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On the probabilistic convergence ring and its natural uniform convergence structure

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Abstract. Introducing the notion of probabilistic convergence ring, and probabilistic limit ring, our motivations among others are, to focus at two vital issues, such as, (a) to provide characterization theorems on probabilistic convergence rings, (b) probabilistic uniformizability of probabilistic limit rings, and discuss some results on probabilistic Cauchy rings, and their relationship with probabilistic convergence rings. In doing so, we produce various examples, particularly, from function space structure of continuous probabilistic convergence. Moreover, we observe that the category of probabilistic convergence ring in the sense of Richardson-Kent is a reflective subcategory of the category of probabilistic convergence rings in our sense.

1. Introduction

Following Menger's statistical metrics [26], the notion of probabilistic metrics got prominence - a notion that can be seen as one of the most influential generalizations of metric spaces. This underlines the importance of probabilistic structures and their applications. The study of compatibility of probabilistic structures with algebraic structures lead to enormous contributions in functional analysis, particularly, the role of probabilistic metric groups are worth mentioning; we quote here a few references for the convenience of the reader, cf. [3, 5, 7, 8, 10–15, 18, 25, 28–30, 35, 36, 38]. The fact of the matter is, the probabilistic metric space serves as a natural example of probabilistic convergence spaces, [19], and probabilistic metric groups used as natural example of probabilistic convergence groups, [3]. The theory of topological rings, [9, 23, 27, 39] is quite rich, and an extensive amount of work has been done in this area but we do not see much work on probabilistic convergence rings as well as probabilistic topological rings. It may be mentioned here that the category of classical convergence spaces, **Conv** is a supercategory of the category of topological spaces, **Top**, and **Conv** is a Cartesian closed category whereas **Top** is not, [31]. Inspired by the work on probabilistic convergence spaces, [19, 20], a Cartesian closed category, several papers were published on the compatibility of probabilistic convergence structures on group structures; and a good number of examples are also provided which are themselves interesting in their own right. The motivation of this work is to introduce the notion of probabilistic convergence ring, and probabilistic limit ring and its natural uniform

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convergence structure together with the notion of probabilistic Cauchy ring, and provide various examples. We arrange our work as follows. In Section 3, after a brief note on probabilistic metric space and its so-called Tardiff neighborhood system, we recall the notion of probabilistic convergence space and probabilistic limit space from [19]. In Section 4, the notion of probabilistic convergence ring and probabilistic limit ring are introduced, and we show that the category of probabilistic convergence rings is a topological category over the category of rings. As every probabilistic convergence ring is homogeneous, we are able to provide two fundamental characterization theorems, including various examples. In Section 5, we provide the notion of probabilistic convergence ring in the sense of Richardson-Kent showing that this category is a reflective subcategory of our category of probabilistic convergence rings. The natural uniform convergence structure for probabilistic limit ring is considered in Section 6. Presenting the category of probabilistic Cauchy rings in Section 7, we show that the category of probabilistic Cauchy rings is topological, and we produce an example relating to function space structure. Furthermore, we show that every probabilistic convergence ring is a probabilistic Cauchy ring.

2. Preliminaries

If $(A, \leq)$ is an ordered set, we denote by $\bigwedge_{j\in J} \alpha_j$ the infimum, while $\bigvee_{j\in J} \alpha_j$ denotes the supremum, if they exist, of the set $\{\alpha_j : j \in J\} \subseteq A$. In case of a two-point set $\{\alpha, \beta\}$ we write $\alpha \wedge \beta$ and $\alpha \vee \beta$, respectively.

A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$, is called a *distance distribution function* [34]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\epsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \epsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

belong to Δ^+. The set Δ^+ is ordered pointwisely, i.e., for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$, we have $\varphi(x) \leq \psi(x)$. The smallest element of Δ^+ is then ϵ_∞ and the largest element is ϵ_0.

The following result is mentioned in Schweizer and Sklar [34].

Lemma 2.1. 1. *If $\varphi, \psi \in \Delta^+$, then also $\varphi \wedge \psi \in \Delta^+$.*
2. *If $\varphi_j \in \Delta^+$ for all $j \in J$, then also $\bigvee_{j\in J} \varphi_j \in \Delta^+$.*

Here, $\varphi \wedge \psi$ denotes the pointwise minimum of φ and ψ in $(\Delta^+, \leq)$ and $\bigvee_{j\in J} \varphi_j$ denotes the pointwise supremum of the family $\{\varphi_j : j \in J\}$ in $(\Delta^+, \leq)$. On the set Δ^+ we consider the *modified Lévy metric* [37], which is defined below for the convenience of the reader.

Let $\varphi, \psi \in \Delta^+$ and $\epsilon > 0$. Consider the following properties

$$A(\varphi, \psi; \epsilon) \Longleftrightarrow \varphi(x - \epsilon) - \epsilon \leq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon});$$
$$\text{and } B(\varphi, \psi; \epsilon) \Longleftrightarrow \varphi(x + \epsilon) + \epsilon \geq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon}).$$

Then the *modified Lévy metric* d_L on $\Delta^+ \times \Delta^+$ is given by

$$d_L(\varphi, \psi) = \bigwedge\{\epsilon > 0 \colon A(\varphi, \psi; \epsilon) \text{ and } B(\varphi, \psi; \epsilon) \text{ hold }\}.$$

Definition 2.2. ([34]) A *triangle function* is a function $\tau \colon \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ such that the following conditions are satisfied for all $\varphi, \psi, \xi \in \Delta^+$:
(i) $\tau(\tau(\varphi, \psi), \xi) = \tau(\varphi, \tau(\psi, \xi))$;
(ii) $\tau(\varphi, \psi) = \tau(\psi, \varphi)$;
(iii) $\varphi \leq \psi \implies \tau(\varphi, \xi) \leq \tau(\psi, \xi)$;
(iv) $\tau(\varphi, \epsilon_0) = \varphi$.

The largest triangle function is the pointwise minimum $\mu(\varphi, \psi) = \varphi \wedge \psi$. It is easy to prove that a triangle function that is idempotent, i.e., for which $\tau(\varphi, \varphi) = \varphi$ for all $\varphi \in \Delta^+$, must be the largest triangle function. A triangle function is called *continuous* [34, 38] if it is a continuous function with respect to the topology and product topology induced by the modified Lévy metric. A triangle function is called *sup-continuous* [34, 38] if $\tau(\bigvee_{j\in J}\varphi_j,\psi) = \bigvee_{j\in J}\tau(\varphi_j,\psi)$ for all $\varphi_j,\psi \in \Delta^+$ $(j \in J)$. For further study on sup-continuity and its relation to continuity, we refer to [34], and for a good survey on triangle functions, see e.g. [33].

An example for a continuous triangle function with a left-continuous t-norm $*$, [34], is τ_* defined by

$$\tau_*(\varphi,\psi)(x) = \bigvee_{s+t=x}\varphi(s) * \psi(t), \text{ for } \varphi,\psi \in \Delta^+.$$

For $\varphi \in \Delta^+$, we define the right-hand limit $\varphi(0^+) = \lim_{x\to 0^+}\varphi(x)$.

Lemma 2.3. ([19]) *For a continuous t-norm* $*$, *the triangle function* τ_* *satisfies* $\tau_*(\varphi,\psi)(0^+) = \varphi(0^+) * \psi(0^+)$.

For a set S, we denote $P(S)$ its power set. We denote the set of filters on the set S by $\mathbb{F}(S)$. We order this set by set inclusion, and we denote for $s \in S$ the point filter by $[s] = \{F \subseteq S : s \in F\}$. If $\mathbb{F} \in \mathbb{F}(S)$ and $\mathbb{G} \in \mathbb{F}(T)$, then the filter on $S\times T$ generated by the sets of the form $\{F\times G : F \in \mathbb{F}, G \in \mathbb{G}\}$ is denoted by $\mathbb{F}\times\mathbb{G}$. If $(R,+,\cdot)$ is a ring, and $\mathbb{F},\mathbb{G} \in \mathbb{F}(R)$, then we define $\mathbb{F}\oplus\mathbb{G}$ as a filter generated by the sets $F+G = \{p+q : p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. For the multiplicative operation, we define $\mathbb{F}\odot\mathbb{G}$ as a filter generated by the sets $F\cdot G = \{pq : p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. The filter $-\mathbb{F}$ is generated by the sets $-F = \{-p : p \in F\}$ for $F \in \mathbb{F}$.

For filters on $R\times R$ we use later the letters Φ, Ψ, etc. If $\Phi \in \mathbb{F}(R\times R)$, then Φ^{-1} is generated by the set $H^{-1} = \{(p,q) : (q,p) \in H\}$ with $H \in \Phi$. If $\Phi,\Psi \in \mathbb{F}(R\times R)$, then $\Phi\circ\Psi$, the composition of Φ and Ψ, is defined to be a filter generated by the filterbasis $\{H\circ K : H \in \Phi, K \in \Psi\}$, where $H\circ K = \{(p,q) \in R\times R : \exists\, r \in R$ such that $(p,r) \in K$ and $(r,q) \in H\}$, and whenever $H\circ K \neq \emptyset$, for all $H \in \Phi, K \in \Psi$.

For notions of category theory we refer to Adámek et. al. [1]. However, we recall a few notions for the convenience of the reader. A *category* C consists of three items; namely,
• a class of *objects* usually denote by S, T,...,
• a class of *morphisms* between objects of C, which is denoted by $f\colon S \longrightarrow T$ having domain and codomain, $dom(f) = S$ and $cod(f) = T$,
• a *composition law* which assigns to each pair of morphisms (f,g) with $dom(f) = cod(g)$, a composite morphism $f\circ g\colon dom(g) \longrightarrow cod(f)$ subject to satisfy (i) associativity: $f\circ(g\circ h) = f\circ(g\circ h)$, and (ii) for each object S, there exists an *identity morphism* $id_S\colon S \longrightarrow S$, such that $f\circ id_S = f$ and $id_S\circ g = g$, whenever the composition is defined.

Examples of categories include **Top**, the category of topological spaces as objects and continuous mappings between them as morphisms; **Conv**, the category of convergence spaces as objects and continuous mappings between them as morphisms. Likewise, the category of all probabilistic metric spaces as objects and non-expansive maps as morphisms is denoted by **PMet**; **Rng**, the category of rings as objects and ring homomorphisms as morphisms, and so on.

A *functor* $\mathfrak{A} : C \longrightarrow \mathcal{D}$ is a morphism between categories, precisely it consists of mappings between objects of C and objects of $\mathcal{D}$ (sometime we write $|C|$ to denote the class of objects of C), and mappings between morphisms of C and morphism of $\mathcal{D}$ such that (i) if $f\colon S \longrightarrow T$, then $\mathfrak{A}(f)\colon \mathfrak{A}(S) \longrightarrow \mathfrak{A}(T)$; (ii) $\mathfrak{A}(f\circ g) = \mathfrak{A}(f)\circ\mathfrak{A}(g)$, whenever $f\circ g$ is defined; (iii) $\mathfrak{A}(id_S) = id_{\mathfrak{A}(S)}$.

A *construct* is a category C whose objects are structured sets (S,ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e., if for any source $\left(f_j\colon S \longrightarrow \left(S_j,\xi_j\right)\right)_{j\in J}$, there is a unique structure ξ on S such that a mapping $g\colon (T,\eta) \longrightarrow (S,\xi)$ is a morphism if and only if for each $j \in J$ the composition $f_j\circ g\colon (T,\eta) \longrightarrow \left(S_j,\xi_j\right)$ is a morphism, where (T,η) is a structured set.

A topological construct is called *Cartesian closed* if for each pair of objects $(S,\xi)\,,(T,\eta)$, there is a structure on the set $C(S,T)$ of morphisms from S to T such that mapping $ev\colon C(S,T)\times S \longrightarrow T$, defined for any $f \in C(S,T)$ and $s \in S$ by $ev(f,s) = f(s)$ (called an *evaluation mapping*) is a morphism, and for each object (Z,ζ), and each morphism $f\colon S\times Z \longrightarrow T$ the mapping $\hat{f}\colon Z \longrightarrow C(S,T)$ defined by $\hat{f}(z)(x) = f(x,z)$ is a morphism.

Let $\mathfrak{C}$ be a subcategory of a category $\mathfrak{A}$. Then $\mathfrak{C}$ is said to be *reflective* in $\mathfrak{A}$ (or $\mathfrak{C}$ is a reflective subcategory of $\mathfrak{A}$) if for each $\mathbb{X} \in |\mathfrak{A}|$ there exists a $\mathfrak{C}$-object $\mathbb{X}_{\mathfrak{C}}$ and an $\mathfrak{A}$-morphism $r_{\mathbb{X}}\colon \mathbb{X} \longrightarrow \mathbb{X}_{\mathfrak{C}}$ such that for each $\mathfrak{C}$-object $\mathbb{C}$ and each $\mathfrak{A}$-morphism $f\colon \mathbb{X} \longrightarrow \mathbb{C}$ there exists unique morphism $f'\colon \mathbb{X}_{\mathfrak{C}} \longrightarrow \mathbb{C}$ such that $f'\circ r_{\mathbb{X}} = f$. The notion of *coreflective subcategory* is defined dually.

3. Probabilistic metric spaces and probabilistic convergence spaces

Definition 3.1. ([34]) *A probabilistic metric space under a triangle function* τ is a pair (S, F), where $\mathsf{F}: S\times S \longrightarrow \Delta^+$ such that for all $p, q \in S$ the following properties hold:
(PM1) $\mathsf{F}(p, q) = \epsilon_0 \Longleftrightarrow p = q$;
(PM2) $\mathsf{F}(p, q) = \mathsf{F}(q, p)$;
(PM3) $\tau(\mathsf{F}(p, q), \mathsf{F}(q, r)) \leq \mathsf{F}(p, r)$.
A mapping $f: (S, \mathsf{F}) \longrightarrow (S', \mathsf{F}')$ is called *non-expansive* if $\mathsf{F}(p, q) \leq \mathsf{F}'(f(p), f(q))$ for all $p, q \in S$.

Definition 3.2. ([19]) Let S be a set. A family of mappings $\left(c_\varphi: \mathbb{F}(S) \longrightarrow P(S)\right)_{\varphi\in\Delta^+}$ which satisfies the axioms
(PC1) $p \in c_\varphi([p]), p \in S, \varphi \in \Delta^+$;
(PC2) if $\mathbb{F} \leq \mathbb{G}$, then $c_\varphi(\mathbb{F}) \subseteq c_\varphi(\mathbb{G})$, $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ and $\forall \varphi \in \Delta^+$;
(PC3) if $\varphi \leq \psi$, then $c_\psi(\mathbb{F}) \subseteq c_\varphi(\mathbb{F})$, $\forall \mathbb{F} \in \mathbb{F}(S)$ and $\forall \varphi, \psi \in \Delta^+$;
(PC4) $p \in c_{\epsilon_\infty}(\mathbb{F})$ $\forall p \in S$, $\mathbb{F} \in \mathbb{F}(S)$,
is called a *probabilistic convergence structure on* S. The pair $\left(S, \overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right)$ is called a (distance distribution function indexed) *probabilistic convergence space*.

If $(S, \overline{c})$ satisfies further the axiom (PC5)

$$c_\varphi(\mathbb{F}) \cap c_\varphi(\mathbb{G}) \subseteq c_\varphi(\mathbb{F} \wedge \mathbb{G}),\ \forall \varphi \in \Delta^+,\ \forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(S),$$

then we speak of a *probabilistic limit space*.

A mapping $f: (S, \overline{c}) \longrightarrow \left(S', \overline{c'}\right)$ between probabilistic convergence spaces (resp. probabilistic limit spaces) is called *continuous* if $f(p) \in c'_\varphi(f(\mathbb{F}))$ whenever $p \in c_\varphi(\mathbb{F})$ for every $p \in S$ and for every $\mathbb{F} \in \mathbb{F}(S)$, and $\varphi \in \Delta^+$. **PConv** denotes the category of probabilistic convergence spaces as objects and continuous maps as morphisms while the category of all probabilistic limit spaces and continuous mappings between them is denoted by **PLim**. Clearly, **PLim** is a full subcategory of **PConv**.

We now recall Tardiff's neighborhood systems [38] that are based on the so-called *profile functions* [16]. A profile function is in fact just an element $\varphi \in \Delta^+$, where $\varphi(x)$, $x > 0$, is interpreted as the maximum probability assigned to the event that the distance between p and q is less than x. Given a $\varphi \in \Delta^+$, $\epsilon > 0$ and $p \in S$, the (φ, ϵ)*-neighborhood of* p is defined by

$$N_p^{\varphi,\epsilon} = \{q \in S\ :\ F_{p,q}(x + \epsilon) + \epsilon \geq \varphi(x), \forall x \in [0, \frac{1}{\epsilon})\}.$$

The set $\{N_p^{\varphi,\epsilon}\ :\ \epsilon > 0\}$ is then a filter basis, and the filter generated by this basis is denoted by $\mathbb{N}_p^\varphi$ [19].

For a probabilistic metric space (S, F), define $p \in c_\varphi^F(\mathbb{F}) \Longleftrightarrow \mathbb{F} \geq \mathbb{N}_p^\varphi$. Then $\left(S, \overline{c^{\mathsf{F}}}\right)$ is a probabilistic convergence space and a non-expansive mapping $f: (S, \mathsf{F}) \longrightarrow (S', \mathsf{F}')$ is continuous as a mapping $f: \left(S, \overline{c^{\mathsf{F}}}\right) \longrightarrow \left(S', \overline{c^{\mathsf{F}'}}\right)$. It follows from [19] that every probabilistic metric space gives rise to a natural probabilistic convergence space.

4. Probabilistic convergence rings

Lemma 4.1. *Let* $(R, +, \cdot), (R', +, \cdot) \in |\boldsymbol{Rng}|$, *and* $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathbb{F}(R)$. *Let* $h: R\times R \longrightarrow R, (x, y) \longmapsto x + y$, $m: R\times R \longrightarrow R, (x, y) \longmapsto xy$, $\jmath: R \longrightarrow R, x \longmapsto -x$ *and* $k: R \times R \longrightarrow R, (x, y) \longmapsto R, (x, y) \longmapsto x - y$ *be mappings and* $f: R \longrightarrow R'$ *be a ring homomorphism. Then:*
(1) $-(-\mathbb{F}) = \mathbb{F}$;
(2) $\mathbb{F} \leq \mathbb{G} \implies -\mathbb{F} \leq -\mathbb{G}$;
(3) $[x] \oplus [-x] = [0]$;
(4) $[x] \oplus [-y] = [x - y]$;

(5) $[x] \oplus [y] = [x + y]$;
(6) $P \in (\mathbb{F} \oplus [x]) \Leftrightarrow (P - x) \in \mathbb{F}$;
(7) $[0] \oplus \mathbb{F} = \mathbb{F}$, $[0] \ominus \mathbb{F} = -\mathbb{F}$;
(8) $\mathbb{F} \oplus (\mathbb{G} \oplus \mathbb{H}) = (\mathbb{F} \oplus \mathbb{G}) \oplus \mathbb{H}$; $\mathbb{F} \odot (\mathbb{G} \odot \mathbb{H}) = (\mathbb{F} \odot \mathbb{G}) \odot \mathbb{H}$;
(9) $\mathbb{F} \oplus (-\mathbb{G}) = \mathbb{F} \ominus \mathbb{G}$
(10) $f(\mathbb{F} \oplus \mathbb{G}) = f(\mathbb{F}) \oplus f(\mathbb{G})$; $f(\mathbb{F} \odot \mathbb{G}) = f(\mathbb{F}) \odot f(\mathbb{G})$.

Definition 4.2. A quadruple $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called a *probabilistic convergence ring* under the triangle function τ if for all $\varphi, \psi \in \Delta^+$ and for all filter $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$
(PCR1) $(R, +, \cdot)$ is a ring;
(PCR2) $(R, \overline{c})$ is a probabilistic convergence space;
(PCRA) $p + q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$ whenever $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$;
(PCRI) $-p \in c_\varphi(-\mathbb{F})$ whenever $p \in c_\varphi(\mathbb{F})$;
(PCRM) $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$ whenever $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$;
Furthermore, if (PCR2) is replaced by $(R, \overline{c})$ the probabilistic limit space, then the quadruple $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called a *probabilistic limit ring* under the triangle function τ.

The category of probabilistic convergence rings under the triangle function τ and continuous ring homomorphisms is denoted by $\mathbf{PConvRng}_\tau$ while $\mathbf{PLimRng}_\tau$ denotes the category of probabilistic limit rings under the triangle function τ whence objects are probabilistic limit rings under the triangle function τ, and morphisms are continuous ring homomorphisms. Clearly, the category of probabilistic limit rings under the triangle function τ, $\mathbf{PLimRng}_\tau$ is a full subcategory of $\mathbf{PConvRng}_\tau$.

Proposition 4.3. *Let* $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\boldsymbol{PConvRng}_\tau|$. *Then for all* $\varphi, \psi \in \Delta^+$ *and for all filter* $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$ *the following are fulfilled:*
(PCRS) $p \in c_\varphi(\mathbb{F})$ *and* $q \in c_\psi(\mathbb{G})$ *implies* $p - q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \ominus \mathbb{G}) \Longleftrightarrow$
(PCRA) and (PCRI′) $p \in c_\varphi(\mathbb{F}) \Rightarrow -p \in c_\varphi(0 \ominus \mathbb{F})$.

Proof. (PCRS) $\Longrightarrow$ (PCRA) and (PCRI′). Let $\mathbb{F} \in \mathbb{F}(R)$. Then $-\mathbb{F} \in \mathbb{F}(R)$. Let $p \in c_\varphi(\mathbb{F})$. Since $[0] \in c_{\epsilon_0}([0])$, we have by Lemma 4.1(7), $-p = 0 - p \in c_{\tau(\epsilon_0,\varphi)}([0] \ominus \mathbb{F}) = c_\varphi([0] \ominus \mathbb{F})$ and hence $-p \in c_\varphi([0] \ominus \mathbb{F})$, which is (PCRI′). For (PCRA), let $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$. Then since $\mathbb{F} \oplus \mathbb{G} = \mathbb{F} \ominus ([0] \ominus \mathbb{G})$, due to (PCRI′) and (PCRS), we have $p + q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$.

(PCRA) and (PCRI′) $\Longrightarrow$ (PCRS). Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$. Then if $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$, we have $-q \in c_\psi([0] \ominus \mathbb{G})$, and hence $p - q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus ([0] \ominus \mathbb{G})$ which by using Lemma 4.1(7) and (8), we get $p - q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \ominus \mathbb{G})$. □

Lemma 4.4. *Let* $(R, +, \cdot) \in |\boldsymbol{Rng}|$ *and* $(R, \overline{c}) \in |\boldsymbol{PConv}|$. *Then the quadruple* $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ *is a probabilistic convergence ring under the largest triangle function* τ, *i.e.* $\tau(\varphi, \varphi) = \varphi$ *for all* $\varphi \in \Delta^+$, *if and only if* $k\colon \left(R \times R, \overline{c \times c}\right) \longrightarrow (R, \overline{c}), (p, q) \mapsto p - q$, *and* $m\colon \left(R \times R, \overline{c \times c}\right) \longrightarrow (R, \overline{c}), (p, q) \mapsto pq$ *are continuous.*

Proof. Let $\Phi \in \mathbb{F}(R \times R)$, $\varphi \in \Delta^+$ and $(p, q) \in (c \times c)_\varphi(\Phi)$. Then $p = pr_1(p, q) \in c_\varphi(pr_1(\Phi))$ and $q = pr_2(p, q) \in c_\varphi(pr_2(\Phi))$, where pr_1 and pr_2 are projection maps. But then $k(p, q) = p - q \in c_{\tau(\varphi,\varphi)}(k(pr_1(\Phi) \times pr_2(\Phi)))$ by (PCRS). Since $pr_1(\Phi) \times pr_2(\Phi) \leq \Phi$ and $\tau(\varphi, \varphi) = \varphi$, we have by Proposition 4.3, $p - q \in c_\varphi(k(\Phi))$, proving that k is continuous. For the converse part, let $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$. As $\tau(\varphi, \psi) \leq \varphi, \psi$, then $p \in c_{\tau(\varphi,\psi)}(\mathbb{F})$ and $q \in c_{\tau(\varphi,\psi)}(\mathbb{G})$. Hence $(p, q) \in (c \times c)_{\tau(\varphi,\psi)}(\mathbb{F} \times \mathbb{G})$, and therefore, by continuity of k, we get $p - q \in c_{\tau(\varphi,\psi)}(k(\mathbb{F} \times \mathbb{G})) = c_{\tau(\varphi,\psi)}(\mathbb{F} \ominus \mathbb{G})$.

For the continuity of m, we proceed as follows. Let $\Phi \in \mathbb{F}(R \times R)$, $\varphi \in \Delta^+$ and $(p, q) \in (c \times c)_\varphi(\Phi)$. Then $p = pr_1(p, q) \in c_\varphi(pr_1(\Phi))$ and $q = pr_2(p, q) \in c_\varphi(pr_2(\Phi))$, where pr_1 and pr_2 are projection maps. But then $m(p, q) = pq \in c_{\tau(\varphi,\varphi)}(m(pr_1(\Phi) \times pr_2(\Phi)))$ by (PCRM). Since $pr_1(\Phi) \times pr_2(\Phi) \leq \Phi$ and $\tau(\varphi, \varphi) = \varphi$, we have by (PCR2) that $pq \in c_\varphi(m(\Phi))$, proving that m is continuous. For the converse part, let $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$. As $\tau(\varphi, \psi) \leq \varphi, \psi$, then $p \in c_{\tau(\varphi,\psi)}(\mathbb{F})$ and $q \in c_{\tau(\varphi,\psi)}(\mathbb{G})$. Hence $(p, q) \in (c \times c)_{\tau(\varphi,\psi)}(\mathbb{F} \times \mathbb{G})$, and therefore, by continuity of m, we get $pq \in c_{\tau(\varphi,\psi)}(m(\mathbb{F} \times \mathbb{G})) = c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$. □

Example 4.5. Let $(R, +, \cdot)$ be a ring equipped with an indiscrete probabilistic convergence structure given by

$$p \in c_{\epsilon_0}(\mathbb{F}), \forall \mathbb{F} \in \mathbb{F}(R) \text{ and } p \in R.$$

Then the quadruple $(R, +, \cdot, \overline{c})$ is a probabilistic convergence ring under the triangle function τ, called *indiscrete probabilistic convergence ring* under the triangle function τ.

Example 4.6. Let $(R, +, \cdot)$ be a ring equipped with a discrete probabilistic convergence structure given for all $\mathbb{F} \in \mathbb{F}(R)$, $\varphi \in \Delta^+$, $\varphi \neq \epsilon_\infty$, and $p \in R$ by

$$p \in c_\varphi(\mathbb{F}) \Leftrightarrow \mathbb{F} \geq [p].$$

Then the quadruple $(R, +, \cdot, \overline{c})$ is a probabilistic convergence ring under the triangle function τ, called discrete probabilistic convergence ring. In fact, one can easily check that $(R, \overline{c})$ is a discrete probabilistic convergence space. If now $p, r \in R$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, then for $p \in c_\varphi(\mathbb{F})$ and $r \in c_\psi(\mathbb{G})$, $\mathbb{F} \geq [p]$ and $\mathbb{G} \geq [r]$, respectively. Since $\mathbb{F} \ominus \mathbb{G} \geq [p] \ominus [r] = [p - r]$, and $\tau(\varphi, \psi) \leq \varphi, \psi$, we have $p \in c_{\tau(\varphi,\psi)}(\mathbb{F})$ and $r \in c_{\tau(\varphi,\psi)}(\mathbb{G})$. Hence $p - r \in c_{\tau(\varphi,\psi)}(\mathbb{F} \ominus \mathbb{G})$. Condition (PCRM) is shown in a similar way.

Example 4.7. Let λ be the Lebesque measure on $[0, 1]$, and T be usual topology on $\mathfrak{R}$. Consider $R = \{f \colon [0, 1] \longrightarrow \mathfrak{R}; f \text{ measurable}\}$. Then $(R, +, \cdot)$ is a ring under the operations: $(f + g)(p) = f(p) + g(p)$, $(-f)(p) = -(f(p))$, and $(f \cdot g)(p) = f(p)g(p)$.

Define for $\mathbb{F} \in \mathbb{F}(\mathfrak{R})$, $f \in R$, and $\varphi \in \Delta^+$,

$$f \in c_\varphi(\mathbb{F}) \Longleftrightarrow \exists A \subseteq [0, 1] \text{ with } \lambda(A) \leq 1 - \varphi(0^+) \text{ and } \mathbb{F}(p) \to f(p) \;\; \forall p \notin A.$$

Then $(R, \overline{c})$ is a probabilistic convergence space under continuous triangle function τ_* induced by the Lukasiewicz t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$. Note that c_{ϵ_0} describes convergence almost everywhere.

Now we check (PCRA) and (PCRM). For, let $f \in c_\varphi(\mathbb{F})$ and $g \in c_\psi(\mathbb{G})$, there are $A, B \subseteq [0, 1]$ with $\lambda(A) \leq 1 - \varphi(0+)$ and $\lambda(B) \leq 1 - \psi(0+)$ such that $\mathbb{F}(p) \longrightarrow f(p)\, \forall p \notin A$, and $\mathbb{G}(p) \longrightarrow g(p)\, \forall p \notin B$. For $p \notin A \cup B$, we have $(\mathbb{F} \oplus \mathbb{G})(p) = \mathbb{F}(p) \oplus \mathbb{G}(p) \xrightarrow{\mathsf{T}} f(p) + g(p) = (f + g)(p)$ and $(\mathbb{F} \odot \mathbb{G})(p) = \mathbb{F}(p) \odot \mathbb{G}(p) \xrightarrow{\mathsf{T}} f(p) \cdot g(p) = (fg)(p)$ and we have $\lambda(A \cup B) \leq \lambda(A) + \lambda(B) = 1 - \varphi(0+) + 1 - \psi(0+) = 1 - \varphi(0+) * \psi(0+)$, upon using Lukasiewicz t-norm, cf. [24]. Therefore, $f + g \in c_{\varphi(0+)*\psi(0+)}(\mathbb{F} \oplus \mathbb{G})$ and $f \cdot g \in c_{\varphi(0+)*\varphi(0+)}(\mathbb{F} \odot \mathbb{G})$. In view of Lemma 2.3, one obtains: $f + g \in c_{\tau_*(\varphi(0+),\psi(0+))}(\mathbb{F} \oplus \mathbb{G})$ and $f \cdot g \in c_{\tau_*(\varphi(0+),\varphi(0+))}(\mathbb{F} \odot \mathbb{G})$ under continuous t-norm $*$. The missing part, i.e., (PCRI), $f \in c_\varphi(\mathbb{F})$ implies $-f \in c_\varphi(-\mathbb{F})$ follows at ease. Thus, the quadruple $(R, +, \cdot, \overline{c})$ is a probabilistic convergence ring under the continuous triangle function τ_*.

Definition 4.8. ([19]) Let $\left(S, \overline{c^S} = (c^S_\varphi)_{\varphi \in \Delta^+}\right), \left(T, \overline{c^T} = (c^T_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{PCONV}|$, and consider $C(S, T) = \{f : S \longrightarrow T,\ f \text{ is continuous}\}$. The probabilistic convergence structure $\overline{\mathfrak{c}} = (\mathfrak{c}^c_\varphi)_{\varphi \in \Delta^+}$ on $C(S, T)$, called the structure of continuous probabilistic convergence, is defined for $\Phi \in \mathbb{F}(C(S, T))$ and $f \in C(S, T)$ by

$$f \in \mathfrak{c}^c_\varphi(\Phi) \Longleftrightarrow f(p) \in c^T_\psi(ev(\Phi \times F)) \text{ whenever } \psi \leq \varphi \text{ and } p \in c^S_\varphi(\mathbb{F}),$$

where $ev \colon C(S, T) \times S \longrightarrow T, (f, s) \mapsto f(s)$ is the evaluation mapping.

Example 4.9. Let $\tau(\varphi, \varphi) = \varphi$, for all $\varphi \in \Delta^+$, i.e., τ is the largest triangle function. Let $(S, \overline{c})$ be a probabilistic convergence space, and $\left(T, +, \cdot, \overline{d}\right)$ be a probabilistic convergence ring under the largest triangle function τ. Then $(C(S, T), +, \cdot, \overline{\mathfrak{c}})$ is a probabilistic convergence ring under the largest triangle function τ, where for any $f, g \in C(S, T)$, $(f+g)(s) = f(s)+g(s)$, $f \cdot g(s) = f(s)g(s)$, and the inverse, written as $f^*(s) = -f(s)$, for all $s \in S$.

We only check the condition (PCRA). For, let $\Phi, \Psi \in \mathbb{F}(C(S, T)$ and $\mathbb{F} \in \mathbb{F}(R)$. Further, let $f \in \mathfrak{c}^c_\varphi(\Phi)$ and $g \in \mathfrak{c}^c_\psi(\Psi)$. Then for $\gamma \leq \varphi$ and $\gamma \leq \psi$, with $s \in c^S_\gamma(\mathbb{F})$, we get $f(s) \in c^T_\gamma(ev(\Phi \times \mathbb{F}))$ and $g(s) \in c^T_\gamma(ev(\Psi \times \mathbb{F}))$, respectively. Since $\left(T, +, \cdot, \overline{d}\right)$ is a probabilistic convergence ring under the largest triangle function τ, we have by (PCRA),

$$(f+g)(s) = f(s) + g(s) \in c^T_\gamma(ev(\Phi \times \mathbb{F}) \oplus ev(\Psi \times \mathbb{F})), \text{ for } \gamma \leq \tau(\varphi, \varphi) \text{ and } s \in c^S_\gamma(\mathbb{F}).$$

But since $ev(\Phi \times \mathbb{F}) \oplus ev(\Psi \times \mathbb{F}) \leq ev((\Phi \oplus \Psi) \times \mathbb{F})$, we have $f + g \in \mathfrak{c}_{\tau(\varphi,\varphi)}(\Phi \oplus \Psi)$. Similarly, one can prove (PCRM) and (PCRI) by using the inequality $ev(\Phi \times \mathbb{F}) \odot ev(\Psi \times \mathbb{F}) \leq ev((\Phi \odot \Psi) \times \mathbb{F})$, and the equality $ev(-\Phi \times \mathbb{F}) = -ev(\Phi \times \mathbb{F})$, respectively.

Lemma 4.10. *Let $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ be a probabilistic convergence ring and $x \in R$. Then*
(a) the left homothety ${}^x\mathcal{H} \colon (R, \overline{c}) \longrightarrow \left(R', \overline{c'}\right), z \longmapsto xz$ (resp. right homothety $\mathcal{H}^x \colon (R, \overline{c}) \longrightarrow \left(R', \overline{c'}\right), z \longmapsto zx$) is continuous. If x is the unit element of R, then each homothety is a homeomorphism.
(b) the translation $\mathcal{T}_x \colon (R, \overline{c}) \longrightarrow (R, \overline{c}), z \longmapsto z + x$, and the inversion $\jmath \colon (R, \overline{c}) \longrightarrow (R, \overline{c}), z \longmapsto -z$ are homeomorphisms.

Proof. We only check (a). To show that the left homothey ${}^x\mathcal{H}$ is continuous, for $p \in R$, let $p \in c_\varphi(\mathbb{F})$. Since $x \in c_{\epsilon_0}([x])$, one obtains: $xp \in c_{\tau(\varphi,\epsilon_0)}([x] \odot \mathbb{F})$ by using Definition 2.2(ii) and (iv) for triangle function τ. But then it follows that ${}^x\mathcal{H}(p) = xp \in c_\varphi({}^x\mathcal{H}(\mathbb{F}))$, i.e., ${}^x\mathcal{H}(p) \in c_\varphi({}^x\mathcal{H}(\mathbb{F}))$. Considering the right homothety, we have for $p \in c_\varphi(\mathbb{F})$ and $x \in c_{\epsilon_0}([x])$, $\mathcal{H}^x(p) = px \in c_\varphi(\mathcal{H}^x(\mathbb{F})) = c_\varphi(\mathcal{H}^x(\mathbb{F}))$, i.e., $\mathcal{H}^x(p) \in c_\varphi(\mathcal{H}^x(\mathbb{F}))$. All other proofs for missing items can be verified analogously by using definitions. □

Proposition 4.11. ***PConvRng**$_\tau$ is topological over **Rng**.*

Proof. Let $\left(f_j \colon (R, +, \cdot) \longrightarrow \left(R_j, +_j, \cdot_j, (c^j_\varphi)_{\varphi \in \Delta^+}\right)\right)_{j \in J}$ be a source, where each f_j, for $j \in J$ is a ring homomorphism. From the initial probabilistic convergence structure on R under a triangle function τ, it follows from [19]:

$$p \in c_\varphi(\mathbb{F}) \Longleftrightarrow \forall j \in J : f_j(p) \in c^j_\varphi\left(f_j(\mathbb{F})\right), \forall p \in R, \forall \mathbb{F} \in \mathbb{F}(R), \forall \varphi \in \Delta^+.$$

If now $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$, then $f_j(p) \in c^j_\varphi(f_j(\mathbb{F}))$ and $f_j(q) \in c^j_\psi(f_j(\mathbb{G}))$, for all $j \in J$. Hence, by (PCRA), for the spaces $\left(R_j, +_j \cdot_j, \overline{c^j}\right)$, $f_j(p+q) = f_j(p) + f_j(q) \in c^j_{\tau(\varphi,\psi)}\left(f_j(\mathbb{F}) \oplus f_j(\mathbb{G})\right) = c^j_{\tau(\varphi,\psi)}\left(f_j(\mathbb{F} \oplus \mathbb{G})\right)$, for all $j \in J$ which in turn implies that $p + q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$, proving that $(R, +, \cdot, \overline{c})$ satisfies (PCRA). Now we verify (PCRI). If $p \in c_\varphi(\mathbb{F})$, then $f_j(p) \in c^j_\varphi(f_j(\mathbb{F}))$ for all $j \in J$ and hence $f_j(-p) = -(f_j(p)) \in c^j_\varphi\left((f_j(-\mathbb{F})\right) = c^j_\varphi\left(f_j(-\mathbb{F})\right)$. Therefore, $-p \in c_\varphi(-\mathbb{F})$. Final part follows almost the same way as in [3]. □

Lemma 4.12. *Let $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ and $\left(R', +, \cdot, \overline{c'} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ be probabilistic convergence rings under the triangle function τ, and $f \colon R \longrightarrow R'$ is a ring homomorphism. Then $f \colon (R, +, \cdot, \overline{c}) \longrightarrow \left(R', +, \cdot, \overline{c'}\right)$ is continuous if and only if it is continuous at 0.*

Proof. Let us assume that f is continuous at $0 \in R$, $\mathbb{F} \in \mathbb{F}(R)$ and $p \in R$. Now let for any $\varphi \in \Delta^+$, $p \in c_\varphi(\mathbb{F})$. Then $0 = -p + p \in c_\varphi(-[p] \oplus \mathbb{F})$. By continuity at 0, we get $f(0) \in (-[f(p)] \oplus f(\mathbb{F}))$. Then $f(p) = f(0) + f(p) \in c_\varphi(f(\mathbb{F}))$, i.e., $f(p) \in c_\varphi(f(\mathbb{F}))$, proving that f is continuous at p. The converse is obviously true. □

Definition 4.13. Let $(R, +, \cdot)$ be a ring with 0 as its identity, and $\left(c_\varphi \colon \mathbb{F}(R) \longrightarrow P(R)\right)_{\varphi \in \Delta^+}$ be a probabilistic convergence structure on R. Then the pair $\left(R, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ is called *homogeneous* if and only if for each filter $\mathbb{F}$ on R and $p \in R$, $0 \in c_\varphi(\mathbb{F}) \iff p \in c_\varphi([p] \oplus \mathbb{F})$.

Proposition 4.14. *Every probabilistic convergence ring under the triangle function τ is homogeneous.*

Proof. Let $(R, +, \cdot, \overline{c}) \in |\mathbf{PConvRng}_\tau|$, $\mathbb{F} \in \mathbb{F}(R)$ and $p \in R$. Now assume that $0 \in c_\varphi(\mathbb{F})$. As $p \in c_{\epsilon_0}([p])$, we have $p = p + 0 \in c_{\tau(\varphi,\epsilon_0)}([p] \oplus \mathbb{F})$ meaning $p \in c_\varphi([p] \oplus \mathbb{F})$. Conversely, from $p \in c_\varphi([p] \oplus \mathbb{F})$, and $-p \in c_{\epsilon_0}(-[p])$, we have $0 = -p + p \in c_{\tau(\varphi,\epsilon_0)}(-[p] \oplus ([p] \oplus \mathbb{F}))$ implies $0 \in c_\varphi(\mathbb{F})$. □

Theorem 4.15. *Let $(R,+,\cdot) \in |\boldsymbol{Rng}|$, and $\left(R, \overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right) \in |\boldsymbol{PConv}|$. Then the quadruple $\left(R,+,\cdot,\overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right) \in |\boldsymbol{PConvRng}_\tau|$ if and only if the following are satisfied:*
(1) $0 \in c_\varphi([0])$ for all $\varphi \in \Delta^+$;
(2) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$ with $\mathbb{F} \le \mathbb{G}$, $0 \in c_\varphi(\mathbb{F})$ implies $0 \in c_\varphi(\mathbb{G})$, for all $\varphi \in \Delta^+$;
(3) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $0 \in c_\varphi(\mathbb{F})$ and $0 \in c_\psi(\mathbb{G})$ imply $0 \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$, for all $\varphi, \psi \in \Delta^+$;
(4) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in c_\varphi(\mathbb{F})$ implies $0 \in c_\varphi(-\mathbb{F})$, for all $\varphi \in \Delta^+$;
(5) (i) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in c_\varphi(\mathbb{F})$ implies $0 \in c_\varphi([p_0] \odot \mathbb{F})$, for all $p_0 \in R$ and $\varphi \in \Delta^+$;
(ii) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in c_\varphi(\mathbb{F})$ implies $0 \in c_\varphi(\mathbb{F} \odot [p_0])$, for all $p_0 \in R$ and $\varphi \in \Delta^+$;
(6) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $0 \in c_\varphi(\mathbb{F})$ and $0 \in c_\psi(\mathbb{G})$ imply $0 \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, for all $\varphi, \psi \in \Delta^+$;
(7) $\forall \mathbb{F} \in \mathbb{F}(R)$ and $p \in R$, and $\varphi \in \Delta^+$, $p \in c_\varphi(\mathbb{F})$ implies $0 \in c_\varphi(\mathbb{F} \ominus [p])$, for all $\varphi \in \Delta^+$ (resp. $0 \in c_\varphi(\mathbb{F}) \iff p \in c_\varphi(\mathbb{F} \oplus [p])$, for all $\varphi \in \Delta^+$).

Proof. If $\left(R,+,\cdot,\overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right) \in |\mathbf{PConvRng}_\tau|$, then items (1)-(7) are fulfilled. Conversely, first, we show that the condition (PCRA) is true. Assume (1)-(7) are true. Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$. Let $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$. Then by using (7), $0 \in c_\varphi(\mathbb{F} \ominus [p])$ and $0 \in c_\psi(\mathbb{G} \ominus [q])$. By (3) these together imply that

$$0 \in c_{\tau(\varphi,\psi)}\left((\mathbb{F} \ominus [p]) \oplus (\mathbb{G} \ominus [q])\right) = c_{\tau(\varphi,\psi)}\left((\mathbb{F} \oplus \mathbb{G}) \ominus ([p] + [q])\right) = c_{\tau(\varphi,\psi)}\left((\mathbb{F} \oplus \mathbb{G}) \ominus [p+q]\right)$$

which implies that $p + q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$. Next, we check condition (PCRM). For, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, and $p_0 \in c_\varphi(\mathbb{F})$ and $q_0 \in c_\psi(\mathbb{G})$. In view of (7), $0 \in c_\varphi(\mathbb{F} \ominus [p_0])$ and $0 \in c_\psi(\mathbb{G} \ominus [q_0])$. Then by (6), $0 \in c_{\tau(\varphi,\psi)}\left((\mathbb{F} \ominus [p_0]) \odot (\mathbb{G} \ominus [q_0])\right)$. As $0 \in c_\varphi(\mathbb{F} \ominus [p_0])$ and $0 \in c_\psi(\mathbb{G} \ominus [q_0])$, using (5)(i), $0 \in c_\psi([p_0] \odot (\mathbb{G} \ominus [q_0]))$ and by 5(ii), $0 \in c_\varphi((\mathbb{F} \ominus [p_0]) \odot [q_0])$ which upon using (3), imply that $0 \in c_{\tau(\varphi,\psi)}\left(([p_0] \odot (\mathbb{G} \ominus [q_0])) \oplus ((\mathbb{F} \ominus [p_0]) \odot [q_0])\right)$. Thus, upon using (1) and (3), we get

$$0 \in c_{\tau(\varphi,\psi)}\left(((\mathbb{F} \ominus [p_0]) \odot (\mathbb{G} \ominus [q_0])) \oplus ([p_0] \odot (\mathbb{G} \ominus [q_0]))\right) \oplus \left((\mathbb{F} \ominus [p_0]) \odot ([q_0] \oplus [p_0 q_0] \ominus [p_0 q_0])\right)$$

which implies that

$$0 \in c_{\tau(\varphi,\psi)}\left(((\mathbb{F} \ominus [p_0]) \oplus [p_0]) \odot ((\mathbb{G} \ominus [q_0]) \oplus [q_0]) \ominus [p_0 q_0]\right),$$

that is, $p_0 q_0 \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$. This end the proof of the theorem. □

Theorem 4.16. *Let $(R,+,\cdot)$ be a ring, and let $\left(d_\varphi \colon \mathbb{F}(R) \longrightarrow P(X)\right)_{\varphi\in\Delta^+}$ be a family of mappings such that the following conditions are fulfilled:*
(1) $0 \in d_\varphi([0])$ for all $\varphi \in \Delta^+$;
(2) (i) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$ with $\mathbb{F} \le \mathbb{G}$, $0 \in d_\varphi(\mathbb{F})$ implies $0 \in d_\varphi(\mathbb{G})$, for all $\varphi \in \Delta^+$;
(ii) if $\varphi \le \psi$, then for $0 \in d_\psi(\mathbb{F})$ implies $0 \in d_\varphi(\mathbb{F})$, for all $\mathbb{F} \in \mathbb{F}(R)$;
(3) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $0 \in d_\varphi(\mathbb{F}) \cap d_\psi(\mathbb{G})$ imply $0 \in d_{\tau(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$, for all $\varphi, \psi \in \Delta^+$;
(4) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in d_\varphi(\mathbb{F})$ implies $0 \in d_\varphi(-\mathbb{F})$, for all $\varphi \in \Delta^+$;
(5) (i) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in d_\varphi(\mathbb{F})$ implies $0 \in d_\varphi([p_0] \odot \mathbb{F})$, for all $p_0 \in R$ and $\varphi \in \Delta^+$;
(ii) $\forall \mathbb{F} \in \mathbb{F}(R)$, $0 \in d_\varphi(\mathbb{F})$ implies $0 \in d_\varphi(\mathbb{F} \odot [p_0])$, for all $p_0 \in R$ and $\varphi \in \Delta^+$;
(6) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $0 \in d_\varphi(\mathbb{F}) \cap d_\psi(\mathbb{G})$ implies $0 \in d_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, for all $\varphi \in \Delta^+$.
Then there exists a unique probabilistic convergence structure $\overline{c} = (c_\varphi)_{\varphi\in\Delta^+}$ on R that satisfies $0 \in c_\varphi(\mathbb{F}) \iff 0 \in d_\varphi(\mathbb{F})$, for all $\mathbb{F} \in \mathbb{F}(R)$ such that the quadruple $\left(R,+,\cdot,\overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right)$ is a homogeneous probabilistic convergence ring.

Proof. Define the mapping $c_\varphi \colon \mathbb{F}(R) \longrightarrow P(R)$ by

$$p \in c_\varphi(\mathbb{F}) \Leftrightarrow 0 \in d_\varphi(\mathbb{F} \ominus [p]), \text{ for all } \mathbb{F} \in \mathbb{F}(R) \text{ and } p \in R.$$

We need to verify the following:
(PC1) In view of (1), $0 \in d_\varphi([0])$ implies $0 \in d_\varphi([p] \ominus [p])$ implies $p \in c_\varphi([p])$.
(PC2) Let $\mathbb{F} \le \mathbb{G}$, and for any $\varphi \in \Delta^+$, let $p \in c_\varphi(\mathbb{F})$. Then $0 \in d_\varphi(\mathbb{F} \ominus [p])$, so, by (2)(i), $0 \in d_\varphi(\mathbb{G} \ominus [p])$ which in turn yields that $p \in c_\varphi(\mathbb{G})$.
(PC3) Let $\varphi, \psi \in \Delta^+$ with $\varphi \le \psi$. If then $p \in c_\psi(\mathbb{F})$, for $\mathbb{F} \in \mathbb{F}(R)$, then $0 \in d_\psi(\mathbb{F} \ominus [p])$ implies by (2)(ii),

$0 \in d_\varphi(\mathbb{F} \ominus [p])$ implies $p \in c_\varphi(\mathbb{F})$.
(PC4) Clearly for any $\mathbb{F} \in \mathbb{F}(R)$, $p \in c_{\epsilon_\infty}(\mathbb{F})$.
Hence $(R, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+})$ is a probabilistic convergence space. That the conditions (PCRA), (PCRI) and (PCRM) follow from the preceding theorem. The uniqueness follows from the construction while the homogeneity is obviously true. Consequently, for any $\mathbb{F} \in \mathbb{F}(R)$ and $p \in R$,

$$0 \in c_\varphi(\mathbb{F}) \Leftrightarrow p \in c_\varphi(\mathbb{F} \oplus [p]) \Leftrightarrow 0 \in d_\varphi\left((\mathbb{F} \oplus [p]) \ominus [p]\right) \Leftrightarrow 0 \in d_\varphi(\mathbb{F}).$$

□

5. Example: Richardson-Kent probabilistic convergence rings

Definition 5.1. ([19, 32]) Let R be a set and $\overline{q} = (\{q_\lambda : \mathbb{F}(R) \longrightarrow P(R)\})_{\lambda \in [0,1]}$ be a family of maps such that
(PCR1*) $p \in q_\alpha([p])$ for all $\alpha \in [0,1]$ and $p \in R$;
(PCR2*) $q_\alpha(\mathbb{F}) \subseteq q_\alpha(\mathbb{G})$ whenever $\mathbb{F} \leq \mathbb{G}$;
(PCR3*) $q_\beta(\mathbb{F}) \subseteq q_\alpha(\mathbb{F})$ whenever $\alpha \leq \beta$;
(PCR4*) $q_0(\mathbb{F}) = R$.
Then the pair $(R, \overline{q})$ is called a *Richardson-Kent probabilistic convergence space*.
If, moreover, the axiom (PCR5*)

$$q_\alpha(\mathbb{F}) \cap q_\alpha(\mathbb{G}) \subseteq q_\alpha(\mathbb{F} \wedge \mathbb{G}), \forall \alpha \in [0,1], \text{ and } \forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(R),$$

is satisfied, then $(R, \overline{q})$ is called a *Richardson-Kent probabilistic limit space*. A map $f : (R, \overline{q}) \longrightarrow \left(R', \overline{q'}\right)$ between probabilistic convergence spaces is called *continuous* if and only if for all $\alpha \in [0,1]$, $\mathbb{F} \in \mathbb{F}(R)$ and for all $p \in R$, $p \in q_\alpha(\mathbb{F})$ implies $f(p) \in q'_\alpha(f(\mathbb{F}))$.

RK-PConv denotes the category of all Richardson-Kent probabilistic convergence spaces and continuous maps.

Definition 5.2. A probabilistic convergence ring under a t-norm $*$ *is a quadruple*
$(R, +, \cdot, \overline{q} = (q_\alpha)_{\alpha \in [0,1]})$ *such that*
(PCR1) $(R, +, \cdot)$ *is a ring;*
(PCR2)* $(R, \overline{q} = (q_\alpha)_{\alpha \in [0,1]})$ *is a Richardson-Kent probabilistic convergence space;*
(PCRA)* $p + q \in q_{\alpha * \beta}(\mathbb{F} \oplus \mathbb{G})$ *whenever* $p \in q_\alpha(\mathbb{F})$ *and* $q \in q_\beta(\mathbb{G})$*;*
(PCRI)* $-p \in q_\alpha(-\mathbb{F})$ *whenever* $p \in q_\alpha(\mathbb{F})$*.*
(PCRM)* $pq \in q_{\alpha * \beta}(\mathbb{F} \odot \mathbb{G})$ *whenever* $p \in q_\alpha(\mathbb{F})$ *and* $q \in q_\beta(\mathbb{G})$*;*

We denote by **PConvRng*** the category of probabilistic convergence rings under a t-norm $*$ and continuous ring homomorphisms.

Given $(R, +, \cdot, \overline{q}) \in |$**PConvRng***$|$, we define $p \in c^{\overline{q}}_\varphi(\mathbb{F})$ iff $p \in q_{\varphi(0^+)}(\mathbb{F})$, $\forall \mathbb{F} \in \mathbb{F}(R)$ and $p \in R$.

Lemma 5.3. *Let* $(R, +, \cdot, \overline{q} = (q_\alpha)_{\alpha \in [0,1]}) \in |$***PConvRng****$|$ *with a continuous t-norm* $*$*. Then* $\left(R, +, \cdot, \overline{c}^{\overline{q}} = (c^{\overline{q}}_\varphi)_{\varphi \in \Delta^+}\right) \in |$***PConvRng***$_{\tau_*}|$.

Proof. In view of [Example 3.3, [19]], we only show that condition (PCRA*) of the Definition 5.2 implies condition (PCRA) of the Definition 4.2, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, and $\varphi, \psi \in \Delta^+$. If $p \in c^{\overline{q}}_\varphi(\mathbb{F})$ and $r \in c^{\overline{q}}_\psi(\mathbb{G})$, then we have $p \in q_{\varphi(0^+)}(\mathbb{F})$ and $r \in q_{\psi(0^+)}(\mathbb{G})$. From (PCRA*), it follows that $p + r \in q_{\varphi(0^+) * \psi(0^+)}(\mathbb{F} \oplus \mathbb{G}) = q_{\tau_*(\varphi,\psi)(0^+)}(\mathbb{F} \oplus \mathbb{G})$. Hence $p + r \in c^{\overline{q}}_{\tau_*(\varphi,\psi)}(\mathbb{F} \oplus \mathbb{G})$; the missing parts follow in a similar way, cf. [3]. □

Lemma 5.4. *Let* $(R,+,\cdot,\overline{q}=(q_\alpha)_{\alpha\in[0,1]}),\left(R',+,\cdot,\overline{q'}=(q'_\alpha)_{\alpha\in[0,1]}\right)\in|\boldsymbol{PConvRng}^*|$ *with a continuous t-norm* $*$, *and let* $f\colon (R,+,\cdot)\longrightarrow(R',+,\cdot,)$ *be a ring homomorphism.*
If $f\colon (R,+,\cdot,\overline{q}=(q_\alpha)_{\alpha\in[0,1]})\longrightarrow\left(R',+,\cdot,\overline{q'}=(q'_\alpha)_{\alpha\in[0,1]}\right)$ *is continuous,*
then $f\colon \left(R,+,\cdot,(c^{\overline{q}}_\varphi)_{\varphi\in\Delta^+}\right)\longrightarrow\left(R',+,\cdot,(c^{\overline{q'}}_\varphi)_{\varphi\in\Delta^+}\right)$ *is continuous.*

Remark 5.5. Lemmas 5.3 and 5.4 show that

$$\mathfrak{E}:\begin{cases}\textbf{PConvRng}^* & \longrightarrow & \textbf{PConvRng}_{\tau^*}\\ (R,+,\cdot,\overline{q}) & \longmapsto & \left(R,+,\cdot,\overline{c^{\overline{q}}}\right)\\ f & \longmapsto & f,\end{cases}$$

is a functor from $\textbf{PConvRng}^*$ to $\textbf{PConvRng}_{\tau^*}$. This functor is injective on objects and hence is an embedding. Indeed, if $(R,+,\cdot,\overline{q})\neq(R,+,\cdot,\overline{q'})$, then there are $\alpha\in[0,1], p\in R$ and $\mathbb{F}\in\mathbb{F}(R)$ such that $p\in q_\alpha(\mathbb{F})$ but $p\notin q'_\alpha(\mathbb{F})$. Then $\varphi_\alpha\in\Delta^+$ defined by $\varphi_\alpha(x)=\alpha$ for $0<x<\infty$ shows that $p\in c^{\overline{q}}_{\varphi_\alpha}(\mathbb{F})$ but $p\notin c^{\overline{q'}}_{\varphi_\alpha}(\mathbb{F})$, i.e., $(R,+,\cdot,\overline{c^{\overline{q}}})\neq(R,+,\cdot,\overline{c^{\overline{q'}}})$.

For $(R,+,\cdot,\overline{c})\in|\textbf{PConvRng}_{\tau_*}|$, define $p\in q^{\overline{c}}_\alpha(\mathbb{F})$ iff $\exists\varphi\in\Delta^+$ such that $\varphi(0^+)=\alpha$ and $p\in c_\varphi(\mathbb{F})$.

Lemma 5.6. *Let* $*$ *be a continuous t-norm and let* $(R,+,\cdot,\overline{c})\in|\boldsymbol{PConvRng}_{\tau_*}|$.
Then $\left(R,+,\cdot,\overline{q}^{\overline{c}}=(q^{\overline{c}}_\alpha)_{\alpha\in[0,1]}\right)\in|\boldsymbol{PConvRng}^*|$.

Proof. Here, we only show that (PCRA) implies (PCRA*). In fact, if for any $\mathbb{F},\mathbb{G}\in\mathbb{F}(R)$, and $\alpha,\beta\in[0,1]$, $p\in q^{\overline{c}}_\alpha(\mathbb{F})$ and $r\in q^{\overline{c}}_\beta(\mathbb{G})$, there are $\varphi,\psi\in\Delta^+$ such that $\varphi(0^+)=\alpha$ and $\psi(0^+)=\beta$ with $p\in c_\varphi(\mathbb{F})$ and $r\in c_\psi(\mathbb{G})$, respectively. Consequently, there are $\varphi,\psi\in\Delta^+$ such that $\varphi(0^+)*\psi(0^+)=\alpha*\beta$ with $p\in c_\varphi(\mathbb{F})$ and $r\in c_\psi(\mathbb{G})$. But then by (PCRA) we get that $p+r\in c_{\tau_*(\varphi,\psi)}(\mathbb{F}\oplus\mathbb{G})$, and hence, because of $\tau_*(\varphi,\psi)(0+)=\varphi(0^+)*\psi(0^+)=\alpha*\beta$, we obtain $p+r\in q_{\alpha*\beta}(\mathbb{F}\oplus\mathbb{G})$. □

Lemma 5.7. *Let* $\left(R,+,\cdot,\overline{c}=(c_\varphi)_{\varphi\in\Delta^+}\right),\left(R',+,\cdot,\overline{c'}=(c'_\varphi)_{\varphi\in\Delta^+}\right)\in|\boldsymbol{PConvRng}_{\tau_*}|$ *and* $*$ *be a continuous t-norm, and let* $f\colon (R,+,\cdot)\longrightarrow(R',+,\cdot)$ *be a ring homomorphism.*
If $f\colon \left(R,+,\cdot,\overline{c}=(c_\varphi)_{\varphi\in\Delta^+}\right)\longrightarrow\left(R',+,\cdot,\overline{c'}=(c'_\varphi)_{\varphi\in\Delta^+}\right)$ *is continuous,*
then $f\colon \left(R,+,\cdot,(q^{\overline{c}}_\alpha)_{\alpha\in[0,1]}\right)\longrightarrow\left(R',+,\cdot,(q^{\overline{c'}}_\alpha)_{\alpha\in[0,1]}\right)$ *is continuous.*

Due to Lemmas 5.6 and 5.7, there is a functor

$$\mathfrak{K}:\begin{cases}\textbf{PConvRng}_{\tau_*} & \longrightarrow & \textbf{PConvRng}^*\\ (R,+,\cdot,\overline{c}) & \longmapsto & \left(R,+,\cdot,\overline{q^{\overline{c}}}\right)\\ f & \longmapsto & f.\end{cases}$$

Theorem 5.8. $\boldsymbol{PConvRng}^*$ *is a reflective subcategory of* $\boldsymbol{PConvRng}_{\tau_*}$.

Proof. From Lemmas 5.3 and 5.4, we obtain the embedding $\textbf{PConvRng}^*\xrightarrow{\mathfrak{E}}\textbf{PConvRng}_{\tau_*}$. Likewise, from Lemmas 5.6 and 5.7 we have the functor $\textbf{PConvRng}_{\tau_*}\xrightarrow{\mathfrak{K}}\textbf{PConvRng}^*$. It follows that $\mathfrak{K}\circ\mathfrak{E}=id_{\textbf{PConvRng}^*}$, while $\mathfrak{E}\circ\mathfrak{K}\geq id_{\textbf{PConvRng}_{\tau_*}}$. □

6. Probabilistic uniformizability of probabilistic convergence rings

Definition 6.1. ([2]) A pair $\left(R,\overline{\Lambda}=\left(\Lambda_\varphi\right)_{\varphi\in\Delta^+}\right)$ is called a *probabilistic uniform convergence space* under the triangle function τ, where $\Lambda_\varphi\subseteq\mathbb{F}(R\times R)$ is such that the following conditions are fulfilled:

(PUC1) $[(p,p)] \in \Lambda_\varphi$, $\forall p \in R$, $\forall \varphi \in \Delta^+$;
(PUC2) $\Phi \in \Lambda_\varphi$, $\Psi \geq \Phi$ implies $\Psi \in \Lambda_\varphi$;
(PUC3) $\Phi, \Psi \in \Lambda_\varphi$ implies $\Phi \wedge \Psi \in \Lambda_\varphi$;
(PUC4) $\Phi \in \Lambda_\varphi$ implies $\Phi^{-1} \in \Lambda_\varphi$;
(PUC5) $\Phi \in \Lambda_\varphi$ and $\Psi \in \Lambda_\psi$ such that $\Phi \circ \Psi \in \mathbb{F}(R \times R)$ implies $\Phi \circ \Psi \in \Lambda_{\tau(\varphi,\psi)}$;
(PUC6) $\varphi \leq \psi$ implies $\Lambda_\psi \subseteq \Lambda_\varphi$;
(PUC7) $\Lambda_{\epsilon_\infty} = \mathbb{F}(R \times R)$.
A mapping $f\colon \left(R, \overline{\Lambda}\right) \longrightarrow \left(R', \overline{\Lambda'}\right)$ between probabilistic uniform convergence spaces under the triangle functions τ is called *uniformly continuous* if for all $\varphi \in \Delta^+$, $\forall \Phi \in \mathbb{F}(R \times R)$, $\Phi \in \Lambda_\varphi$ implies $(f \times f)(\Phi) \in \Lambda'_\varphi$.

If $\left(R, \overline{\Lambda}\right)$ is a probabilistic uniform convergence space under the triangle function τ, then the probabilistic convergence structure $c^{\overline{\Lambda}}$ is defined by $p \in c^{\Lambda}_\varphi(\mathbb{F}) \Longleftrightarrow \mathbb{F} \times [p] \in \Lambda_\varphi$ [2].

It follows from [2] that every probabilistic metric space under the continuous triangle function τ can serve as a natural example of probabilistic uniform convergence space under τ.

The category of probabilistic uniform convergence spaces under the triangle function τ, and uniformly continuous mappings is denoted by $\mathbf{PUConv}_\tau$.

Let $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right) \in |\mathbf{PConvRng}_\tau|$. Define a mapping $\omega\colon R \times R \longrightarrow R$, by $\omega(p,q) = p - q$.

Lemma 6.2. *Let $(R,+)$, and $(R',+')$ be additive groups, and $f\colon R \longrightarrow R'$ be group homomorphism. Then for any $p \in R$, $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $\Phi, \Psi \in \mathbb{F}(R \times R)$, we have*
(1) $\omega([p],[p]) = [0]$;
(2) $\omega(\mathbb{F} \times \mathbb{G}) = \mathbb{F} \ominus \mathbb{G}$;
(3) $\omega(-\Phi) = -\omega(\Phi)$;
(4) $\omega(\Phi) \oplus \omega(\Psi) \leq \omega(\Phi \circ \Psi)$;
(5) $f(\omega(\Phi)) = \omega(f \times f(\Phi))$.

Proof. (1) $\omega([p],[p]) = ([p] \ominus [p]) = [0]$;

(2) $\omega(\mathbb{F} \times \mathbb{G}) = \{\omega(F \times G)\colon F \in \mathbb{F}, G \in \mathbb{G}\} = \{F - G\colon F \in \mathbb{F}, G \in \mathbb{G}\} = \mathbb{F} \ominus \mathbb{G}$.

(3) Let $H \subseteq R \times R$. Then $z \in -\omega(H)$ if and only if $z = -u$ with $u \in \omega(H)$ if and only if there exists $(x,y) \in H$ such that $z = -u = -(x-y) = y - x = \omega(y,x)$, i.e., $z \in \omega(-H)$.

(4) Let $H \in \omega(\Phi)$ and $K \in \omega(\Psi)$. We show that $\omega(H \circ K) \subseteq \omega(H) + \omega(K)$. Let $z \in \omega(H \circ K)$, then there is $(x,y) \in H \circ K$ such that $z = x - y$; hence there is $u \in R$ such that $(x,u) \in K$ and $(u,y) \in H$ such that $z = x - y$. Consequently, $\omega(x,u) + \omega(u,y) = x - u + u - y = x - y = z$ so that $z \in \omega(H) + \omega(K)$. □

Theorem 6.3. *Every probabilistic limit ring under the triangle function τ is a probabilistic uniform convergence space under the triangle function τ.*

Proof. Let $\left(R, +, \cdot, \overline{c} = (c_\varphi)_{\varphi \in \Delta^+}\right)$ be a probabilistic limit ring under the triangle function τ. Define Λ^c_φ by $\Phi \in \Lambda^c_\varphi \Leftrightarrow 0 \in c_\varphi(\omega(\Phi))$. We show that $\left(R, \overline{\Lambda} = (\Lambda^c_\varphi)_{\varphi \in \Delta^+}\right)$ is a probabilistic uniform convergence space under the triangle function τ. (PUC1)-(PUC4) and (PUC7) are easy to prove. We only check (PUC5) and (PUC6). To prove (PUC5), let $\Phi \in \Lambda^c_\varphi$ and $\Psi \in \Lambda^c_\psi$ such that $\Phi \circ \Psi \in \mathbb{F}(R \times R)$. Then $0 \in c_\varphi(\omega(\Phi))$ and $0 \in c_\psi(\omega(\Psi))$. These together imply that $0 = 0 + 0 \in c_{\tau(\varphi,\psi}(\omega(\Phi) \oplus \omega(\Psi))$. But then by the preceding Lemma 6.2, $0 \in c_{\tau(\varphi,\psi)}(\omega(\Phi \circ \Psi))$ implies $\Phi \circ \Psi \in \Lambda^c_{\tau(\varphi,\psi)}$. For (PUC6), we let $\varphi, \psi \in \Delta^+$ with $\varphi \leq \psi$, and let $\Phi \in \Lambda^c_\psi$. Then $0 \in c_\psi(\omega(\Phi))$ implies $0 \in c_\varphi(\omega(\Psi))$ and this yields that $\Phi \in \Lambda^c_\varphi$. □

Proposition 6.4. *Let $(R,+,\cdot,\overline{c})$, $(R',+',\cdot',\overline{c'})$ be probabilistic convergence rings under the triangle function τ, and $f\colon (R,+,\cdot) \longrightarrow (R',+',\cdot')$ be a ring homomorphism. Then the following statements are equivalent:*
(a) $f\colon (R,\overline{c}) \longrightarrow (R',\overline{c'})$ is continuous;
(b) $f\colon (R,\overline{\Lambda^c}) \longrightarrow (R,\overline{\Lambda^{c'}})$ is uniformly continuous.

Proof. Let $f\colon (R,\overline{c}) \longrightarrow (R',\overline{c'})$ be continuous, and $\Phi \in \Lambda^c_\varphi$. Then $0 \in c_\varphi(\omega(\Phi))$ and hence $0 = f(0) \in c'_\varphi(f(\omega(\Phi))) = c_\varphi(\omega(f \times f(\Phi)))$. Hence $(f \times f)(\Phi) \in \Lambda^{c'}_\varphi$.

Conversely, assume that $f\colon (R,\overline{\Lambda^c}) \longrightarrow (R,\overline{\Lambda^{c'}})$ is uniformly continuous, and $p \in c_\varphi(\mathbb{F})$. Then $0 \in (\mathbb{F}\ominus[p]) = c_\varphi(\omega(\mathbb{F}\times[p]))$ implies $\mathbb{F}\times[p] \in \Lambda^c_\varphi$ implies $(f\times f)(\mathbb{F}\times[p]) \in \Lambda^{c'}_\varphi$. This implies $0 \in c'_\varphi(\omega(f(\mathsf{F})\times[f(p)]))$ implying $0 \in c'_\varphi(\omega(f(\mathbb{F}) \ominus [f(p)]))$ which implies that $f(p) \in c'_\varphi(f(\mathbb{F}))$. □

Then there is a functor

$$\mathfrak{B} : \begin{cases} \mathbf{PLimRng}_\tau & \longrightarrow & \mathbf{PUConv}_\tau \\ (R,+,\cdot,\overline{c}) & \longmapsto & (R,\overline{\Lambda^c}) \\ f & \longmapsto & f \end{cases}$$

Proposition 6.5. *The functor $\mathfrak{B}$ preserves initial constructions.*

Proof. Let $(f_j : (R,+,\cdot) \longrightarrow (R_j,+_j,\cdot_j,\overline{c^j}))_{j\in J}$ be a family of ring homeomorphisms and denote the initial structure on R with respect to this source by $init(\overline{c^j})$. Then $(f_j : R \longrightarrow (R_j,\overline{\Lambda^{\overline{c^j}}}))_{j\in J}$ is a source in $\mathbf{PUConv}_\tau$ and we denote the initial structure on R with respect to this source by $init(\overline{\Lambda^{\overline{c^j}}})$.

We further define for $j \in J$ the mapping $\omega_j : R_j \times R_j \longrightarrow R_j$, $(p_j,q_j) \longmapsto p_j -_j q_j$, and we denote $\omega : R \times R \longrightarrow R$, $(p,q) \longmapsto p - q$. It is not difficult to see that $f_j \circ \omega = \omega_j \circ (f_j \times f_j)$. Hence, $\Phi \in init(\overline{\Lambda^{\overline{c^j}}})_\varphi$ if, and only if, $(f_j \times f_j)(\Phi) \in \Lambda^{\overline{c^j}}_\varphi$ for all $j \in J$. This is equivalent to $0 = f_j(0) \in c^j_\varphi(\omega_j((f_j \times f_j)(\Phi))) = c^j_\varphi(f_j(\omega(\Phi)))$ for all $j \in J$, i.e., to $0 \in init(\overline{c^j})_\varphi(\omega(\Phi))$ which means to $\Phi \in \Lambda^{init(\overline{c^j})}_\varphi$. □

7. Category of probabilistic Cauchy rings and its relationship with the category of probabilistic limit rings

Definition 7.1. A pair $\left(R, \overline{\mathfrak{p}} = (\mathfrak{p}_\varphi)_{\varphi\in\Delta^+}\right)$ is called a *probabilistic Cauchy space* under the triangle function τ if the following conditions are fulfilled:
(PChy1) $[p] \in \mathfrak{p}_\varphi$ for all $p \in R$ and $\varphi \in \Delta^+$;
(PChy2) $\mathbb{F} \in \mathfrak{p}_\varphi$ and $\mathbb{F} \le \mathbb{G}$, implies $\mathbb{G} \in \mathfrak{p}_\varphi$;
(PChy3) $\varphi \le \psi$, $\mathbb{F} \in \mathfrak{p}_\psi$ implies $\mathbb{F} \in \mathfrak{p}_\varphi$;
(PChy4) $\mathfrak{p}_{\epsilon_\infty} = \mathbb{F}(R)$;
(PChy5) $\mathbb{F} \in \mathfrak{p}_\varphi$, $\mathbb{G} \in \mathfrak{p}_\psi$, $\mathbb{F} \vee \mathbb{G}$ exits, implies $\mathbb{F} \wedge \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$.
A mapping $f\colon (R,\overline{\mathfrak{p}}) \longrightarrow (R',\overline{\mathfrak{p}'})$ is called probabilistic *Cauchy-continuous* if for all $\mathbb{F} \in \mathbb{F}(R)$, for all $\varphi \in \Delta^+$, $\mathbb{F} \in \mathfrak{p}_\varphi$ implies $f(\mathbb{F}) \in \mathfrak{p}'_\varphi$.

The category of all probabilistic Cauchy spaces under the triangle function τ and Cauchy-continuous mappings is denoted by $\mathbf{PChy}_\tau$. Given a probabilistic uniform convergence space (R,Λ) under the triangle function τ, we define probabilistic Cauchy structure $\overline{\mathfrak{p}}$ by $\mathbb{F} \in \mathfrak{p}^\Lambda_\varphi \Leftrightarrow \mathbb{F} \times \mathbb{F} \in \Lambda_\varphi$, for all $\varphi \in \Delta^+$.

Definition 7.2. Let $\left(R, \overline{\mathfrak{p}} = (\mathfrak{p}_\varphi)_{\varphi\in\Delta^+})\right)$ be a probabilistic Cauchy space under the triangle function τ. Then a quadruple $\left(R, +, \cdot, \overline{\mathfrak{p}} = (\mathfrak{p}_\varphi)_{\varphi\in\Delta^+})\right)$ is called a *probabilistic Cauchy ring* under the triangle function τ provided the following are satisfied.
(PChyR) $(R, +, \cdot)$ is a ring;
(PChyRA) $\forall\varphi, \psi \in \Delta^+$, $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $\mathbb{F} \in \mathfrak{p}_\varphi, \mathbb{G} \in \mathfrak{p}_\psi$, $\mathbb{F} \oplus \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$;
(PChyRI) $\forall\varphi \in \Delta^+$, $\forall\mathbb{F} \in \mathbb{F}(R)$, $\mathbb{F} \in \mathfrak{p}_\varphi$, $-\mathbb{F} \in \mathfrak{p}_\varphi$;
(PChyRM) $\forall\varphi, \psi \in \Delta^+$, $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, $\mathbb{F} \in \mathfrak{p}_\varphi, \mathbb{G} \in \mathfrak{p}_\psi$, $\mathbb{F} \odot \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$.
The category of probabilistic Cauchy rings under the triangle function τ and Cauchy-continuous ring homomorphisms is denoted by $\mathbf{PChyRng}_\tau$.

Proposition 7.3. *Let $(R, +, \cdot, \overline{\mathfrak{p}}) \in |\mathbf{PChyRng}_\tau|$. Then for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, and for all $\varphi, \psi \in \Delta^+$ the following are fulfilled.*
(PChyRS) $\mathbb{F} \in \mathfrak{p}_\varphi$ and $\mathbb{G} \in \mathfrak{p}_\psi$ implies $\mathbb{F} \ominus \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)} \Longleftrightarrow$
(PChyRA), and (PChyRI′) $\mathbb{F} \in \mathfrak{p}_\varphi$ implies $([0] \ominus \mathbb{F}) \in \mathfrak{p}_\varphi$.

Proof. (PChyRS) $\Longrightarrow$ (PChyA) and(PChyRI′). Let $\mathbb{F} \in \mathbb{F}(R)$. Then $-\mathbb{F} \in \mathbb{F}(R)$, and so, let $\mathbb{F} \in \mathfrak{p}_\varphi$. Since $[0] \in \mathfrak{p}_{\epsilon_0}$, we have by Lemma 4.1(7), $([0] \ominus \mathbb{F}) \in \mathfrak{p}_{\tau(\epsilon_0,\varphi)} = \mathfrak{p}_\varphi$ and hence $([0] \ominus \mathbb{F}) \in \mathfrak{p}_\varphi$ which is (PChyRI′). For (PChyRA), let $\mathbb{F} \in \mathfrak{p}_\varphi$ and $\mathbb{G} \in \mathfrak{p}_\psi$. Since $\mathbb{F} \oplus \mathbb{G} = \mathbb{F} \ominus ([0] \ominus \mathbb{G})$, due to (PChyRI′) and (PChyRS), we have $\mathbb{F} \oplus \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$.

(PChyRA) and (PChyRI′) $\Longrightarrow$ (PChyRS). Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$. Then if $\mathbb{F} \in \mathfrak{p}_\varphi$ and $\mathbb{G} \in \mathfrak{p}_\psi$, then as $\mathbb{G} \in \mathfrak{p}_\psi$ and $([0] \ominus \mathbb{G}) \in \mathfrak{p}_\varphi$, by applying symmetry of τ in Definition 2.2(ii), one obtains: $\mathbb{F} \oplus ([0] \ominus \mathbb{G}) \in \mathfrak{p}_{\tau(\varphi,\psi)}$ which by using Lemma 4.1(7) and (8), we get $\mathbb{F} \ominus \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$. □

Example 7.4. In view of [21], the category of probabilistic Cauchy spaces under the largest triangle function τ is a Cartesian closed; therefore, it has function spaces structure. Let $\left(S, \overline{\mathfrak{p}} = (\mathfrak{p}_\varphi)_{\varphi\in\Delta^+}\right) \in |\mathbf{PChy}_\tau|$, and $\left(T, +, \cdot, \overline{\mathfrak{p}'} = (\mathfrak{p}'_\varphi)_{\varphi\in\Delta^+}\right) \in |\mathbf{PChyRng}_\tau|$. Consider $\mathsf{C}(S,T) = \{f\colon (S, \overline{\mathfrak{p}}) \longrightarrow (R, +, \cdot, \overline{\mathfrak{p}'}); f$ is Cauchy-continuous $\}$. Then one can check that $(\mathsf{C}(S,T), +, \cdot)$ is a ring (for classical case, cf. [6, 17]). We show that $(\mathsf{C}(S,T), +, \cdot, \overline{\mathfrak{c}})$ is a probabilistic Cauchy ring under the largest triangle function τ. In accordance to the Proposition 3.2[21], define for $\Phi \in \mathbb{F}(\mathsf{C}(S,T))$,

$$\Phi \in \mathfrak{c}_\varphi^{\mathsf{C}(S,T)} \Leftrightarrow \forall\psi \leq \varphi \;\; \forall\mathbb{F} \in \mathfrak{p}_\psi : ev(\Phi \times \mathbb{F}) \in \mathfrak{p}'_\psi$$

where $ev\colon \mathsf{C}(S,T) \times S \longrightarrow T, (f, x) \longmapsto f(x)$, is the evaluation mapping. Following the Proposition 3.2[21](see also, [22]), one can prove that $(\mathsf{C}(S,T), \overline{\mathfrak{c}})$ is a probabilistic Cauchy space under the largest triangle function τ. It suffices to check conditions (PChyRS) and (PChyRM). Fist, let us verify condition (PChyRM). For, let $\varphi, \psi, \gamma \in \Delta^+$, $\Phi, \Psi \in \mathbb{F}(\mathsf{C}(S,T))$, $\mathbb{F} \in \mathbb{F}(S)$, $\Phi \in \mathfrak{c}_\varphi^{\mathsf{C}(S,T)}$ and $\Psi \in \mathfrak{c}_\psi^{\mathsf{C}(S,T)}$. Let $\mathbb{F} \in \mathfrak{p}_\gamma$ with $\gamma \leq \varphi, \gamma \leq \psi$. Then $ev(\Phi \times \mathbb{F}) \in \mathfrak{p}'_\psi$ and $ev(\Psi \times \mathbb{F}) \in \mathfrak{p}'_\psi$. Since $(T, +, \cdot, \mathfrak{p}'_\psi)$ is a probabilistic Cauchy ring under the largest triangle function τ, $ev(\Phi \times \mathbb{F}) \odot ev(\Psi \times \mathbb{F}) \in \mathfrak{p}'_{\tau(\varphi,\psi)}$. But $ev(\Phi \times \mathbb{F}) \odot ev(\Psi \times \mathbb{F}) \leq ev((\Phi \odot \Psi) \times \mathbb{F})$, hence by (Pchy2), $ev((\Phi \odot \Psi) \times \mathbb{F}) \in \mathfrak{p}'_{\tau(\varphi,\psi)}$. Now $\gamma \leq \varphi \wedge \psi = \tau(\varphi, \psi)$, and by (PChy3), $ev((\Phi \odot \Psi) \times \mathbb{F}) \in \mathfrak{p}'_\gamma$. Consequently, $\Phi \odot \Psi \in \mathfrak{c}_{\tau(\varphi,\psi)}$. In a similar fashion one can prove (PChyRS).

In view of [21], we have the following

Proposition 7.5. *Every probabilistic uniform convergence space $(R, \overline{\Lambda})$ under the triangle τ is a probabilistic Cauchy space $(R, \overline{\mathfrak{p}^{\overline{\Lambda}}})$ under the triangle function τ.*

Proposition 7.6. *Every probabilistic limit ring under the triangle function τ is a probabilistic uniform convergence space under the triangle function τ, and hence a probabilistic Cauchy space under the triangle function τ.*

Proof. Let $(R, +, \cdot, \overline{c})$ be a probabilistic limit ring under the triangle function τ. Then it follows from Theorem 6.3, $(R, \overline{\Lambda^{\overline{c}}})$ is a probabilistic uniform convergence space under the triangle function τ. Define a probabilistic Cauchy filter in the probabilistic uniform space $(R, \Lambda^{\overline{c}})$ as follows:

$$\mathbb{F} \in \mathfrak{p}^{\bar{c}}_{\varphi} \Leftrightarrow \mathbb{F} \times \mathbb{F} \in \Lambda^{\bar{c}}_{\varphi}.$$

(Pchy1) This follows from the definition.

(PChy2) Obvious.

(PChy3) Let $\varphi, \psi \in \Delta^+$ with $\varphi \le \psi$ and $\mathbb{F} \in \mathfrak{p}^{\bar{c}}_{\psi}$. Then $\mathbb{F} \times \mathbb{F} \in \Lambda^{\bar{c}}_{\psi}$. But then $\mathbb{F} \times \mathbb{F} \in \Lambda^{\bar{c}}_{\varphi}$, and hence $\mathbb{F} \in \mathfrak{p}^{\bar{c}}_{\varphi}$.

(PChy4) Obvious.

(PChy5) Let $\mathbb{F} \in \mathfrak{p}^{\bar{c}}_{\varphi}$ and $\mathbb{G} \in \mathfrak{p}^{\bar{c}}_{\psi}$ such that $\mathbb{F} \vee \mathbb{G}$ exists. These imply $\mathbb{F} \times \mathbb{F} \in \Lambda^{\bar{c}}_{\varphi}$ and $\mathbb{G} \times \mathbb{G} \in \Lambda^{\bar{c}}_{\psi}$. Then $\mathbb{F}\times\mathbb{G} = (\mathbb{F}\times\mathbb{G})\circ(\mathbb{F}\times\mathbb{G}) \in \Lambda_{\tau(\varphi,\psi)}$. Also, $\mathbb{G}\times\mathbb{F} \in \Lambda^{\bar{c}}_{\tau(\varphi,\psi)}$. Since $(\mathbb{F}\wedge\mathbb{G})\times(\mathbb{F}\wedge\mathbb{G}) = (\mathbb{F}\times\mathbb{F})\wedge(\mathbb{F}\times\mathbb{G})\wedge(\mathbb{G}\times\mathbb{F})\wedge(\mathbb{G}\times\mathbb{G})$. We get $(\mathbb{F} \wedge \mathbb{G}) \times (\mathbb{F} \wedge \mathbb{G}) \in \Lambda^{\bar{c}}_{\tau(\varphi,\psi)}$, and hence $\mathbb{F} \wedge \mathbb{G} \in \mathfrak{p}^{\bar{c}}_{\tau(\varphi,\psi)}$. □

Note that we can describe the probabilistic Cauchy structure of a probabilistic limit ring directly by $\mathbb{F} \in \mathfrak{p}_{\varphi}$ if, and only if, $0 \in c_{\varphi}(\mathbb{F} \ominus \mathbb{F})$.

Theorem 6.3 in conjunction with Propositions 7.5 and 7.6 lead to the following situation.

$$\begin{array}{ccc} \mathbf{PLimRng}_{\tau} & \xrightarrow{\mathfrak{I}} & \mathbf{PUConv}_{\tau} \\ & {\scriptstyle \mathfrak{K}\circ\mathfrak{I}}\searrow & \mathfrak{K}\downarrow \\ & & \mathbf{PChy}_{\tau} \end{array}$$

Proposition 7.7. *$\mathbf{PChyRng}_{\tau}$ is topological over **Rng**.*

Proof. Invoking Proposition 3.2[21], we only describe the initial construction that involves ring structure. Let $(R, +, \cdot)$ be a ring, and for each $j \in J$, let $f_j \colon R \longrightarrow R_j$ be a ring homomorphism and $(R_j, +, \cdot, \overline{\mathfrak{p}}_j)_{j\in J}$ be a family of probabilistic Cauchy rings under the triangle function τ. If $\mathcal{S} = (f_j \colon R \longrightarrow (R_j, +, \cdot, \overline{\mathfrak{p}}_j)_{j\in J})$ is a source, define for $\mathbb{F} \in \mathbb{F}(R)$, $\mathbb{F} \in \mathfrak{p}_{\varphi} \iff f_j(\mathbb{F}) \in \mathfrak{p}^j_{\varphi}$ for all $j \in J$. Then $(R, +, \cdot, \overline{\mathfrak{p}})$ is a probabilistic Cauchy space under the triangle function τ. Now let $\mathbb{F} \in \mathfrak{p}_{\varphi}$, and $\mathbb{G} \in \mathfrak{p}_{\psi}$. Then for each $j \in J$, $f_j(\mathbb{F}) \in \mathfrak{p}^j_{\varphi}$, $f_j(\mathbb{G}) \in \mathfrak{p}^j_{\psi}$. Thus for all $j \in J$, $f_j(\mathbb{F}) \ominus f_j(\mathbb{G}) \in \mathfrak{p}_{\tau(\varphi,\psi)}$ implies by applying Lemma 4.1 that $f_j(\mathbb{F} \ominus \mathbb{G}) \in \mathfrak{p}^j_{\tau(\varphi,\psi)}$ for all $j \in J$. Hence $\mathbb{F} \ominus \mathbb{G} \in p_{\tau(\varphi,\psi)}$. Similarly, one can verify (PChyRM). Finally, it is easy to show that a ring homomorphism $g \colon (R, +, \cdot, \overline{\mathfrak{p}}) \longrightarrow (R', +, \cdot, \overline{\mathfrak{p}'})$ is Cauchy-continuous if and only if $f_j \circ g \colon (R, +, \cdot, \overline{\mathfrak{p}}) \longrightarrow (R_j, +, \cdot, \overline{\mathfrak{p}'})$ is Cauchy continuous for all $j \in J$. □

Theorem 7.8. *Every probabilistic Cauchy ring under the triangle function τ is a probabilistic convergence ring under the triangle function τ; it is also a probabilistic limit ring under the triangle function τ.*

Proof. Let $(R, +, \cdot, \overline{\mathfrak{p}})$ be a probabilistic Cauchy ring under the triangle function τ, $\mathbb{F}, \mathbb{G} \in \mathbb{F}(R)$, and $p, q \in R$. In view of [21], the underlying probabilistic convergence space $(R, c^{\overline{\mathfrak{p}}})$ is described as follows:

$$p \in c^{\overline{\mathfrak{p}}}_{\varphi}(\mathbb{F}) \Leftrightarrow \mathbb{F} \wedge [p] \in \mathfrak{p}_{\varphi}, \text{ for all } \varphi \in \Delta^+.$$

In view of (PChyRA), for $\mathbb{F} \in \mathfrak{p}_{\varphi}$ and $\mathbb{G} \in \mathfrak{p}_{\psi}$, we have $\mathbb{F} \ominus \mathbb{G} \in \mathfrak{p}_{\tau(\varphi,\psi)}$. Let $p \in c^{\overline{\mathfrak{p}}}_{\varphi}(\mathbb{F})$ and $q \in c^{\overline{\mathfrak{p}}}_{\psi}(\mathbb{G})$. Then $([p] \wedge \mathbb{F}) \in \mathfrak{p}_{\varphi}$ and $([q] \wedge \mathbb{G}) \in \mathfrak{p}_{\psi}$ implying $([p] \wedge \mathbb{F}) \ominus ([q] \wedge \mathbb{G}) \in \mathfrak{p}_{\tau(\varphi,\psi)}$. But then $([p] - [q]) \wedge (\mathbb{F} \ominus \mathbb{G}) \in \mathfrak{p}_{\tau(\varphi,\psi)}$ implying $(([p-q]) \wedge (\mathbb{F}\ominus\mathbb{G})) \in \mathfrak{p}_{\tau(\varphi,\psi)}$. Hence $p-q \in c^{\overline{\mathfrak{p}}}_{\tau(\varphi,\psi)}(\mathbb{F}\ominus\mathbb{G})$. Similarly, one can check item (PCRM). That $(R, c^{\overline{\mathfrak{p}}})$ is a probabilistic limit space follows at once from the definition and hence $(R, +, \cdot, \overline{c^{\mathfrak{p}}})$ is a probabilistic limit ring under the triangle function τ as all other conditions remain the same, see for instance, Definitions 3.2 and 4.2. □

Proposition 7.9. *Let* $f\colon (R,+,\cdot,\overline{\mathfrak{p}}) \longrightarrow (R',+,\cdot,\overline{\mathfrak{p}'})$ *be Cauchy-continuous ring homomorphism between probabilistic Cauchy rings under the triangle function* τ. *Then* $f\colon (R,+,\cdot,\overline{c^{\overline{\mathfrak{p}}}}) \longrightarrow (R',+,\cdot,\overline{c^{\overline{\mathfrak{p}'}}})$ *is continuous ring homomorphism between probabilistic convergence rings under the triangle function* τ.

Proof. Let $\mathbb{F} \in \mathbb{F}(R)$, and $\varphi \in \Delta^+$. Consider $p \in c_\varphi^{\mathfrak{p}_\varphi}(\mathbb{F})$. Then $\mathbb{F} \wedge [p] \in \mathfrak{p}_\varphi$. By Cauchy-continuity, we get $f(\mathbb{F} \wedge [p]) \in \mathfrak{p}'_\varphi$. But then $f(\mathbb{F}) \wedge [f(p)] \in \mathfrak{p}'_\varphi$ and hence $f(p) \in c_\varphi^{\mathfrak{p}'_\varphi}(f(\mathbb{F}))$. Since ring homomorphism remains the same we are done. $\square$

As a consequence of Theorem 7.8 and Proposition 7.9 above, we have the following functor as given below.

$$\mathfrak{I} : \begin{cases} \mathbf{PChyRng}_\tau & \longrightarrow & \mathbf{PConvRng}_\tau \\ (R,+,\cdot,\overline{\mathfrak{p}}) & \longmapsto & \left(R,+,\cdot,\overline{c^{\overline{\mathfrak{p}}}}\right) \\ f & \longmapsto & f \end{cases}$$

Proposition 7.10. *The functor* $\mathfrak{I}$ *preserves initial constructions.*

Proof. For a source $(f_j : R \longrightarrow (R_j, \mathfrak{p}^j))_{j\in J}$ we have the initial structure $init(\mathfrak{p}^j)$ and from this the prob limit structure $c^{init(\mathfrak{p}^j)}$. Likewise, the source $(f_j\colon R \longrightarrow (R_j, c^{\mathfrak{p}^j}))_{j\in J}$ has the initial structure $init(c^{\mathfrak{p}^j})$ and we have $p \in init(c^{\mathfrak{p}^j})_\varphi(\mathbb{F})$ if, and only if, $f_j(p) \in c_\varphi^{\mathfrak{p}^j}(f_j(\mathbb{F}))$ for all $j \in J$ if, and only if, $f_j([p] \wedge \mathbb{F}) = [f_j(p)] \wedge f_j(\mathbb{F}) \in \mathfrak{p}_\varphi^f$ for all $j \in J$ if, and only, $[p] \wedge \mathbb{F} \in init(\mathfrak{p}^j)$ if, and only if, $\mathbb{F} \in c^{init(\mathfrak{p}^j)}$. $\square$

8. Conclusion

In this paper, we have shown that every probabilistic limit ring gives rise to a natural probabilistic uniform convergence structure. Various natural and interesting examples are provided for the notions of probabilistic convergence rings, and probabilistic Cauchy rings that we considered in this text. As we pointed out that probabilistic metric spaces are influential generalization of classical metric spaces, we obtained in our previous work numerous results on probabilistic metric groups and obtained probabilistic metrization of probabilistic convergence groups, and much beyond, cf. [3, 4, 19]. Unfortunately, at this moment we are unable to add a well formulated notion of probabilistic metric ring that can lead to an arbitrary probabilistic convergence ring, and also, unable to provide probabilistic metrizability of probabilistic convergence ring. We intend to pursue these open problems in our future research.

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