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Improving Jensen-type inequalities via the Taylor interpolation formula

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Abstract. The main objective of this paper is to establish some general improvements of the Jensen inequality for the classes of absolutely and completely monotonic functions. The key role in this work is played by the Taylor interpolation formula. Besides the improvement of the Jensen inequality, we also derive more accurate superadditivity and monotonicity relations for the Jensen functional. As an application, we obtain improved versions of power mean inequalities and the Hölder inequality. Finally, we obtain more accurate form of the Lah-Ribarič inequality for the aforementioned classes of functions. In particular, by using the developed method, we also get a non-trivial lower bound for the non-weighted Jensen functional.

1. Introduction

Infinitely-differentiable functions on an interval I that are non-negative on I, as well as all their derivatives, are called absolutely monotonic functions, according to early investigations done by S. Bernstein (see [1] and [2]). A companion definition says that a function f, infinitely differentiable on an interval I, is completely monotonic on I if for all non-negative integers n, $(-1)^n f^{(n)}(x) \geq 0$ on I. The importance of absolutely monotonic functions stems from two important results. The first refers to the analytical extension of such functions, while the second one refers to their representation in the form of the Laplace integral (see [2]).

However, in this paper we use these classes of functions in apparently different context. Namely, both absolutely and completely monotonic functions are convex. It turns out that the Jensen inequality can be significantly improved for these classes of functions. Therefore, this is the main task of this paper.

The Jensen inequality can be rewritten in the form of the corresponding functional, i.e.

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{m} p_i f(x_i) - P_m f\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \geq 0, \tag{1}$$

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where the function $f : I \to \mathbb{R}$ is convex and $\mathbf{x} = (x_1, x_2, \dots, x_m) \in I^m$, $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_+^m$, $P_m = \sum_{i=1}^m p_i > 0$. Dragomir et al. [3], noticed that the Jensen functional is superadditive, that is,

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) + \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}), \tag{2}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^m$. In the years that followed, this relation took the role of the starting point for the improvements of the Jensen-type inequalities, since it also implied the property that was referred to as monotonicity of the Jensen functional:

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \geq 0, \tag{3}$$

whenever $\mathbf{p} \geq \mathbf{q}$, i.e. $p_i \geq q_i$, $i = 1, 2, \dots, m$ (see also [10], p.717).

By virtue of (3), Krnić et al. [8], established the mutual bounds for the Jensen functional expressed in terms of the corresponding non-weighted functional. More precisely, they proved that

$$m p_{\max} \mathcal{I}_m(f, \mathbf{x}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq m p_{\min} \mathcal{I}_m(f, \mathbf{x}), \tag{4}$$

where $p_{\min} = \min_{1 \leq i \leq m} p_i$, $p_{\max} = \max_{1 \leq i \leq m} p_i$, and where $\mathcal{I}_m(f, \mathbf{x})$ stands for the associated non-weighted functional, i.e.

$$\mathcal{I}_m(f, \mathbf{x}) = \frac{\sum_{i=1}^m f(x_i)}{m} - f\left(\frac{\sum_{i=1}^m x_i}{m}\right).$$

The lower bound in (4) represents the refinement, while the upper one is the reverse of the Jensen inequality. Based on this property, numerous inequalities such as the Young inequality, the Hölder inequality, power mean inequalities, etc. have been refined (see, e.g. [7, 8] and the references cited therein). In addition, for a systematic overview of the classical and new results in connection to the Jensen inequality, the reader is referred to monographs [6, 10, 11] and the references cited therein.

As we have already announced, the main goal of this paper is to establish some general improvements of the Jensen inequality for a classes of absolutely and completely monotonic functions. The key role in our work is played by the well-known Taylor formula. Let $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on I and let $a, b \in I$, $a < b$. Then for $c, x \in [a, b]$, the following Taylor expansion at the point c holds

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k + \frac{1}{(n-1)!} \int_c^x (x - s)^{n-1} f^{(n)}(s) ds, \tag{5}$$

where n is a positive integer and the remainder is given in the form of the integral. Since we will deal here with infinitely differentiable functions, we may omit mentioning the property of absolute continuity.

The outline of the paper is as follows: after this Introduction, in Section 2 we establish more accurate Jensen-type inequalities. The crucial step in deriving our results is the Taylor expansion at the endpoints of an interval. While the Taylor expansion at the left endpoint of the interval is suitable for absolutely monotonic functions, the Taylor expansion at the right endpoint of the interval corresponds to a class of completely monotonic functions. Besides improvements of the basic Jensen inequality for these classes of functions, we derive more accurate superadditivity and monotonicity relations, as well as the mutual bounds for the Jensen functional in this setting. It is important to point out that with the method presented in this paper, we get bounds for the non-weighted Jensen functional as well, which was not the case in our earlier paper [8]. As an application, in Section 3 our main results are applied while deriving refinements of power mean inequalities and in Section 4 several refinements of the Hölder inequality are obtained. Finally, in Section 5 we derive more accurate Lah-Ribarič-type inequalities for absolutely and completely monotonic functions.

2. Main results

In order to make our further discussion concise, let's introduce some notation that will be used throughout the paper. Namely, if $\mathbf{x} = (x_1, x_2, \dots, x_m) \in [a,b]^m \subset I^m$ and $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_+^m$, then x_{P_m} and $\overline{x}_M$ stand for $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$ and $\overline{x}_M = \frac{1}{m}\sum_{i=1}^m x_i$, respectively. In addition, we denote $p_{\min} = \min_{1\le i\le m} p_i$ and $p_{\max} = \max_{1\le i\le m} p_i$.

In this paper, we consider functions with non-negative derivatives, as well as with alternating derivatives, on the interval $[a,b]$. Consequently, we use Taylor expansions at the endpoints of this interval, so we define

$$T_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad n \ge 1,$$

$$T^*_{n-1}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k, \quad n \ge 1.$$

Moreover, let $e_k(x) = \frac{(x-a)^k}{k!}$ and $e^*_k(x) = \frac{(b-x)^k}{k!}$, $k = 0,1,2,\dots,n-1$, so that $T_{n-1}(x) = \sum_{k=0}^{n-1} f^{(k)}(a)e_k(x)$ and $T^*_{n-1}(x) = \sum_{k=0}^{n-1}(-1)^k f^{(k)}(b)e^*_k(x)$. At the beginning, we have to adapt the reminder in the Taylor formula (5), as it has been done in [4]. Namely, since

$$\int_a^x (x-s)^{n-1} f^{(n)}(s)ds = \int_a^b G_{n-1}(x,s) f^{(n)}(s)ds,$$

where

$$G_{n-1}(x,s) = \begin{cases} (x-s)^{n-1}, & x \ge s, \\ 0, & x < s, \end{cases}$$

the Taylor expansion at the point a can be rewritten as

$$f(x) = T_{n-1}(x) + \frac{1}{(n-1)!}\int_a^b G_{n-1}(x,s) f^{(n)}(s)ds. \tag{6}$$

Similarly, the Taylor formula at the point b can be rewritten as

$$f(x) = T^*_{n-1}(x) + \frac{(-1)^n}{(n-1)!}\int_a^b G^*_{n-1}(x,s) f^{(n)}(s)ds, \tag{7}$$

where

$$G^*_{n-1}(x,s) = \begin{cases} (s-x)^{n-1}, & x \le s, \\ 0, & x > s. \end{cases}$$

We will now utilize identities (6) and (7) in obtaining suitable forms of the Jensen functional $\mathcal{J}_m(f,\mathbf{x},\mathbf{p})$. Namely, let $f : I \to \mathbb{R}$ be n-times differentiable function, and let $\mathbf{x} = (x_1, x_2, \dots, x_m) \in [a,b]^m \subset I^m$, $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_+^m$. Then, we have that

$$p_i f(x_i) = p_i T_{n-1}(x_i) + \frac{1}{(n-1)!}\int_a^b p_i G_{n-1}(x_i,s) f^{(n)}(s)ds, \ i = 1,2,\dots,m,$$

and

$$P_m f(x_{P_m}) = P_m T_{n-1}(x_{P_m}) + \frac{1}{(n-1)!}\int_a^b P_m G_{n-1}(x_{P_m},s) f^{(n)}(s)ds.$$

Since $\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^m p_i f(x_i) - P_m f(x_{P_m})$, we arrive at the identity

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) = \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) + \frac{1}{(n-1)!}\int_a^b \mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{p}) f^{(n)}(s)ds. \tag{8}$$

In the same way we have

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) = \mathcal{J}_m(T^*_{n-1}, \mathbf{x}, \mathbf{p}) + \frac{(-1)^n}{(n-1)!}\int_a^b \mathcal{J}_m(G^*_{n-1}, \mathbf{x}, \mathbf{p}) f^{(n)}(s)ds. \tag{9}$$

Identities (8) and (9) will be crucial in establishing improved variants of the Jensen inequality. It turns out that (8) is suitable for the class of absolutely monotonic functions, while (9) fits to completely monotonic functions. More precisely, since $T''_{n-1}(x) = \sum_{k=2}^{n-1} \frac{f^{(k)}(a)}{(k-2)!}(x-a)^{k-2}$, it follows that the polynomial T_{n-1} is convex on $[a, b]$, provided that f is an absolutely monotonic function. In the same way, $T^{*\prime\prime}_{n-1}(x) = \sum_{k=2}^{n-1} \frac{(-1)^k f^{(k)}(b)}{(k-2)!}(b-x)^{k-2}$, so T^*_{n-1} is convex on $[a, b]$ for a completely monotonic function f. Note also that the functions e_k and e^*_k are convex on $[a, b]$, for every integer k.

In order to state our first result, we need to introduce a few more definitions. Namely, if f is absolutely monotonic function on I, we define

$$m_{n-1} = \min_{0\le k\le n-1} f^{(k)}(a),$$

while for completely monotonic function $f : I \to \mathbb{R}$, we define

$$m^*_{n-1} = \min_{0\le k\le n-1} \left|f^{(k)}(b)\right|.$$

Furthermore, we define polynomials

$$t_{n-1}(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} = \sum_{k=0}^{n-1} e_k(x), \quad n \ge 1,$$
$$t^*_{n-1}(x) = \sum_{k=0}^{n-1} \frac{(b-x)^k}{k!} = \sum_{k=0}^{n-1} e^*_k(x), \quad n \ge 1.$$

Note that both polynomials t_{n-1} and t^*_{n-1} are also convex on $[a, b]$. Now, we are ready to establish our first result.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be absolutely monotonic function and let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, b]^m \subset I^m$, $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m_+$. Then hold the inequalities*

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \ge \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) \ge m_{n-1}\mathcal{J}_m(t_{n-1}, \mathbf{x}, \mathbf{p}) \ge 0. \tag{10}$$

On the other hand, if $f : I \to \mathbb{R}$ is completely monotonic function, then

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \ge \mathcal{J}_m(T^*_{n-1}, \mathbf{x}, \mathbf{p}) \ge m^*_{n-1}\mathcal{J}_m(t^*_{n-1}, \mathbf{x}, \mathbf{p}) \ge 0. \tag{11}$$

Proof. We first prove (10). It is easy to see that the function G_{n-1} is convex on $[a, b]$ for every fixed value $s \in [a, b]$. Namely, we have that

$$\frac{\partial G_{n-1}}{\partial x}(x, s) = \begin{cases} (n-1)(x-s)^{n-2}, & x \ge s, \\ 0, & x < s, \end{cases}$$

and

$$\frac{\partial^2 G_{n-1}}{\partial^2 x}(x, s) = \begin{cases} (n-1)(n-2)(x-s)^{n-3}, & x \ge s, \\ 0, & x < s, \end{cases}$$

which provides convexity (note that both derivatives are equal to zero at $x = s$). Hence, if f is absolutely monotonic, it follows that the integral in (8) is non-negative, so it follows that $\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p})$. Now, since the Jensen functional is obviously linear with respect to a function, we have that

$$\begin{aligned}\mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) =& \mathcal{J}_m(\sum_{k=0}^{n-1} f^{(k)}(a)e_k, \mathbf{x}, \mathbf{p}) = \sum_{k=0}^{n-1} f^{(k)}(a)\mathcal{J}_m(e_k, \mathbf{x}, \mathbf{p}) \\ &\geq m_{n-1} \sum_{k=0}^{n-1} \mathcal{J}_m(e_k, \mathbf{x}, \mathbf{p}) = m_{n-1}\mathcal{J}_m(\sum_{k=0}^{n-1} e_k, \mathbf{x}, \mathbf{p}) = m_{n-1}\mathcal{J}_m(t_{n-1}, \mathbf{x}, \mathbf{p}).\end{aligned}$$

Clearly, the inequality sign in the above relation holds due to $\mathcal{J}_m(e_k, \mathbf{x}, \mathbf{p}) \geq 0$, for every integer k. Finally, the last inequality sign in (10) holds due to convexity of the polynomial t_{n-1} on $[a, b]$.

In the same way as above, it follows that G^*_{n-1} is also convex on $[a, b]$, for every fixed value $s \in [a, b]$. Then, the proof of (11) follows the lines of the above proof except that we use identity (9) instead of (8). □

Remark 2.2. *If $n = 1$ and $n = 2$, inequalities (10) and (11) reduce to the classical Jensen inequality, while for $n \geq 3$ we obtain refinement of the Jensen inequality. Clearly, as n increases, the precision becomes better.*

Remark 2.3. *The conditions in Theorem 2.1 can be slightly relaxed. Namely, it is not necessary to demand that the function f is absolutely monotonic. It suffices to assume that the first n derivatives of that function are non-negative. The same conclusion can be drawn for completely monotonic functions. However, to make our further discussion concise, we deal with absolutely and completely monotonic functions.*

Remark 2.4. *Generally speaking, an n-convex function is defined via the n-th order divided difference (see, e.g. [11]). The simplest characterization of the n-convexity asserts that if the n-th order derivative $f^{(n)}$ exists on the given interval, then the function f is n-convex if and only if $f^{(n)} \geq 0$ on that interval. This means that the first inequality signs in (10) and (11) also hold for n-times differentiable n-convex functions. However, the remaining inequality signs in (10) and (11) do not have to hold, that is, we can get a weaker inequality than the basic Jensen inequality.*

Our next goal is to show superadditivity of the Jensen functional that corresponds to the classes of absolutely and completely monotonic functions. It turns out that this superadditivity is bounded by the superadditivity of the corresponding Taylor polynomial, and hence, it is more accurate than the classical superadditivity stated in the Introduction.

Theorem 2.5. *Let $f : I \to \mathbb{R}$ be absolutely monotonic function and let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, b]^m \subset I^m$, $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, $\mathbf{q} = (q_1, q_2, \ldots, q_m) \in \mathbb{R}^m_+$. Then holds the inequality*

$$\begin{aligned}&\mathcal{J}_m(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \\ \geq& P_m T_{n-1}(x_{P_m}) + Q_m T_{n-1}(x_{Q_m}) - (P_m + Q_m)T_{n-1}(x_{P_m+Q_m}) \geq 0.\end{aligned} \tag{12}$$

In addition, if $f : I \to \mathbb{R}$ is completely monotonic function, then

$$\begin{aligned}&\mathcal{J}_m(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \\ \geq& P_m T^*_{n-1}(x_{P_m}) + Q_m T^*_{n-1}(x_{Q_m}) - (P_m + Q_m)T^*_{n-1}(x_{P_m+Q_m}) \geq 0.\end{aligned} \tag{13}$$

Proof. First, let f be absolutely monotonic function. Then, utilizing (8), we have that

$$\begin{aligned}&\mathcal{J}_m(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) - \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \\ =& \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p} + \mathbf{q}) - \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) - \mathcal{J}_m(T_{n-1}, \mathbf{x}, \mathbf{q}) \\ &+ \frac{1}{(n-1)!} \int_a^b \left(\mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{p} + \mathbf{q}) - \mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{p}) - \mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{q})\right) f^{(n)}(s)ds.\end{aligned}$$

Now, since $G_{n-1}(x, s)$ is convex on $[a, b]$ for every fixed value s, it follows that

$$\mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq \mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{p}) + \mathcal{J}_m(G_{n-1}, \mathbf{x}, \mathbf{q}),$$

due to superadditivity (2) of the Jensen functional. Consequently, the integral on the right-hand side of the above identity is non-negative, so

$$\begin{aligned}&\mathcal{J}_m(f,\mathbf{x},\mathbf{p}+\mathbf{q})-\mathcal{J}_m(f,\mathbf{x},\mathbf{p})-\mathcal{J}_m(f,\mathbf{x},\mathbf{q})\\ &\geq \mathcal{J}_m(T_{n-1},\mathbf{x},\mathbf{p}+\mathbf{q})-\mathcal{J}_m(T_{n-1},\mathbf{x},\mathbf{p})-\mathcal{J}_m(T_{n-1},\mathbf{x},\mathbf{q}).\end{aligned}$$

Clearly, the right-hand side of the latter inequality is nonnegative, again by the superadditivity of the Jensen functional. Moreover, it can easily be transformed to (12). Inequality (13) is proved in the same way except that we use identity (9) instead of (8). □

By virtue of Theorem 2.5, we are able to derive a variant of monotonicity of the Jensen functional in this setting.

Corollary 2.6. *Let $f : I \to \mathbb{R}$ be absolutely monotonic function and let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, b]^m \subset I^m$. If $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^m$ are such that $\mathbf{p} \geq \mathbf{q}$, then holds the inequality*

$$\begin{aligned}&\mathcal{J}_m(f,\mathbf{x},\mathbf{p})-\mathcal{J}_m(f,\mathbf{x},\mathbf{q})\\ &\geq \sum_{i=1}^{m}(p_i-q_i)T_{n-1}(x_i)+Q_mT_{n-1}(x_{Q_m})-P_mT_{n-1}(x_{P_m})\geq 0.\end{aligned} \tag{14}$$

Furthermore, if f is completely monotonic, then

$$\begin{aligned}&\mathcal{J}_m(f,\mathbf{x},\mathbf{p})-\mathcal{J}_m(f,\mathbf{x},\mathbf{q})\\ &\geq \sum_{i=1}^{m}(p_i-q_i)T^*_{n-1}(x_i)+Q_mT^*_{n-1}(x_{Q_m})-P_mT^*_{n-1}(x_{P_m}).\end{aligned} \tag{15}$$

Proof. We only prove (14). Relation (15) is proved similarly. Rewriting (12) with $\mathbf{p}-\mathbf{q}$ instead of $\mathbf{p}$, we arrive at the inequality

$$\begin{aligned}&\mathcal{J}_m(f,\mathbf{x},\mathbf{p})-\mathcal{J}_m(f,\mathbf{x},\mathbf{p}-\mathbf{q})-\mathcal{J}_m(f,\mathbf{x},\mathbf{q})\\ &\geq (P_m-Q_m)T_{n-1}(x_{P_m-Q_m})+Q_mT_{n-1}(x_{Q_m})-P_mT_{n-1}(x_{P_m}).\end{aligned}$$

On the other hand, due to (10), it follows that

$$\begin{aligned}\mathcal{J}_m(f,\mathbf{x},\mathbf{p}-\mathbf{q})&\geq \mathcal{J}_m(T_{n-1},\mathbf{x},\mathbf{p}-\mathbf{q})\\ &=\sum_{i=1}^{m}(p_i-q_i)T_{n-1}(x_i)-(P_m-Q_m)T_{n-1}(x_{P_m-Q_m}).\end{aligned}$$

Finally, combining the last two inequalities we obtain (14), as claimed. □

Corollary 2.6 can be utilized in deriving mutual bounds for the Jensen functional expressed in terms of the corresponding non-weighted functional. Of course, these bounds are more precise than the ones, previously described in Introduction.

Corollary 2.7. *Let $f : I \to \mathbb{R}$ be absolutely monotonic function and let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, b]^m \subset I^m$. If $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$, then hold the inequalities*

$$\begin{aligned}&\mathcal{J}_m(f,\mathbf{x},\mathbf{p})-mp_{\min}\mathcal{I}_m(f,\mathbf{x})\\ &\geq \sum_{i=1}^{m}(p_i-p_{\min})T_{n-1}(x_i)-P_mT_{n-1}(x_{P_m})+mp_{\min}T_{n-1}(\overline{x}_M)\geq 0\end{aligned} \tag{16}$$

and

$$
\begin{aligned}
& mp_{\max}\mathcal{I}_m(f,\mathbf{x}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{p}) \\
& \geq \sum_{i=1}^{m}(p_{\max}-p_i)T_{n-1}(x_i) - mp_{\max}T_{n-1}(\overline{x}_M) + P_mT_{n-1}(x_{P_m}) \geq 0.
\end{aligned} \tag{17}
$$

Similarly, if f is completely monotonic, then hold the relations

$$
\begin{aligned}
& \mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - mp_{\min}\mathcal{I}_m(f,\mathbf{x}) \\
& \geq \sum_{i=1}^{m}(p_i-p_{\min})T^*_{n-1}(x_i) - P_mT^*_{n-1}(x_{P_m}) + mp_{\min}T^*_{n-1}(\overline{x}_M) \geq 0
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
& mp_{\max}\mathcal{I}_m(f,\mathbf{x}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{p}) \\
& \geq \sum_{i=1}^{m}(p_{\max}-p_i)T^*_{n-1}(x_i) - mp_{\max}T^*_{n-1}(\overline{x}_M) + P_mT^*_{n-1}(x_{P_m}) \geq 0.
\end{aligned} \tag{19}
$$

Proof. All four relations (16), (17), (18) and (19) follow from Corollary 2.6 by comparing the n-tuple $\mathbf{p}$ with constant n-tuples $\mathbf{p}_{\min} = (p_{\min}, p_{\min}, \ldots, p_{\min})$ and $\mathbf{p}_{\max} = (p_{\max}, p_{\max}, \ldots, p_{\max})$. □

Remark 2.8. *According to Remark 2.4, inequalities in Theorem 2.5, Corollary 2.6 and Corollary 2.7 also hold for n-times differentiable n-convex functions, but their right-hand sides need not be non-negative.*

In order to conclude this section, let's emphasize another interesting feature in connection to our Theorem 2.1.

Remark 2.9. *It should be noticed here that the inequalities in (10) and (11) are homogeneous with respect to the m-tuple $\mathbf{p}$. In particular, since $\mathcal{J}_m(f,\mathbf{x},\mathbf{1}) = m\mathcal{I}_m(f,\mathbf{x})$, where $\mathbf{1} = (1,1,\ldots,1)$, relations (10) and (11) provide the inequalities*

$$
\mathcal{I}_m(f,\mathbf{x}) \geq \mathcal{I}_m(T_{n-1},\mathbf{x}) \geq m_{n-1}\mathcal{I}_m(t_{n-1},\mathbf{x}) \geq 0. \tag{20}
$$

and

$$
\mathcal{I}_m(f,\mathbf{x}) \geq \mathcal{I}_m(T^*_{n-1},\mathbf{x}) \geq m^*_{n-1}\mathcal{I}_m(t^*_{n-1},\mathbf{x}) \geq 0. \tag{21}
$$

The importance of relations (20) and (21) is the fact that they provide lower bounds for the non-weighted functional in terms of the corresponding Taylor polynomial. It is important to point out that the method developed in our earlier paper [8] refers only to the bounds of the weighted functional in terms of the non-weighted functional.

3. More accurate power mean inequalities based on the Taylor interpolation formula

In this section, we derive improved power mean inequalities based on the Jensen-type inequalities (10) and (11), established in Theorem 2.1. Recall that a power mean is defined by

$$
M_r(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i^{\,r}\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^{m} x_i^{\,p_i}\right)^{\frac{1}{P_m}}, & r = 0, \end{cases}
$$

while the case of $p_1 = p_2 = \cdots = p_m$ yields the corresponding non-weighted power mean

$$
m_r(\mathbf{x}) = \begin{cases} \left(\frac{1}{m}\sum_{i=1}^{m} x_i^{\,r}\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod_{i=1}^{m} x_i\right)^{\frac{1}{m}}, & r = 0. \end{cases}
$$

Here, and throughout this section, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ stands for a positive m-tuple, i.e. $x_i > 0$, $i = 1, 2, \ldots, m$. In particular, for $r = -1, 0, 1$, we obtain the harmonic, geometric and arithmetic mean, respectively. The most important power mean inequality asserts that if $r < s$, then

$$M_r(\mathbf{x}, \mathbf{p}) \leq M_s(\mathbf{x}, \mathbf{p}). \tag{22}$$

Inequality (22) describes monotonic behavior of means and is still of interest to numerous mathematicians. For a comprehensive study of power means including refinements and generalizations, the reader is referred to monographs [10, 11], as well as to papers [5, 7, 8] and the references cited therein.

In order to derive the corresponding power mean inequalities based on our previous results, we need to adapt the Jensen functional $\mathcal{J}_m(f, \mathbf{x}, \mathbf{p})$ by a suitable choice of a function f and variable $\mathbf{x}$. First, let $f(t) = t^{\frac{s}{r}}$, $t > 0$, and $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_m^r)$, where $r, s \neq 0$ and $x_i > 0$, $i = 1, 2, \ldots, m$. Then, the Jensen functional can be rewritten in the following way:

$$\mathcal{J}_m(f, \mathbf{x}^r, \mathbf{p}) = \sum_{i=1}^{m} p_i x_i^s - P_m \left(\frac{1}{P_m} \sum_{i=1}^{m} p_i x_i^r \right)^{\frac{s}{r}} = P_m \left[M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \right].$$

Another detail is important to note here. Namely, the function f is not absolutely monotonic for any value of $\frac{s}{r}$, because by successive differentiation we arrive at the negative derivative at one moment. However, since

$$f^{(k)}(t) = \prod_{j=1}^{k} \left(\frac{s}{r} - j + 1 \right) t^{\frac{s}{r} - k},$$

it follows that $f^{(k)}(t) \geq 0$, $t \in \mathbb{R}_+$, for $k = 0, 1, 2, \ldots, n$, if and only if $\frac{s}{r} \geq n - 1$. Hence, according to Remark 2.3 we can also apply Theorem 2.1 in this case. On the other hand, as soon as $\frac{s}{r}$ is negative, f is completely monotonic function. Also, it should be noted that if $x_i \in [a, b]$, then $x_i^r \in [\min\{a^r, b^r\}, \max\{a^r, b^r\}]$. In fact, since f is defined on $\mathbb{R}_+$, we deal here with one-sided intervals $(0, b]$ and $[a, \infty)$, where $a, b > 0$. Finally, taking into account the above discussion and denoting $x_{P_m}^r = \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i^r$, Theorem 2.1 provides the following class of refined power mean inequalities.

Corollary 3.1. *Let n be positive integer and let $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$. Further, let $s, r \neq 0$ be real numbers. If $s \geq (n-1)r \geq 0$ or $r < 0 < s$, then the inequalities*

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \geq \sum_{k=0}^{n-1} \binom{\frac{s}{r}}{k} a^{s-kr} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} (x_i^r - a^r)^k - (x_{P_m}^r - a^r)^k \right] \geq 0 \tag{23}$$

hold for $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, \infty)^m$. Otherwise, if $s \leq (n-1)r \leq 0$ or $s < 0 < r$, then the inequalities

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \geq \sum_{k=0}^{n-1} \binom{\frac{s}{r}}{k} b^{s-kr} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} (x_i^r - b^r)^k - (x_{P_m}^r - b^r)^k \right] \geq 0 \tag{24}$$

hold for every $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in (0, b]^m$.

Proof. The condition $s \geq (n-1)r \geq 0$ is equivalent to $\frac{s}{r} \geq n - 1$ and $r > 0$. So, putting $f(t) = t^{\frac{s}{r}}$ and $\mathbf{x}^r$ in (10), we have

$$\begin{aligned} &M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \geq \frac{1}{P_m} \mathcal{J}_m(T_{n-1}, \mathbf{x}^r, \mathbf{p}) \\ =&\frac{1}{P_m} \sum_{i=1}^{m} p_i \sum_{k=0}^{n-1} \binom{\frac{s}{r}}{k} a^{s-kr} (x_i^r - a^r)^k - \sum_{k=0}^{n-1} \binom{\frac{s}{r}}{k} a^{s-kr} (x_{P_m}^r - a^r)^k, \end{aligned}$$

which reduces to (23) after changing the order of summation in the double sum. Similarly, if $s \leq (n-1)r \leq 0$, then $\frac{s}{r} \geq n-1$ and $r < 0$. In this case the corresponding interval is $[b^r, \infty)$, so we obtain (24) in the same way.

It remains to consider the cases when $\frac{s}{r} < 0$, i.e. when $f(t) = t^{\frac{s}{r}}$ is a completely monotonic function. If $r < 0 < s$, then utilizing (11) on the interval $(0, a^r]$, we have

$$\begin{aligned}&M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \geq \frac{1}{P_m}\mathcal{J}_m(T^*_{n-1}, \mathbf{x}^r, \mathbf{p})\\ =&\frac{1}{P_m}\sum_{i=1}^{m} p_i \sum_{k=0}^{n-1}(-1)^k\binom{\frac{s}{r}}{k}a^{s-kr}(a^r - x_i^r)^k - \sum_{k=0}^{n-1}(-1)^k\binom{\frac{s}{r}}{k}a^{s-kr}(a^r - x_{P_m}^r)^k,\end{aligned}$$

which evidently reduces to (23). The remaining case $s < 0 < r$ is treated in the same way. □

Remark 3.2. *Note that in Corollary 3.1 we did not use the second inequality sign in (10) and (11). Of course, the corresponding inequalities are also valid in this setting, but we are not able to determine the minimum of the derivatives in general, since it depends on the corresponding interval.*

We proceed with power mean inequalities where one of parameters r and s is equal to zero. If $s = 0$, then we set $f(t) = \frac{1}{r}\log t$ and $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_m^r)$, where $r \neq 0$. Since $f^{(k)}(t) = \frac{(-1)^{k-1}(k-1)!}{r}t^{-k}$, we have that $(-1)^k f^{(k)}(t) = -\frac{(k-1)!}{r}t^{-k} \geq 0$, provided that $r < 0$. Note that complete monotonicity of this function is ruined by the zeroth term $f(t) = \frac{1}{r}\log t$, that can take negative values. However, that term cancels in the Jensen functional, so it can be neglected. In other words, we can also apply Theorem 2.1 to this setting.

Corollary 3.3. *Let $n \geq 2$ and let $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$. If $r < 0$, then the inequality*

$$\log\frac{M_0(\mathbf{x}, \mathbf{p})}{M_r(\mathbf{x}, \mathbf{p})} \geq \frac{1}{r}\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}a^{-kr}\Big[\sum_{i=1}^{m}\frac{p_i}{P_m}(x_i^r - a^r)^k - (x_{P_m}^r - a^r)^k\Big] \geq 0 \tag{25}$$

holds for every $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, \infty)^m$. Otherwise, if $r > 0$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in (0, b]^m$, then holds the inequality

$$\log\frac{M_r(\mathbf{x}, \mathbf{p})}{M_0(\mathbf{x}, \mathbf{p})} \geq \frac{1}{r}\sum_{k=1}^{n-1}\frac{(-1)^{k}}{k}b^{-kr}\Big[\sum_{i=1}^{m}\frac{p_i}{P_m}(x_i^r - b^r)^k - (x_{P_m}^r - b^r)^k\Big] \geq 0. \tag{26}$$

Proof. Let $r < 0$. Then, for $f(t) = \frac{1}{r}\log t$ and $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_m^r)$, the Jensen functional reduces to

$$\mathcal{J}_m(f, \mathbf{x}^r, \mathbf{p}) = \sum_{i=1}^{m} p_i \log x_i - P_m \log\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i^r\right)^{\frac{1}{r}} = P_m \log\frac{M_0(\mathbf{x}, \mathbf{p})}{M_r(\mathbf{x}, \mathbf{p})}.$$

Furthermore, in this setting, inequality (11) reads

$$\log\frac{M_0(\mathbf{x}, \mathbf{p})}{M_r(\mathbf{x}, \mathbf{p})} \geq \frac{1}{P_m}\mathcal{J}_m(T^*_{n-1}, \mathbf{x}^r, \mathbf{p}),$$

which reduces to (25). Inequality (26) is proved similarly except that we consider the function $f(t) = -\frac{1}{r}\log t$, $r > 0$. □

Remark 3.4. *In particular, if $r = 1$, relation (26) provides the refinement of the arithmetic-geometric mean inequality in a quotient form:*

$$\begin{aligned}\log\frac{M_1(\mathbf{x}, \mathbf{p})}{M_0(\mathbf{x}, \mathbf{p})} &\geq \sum_{k=1}^{n-1}\frac{(-1)^{k}}{k}b^{-k}\Big[\sum_{i=1}^{m}\frac{p_i}{P_m}(x_i - b)^k - (x_{P_m} - b)^k\Big]\\ &= \sum_{k=1}^{n-1}\frac{b^{-k}}{k}\Big[\sum_{i=1}^{m}\frac{p_i}{P_m}(b - x_i)^k - (b - x_{P_m})^k\Big] \geq 0.\end{aligned}$$

This inequality holds for every $\mathbf{x} \in (0, b]^m$. *In particular, the corresponding non-weighted form reduces to*

$$\log \frac{m_1(\mathbf{x})}{m_0(\mathbf{x})} \geq \sum_{k=1}^{n-1} \frac{b^{-k}}{k} \left[\frac{1}{m} \sum_{i=1}^{m} (b - x_i)^k - (b - m_1(\mathbf{x}))^k \right] \geq 0. \tag{27}$$

The last case we need to consider is $r = 0$. Then, we set $f(t) = e^{st}$ and $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_m)$. Clearly, $f^{(k)}(t) = s^k e^{st}$, $k = 0, 1, 2, \ldots$, which means that f is absolutely monotonic for $s > 0$ and completely monotonic for $s < 0$. The corresponding result reads as follows:

Corollary 3.5. *Let n be positive integer and* $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$. *If $s > 0$ and* $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a, \infty)^m$, *then hold the inequalities*

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \geq a^s \sum_{k=0}^{n-1} \frac{s^k}{k!} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} \log^k \frac{x_i}{a} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{a} \right] \geq 0. \tag{28}$$

On the other hand, if $s < 0$, then the inequalities

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \geq b^s \sum_{k=0}^{n-1} \frac{s^k}{k!} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} \log^k \frac{x_i}{b} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{b} \right] \geq 0 \tag{29}$$

hold for $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in (0, b]^m$.

Proof. If $f(t) = e^{st}$, then $\mathcal{J}_m(f, \log \mathbf{x}, \mathbf{p}) = P_m \left[M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \right]$, so (28) and (29) follow from (10) and (11), respectively. □

Remark 3.6. *Related to the previous corollary, it is easy to find the minimum of the derivatives* $f^{(k)}(t) = s^k e^{st}$, $k = 0, 1, 2, \ldots, n-1$. *Namely, if $s > 0$, we have*

$$m_{n-1} = \min_{0 \leq k \leq n-1} f^{(k)}(\log a) = \min_{0 \leq k \leq n-1} s^k a^s = \begin{cases} a^s, & s \geq 1, \\ s^{n-1} a^s, & 0 < s < 1, \end{cases}$$

while for $s < 0$ holds

$$m_{n-1}^* = \min_{0 \leq k \leq n-1} f^{(k)}(\log b) = \min_{0 \leq k \leq n-1} \left| s^k b^s \right| = \begin{cases} b^s, & s \leq -1, \\ (-s)^{n-1} b^s, & -1 < s < 0. \end{cases}$$

This means that we can utilize the second inequality sign in relations (10) and (11). In other words, if $s > 0$, then holds the inequality

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \geq \min\{1, s^{n-1}\} a^s \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} \log^k \frac{x_i}{a} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{a} \right] \geq 0,$$

while for $s < 0$ holds

$$M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \geq \min\{1, (-s)^{n-1}\} b^s \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} \log^k \frac{x_i}{b} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{b} \right] \geq 0.$$

Remark 3.7. *If $s = 1$ and* $\mathbf{x} \in [a, \infty)^m$, *then (28) again yields the refinement of the arithmetic-geometric inequality, this time in a difference form:*

$$M_1(\mathbf{x}, \mathbf{p}) - M_0(\mathbf{x}, \mathbf{p}) \geq a \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^{m} \frac{p_i}{P_m} \log^k \frac{x_i}{a} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{a} \right] \geq 0. \tag{30}$$

In particular, the corresponding non-weighted form reads:

$$m_1(\mathbf{x}) - m_0(\mathbf{x}) \geq a \sum_{k=0}^{n-1} \frac{1}{k!} \left[\frac{1}{m} \sum_{i=1}^{m} \log^k \frac{x_i}{a} - \log^k \frac{m_0(\mathbf{x})}{a} \right] \geq 0. \tag{31}$$

Inequality (30) allows us to improve the Hölder inequality, which will be done in the next section.

Remark 3.8. *Given a positive n-tuple* $\mathbf{x} = (x_1, x_2, \dots, x_m)$, *the limit values for the endpoints of one-sided intervals* $(0, b]$ *and* $[a, \infty)$ *are* $a = \min_{1 \leq i \leq m} x_i$ *and* $b = \max_{1 \leq i \leq m} x_i$. *In other words, these values can be chosen in Corollaries 3.1, 3.3 and 3.5. For illustration, consider the non-weighted inequalities (27) and (31) in their simplest form, that is, for* $m = 2$. *Consequently, we obtain the following refinements of the non-weighted arithmetic-geometric mean inequality in both quotient and difference form, provided that* $n \geq 2$:

$$\log \frac{x_1 + x_2}{2\sqrt{x_1 x_2}} \geq \sum_{k=1}^{n-1} \frac{(2^{k-1} - 1) \min\{x_1^{-k}, x_2^{-k}\}}{k 2^k} |x_1 - x_2|^k \geq 0$$

and

$$x_1 + x_2 - 2\sqrt{x_1 x_2} \geq \min\{x_1, x_2\} \sum_{k=1}^{n-1} \frac{2^{k-1} - 1}{k! 2^k} \log^k \max\left\{ \frac{x_1}{x_2}, \frac{x_2}{x_1} \right\} \geq 0.$$

In this section we have established refinements of power mean inequalities based on Theorem 2.1. Of course, by using Corollary 2.7, we can obtain even more precise estimates. In this way, we can obtain mutual bounds for the differences of power means in terms of the corresponding non-weighted means. For illustration, we give here only the strengthened version of Corollary 3.5 in the case of $s > 0$.

Corollary 3.9. *Let n be positive integer and* $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_+^m$. *If* $s > 0$ *and* $\mathbf{x} = (x_1, x_2, \dots, x_m) \in [a, \infty)^m$, *then hold the inequalities*

$$\begin{aligned} &M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) - \frac{m p_{\min}}{P_m} \left(m_s^s(\mathbf{x}) - m_0^s(\mathbf{x}) \right) \\ &\geq a^s \sum_{k=0}^{n-1} \frac{s^k}{k!} \Big[\sum_{i=1}^{m} \frac{p_i - p_{\min}}{P_m} \log^k \frac{x_i}{a} - \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{a} + \frac{m p_{\min}}{P_m} \log^k \frac{m_0(\mathbf{x})}{a} \Big] \geq 0 \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\frac{m p_{\max}}{P_m} \left(m_s^s(\mathbf{x}) - m_0^s(\mathbf{x}) \right) - \left(M_s^s(\mathbf{x}, \mathbf{p}) - M_0^s(\mathbf{x}, \mathbf{p}) \right) \\ &\geq a^s \sum_{k=0}^{n-1} \frac{s^k}{k!} \Big[\sum_{i=1}^{m} \frac{p_{\max} - p_i}{P_m} \log^k \frac{x_i}{a} + \log^k \frac{M_0(\mathbf{x}, \mathbf{p})}{a} - \frac{m p_{\max}}{P_m} \log^k \frac{m_0(\mathbf{x})}{a} \Big] \geq 0. \end{aligned} \tag{33}$$

4. Several Hölder-type inequalities in a strengthened form

Let us recall one of the most important consequences of the Jensen inequality. Let (Ω, Σ, μ) be σ-finite measure space and let $\sum_{i=1}^{m} \frac{1}{q_i} = 1, q_i > 1$. If $f_i \in L^{q_i}(\Omega)$, $i = 1, 2, \dots, m$, are non-negative measurable functions, then holds the inequality

$$\int_\Omega \prod_{i=1}^{m} f_i(x) d\mu(x) \leq \prod_{i=1}^{m} \|f_i\|_{q_i}. \tag{34}$$

The Hölder inequality can be proved in several ways, among others, via the arithmetic-geometric mean inequality, i.e. the Young inequality (for more details, see [10, 11]). Taking into account this fact, the arithmetic-geometric mean inequality (30) can be used in obtaining some improved Hölder-type inequalities, based on the Taylor interpolation formula. It is important to note that, since (30) holds for $\mathbf{x} \in [a, \infty)^m$, we need to impose some additional conditions on non-negative measurable functions $f_i \in L^{q_i}(\Omega)$, $i = 1, 2, \dots, m$.

Corollary 4.1. *Let (Ω, Σ, μ) be σ-finite measure space and let $\sum_{i=1}^m \frac{1}{q_i} = 1$, $q_i > 1$, $i = 1, 2, \dots, m$. Further, suppose that $f_i \in L^{q_i}(\Omega)$, $i = 1, 2, \dots, m$, are non-negative measurable functions such that*

$$f_i(x) \geq a^{\frac{1}{q_i}} \|f_i\|_{q_i} > 0,\ x \in \Omega,\ i = 1, 2, \dots, m. \tag{35}$$

Then holds the inequality

$$\begin{aligned} &\prod_{i=1}^m \|f_i\|_{q_i} - \int_\Omega \prod_{i=1}^m f_i(x) d\mu(x) \\ &\geq a \prod_{i=1}^m \|f_i\|_{q_i} \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^m \frac{1}{q_i} \int_\Omega \log^k \left(\frac{f_i^{q_i}(x)}{a \|f_i\|_{q_i}^{q_i}} \right) d\mu(x) - \int_\Omega \log^k \left(\frac{1}{a} \prod_{i=1}^m \frac{f_i(x)}{\|f_i\|_{q_i}} \right) d\mu(x) \right] \geq 0. \end{aligned} \tag{36}$$

Proof. The inital point is to rewrite (30) in the Young form. More precisely, by putting $q_i = \frac{P_m}{p_i}$, $i = 1, 2, \dots, m$ in (30), it follows that

$$\sum_{i=1}^m \frac{x_i}{q_i} - \prod_{i=1}^m x_i^{\frac{1}{q_i}} \geq a \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^m \frac{1}{q_i} \log^k \frac{x_i}{a} - \log^k \left(\frac{1}{a} \prod_{i=1}^m x_i^{\frac{1}{q_i}} \right) \right] \geq 0.$$

The next step is to substitute $f_i^{q_i}(x)/\|f_i\|_{q_i}^{q_i}$, $x \in \Omega$, instead of x_i, $i = 1, 2, \dots, m$, in the above inequality. Of course, this is meaningful due to assumptions in (35). Therefore, we arrive at the relation

$$\begin{aligned} &\sum_{i=1}^m \frac{f_i^{q_i}(x)}{q_i \|f_i\|_{q_i}^{q_i}} - \prod_{i=1}^m \frac{f_i(x)}{\|f_i\|_{q_i}} \\ &\geq a \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^m \frac{1}{q_i} \log^k \left(\frac{f_i^{q_i}(x)}{a \|f_i\|_{q_i}^{q_i}} \right) - \log^k \left(\frac{1}{a} \prod_{i=1}^m \frac{f_i(x)}{\|f_i\|_{q_i}} \right) \right] \geq 0. \end{aligned}$$

It remains to integrate the above inequality over Ω, with respect to the measure μ. Consequently, we have

$$\begin{aligned} &\sum_{i=1}^m \frac{1}{q_i} - \frac{\int_\Omega \prod_{i=1}^m f_i(x) d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}} \\ &\geq a \sum_{k=0}^{n-1} \frac{1}{k!} \left[\sum_{i=1}^m \frac{1}{q_i} \int_\Omega \log^k \left(\frac{f_i^{q_i}(x)}{a \|f_i\|_{q_i}^{q_i}} \right) d\mu(x) - \int_\Omega \log^k \left(\frac{1}{a} \prod_{i=1}^m \frac{f_i(x)}{\|f_i\|_{q_i}} \right) d\mu(x) \right] \geq 0, \end{aligned}$$

which provides (36), due to $\sum_{i=1}^m \frac{1}{q_i} = 1$. □

Obviously, relation (36) improves the Hölder inequality (34). Even more accurate estimates can be achieved through Corollary 2.7, i.e. Corollary 3.9. Based on Corollary 3.9, we obtain mutual bounds for the Hölder inequality in a difference form, with which we conclude this section.

Corollary 4.2. *Assume that the conditions of Corollary* 4.1 *are fulfilled. Then hold the inequalities*

$$
\begin{aligned}
&1-\frac{\int_\Omega \prod_{i=1}^m f_i(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}}-\frac{m}{q_{\max}}\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i^{\frac{q_i}{m}}(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)\\
&\geq a\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg[\sum_{i=1}^m\Big(\frac{1}{q_i}-\frac{1}{q_{\max}}\Big)\int_\Omega \log^k\Big(\frac{f_i^{q_i}(x)}{a\|f_i\|_{q_i}^{q_i}}\Big)d\mu(x)\\
&\qquad -\int_\Omega \log^k\left(\frac{1}{a}\prod_{i=1}^m\frac{f_i(x)}{\|f_i\|_{q_i}}\right)d\mu(x)+\frac{m}{q_{\max}}\int_\Omega \log^k\left(\frac{1}{a}\prod_{i=1}^m\frac{f_i^{\frac{q_i}{m}}(x)}{\|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)d\mu(x)\Bigg]\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{m}{q_{\min}}\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i^{\frac{q_i}{m}}(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)-\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}}\right)\\
&\geq a\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg[\sum_{i=1}^m\Big(\frac{1}{q_{\min}}-\frac{1}{q_i}\Big)\int_\Omega \log^k\Big(\frac{f_i^{q_i}(x)}{a\|f_i\|_{q_i}^{q_i}}\Big)d\mu(x)\\
&\qquad +\int_\Omega \log^k\left(\frac{1}{a}\prod_{i=1}^m\frac{f_i(x)}{\|f_i\|_{q_i}}\right)d\mu(x)-\frac{m}{q_{\min}}\int_\Omega \log^k\left(\frac{1}{a}\prod_{i=1}^m\frac{f_i^{\frac{q_i}{m}}(x)}{\|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)d\mu(x)\Bigg]\geq 0.
\end{aligned}
$$

Proof. By putting $s = 1$ and $q_i = \frac{P_m}{p_i}$, $i = 1, 2, \ldots, m$, in (32) and (33), we obtain the Young-type inequalities

$$
\begin{aligned}
&\sum_{i=1}^m\frac{x_i}{q_i}-\prod_{i=1}^m x_i^{\frac{1}{q_i}}-\frac{m}{q_{\max}}\left(\frac{1}{m}\sum_{i=1}^m x_i-\Big(\prod_{i=1}^m x_i\Big)^{\frac{1}{m}}\right)\\
&\geq a\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg[\sum_{i=1}^m\Big(\frac{1}{q_i}-\frac{1}{q_{\max}}\Big)\log^k\frac{x_i}{a}-\log^k\Big(\frac{1}{a}\prod_{i=1}^m x_i^{\frac{1}{q_i}}\Big)+\frac{m}{q_{\max}}\log^k\Big(\frac{1}{a}\prod_{i=1}^m x_i^{\frac{1}{m}}\Big)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{m}{q_{\min}}\left(\frac{1}{m}\sum_{i=1}^m x_i-\Big(\prod_{i=1}^m x_i\Big)^{\frac{1}{m}}\right)-\left(\sum_{i=1}^m\frac{x_i}{q_i}-\prod_{i=1}^m x_i^{\frac{1}{q_i}}\right)\\
&\geq a\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg[\sum_{i=1}^m\Big(\frac{1}{q_{\min}}-\frac{1}{q_i}\Big)\log^k\frac{x_i}{a}+\log^k\Big(\frac{1}{a}\prod_{i=1}^m x_i^{\frac{1}{q_i}}\Big)-\frac{m}{q_{\min}}\log^k\Big(\frac{1}{a}\prod_{i=1}^m x_i^{\frac{1}{m}}\Big)\Bigg].
\end{aligned}
$$

Now, the rest of the proof follows the lines of the proof of Corollary 4.1. □

5. Improved Lah-Ribarič inequality for absolutely and completely monotonic functions

In order to complete the paper, let us consider the Lah-Ribarič inequality, one of the most interesting reverses of the Jensen inequality. The Lah-Ribarič inequality asserts that if $f : [a, b] \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [\alpha, \beta]^m \subseteq [a, b]$, $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$, then

$$
\frac{1}{P_m}\sum_{i=1}^m p_i f(x_i) \leq \frac{\beta - x_{P_m}}{\beta-\alpha} f(\alpha) + \frac{x_{P_m}-\alpha}{\beta-\alpha} f(\beta). \tag{37}
$$

The geometric interpretation of this inequality should be pointed out here. Namely, if $m = 1$, then the right-hand side of (37) represents a linear function limiting convex function $f(x)$ on interval $[\alpha, \beta]$ from the above (for more details, see [9, 11]).

We can approach this inequality in the same way as we have studied the Jensen inequality in Section 2, via the Taylor interpolation. Hence, we define the Lah-Ribarič functional as the difference between the the right-hand side and the left-hand side of (37) multiplied by P_m:

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) = P_m\left(\frac{\beta - x_{P_m}}{\beta - \alpha} f(\alpha) + \frac{x_{P_m} - \alpha}{\beta - \alpha} f(\beta)\right) - \sum_{i=1}^{m} p_i f(x_i). \tag{38}$$

Clearly, this functional is non-negative, so by using method as in Section 2, we obtain improved lower bound for this functional. In fact, we will show that this functional is bounded by the Lah-Ribarič functional that corresponds to the associated Taylor polynomial. Again, we will have two types of results: the first corresponds to absolutely monotonic functions, while the second is suitable for completely monotonic functions. But first, we have to transform functional (38) to a suitable form. More precisely, considering (6) with $x = \alpha$ and $x = \beta$, we arrive at the following identity:

$$\begin{aligned} &P_m\left(\frac{\beta - x_{P_m}}{\beta - \alpha} f(\alpha) + \frac{x_{P_m} - \alpha}{\beta - \alpha} f(\beta)\right) \\ &= P_m\left(\frac{\beta - x_{P_m}}{\beta - \alpha} T_{n-1}(\alpha) + \frac{x_{P_m} - \alpha}{\beta - \alpha} T_{n-1}(\beta)\right) + \frac{P_m}{(n-1)!}\int_a^b \left(\frac{\beta - x_{P_m}}{\beta - \alpha} G_{n-1}(\alpha, s) + \frac{x_{P_m} - \alpha}{\beta - \alpha} G_{n-1}(\beta, s)\right) f^{(n)}(s)ds. \end{aligned}$$

In the same way, we have that

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{i=1}^{m} p_i T_{n-1}(x_i) + \frac{1}{(n-1)!}\int_a^b \left(\sum_{i=1}^{m} p_i G_{n-1}(x_i, s)\right) f^{(n)}(s)ds.$$

Subtracting the previous two relations and taking into account definition (38), we obtain the identity

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) = \mathcal{L}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) + \frac{1}{(n-1)!}\int_a^b \mathcal{L}_m(G_{n-1}, \mathbf{x}, \mathbf{p}) f^{(n)}(s)ds. \tag{39}$$

In the same way, utilizing (7), we also obtain

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) = \mathcal{L}_m(T^*_{n-1}, \mathbf{x}, \mathbf{p}) + \frac{(-1)^n}{(n-1)!}\int_a^b \mathcal{L}_m(G^*_{n-1}, \mathbf{x}, \mathbf{p}) f^{(n)}(s)ds. \tag{40}$$

Finally, we are ready to state and prove the refinements of the Lah-Ribarič inequality (37) that correspond to the classes of absolutely and completely monotonic functions.

Theorem 5.1. *Let $f : I \to \mathbb{R}$ be absolutely monotonic function and let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [\alpha, \beta]^m \subset [a, b]^m \subset I^m$, $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m_+$. Then hold the inequalities*

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) \geq \mathcal{L}_m(T_{n-1}, \mathbf{x}, \mathbf{p}) \geq 0. \tag{41}$$

On the other hand, if $f : I \to \mathbb{R}$ is completely monotonic function, then

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) \geq \mathcal{L}_m(T^*_{n-1}, \mathbf{x}, \mathbf{p}) \geq 0. \tag{42}$$

Proof. Let f be absolutely monotonic function. Since $G_{n-1}(\cdot, s)$ is convex for every fixed value s (see the proof of Theorem 2.1), it follows that $\mathcal{L}_m(G_{n-1}, \mathbf{x}, \mathbf{p}) \geq 0$. Consequently, the integral in (39) is non-negative, which yields the first inequality sign in (41). The second inequality sign holds due to convexity of Taylor polynomial T_{n-1}. Relation (42) is proved in the same way except that we use identity (40) instead of (39). □

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