \mathrm{sol}\Big(\{Tx_n : n \in \mathbb{N}\}\Big)$. By $\mathbb{omo}$-L-weakly compactness of T, we have $Tx_n \xrightarrow{\mathbb{mo}} 0$ in E. Therefore, T is $\mathbb{omo}$-M-weakly compact. □

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