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A study of spaces and mappings in the sense of ideal convergence

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Abstract. The ideal convergence of sequences in topological spaces not only includes the usual convergence of sequences, but also extends the statistical convergence of sequences with strong applying background. This paper discusses the subject of spaces and mappings in the sense of ideal convergence, and studies the spaces defined by ideal convergence and how to represent them as the images of metric spaces under certain mappings. The following main results are obtained for an admissible ideal $\mathcal{I}$ on the set $\mathbb{N}$ of natural numbers.

(1) A topological space X is a seq-$\mathcal{I}$-space if and only if it is an $\mathcal{I}$-quotient image of a metric space.
(2) A topological space X is a seq-$\mathcal{I}_{sn}$-space if and only if it is an $\mathcal{I}_{sn}$-quotient image of a metric space.

These show the unique role of $\mathcal{I}$-open sets and $\mathcal{I}_{sn}$-open sets in topological spaces, and present a version using the notion of ideals.

1. Introduction

K. Kuratowski et al. [11, 23] introduced and studied ideals in topological spaces. An ideal $\mathcal{I}$ on a set S is a family of subsets of S closed under the operations of taking finite unions and subsets of their elements. The primary concept of topological spaces is open sets. For a topological space X and an ideal $\mathcal{I}$ on $\mathbb{N}$, one defines the ideal convergence of sequences in X and introduces the $\mathcal{I}$-open sets of X [10, 12].

The ideal convergence of sequences in topological spaces not only includes the usual convergence of sequences, but also generalizes the statistical convergence of sequences with extensive background in many domains [7]. For a topological space X and an ideal $\mathcal{I}$ on $\mathbb{N}$, the $\mathcal{I}$-open sets and $\mathcal{I}$-continuity are extensions of sequentially open sets and sequence-continuity, respectively. Compared to the open sets of topological spaces, the $\mathcal{I}$-open sets have the following two significant properties. The first is that the family of all $\mathcal{I}$-open subsets of a topological space X constitutes a generalized topology on X [6, 27], whereby the $\mathcal{I}$-continuity can be regarded as a kind of generalized continuity. The other is that an $\mathcal{I}$-open set is defined by the ideal convergence of sequences. We introduced $\mathcal{I}_{sn}$-open sets and $\mathcal{I}_{sn}$-continuity, where the family of all $\mathcal{I}_{sn}$-open sets of a topological space X constitutes a topology on X [15]. Therefore, we can better study the topological properties of spaces defined by ideal convergence, and further discuss how the continuity

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related to ideal convergence organically combines topology and algebra. Through the preliminary study of ideal convergence in topological spaces, we have reason to believe that the exploration of ideal convergence in topological algebra will exhibit good prospects [3].

This paper discusses the subject of spaces and mappings in the sense of ideal convergence. We study the spaces defined by ideal convergence and how to represent them as the images of metric spaces under certain mappings. Based on the $\mathcal{I}$-interior operator and the $\mathcal{I}_{sn}$-interior operator, in Section 2, we introduce the spaces of FU-type and seq-type defined by ideal convergence, and discuss some relationships and intrinsic characterizations between them (see Theorems 2.5 and 2.6). As a continuation of the research on mutual classifications between spaces and mappings, in Section 3, we discuss how some spaces defined in Section 2 can be represented as the images of metric spaces and analyze their causes. We obtain the characterizations of $\mathcal{I}$-quotient (resp., $\mathcal{I}_{sn}$-quotient) images and continuous pseudo-$\mathcal{I}$-open (resp., pseudo-$\mathcal{I}_{sn}$-open) images of metric spaces (see Theorems 3.7 and 3.8).

Readers may refer to [8] for some terminology unstated here.

2. Spaces defined by $\mathcal{I}$-convergence

In this paper, $\mathcal{I}$ is always an admissible ideal on $\mathbb{N}$, i.e., $\mathcal{I}$ is a hereditary family of subsets of $\mathbb{N}$ which is stable under finite unions and covers $\mathbb{N}$, and $\mathbb{N} \notin \mathcal{I}$. The smallest ideal $\{F \subset \mathbb{N} : |F| < \omega\}$ on $\mathbb{N}$ is denoted by $\mathcal{I}_{fin}$.

Sequential neighborhoods, $\mathcal{I}$-neighborhoods and $\mathcal{I}_{sn}$-neighborhoods at a point in topological spaces can be defined by convergent sequences or $\mathcal{I}$-convergent sequences. Thus sequential spaces [9], $\mathcal{I}$-sequential spaces [20] and $\mathcal{I}_{sn}$-sequential spaces [27] which are defined by convergence or $\mathcal{I}$-convergence are introduced. They are essentially determined by the relationship between interior operators defined on subsets of topological spaces. The research on this aspect comes from both the inherent requirements of logical reasoning and the external reflections of seeking topological properties or solving mathematical problems. This idea is illustrated by the following three examples.

The first example is that we studied $\mathcal{I}$-quotient mappings [25, 27, 29], but it still doesn't know how to characterize the images of metric spaces under $\mathcal{I}$-quotient mappings. We will prove that this is determined by the consistency between sequentially open sets and $\mathcal{I}$-open sets in topological spaces (see Theorem 3.7).

The second example is that we know that each convergent sequence in a topological space is an $\mathcal{I}$-convergent sequence, and not vice versa. However, each $\mathcal{I}$-convergent sequence in metric spaces has a convergent subsequence, which allows us to better discuss $\mathcal{I}$-convergent properties by virtue of convergent properties of sequences. How to characterize the topological properties that each $\mathcal{I}$-convergent sequence has a convergent subsequence? We will prove that this is determined by the relationship between sequentially interior operators and $\mathcal{I}$-interior operators (see Theorem 2.6).

The third example is that networks in topological spaces, as a generalization of topological bases, are an important tool for studying topological properties [2]. Corresponding to the case of dealing with convergent sequences, *cs*-networks or *sn*-networks in topological spaces have better topological properties than spaces determined by networks [14]. Combining with ideal convergence, the concepts of $\mathcal{I}$-*cs*-networks and $\mathcal{I}$-*sn*-networks have emerged [27]. For an ideal $\mathcal{I}$ on $\mathbb{N}$ and a topological space X, we proved that each $\mathcal{I}$-*sn*-network of X is an *sn*-network [27, Lemma 5.2]. It is clear that each *sn*-network in metric spaces is an $\mathcal{I}$-*sn*-network. However, not every *sn*-network in topological spaces is an $\mathcal{I}$-*sn*-network [27, Example 5.8]. What condition is concerned if each *sn*-network in topological spaces is an $\mathcal{I}$-*sn*-network? We will prove that the answer to this question is determined by the relationship between sequentially interior operators and $\mathcal{I}_{sn}$-interior operators (see Theorem 2.6).

On the basis of sequentially interior operators, $\mathcal{I}$-interior operators, $\mathcal{I}_{sn}$-interior operators and interior operators, in this section we define the spaces of FU-type and seq-type, and discuss some relationships and intrinsic characterizations among these spaces, which are prepared for studying the images of metric spaces in next section.

Let X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-*eventually* in $P \subset X$ if the set $\{n \in \mathbb{N} : x_n \notin P\} \in \mathcal{I}$ [29, Definition 3.15]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-*convergent* to $x \in X$ if $\{x_n\}_{n\in\mathbb{N}}$ is

$\mathcal{I}$-eventually in each neighborhood U of x in X, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$, and the point x is called the *$\mathcal{I}$-limit point* of the sequence $\{x_n\}_{n\in\mathbb{N}}$ [12]. A subset P of X is said to be an *$\mathcal{I}$-sequential neighborhood* of $x \in X$ if every sequence which is $\mathcal{I}$-convergent to x is $\mathcal{I}$-eventually in P [28]. A subset U of X is said to be an *$\mathcal{I}$-open set* if there is no sequence in $X \setminus U$ which is $\mathcal{I}$-convergent to some point in U [29, Definition 3.1]. A subset U of X is said to be an *$\mathcal{I}_{sn}$-open set* if U is an $\mathcal{I}$-sequential neighborhood of x for each $x \in U$ [15], and then U is also said to be an *$\mathcal{I}_{sn}$-open neighborhood* of each point in U. $\mathcal{I}_{fin}$-sequential neighborhoods, $\mathcal{I}_{fin}$-open sets ($= (\mathcal{I}_{fin})_{sn}$-open sets) are called *sequential neighborhoods* and *sequentially open sets*, respectively [9].

In the following paragraphs, we introduce several interior operators in topological spaces formed by sequential convergence or $\mathcal{I}$-convergence [15]. Let X be a topological space and $A \subset X$. Put

$$(A)_{seq} = \{x \in X : A \text{ is a sequential neighborhood of } x\},$$
$$(A)_{\mathcal{I}} = \{x \in X : \text{there is no sequence } \{x_n\}_{n\in\mathbb{N}} \text{ in } X \setminus A \text{ such that } x_n \xrightarrow{\mathcal{I}} x\},$$
$$(A)_{\mathcal{I}_{sn}} = \{x \in X : A \text{ is an } \mathcal{I}\text{-sequential neighborhood of } x\}.$$

Lemma 2.1. *Let X be a topological space and $A \subset X$.*

(1) *Open sets $\Rightarrow$ $\mathcal{I}_{sn}$-open sets $\Rightarrow$ $\mathcal{I}$-open sets $\Rightarrow$ sequentially open sets [15, Lemma 2.1].*

(2) *$A^\circ \subset (A)_{\mathcal{I}_{sn}} \subset (A)_{\mathcal{I}} \subset (A)_{seq} \subset A$ [15, Lemma 2.6].*

(3) *A is $\mathcal{I}$-open $\Leftrightarrow A = (A)_{\mathcal{I}}$ [16, Corollary 3.6].*

(4) *A is $\mathcal{I}_{sn}$-open $\Leftrightarrow A = (A)_{\mathcal{I}_{sn}}$ [15, p.1986].*

According to the definitions of Fréchet-Urysohn spaces and sequential spaces [9], we draw into the spaces of FU-type and seq-type, in which the four newly defined spaces have evidence to prove their effectiveness (see Theorems 2.6 and 3.7).

Definition 2.2. Let X be a topological space.

(1) X is called an *Fréchet-Urysohn space* (or FU-space for short) [9] (resp., *$\mathcal{I}$-FU-space* [21], or *$\mathcal{I}_{sn}$-FU-space* [27]), provided $(A)_{seq} \subset A^\circ$ (resp., $(A)_{\mathcal{I}} \subset A^\circ$, or $(A)_{\mathcal{I}_{sn}} \subset A^\circ$) for each $A \subset X$; X is called an *FU-$\mathcal{I}_{sn}$-space* (resp., *FU-$\mathcal{I}$-space*), provided $(A)_{seq} \subset (A)_{\mathcal{I}_{sn}}$ (resp., $(A)_{seq} \subset (A)_{\mathcal{I}}$) for each $A \subset X$.

(2) X is called a *sequential space* (or seq-space for short) [9] (resp., an *$\mathcal{I}$-sequential space* (or $\mathcal{I}$-seq-space for short) [20], or *an $\mathcal{I}_{sn}$-sequential space* (or $\mathcal{I}_{sn}$-seq-space for short) [27]), provided each sequentially open (resp., $\mathcal{I}$-open, or $\mathcal{I}_{sn}$-open) subset in X is open; X is called a *seq-$\mathcal{I}_{sn}$-space* (resp., *seq-$\mathcal{I}$-space*), provided each sequentially open subset in X is $\mathcal{I}_{sn}$-open (resp., $\mathcal{I}$-open).

(3) X is called an *$\mathcal{I}$-neighborhood space* (or $\mathcal{I}$-nbhd-space for short) [15, Definition 3.1], provided each $\mathcal{I}$-open subset in X is $\mathcal{I}_{sn}$-open.

According to Definition 2.2, we have the following relationships.

Lemma 2.3. *Let X be a topological space.*

(1) *X is an FU-space if and only if X is an FU-$\mathcal{I}_{sn}$-space and an $\mathcal{I}_{sn}$-FU-space, if and only if X is an FU-$\mathcal{I}$-space and an $\mathcal{I}$-FU-space.*

(2) *X is a seq-space if and only if X is a seq-$\mathcal{I}_{sn}$-space and an $\mathcal{I}_{sn}$-seq-space, if and only if X is a seq-$\mathcal{I}$-space and an $\mathcal{I}$-seq-space.*

(3) *X is a seq-$\mathcal{I}_{sn}$-space if and only if X is a seq-$\mathcal{I}$-space and an $\mathcal{I}$-nbhd-space.*

(4) *X is an $\mathcal{I}$-seq-space if and only if X is an $\mathcal{I}$-nbhd space and an $\mathcal{I}_{sn}$-seq-space.*

A family $\mathscr{P}$ of subsets of a topological space X is called a *network* at x in X if $x \in \bigcap \mathscr{P}$ and whenever U is a neighborhood of x in X, then $P \subset U$ for some $P \in \mathscr{P}$. If each element of the family $\mathscr{P}$ mentioned above is a sequential neighborhood of x in X, then $\mathscr{P}$ is called an *sn-network* at x in X. A family $\mathscr{P}$ of subsets of X is a *network* (resp., *an sn-network*) of X if $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ and each $\mathscr{P}_x$ is a network [2] (resp., an *sn*-network [13]) at x in X.

Definition 2.4. A topological space X is of an $\mathcal{I}$-*csf-network* [26], if a sequence S in X is $\mathcal{I}$-convergent to a point $x \in X$, then there is a countable network $\mathscr{P}$ at x in X such that S is $\mathcal{I}$-eventually in each element of $\mathscr{P}$. A family $\mathscr{P}$ of subsets of X is called an $\mathcal{I}$-*sn-network* [27] of X, if $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$ satisfies that each $\mathscr{P}_x$ is a network at x in X and each element of $\mathscr{P}_x$ is an $\mathcal{I}$-sequential neighborhood of x in X, in which the family $\mathscr{P}_x$ is called an $\mathcal{I}$-*sn-network* at x in X.

Theorem 2.5. *Each space of $\mathcal{I}$-csf-networks is an FU-$\mathcal{I}_{sn}$-space.*

Proof. Let X be of $\mathcal{I}$-csf-networks. Assume that there is a point $x \in (A)_{seq} \setminus (A)_{\mathcal{I}_{sn}}$ for some $A \subset X$. Then there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X which is $\mathcal{I}$-convergent to x but not $\mathcal{I}$-eventually in A. Since X is of $\mathcal{I}$-csf-networks, there is a countable network $\{P_m\}_{m\in\mathbb{N}}$ at x such that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in each P_m. For each $k \in \mathbb{N}$, put $Q_k = \bigcap_{m \le k} P_m$. If $I_1, I_2 \in \mathcal{I}$, then

$$(\{x\} \cup \{a_n : n \in \mathbb{N} \setminus I_1\}) \cap (\{x\} \cup \{a_n : n \in \mathbb{N} \setminus I_2\}) = \{x\} \cup \{a_n : n \in \mathbb{N} \setminus (I_1 \cup I_2)\}.$$

It follows that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is still $\mathcal{I}$-eventually in Q_k. Thus the set $Q_k \not\subset A$, and there is $x_k \in Q_k \setminus A$. Note that $\{Q_k\}_{k\in\mathbb{N}}$ is a decreasing network at x in X. Hence the sequence $\{x_k\}_{k\in\mathbb{N}}$ is convergent to x. This contradicts to A being a sequential neighborhood of x. Thus X is an FU-$\mathcal{I}_{sn}$-space. □

Figure 2.1 illustrates the basic relationships among these spaces introduced in Definitions 2.2 and 2.4, which enriches [27, Figure 3.1].

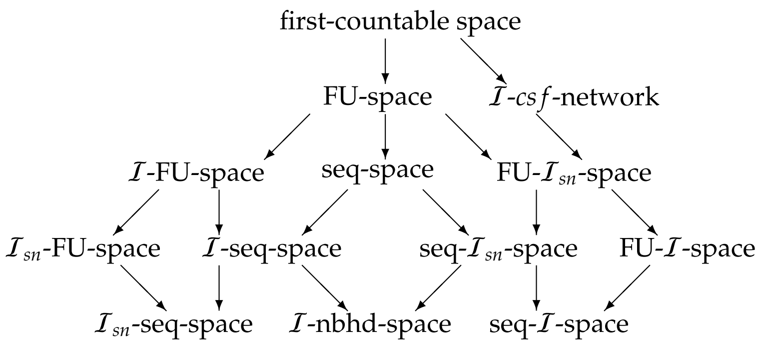

Figure 2.1 The relationships among spaces defined by certain $\mathcal{I}$-open sets

Each topological space has an $\mathcal{I}$-csf-network for the ideal $\mathcal{I}_{fin}$ on $\mathbb{N}$ [26, Example 5.5]. With further research of ideal convergence, topological spaces begin to bloom and reap great benefits.

Let $\mathcal{I}$ be a K-uniform ideal on $\mathbb{N}$ [24, p.2]. Then each topological space is an $\mathcal{I}$-neighborhood space [27, Theorem 5.6], and thus $\mathcal{I}$-open sets are coincident with $\mathcal{I}_{sn}$-open sets. And hence $\mathcal{I}$-FU-spaces are coincident with $\mathcal{I}_{sn}$-FU-spaces; $\mathcal{I}$-seq-spaces are coincident with $\mathcal{I}_{sn}$-seq-spaces; FU-$\mathcal{I}$-spaces are coincident with FU-$\mathcal{I}_{sn}$-spaces; and seq-$\mathcal{I}$-spaces are coincident with seq-$\mathcal{I}_{sn}$-spaces. The smallest ideal $\mathcal{I}_{fin}$, asymptotic density zero ideals and maximal ideals on $\mathbb{N}$ are all K-uniform ideals [24, Example 2.4].

Theorem 2.6. *Let X be a topological space.*

(1) *X is an FU-$\mathcal{I}_{sn}$-space if and only if each sn-network at each point in X is an $\mathcal{I}$-sn-network at the point.*

(2) *X is an FU-$\mathcal{I}$-space if and only if each $\mathcal{I}$-convergent sequence in X has a subsequence converging to the same limit.*

Proof. (1) The necessity is obvious. The sufficiency is proved as follows. Suppose that each *sn*-network at each point in (X, τ) is an $\mathcal{I}$-*sn*-network at the point. Assume that $U \subset X$ and $x \in (U)_{seq}$. Let $\mathscr{P}_x = \{U\} \cup \{O \in \tau : x \in O\}$. Then $\mathscr{P}_x$ is an *sn*-network at x in X, thus $\mathscr{P}_x$ is also an $\mathcal{I}$-*sn*-network at x, and further $x \in (U)_{\mathcal{I}_{sn}}$. This implies that X is an FU-$\mathcal{I}_{sn}$-space.

(2) Necessity. Let (X, τ) be an FU-$\mathcal{I}$-space and a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X be $\mathcal{I}$-convergent to x. Put $O_x = \bigcap\{O \in \tau : x \in O\}$, and $I_1 = \{n \in \mathbb{N} : a_n \in O_x\}$. If I_1 is an infinite set, then there is a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ converging to x. Now, assume that I_1 is a finite set. Set $V = X \setminus \{a_n \notin O_x : n \in \mathbb{N}\}$. Then $x \in V \subsetneq X$. Take a point $y \in X \setminus V$ and define a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $X \setminus V$ by $b_n = y$, if $n \in I_1$; $b_n = a_n$, if $n \notin I_1$. It is easy to see that the sequence $b_n \xrightarrow{\mathcal{I}} x$, hence $x \notin (V)_{\mathcal{I}}$. By the hypothesis, we have $x \notin (V)_{seq}$. Hence there is a sequence $\{c_k\}_{k\in\mathbb{N}} \subset \{a_n \notin O_x : n \in \mathbb{N}\}$ converging to x. Then the set $\{k \in \mathbb{N} : c_k = c_m\}$ is finite for each $m \in \mathbb{N}$ (otherwise, $c_m \in O_x$). Thus there is a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ converging to x.

Sufficiency. Assume that each $\mathcal{I}$-convergent sequence in X has a subsequence converging to the same limit. If a point $x \in X \setminus (A)_{\mathcal{I}}$ for some $A \subset X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus A$, $\mathcal{I}$-converging to x. Thus there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converging to x, and further $x \notin (A)_{seq}$. This implies that X is an FU-$\mathcal{I}$-space. □

Example 2.7. (1) $\mathcal{I}$-csf-networks $\nRightarrow$ $\mathcal{I}_{sn}$-seq-spaces.

Let X be the maximal compactification $\beta\mathbb{N}$ of $\mathbb{N}$ and take $\mathcal{I} = \mathcal{I}_{fin}$. Since X has no non-trivial convergent sequence [8, Corollary 3.6.15], the family $\{\{x\} : x \in X\}$ of subsets of X forms an $\mathcal{I}$-csf-network of X, and every subset of X is sequentially open in X, i.e., $\mathcal{I}_{sn}$-open. But X is not a discrete space, hence X is not an $\mathcal{I}_{sn}$-seq-space.

(2) $\mathcal{I}$-FU-spaces $\nRightarrow$ seq-$\mathcal{I}$-spaces.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and set $X = \mathbb{N} \cup \{\infty\}$. The set X endowed with the following topology is denoted by $X(\mathcal{I})$.

(a) Each point $n \in \mathbb{N}$ is isolated.

(b) Each open neighborhood U of ∞ is of the form $(\mathbb{N} \setminus I) \cup \{\infty\}$, for each $I \in \mathcal{I}$.

There exists a maximal ideal $\mathcal{I}$ on $\mathbb{N}$ such that the space $X(\mathcal{I})$ has neither non-trivial convergent sequence nor an $\mathcal{I}$-FU-space [29, Examples 2.7 and 6.5]. Since $X(\mathcal{I})$ is not a seq-space [29, Example 3.9], it follows from part (2) of Lemma 2.3 that $X(\mathcal{I})$ is not a seq-$\mathcal{I}$-space.

(3) $\mathcal{I}_{sn}$-FU-spaces $\nRightarrow$ seq-$\mathcal{I}$, $\mathcal{I}$-nbhd-spaces.

H. Zhang and S.G. Zhang proved that the space $X(\mathcal{I})$ for some ideal $\mathcal{I}$ on $\mathbb{N}$ has two $\mathcal{I}$-open subsets such that their intersection is not $\mathcal{I}$-open [24, Theorem 2.9]. Then $X(\mathcal{I})$ is an $\mathcal{I}_{sn}$-FU-space, but it is not an $\mathcal{I}$-nbhd-space [27, Example 3.4]. And thus sequentially open subsets in $X(\mathcal{I})$ are not necessarily $\mathcal{I}$-open, hence $X(\mathcal{I})$ is not a seq-$\mathcal{I}$-space.

(4) Seq-spaces $\nRightarrow$ $\mathcal{I}_{sn}$-FU-spaces.

Let $X = \{0\} \cup \bigcup_{i\in\mathbb{N}} X_i$, where each $X_i = \{1/i\} \cup \{1/i + 1/k : k \in \mathbb{N}, k \geq i^2\}$. The set X is endowed with the following topology.

(a) Each point of the form $1/i + 1/j$ is isolated.

(b) Each neighborhood of each point of the form $1/i$ contains a set of the form $\{1/i\} \cup \{1/i + 1/k : k \geq j\}$, where each $j \geq i^2$.

(c) Each neighborhood of the point 0 contains a set obtained from X by removing a finite number of X_i's and a finite number of points of the form $1/i + 1/j$ in all the remaining X_i's.

The topological space X is called Arens' space and is denoted by S_2 [8, Example 1.6.19]. Then S_2 is a seq-space instead of an FU-space. Take $\mathcal{I} = \mathcal{I}_{fin}$. Then S_2 is an FU-$\mathcal{I}_{sn}$-space. Put $U = \{0\} \cup \{1/i : i \in \mathbb{N}\}$. Then $0 \in (U)_{seq}$. However, there is no sequentially open set V such that $0 \in V \subset U$. Hence S_2 is not an $\mathcal{I}_{sn}$-FU-space.

In Example 2.7, the four examples are constructed for special ideal $\mathcal{I}$. The following questions are raised: Can we construct related examples for each admissible ideal $\mathcal{I}$?

Problem 2.8. (1) *Is there a space of $\mathcal{I}$-csf-networks which is not an $\mathcal{I}_{sn}$-seq-space for each admissible ideal $\mathcal{I}$?*

(2) *Is there an $\mathcal{I}$-FU-space which is not a seq-$\mathcal{I}$-space for each admissible ideal $\mathcal{I}$?*

(3) *Is there an $\mathcal{I}_{sn}$-FU-space which is not a seq-$\mathcal{I}$, $\mathcal{I}$-nbhd-space for each admissible ideal $\mathcal{I}$?*

(4) *Is there a seq-space which is not an $\mathcal{I}_{sn}$-FU-space for each admissible ideal $\mathcal{I}$?*

To obtain further relationships in Figure 2.1, we have the following questions.

Problem 2.9. (1) *Is there an FU-space having no $\mathcal{I}$-csf-network [26, Question 5.6]?*
(2) *Is each seq-space an FU-$\mathcal{I}$-space?*
(3) *Is each FU-$\mathcal{I}$-space an $\mathcal{I}$-nbhd-space?*

A topological space X is of *countable tightness* [19] if whenever $A \subset X$ and $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable subset C of A. Every $\mathcal{I}$-sequential space is of countable tightness [29, Theorem 3.8]. We have the following question.

Problem 2.10. *Does each $\mathcal{I}_{sn}$-sequential space have countable tightness?*

3. Seq-$\mathcal{I}$-spaces, seq-$\mathcal{I}_{sn}$-spaces and the images of metric spaces

Mappings are effective methods to reveal the relationships between spaces [2]. This section will discuss how the spaces defined by ideal convergence in Section 2 are characterized as the images of metric spaces. We obtain intrinsic characterizations of $\mathcal{I}$-quotient and $\mathcal{I}_{sn}$-quotient images of metric spaces, and systematically describe the functions of the spaces of FU-type and seq-type in mutual classifications between spaces and mappings.

Mappings that preserve or inversely preserve convergent sequences play positive roles in discussing the spaces determined by sequences [4, 5, 14]. Let $f : X \to Y$ be a mapping. f is said to be *sequentially continuous* [5], provided V is sequentially open in Y, then $f^{-1}(V)$ is sequentially open in X. f is said to be *sequentially quotient* [5], provided V is sequentially open in Y if and only if $f^{-1}(V)$ is sequentially open in X. f is said to be *preserving convergent sequences* [5], provided the image of each convergent sequence in X under f is a convergent sequence in Y. f is said to be *sequence-covering* [22], provided each convergent sequence in Y is the image of some convergent sequence in X under f. It is well-known that sequentially continuous mappings coincide with the mappings preserving convergent sequences [5].

According to the definitions of continuous, quotient, pseudo-open and sequence-covering mappings, we draw into several classes of mappings related to $\mathcal{I}$-convergence, in which the two newly defined spaces have evidence to prove their roles (see Theorem 3.6).

Definition 3.1. Let $f : X \to Y$ be a mapping.
(1) f is said to be *$\mathcal{I}$-continuous* [29, Definition 4.1] (resp., *$\mathcal{I}_{sn}$-continuous* [15, Definition 2.7]), provided V is $\mathcal{I}$-open (resp., $\mathcal{I}_{sn}$-open) in Y, then $f^{-1}(V)$ is $\mathcal{I}$-open (resp., $\mathcal{I}_{sn}$-open) in X.

(2) f is said to be *preserving $\mathcal{I}$-convergent* [12], provided for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{I}} x$, the sequence $\{(f(x_n)\}_{n\in\mathbb{N}}$ in Y is $\mathcal{I}$-convergent to $f(x)$; f is said to be *$\mathcal{I}$-covering* [29, Definition 5.1], provided $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y with $y_n \xrightarrow{\mathcal{I}} y$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, $\mathcal{I}$-converging to $x \in f^{-1}(y)$ with each $f(x_n) = y_n$.

(3) f is said to be *pseudo-open* [1] (resp., *pseudo-$\mathcal{I}$-open*, or *pseudo-$\mathcal{I}_{sn}$-open*), provided $f^{-1}(y) \subset U^\circ$ (resp., $f^{-1}(y) \subset (U)_{\mathcal{I}}$, or $f^{-1}(y) \subset (U)_{\mathcal{I}_{sn}}$) for some $y \in Y$ and $U \subset X$, then $y \in (f(U))^\circ$ (resp., $y \in (f(U))_{\mathcal{I}}$, or $y \in (f(U))_{\mathcal{I}_{sn}}$).

(4) f is said to be *quotient* [8] (resp., *$\mathcal{I}$-quotient* [29, Definition 5.1], or *$\mathcal{I}_{sn}$-quotient* [15, Definition 4.1]), provided f is surjective and V is open (resp., $\mathcal{I}$-open, or $\mathcal{I}_{sn}$-open) in Y if and only if $f^{-1}(V)$ is open (resp., $\mathcal{I}$-open, or $\mathcal{I}_{sn}$-open) in X.

$\mathcal{I}$-covering mappings and mappings satisfying the condition of part (3) of Definition 3.1 are surjective. For the ideal $\mathcal{I}_{fin}$ on $\mathbb{N}$, the following mappings are consistent with sequentially quotient mappings if the mappings are $\mathcal{I}$-continuous: pseudo-$\mathcal{I}$-open mappings, pseudo-$\mathcal{I}_{sn}$-open mappings, $\mathcal{I}$-quotient mappings, and $\mathcal{I}_{sn}$-quotient mappings [17, Lemma 3.2].

Lemma 3.2. *Let X be a topological space.*
(1) *X is a continuous and sequence-covering image of some metric space [18, Lemma 3.6].*
(2) *X is a seq-space if and only if X is a quotient image of some metric space [9].*
(3) *X is an FU-space if and only if X is a continuous and pseudo-open image of some metric space [9].*

Corresponding to ideal convergence, the continuous and $\mathcal{I}$-covering images of metric spaces are characterized by the spaces of $\mathcal{I}$-csf-networks [26, Theorem 3.5]. Parts (2) and (3) of Examples 2.7 show that not every topological space is a continuous and $\mathcal{I}$-covering image of a metric space. The main purpose of this section is to explore $\mathcal{I}$-convergent versions of parts (2) and (3) of Lemma 3.2. Spaces and mappings introduced in Definitions 2.2 and 3.1 are the main objects of our discussion.

The following results are known.

Lemma 3.3. *Let X, Y be topological spaces and $f : X \to Y$ be a mapping.*

(1) *f is continuous $\Rightarrow$ f is $\mathcal{I}_{sn}$-continuous $\Leftrightarrow$ f preserves $\mathcal{I}$-convergence $\Rightarrow$ f is $\mathcal{I}$-continuous [15, Lemma 2.8 and Theorem 3.12].*

(2) *If X is a seq-space, then f is continuous $\Leftrightarrow$ f is $\mathcal{I}_{sn}$-continuous $\Leftrightarrow$ f is $\mathcal{I}$-continuous $\Leftrightarrow$ f is sequentially continuous [29, Corollary 4.6].*

The following further results are obtained

Lemma 3.4. *Let X, Y be topological spaces and $f : X \to Y$ be a mapping.*

(1) *If f is $\mathcal{I}$-covering, then f is a pseudo-$\mathcal{I}$-open mapping and a pseudo-$\mathcal{I}_{sn}$-open mapping.*

(2) *If f is an $\mathcal{I}$-continuous* (*resp., $\mathcal{I}_{sn}$-continuous*) *pseudo-$\mathcal{I}$-open* (*resp., pseudo-$\mathcal{I}_{sn}$-open*) *mapping, then f is an $\mathcal{I}$-quotient* (*resp., $\mathcal{I}_{sn}$-quotient*) *mapping.*

Proof. (1) Suppose that $f : X \to Y$ is an $\mathcal{I}$-covering mapping. Let $y \in Y \setminus (f(U))_{\mathcal{I}}$ for some $U \subset X$. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $Y \setminus f(U)$ such that it is $\mathcal{I}$-convergent to y. Since f is an $\mathcal{I}$-covering mapping, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that it is $\mathcal{I}$-convergent to some point $x \in f^{-1}(y)$ and each $f(x_n) = y_n$. Thus each $x_n \notin U$, and so $x \notin (U)_{\mathcal{I}}$. It follows that $f^{-1}(y) \not\subset (U)_{\mathcal{I}}$. Hence f is a pseudo-$\mathcal{I}$-open mapping.

Suppose that $f^{-1}(y) \subset (V)_{\mathcal{I}_{sn}}$ for some $y \in Y$ and $V \subset X$. If a sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y is $\mathcal{I}$-convergent to y, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that it is $\mathcal{I}$-convergent to some point $x \in f^{-1}(y)$ and each $f(x_n) = y_n$. Thus the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in V, and hence $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in $f(V)$. It follows that the set $f(V)$ is an $\mathcal{I}$-sequential neighborhood of y, i.e., $y \in (f(V))_{\mathcal{I}_{sn}}$. This implies that f is a pseudo-$\mathcal{I}_{sn}$-open mapping.

(2) We only show the case of $\mathcal{I}$-continuous mappings. Let $f : X \to Y$ be an $\mathcal{I}$-continuous pseudo-$\mathcal{I}$-open mapping and $f^{-1}(U)$ be $\mathcal{I}$-open in X for some $U \subset Y$. If $y \in U$, then $f^{-1}(y) \subset f^{-1}(U) = (f^{-1}(U))_{\mathcal{I}}$, and hence $y \in (U)_{\mathcal{I}}$. This implies that $U = (U)_{\mathcal{I}}$ is $\mathcal{I}$-open in Y. Thus f is an $\mathcal{I}$-quotient mapping. □

Figure 3.1 illustrates the relationships between these mapping classes introduced in Definition 3.1.

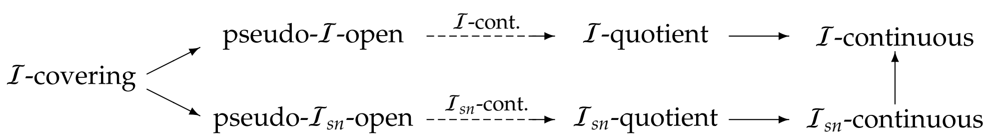

Figure 3.1 $\mathcal{I}$-quotient mappings

In Section 2, we mention the question to characterize the images of metric spaces under $\mathcal{I}$-quotient mappings. Similarly, [27, p.6] discussed the $\mathcal{I}_{sn}$-quotient images of metric spaces, which showed that it was not necessarily an $\mathcal{I}_{sn}$-sequential space, but its characterization was not presented. We will use sequential coreflections to study the $\mathcal{I}$-quotient and $\mathcal{I}_{sn}$-quotient images of metric spaces.

Definition 3.5. Let (X, τ) be a topological space.

(1) The *sequential coreflection* of the space X, denoted by sX, is the set X with *sequentially interior topology* (or *sequentially closure topology*) τ_s as follows: $U \in \tau_s$ if and only if U is a sequentially open subset of (X, τ) [4, 9].

(2) The *$\mathcal{I}_{sn}$-coreflection* of the space X, denoted by $X_{\mathcal{I}_{sn}}$, is the set X with *$\mathcal{I}_{sn}$-topology* $\tau_{\mathcal{I}_{sn}}$ as follows: $U \in \tau_{\mathcal{I}_{sn}}$ if and only if U is an $\mathcal{I}_{sn}$-open subset of (X, τ) [15, Definition 3.1].

(3) The family of all $\mathcal{I}$-open subsets of the space X is called an *$\mathcal{I}$-topology* if it is a topology of X, which is denoted by $\tau_{\mathcal{I}}$. The space $(X, \tau_{\mathcal{I}})$ is called an *$\mathcal{I}$-coreflection* of the space X, which is denoted by $X_{\mathcal{I}}$ [15, Definition 5.2].

It follows from [15, Lemma 3.2] that both spaces (X, τ) and $(X, \tau_{\mathcal{I}_{sn}})$ have the same $\mathcal{I}$-convergent sequences, hence they have the same $\mathcal{I}_{sn}$-open sets. By part (1) of Lemma 2.1, we have that $\tau \subset \tau_{\mathcal{I}_{sn}} \subset \tau_{\mathcal{I}} \subset \tau_s$.

Remark 3.6. Let (X, τ) be a topological space. We have the following three facts.

(i) Both τ and τ_s have the same convergent sequences [4, 9].

(ii) Let M be an $\mathcal{I}_{sn}$-seq-space. Then $f : M \to (X, \tau)$ is $\mathcal{I}_{sn}$-quotient $\Leftrightarrow$ $f : M \to (X, \tau_{\mathcal{I}_{sn}})$ is quotient.

In fact, for each $V \subset X$, the preceding formula is equivalent to that $f^{-1}(V)$ is $\mathcal{I}_{sn}$-open in M if and only if $V \in \tau_{\mathcal{I}_{sn}}$. The latter formula is equivalent to that $f^{-1}(V)$ is open in M if and only if $V \in \tau_{\mathcal{I}_{sn}}$. Since M is an $\mathcal{I}_{sn}$-seq-space, it follows that the preceding formula coincides with the latter formula.

(iii) Let M be a seq-space. Then $f : M \to (X, \tau)$ is $\mathcal{I}$-quotient $\Leftrightarrow$ the family of all $\mathcal{I}$-open subsets of X forms a topology, and $f : M \to (X, \tau_{\mathcal{I}})$ is quotient.

In fact, for each $V \subset X$, the preceding formula is equivalent to that $f^{-1}(V)$ is $\mathcal{I}$-open in M if and only if V is $\mathcal{I}$-open in (X, τ). If $V \in \tau_s$, since every $\mathcal{I}$-quotient mapping is $\mathcal{I}$-continuous, it follows from part (2) of Lemma 3.3 that f is sequentially continuous, thus $f^{-1}(V)$ is sequentially open in M, and so $f^{-1}(V)$ is $\mathcal{I}$-open in M. Hence this formula derives that the space (X, τ) is a seq-$\mathcal{I}$-space, therefore $\tau_{\mathcal{I}} = \tau_s$ is an $\mathcal{I}$-topology. The latter formula is equivalent to that $f^{-1}(V)$ is open in M if and only if $V \in \tau_{\mathcal{I}}$. Since M is a seq-space, it follows that the preceding formula coincides with the latter formula.

As an extension of part (2) of Lemma 3.2, the following theorem gives intrinsic characterizations of the $\mathcal{I}$-quotient and $\mathcal{I}_{sn}$-quotient images of metric spaces.

Theorem 3.7. *Let X be a topological space.*

(1) *X is a seq-$\mathcal{I}_{sn}$-space if and only if X is an $\mathcal{I}_{sn}$-quotient image of a metric space.*

(2) *X is a seq-$\mathcal{I}$-space if and only if X is an $\mathcal{I}$-quotient image of a metric space.*

Proof. (1) (X, τ) is a seq-$\mathcal{I}_{sn}$-space, by part (i) of Remark 3.6
$\Leftrightarrow$ $(X, \tau_{\mathcal{I}_{sn}})$ is a seq-space, by Lemma 3.2
$\Leftrightarrow$ there are a metric space M and a quotient mapping $f : M \to (X, \tau_{\mathcal{I}_{sn}})$, by part (ii) of Remark 3.6
$\Leftrightarrow$ there are a metric space M and an $\mathcal{I}_{sn}$-quotient mapping $f : M \to (X, \tau)$.

(2) (X, τ) is a seq-$\mathcal{I}$-space, by (i) of Remark 3.6
$\Leftrightarrow$ $(X, \tau_{\mathcal{I}})$ is a seq-space, by Lemma 3.2
$\Leftrightarrow$ there are a metric space M and a quotient mapping $f : M \to (X, \tau_{\mathcal{I}})$, by (iii) of Remark 3.6
$\Leftrightarrow$ there are a metric space M and an $\mathcal{I}$-quotient mapping $f : M \to (X, \tau)$. □

As a further development of part (3) of Lemma 3.2, the following theorem gives intrinsic characterizations of the pseudo-$\mathcal{I}$-open and pseudo-$\mathcal{I}_{sn}$-open images of metric spaces.

Theorem 3.8. *Let X be a topological space.*

(1) *X is an FU-$\mathcal{I}_{sn}$-space if and only if X is a continuous pseudo-$\mathcal{I}_{sn}$-open image of a metric space.*

(2) *X is an FU-$\mathcal{I}$-space if and only if X is a continuous pseudo-$\mathcal{I}$-open image of a metric space.*

Proof. Since the proofs of (1) and (2) are similar, we only prove that (1) is true.

Necessity. Let X be an FU-$\mathcal{I}_{sn}$-space. By part (1) of Lemma 3.2, there are a metric space M and a continuous sequence-covering mapping $f : M \to X$. Assume that $x \in X \setminus (f(U))_{\mathcal{I}_{sn}}$ for some $U \subset M$. Since X is an FU-$\mathcal{I}_{sn}$-space, it follows that $x \notin (f(U))_{seq}$, and that there is a sequence T in $X \setminus f(U)$ such that it is convergent to x. Note that f is a sequence-covering mapping. There is a sequence S in M such that $f(S) = T$ and S is convergent to some point $z \in f^{-1}(x)$. Thus $S \cap U = \emptyset$, and $f^{-1}(x) \not\subset (U)_{seq} = (U)_{\mathcal{I}_{sn}}$. This shows that $f : M \to X$ is a pseudo-$\mathcal{I}_{sn}$-open mapping.

Sufficiency. By part (2) of Lemma 3.2, it suffices to show that each FU-$\mathcal{I}_{sn}$-space is preserved by a sequentially continuous and pseudo-$\mathcal{I}_{sn}$-open mapping. Let $f : M \to X$ be a sequentially continuous and pseudo-$\mathcal{I}_{sn}$-open mapping, where M is an FU-$\mathcal{I}_{sn}$-space. Suppose that $x \in X \setminus (U)_{\mathcal{I}_{sn}}$ for some $U \subset X$. Since f is a pseudo-$\mathcal{I}_{sn}$-open mapping, there exists $z \in f^{-1}(x) \setminus (f^{-1}(U))_{\mathcal{I}_{sn}}$. Note that M is an FU-$\mathcal{I}_{sn}$-space. Then $z \notin (f^{-1}(U))_{seq}$, and there is a sequence S in $M \setminus f^{-1}(U)$ such that it is convergent to z. And since every convergent sequence is preserved by a sequentially continuous mapping [5, Theorem 3.1], it follows that the sequence $f(S)$ in $X \setminus U$ is convergent to x, and $x \notin (U)_{seq}$. Therefore X is an FU-$\mathcal{I}_{sn}$-space. □

Remark 3.9. Compare to Definition 2.2 and the properties of the images of metric spaces obtained in this section, there are the following questions. Are there similar characterizations as Theorems 3.7 or 3.8 in the following spaces: $\mathcal{I}$-seq-spaces, $\mathcal{I}_{sn}$-seq-spaces, $\mathcal{I}$-FU-spaces or $\mathcal{I}_{sn}$-FU-spaces? The information extracted from Theorem 3.7 can be expressed as follows: Find a property Q of subsets of a topological space satisfying the following requirements. Let X be an $\mathcal{I}$-seq or $\mathcal{I}_{sn}$-seq-space. There are a metric space M and a mapping $f : M \to X$ such that:

(a) each open subset has property Q;

(b) a subset V of X has property Q if and only if $f^{-1}(V)$ has property Q in M;

(c) each subset having property Q is open in M and X.

Suppose that Q has the property. Then f is a quotient mapping, and X is a seq-space. Thus the $\mathcal{I}$-FU-space $X(\mathcal{I})$ in part (2) of Example 2.7 does not satisfy the requirements. This seems to explain why $\mathcal{I}$-seq, $\mathcal{I}_{sn}$-seq, $\mathcal{I}$-FU or $\mathcal{I}_{sn}$-FU-spaces do not have characterizations similar to Theorems 3.7 or 3.8.

At the end of this section, we will show the role of the four spaces listed in Remark 3.9 in mutual classifications between spaces and mappings.

The following results are known.

Theorem 3.10. *Let X be a topological space.*

(1) *X is an $\mathcal{I}_{sn}$-seq-space if and only if every continuous $\mathcal{I}_{sn}$-quotient mapping onto X is quotient [27, Theorem 3.10].*

(2) *X is an $\mathcal{I}$-seq-space if and only if every continuous $\mathcal{I}$-quotient mapping onto X is quotient, and X is an $\mathcal{I}$-nbhd-space [15, Theorem 4.10].*

Lemma 3.11. ([27, Theorem 4.7]) *A topological space X is an $\mathcal{I}_{sn}$-FU-space if and only if every $\mathcal{I}$-covering mapping onto X is pseudo-open.*

Theorem 3.12. *Let X be a topological space.*

(1) *X is an $\mathcal{I}_{sn}$-FU-space if and only if every pseudo-$\mathcal{I}_{sn}$-open mapping onto X is pseudo-open.*

(2) *X is an $\mathcal{I}$-FU-space if and only if every pseudo-$\mathcal{I}$-open mapping onto X is pseudo-open, and $(A)_{\mathcal{I}} \subset (A)_{\mathcal{I}_{sn}}$ for each $A \subset X$.*

Proof. (1) The sufficiency follows from Lemmas 3.4 and 3.11. The necessity is proved as follows. Suppose that X is an $\mathcal{I}_{sn}$-FU-space and $f : Z \to X$ is a pseudo-$\mathcal{I}_{sn}$-open mapping. Let $f^{-1}(x) \subset U^\circ$ for some $x \in X$ and $U \subset Z$. Then $f^{-1}(x) \subset (U)_{\mathcal{I}_{sn}}$. Since f is pseudo-$\mathcal{I}_{sn}$-open, it follows that $x \in (f(U))_{\mathcal{I}_{sn}}$. Note that X is an $\mathcal{I}_{sn}$-FU-space. Hence $x \in (f(U))^\circ$. This implies that f is pseudo-open.

(2) Sufficiency. Suppose that every pseudo-$\mathcal{I}$-open mapping onto X is pseudo-open, and $(A)_{\mathcal{I}} \subset (A)_{\mathcal{I}_{sn}}$ for each $A \subset X$. It follows from Lemmas 3.4 and 3.11 that X is an $\mathcal{I}_{sn}$-FU-space. Since $(A)_{\mathcal{I}} \subset (A)_{\mathcal{I}_{sn}} \subset A^\circ$ for each $A \subset X$, the space X is an $\mathcal{I}$-FU-space.

Necessity. Let X be an $\mathcal{I}$-FU-space. It is clear that $(A)_{\mathcal{I}} \subset (A)_{\mathcal{I}_{sn}}$ for each $A \subset X$. Assume that $f : Z \to X$ is pseudo-$\mathcal{I}$-open. Let $f^{-1}(x) \subset U^\circ$ for some $x \in X$ and $U \subset Z$. Then $f^{-1}(x) \subset (U)_{\mathcal{I}}$. Since f is pseudo-$\mathcal{I}$-open, it follows that $x \in (f(U))_{\mathcal{I}}$. Note that X is an $\mathcal{I}$-FU-space. Hence $x \in (f(U))^\circ$. Therefore f is pseudo-open. □

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