Filomat 38:20 (2024), 7127–7141
https://doi.org/10.2298/FIL2420127A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Soft $_{(pre)}$-expandable spaces

Heyam H. Al-Jarrah[a,*], Amani A. Rawshdeh[b], Tareq M. Al-shami[c,d]

[a]*Department of Mathematics, Faculty of Science, Yarmouk University, Jordan*
[b]*Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Jordan*
[c]*Department of Mathematics, Sana'a University, Sana'a, Yemen*
[d]*Department of Engineering Mathematics & Physics, Faculty of Engineering & Technology, Future University, New Cairo, Egypt*

Abstract. Soft set theory is a topic of interest for many researches working in a various of fields. In order to advance this field of study, we invest the soft pre-open sets to introduce a new generalization for soft-expandable, namely soft$_{(pre)}$-expandable. We used the soft pre-open sets in various locations in the definition of soft-expandable spaces to give either equivalent definitions of soft expandable spaces or to give different types of soft spaces, we will show the connections between them and based on these connections, the definition of soft$_{(pre)}$-expandable space was chosen. Relationships between this soft topological space with other some know soft topological spaces are examined. Also, we show soft$_{(pre)}$-expandable and soft expandable are equivalent if the soft topological space is either soft-$\mathcal{QSM}$ or ω_0-soft$_{(pre)}$-Compact, and some other features of this spaces with helpful examples are discussed to understand this space more. Finally, we finished this work with our study of two concepts called soft$_{(pre)_\alpha}$-expandable set and soft$_{(pre)_\beta}$-expandable set.

1. Introduction

Several practical issues in a variety of scientific areas, including engineering, environment, economics and medical science, require technical approaches rather than dealing with them in a traditional manner. Many researchers went on to create soft set theory as a new mathematical tool to deal with problems containing uncertainty after the pioneering monograph of Molodtsov [31]. The first attempt was made by Maji et al. [29]. He introduced null and absolute soft sets in 2003, as well as the complement of a soft set, soft intersection, and soft union between two soft sets. Some of these notions have been reformulated to be fit for abstract and applied aspects as well as to get rid of some shortcomings concerning onology. This gives rise to conducting of some studies such as Ali et al. [3] and Al-shami and El-shafei [9]. They proposed new soft operators and redefined certain existing soft operators.

Shabir and Naz [39] introduced the analysis of soft topological spaces in 2011. They used the concept of soft sets to describe soft topology and developed fundamental concepts for soft topological spaces, such as soft open, soft closed, soft closure, soft neighbourhood of a point, soft subspace, soft T_i-spaces, for

2020 *Mathematics Subject Classification*. Primary 54A40; Secondary 06D72, 54E05.
Keywords. Soft set, soft topology, soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$, soft-expandable, soft$_{(pre)}$-expandable.
Received: 14 November 2023; Revised: 18 March 2024; Accepted: 08 April 2024
Communicated by Ljubiša D. R. Kočinac
The publication of this paper was supported by Yarmouk University Research council.
* Corresponding author: Heyam H. Al-Jarrah
Email addresses: hiamaljarah@yahoo.com; heyam@yu.edu.jo (Heyam H. Al-Jarrah), amanirawshdeh@bau.edu.jo (Amani A. Rawshdeh), tareqalshami83@gmail.com; t.alshami@su.edu.ye (Tareq M. Al-shami)

$i = 1, 2, 3, 4$, soft regular and normal spaces. Min [30] conducted additional research on soft T_i-spaces and proved that a soft T_3-space is a soft T_2. The fundamental characteristics of soft closure and soft interior operators were defined by Hussain and Ahmad [25]. Covering properties, i.e., nearly soft Menger, almost soft Menger and weakly soft Menger spaces were introduced and explored by Al-shami and Kočinac [12, 13].

A brilliant idea developed by the authors of [23, 32] was the "soft point," which was utilized to analyze various characteristics of soft interior points and soft neighborhood systems. Kharal and Ahmad [27] identified soft mapping and developed its master qualities, while Zorlutuna and Çakir [46] discussed the idea of soft continuous mapping. Al-shami [7] provided a new formulas to compute the image and pre-image of a soft set depending on a soft point. Aygünoğlu and Aygün [21] established the concept of soft compact spaces and Al-shami et al. [10] proposed the notions of almost soft compact and approximately soft Lindelöf spaces, while Aljarrah et al. [4] employed the soft regular closed sets to introduce a new type of soft compact and soft Lindelöf spaces namely, soft_{int}-compact and soft_{int}-Lindelöf. Furthermore, Rawshdeh et al. [37] introduced soft-expandable spaces as a generalization of soft paracompact and countably soft compact spaces and they went through the condition in which a countably soft-expandable space becomes soft paracompact where a soft-$\mathcal{TS}$ $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-expandable if for each Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$, there is a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. Also, they demonstrated there is no connection between soft topology and its parametric topologies in terms of having the property of being a soft-expandable space. Moreover, they defined soft s-expandable spaces which are stronger than soft-expandable spaces, they looked into some concepts that are equivalent to soft s-expandable spaces and they investigated how soft s-expandable spaces behave when certain soft mappings are applied.

In general, the soft set theory was still being used in numerous fields such as [11, 38, 42] and from these works was introduced by Arockiarani and Lancy [19] where they defined the concept of soft pre-open sets and demonstrated some of their features. Additionally, they studied the prerequisite for a collection of soft pre-open sets to be a soft topology. Many researchers invest soft pre-open sets to introduce different ideas in soft structures such as [8, 20, 35, 36]. We continue our investigation into the characteristic of soft set theory in particular soft pre-open sets to define the notion of $\text{soft}_{(pre)}$-expandable which is a generalization for soft expandable [37]. We provide various examples to illustrate the relationship a among these spaces and we give the prerequisite for $\text{soft}_{(pre)}$-expandable and soft-expandable are equivalent.

Following this brief introduction, we recollect some preliminaries concepts in Section 2. Then, In Section 3, we introduce a generalization of soft submaximal namely soft quasi-submaximal soft-$\mathcal{TS}$ and study some main properties of this notion. In Section 4, we used the soft pre-open sets in various locations in the definition of soft expandable spaces [37] to give either equivalent definitions of soft expandable spaces or to give different types of soft spaces, we will show the connections between them and based on these connections we give the definition of $\text{soft}_{(pre)}$-expandable space. In Section 5, we define the concepts of $\text{soft}_{(pre)_\alpha}$-expandable set and $\text{soft}_{(pre)_\beta}$-expandable set. Finally, some conclusions and upcoming works are given in Section 6.

2. Preliminaries

The purpose of this section is to provide a brief overview of some of the fundamental definitions and results that we will need in our future studies. The set of alternatives and its power set are represented as $\mathcal{Z}$ and $2^{\mathcal{Z}}$, respectively.

Definition 2.1. Let Γ be a parameters set with a function $\mathcal{H} : \Gamma \to 2^{\mathcal{Z}}$. Then $(\mathcal{H}, \Gamma) = \{(\gamma, \mathcal{H}(\gamma)) : \gamma \in \Gamma$ and $\mathcal{H}(\gamma) \in 2^{\mathcal{Z}}\}$ is said to be a soft set over $\mathcal{Z}$ [31]. Moreover:

1. If $\mathcal{H}(\gamma) = \mathcal{Z}$ (resp. $\mathcal{H}(\gamma) = \emptyset$) for each $\gamma \in \Gamma$, then $(\mathcal{H}, \Gamma)$ is said to be absolute (resp. null) soft set, denoted by $(\mathcal{Z}, \Gamma)$ (resp. Φ) [29].
2. If there is $z \in \mathcal{Z}$ and $\gamma \in \Gamma$ with $\mathcal{P}(\gamma) = \{z\}$ and $\mathcal{P}(\gamma^*) = \emptyset$ for each $\gamma^* \in \Gamma - \{\gamma\}$, then $(\mathcal{H}, \Gamma)$ is said to be soft point , denoted by $\mathcal{P}^z_\gamma$ [45].

Definition 2.2. Let $(\mathcal{H}, \Gamma)$ and $(\mathcal{H}^*, \Gamma)$ be soft sets over $\mathcal{Z}$. Then:

1. $p \in (\mathcal{H}, \Gamma)$ if $p \in \mathcal{H}(\gamma)$ for each $\gamma \in \Gamma$ [39].
2. $p \Subset (\mathcal{H}, \Gamma)$ if $p \in \mathcal{H}(\gamma)$ for some $\gamma \in \Gamma$ [24].
3. $(\mathcal{H}, \Gamma) \lll (\mathcal{H}^*, \Gamma)$ if $\mathcal{H}(\gamma) \subseteq \mathcal{H}^*(\gamma)$ for each $\gamma \in \Gamma$ [29].
4. $(\mathcal{H}, \Gamma) \sqcap (\mathcal{H}^*, \Gamma) = (\mathcal{M}, \Gamma)$, where $\mathcal{M}(\gamma) = \mathcal{H}(\gamma) \cap \mathcal{H}^*(\gamma)$ for each $\gamma \in \Gamma$ [33].
5. $(\mathcal{H}, \Gamma) \sqcup (\mathcal{H}^*, \Gamma) = (\mathcal{M}, \Gamma)$, where $\mathcal{M}(\gamma) = \mathcal{H}(\gamma) \cup \mathcal{H}^*(\gamma)$ for each $\gamma \in \Gamma$ [29].
6. $(\mathcal{H}, \Gamma)^c = (\mathcal{H}^c, \Gamma)$, where a function $\mathcal{H}^c : \Gamma \to 2^{\mathcal{Z}}$ is defined by $\mathcal{H}^c(\gamma) = \mathcal{Z} - \mathcal{H}(\gamma)$ for each $\gamma \in \Gamma$ [29].

Definition 2.3. ([39]) Let $\mathfrak{A}$ be a family of soft sets over $\mathcal{Z}$ with a fixed parameters set Γ. Then $\mathfrak{A}$ is said to be soft topology on $\mathcal{Z}$ if it satisfies the following:

1. Φ and $(\mathcal{Z}, \Gamma)$ belong to $\mathfrak{A}$.
2. The union of an arbitrary number of soft sets in $\mathfrak{A}$ belongs to $\mathfrak{A}$.
3. The intersection of a finite number of soft sets in $\mathfrak{A}$ belongs to $\mathfrak{A}$.

And $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is said to be a soft topological space, denoted by soft-$\mathcal{TS}$. The members of $\mathfrak{A}$ are said to be soft open and its complement are said to be soft closed where the family of all soft open (resp. soft closed) subset of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\mathcal{SC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$). For $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$, $Int(\mathcal{H}, \Gamma)$ is the union of all soft open sets contained in $(\mathcal{H}, \Gamma)$ and $Cl(\mathcal{H}, \Gamma)$ is the intersection of all soft closed super sets of $(\mathcal{H}, \Gamma)$. If $Cl(\mathcal{H}, \Gamma) = (\mathcal{Z}, \Gamma)$, then $(\mathcal{H}, \Gamma)$ is said to be soft dense [26] and the family of all soft dense will be denoted by $\mathcal{SD}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.

Definition 2.4. ([32]) Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a a soft-$\mathcal{TS}$ and $(\mathcal{H}, \Gamma)$ be a non-null soft subset of $(\mathcal{Z}, \Gamma)$. Then a relative soft topology on $(\mathcal{H}, \Gamma)$ is defined by $\mathfrak{A}_{(\mathcal{H},\Gamma)} = \{(\mathcal{H}, \Gamma) \sqcap (\mathcal{U}, \Gamma) : (\mathcal{U}, \Gamma) \in \mathfrak{A}\}$ and $((\mathcal{H}, \Gamma), \mathfrak{A}_{(\mathcal{H},\Gamma)}, \Gamma)$ is said to be a soft subspace of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$.

Definition 2.5. ([21]) Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a Soft-$\mathcal{TS}$. A soft topology $\mathfrak{A}$ on $\mathcal{Z}$ is said to be an extended soft topology if $\mathfrak{A} = \{(\mathcal{U}, \Gamma) : \mathcal{U}(\gamma) \in \mathfrak{A}_\gamma$ for each $\gamma \in \Gamma\}$ where $\mathfrak{A}_\gamma$ is a parametric topology on $\mathcal{Z}$.

Lemma 2.6. ([6]) *If* $(\mathcal{U}, \Gamma) \in \mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, *then* $(\mathcal{U}, \Gamma) \sqcap Cl(\mathcal{H}, \Gamma) \sqsubseteq Cl((\mathcal{U}, \Gamma) \sqcap (\mathcal{H}, \Gamma))$ *for each* $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$.

Definition 2.7. Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$. Then $(\mathcal{H}, \Gamma)$ is said to be:

1. Soft semi-open [22] if $(\mathcal{H}, \Gamma) \lll Cl(Int(\mathcal{H}, \Gamma))$. The complement of soft semi-open is called a soft semi-closed set. The family of all soft semi-open (resp. soft-semi closed) subsets of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\mathcal{SSC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$). The intersection of all $\mathcal{SSC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ super sets of $(\mathcal{H}, \Gamma)$ is $Cl_{\mathcal{SSO}}(\mathcal{H}, \Gamma)$.
2. Soft pre-open [19] if $(\mathcal{H}, \Gamma) \lll Int(Cl(\mathcal{H}, \Gamma))$. The complement of soft pre-open is called a soft pre-closed set. The family of all soft pre-open (resp. soft pre-closed) subsets of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\mathcal{SPC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$). If $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma) \sqcap \mathcal{SPC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, then $(\mathcal{H}, \Gamma)$ is said to be soft pre-clopen [8], denoted by $\mathcal{SPCO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.
3. Soft semi-pre open [19] if $(\mathcal{H}, \Gamma) \lll Cl(Int(Cl(\mathcal{H}, \Gamma)))$. The complement of soft semi pre-open is called a soft semi-pre closed set. The family of all soft semi-pre open (resp. soft semi-pre closed) subsets of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\mathcal{SSPC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$).
4. Soft α-open [1] if $(\mathcal{H}, \Gamma) \lll Int(Cl(Int(\mathcal{H}, \Gamma)))$. The complement of soft α-open is called a soft α-closed set. The family of all soft α-open (resp. soft α-closed) subsets of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\alpha\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\alpha\mathcal{SC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$). In [40] show that the family $\alpha\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is a soft-$\mathcal{TS}$.
5. Soft regular-open [43] if $(\mathcal{H}, \Gamma) = Int(Cl(\mathcal{H}, \Gamma))$. The complement of soft regular-open is called a soft regular-closed set. The family of all soft regular-open (resp. soft regular-closed) subsets of $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ will be denoted $\mathcal{SRO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ (resp. $\mathcal{SRC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$).

Theorem 2.8. *Let* $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ *be a soft-*$\mathcal{TS}$. *Then:*

1. $\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma) \subseteq \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ *[2].*
2. $\bigsqcup_{\beta \in \Omega} (\mathcal{H}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ *whenever* $(\mathcal{H}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ *for each* $\beta \in \Omega$ *[2].*

3. $\mathcal{SRO}(\mathcal{Z},\mathfrak{A},\Gamma) \subseteq \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [43].
4. $\mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma) \cup \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma) \subseteq \mathcal{SSPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [44].
5. $\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma) \subseteq \alpha\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [1].
6. $Cl(\mathcal{H},\Gamma) \in \mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)$ *iff* $(\mathcal{H},\Gamma) \in \mathcal{SSPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [44].
7. $Cl_{\mathcal{SSO}}(\mathcal{H},\Gamma) = Cl_{\mathcal{SPO}}(\mathcal{H},\Gamma)$ *whenever* $(\mathcal{H},\Gamma) \in \mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [20].

Proposition 2.9. ([22]) *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H},\Gamma) \lll (\mathcal{Z},\Gamma)$. Then:*

1. *A soft set $(\mathcal{H},\Gamma) \in \mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$ iff there is $(\mathcal{U},\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $(\mathcal{U},\Gamma) \lll (\mathcal{H},\Gamma) \lll Cl(\mathcal{U},\Gamma)$.*
2. *A soft set $(\mathcal{H},\Gamma)$ is soft semi preopen iff there is $(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $(\mathcal{U},\Gamma) \lll (\mathcal{H},\Gamma) \lll Cl(\mathcal{U},\Gamma)$.*

Theorem 2.10. ([26]) *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then $(\mathcal{H},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ iff $(\mathcal{H},\Gamma) = (\mathcal{U},\Gamma) \sqcap (\mathcal{D},\Gamma)$ where $(\mathcal{U},\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Lemma 2.11. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H}_1,\Gamma)$, $(\mathcal{H}_2,\Gamma) \lll (\mathcal{Z},\Gamma)$.*

1. *If $(\mathcal{H}_1,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{H}_2,\Gamma) \in \mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$, then $(\mathcal{H}_1,\Gamma) \sqcap (\mathcal{H}_2,\Gamma) \in \mathcal{SPO}((\mathcal{H}_2,\Gamma),\mathfrak{A}_{(\mathcal{H}_2,\Gamma)},\Gamma))$ [8].*
2. *If $(\mathcal{H}_1,\Gamma) \in \mathcal{SPO}((\mathcal{H}_2,\Gamma),\mathfrak{A}_{(\mathcal{H}_2,\Gamma)},\Gamma))$ and $(\mathcal{H}_2,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$, then $(\mathcal{H}_1,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ [35].*
3. *If $(\mathcal{H}_1,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{H}_2,\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$, then $(\mathcal{H}_1,\Gamma) \sqcap (\mathcal{H}_2,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$. [36].*

Definition 2.12. ([28]) Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then a family $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) : \beta \in \Omega\}$ is said to be soft locally finite in $(\mathcal{Z},\mathfrak{A},\Gamma)$, denoted by Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$, if for each $\mathcal{P}^z_\gamma \in (\mathcal{Z},\Gamma)$, there is $(\mathcal{U},\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ containing $\mathcal{P}^z_\gamma$ and $(\mathcal{U},\Gamma)$ meets at most finite members of $\mathfrak{S}$.

Theorem 2.13. ([28]) *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with the family $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$. Then:*

1. *$\mathfrak{S}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ iff $\{Cl(\mathcal{S}_\beta,\Gamma) : \beta \in \Omega\}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$.*
2. *$Cl(\bigsqcup_{\beta\in\Omega}(\mathcal{S}_\beta,\Gamma)) = \bigsqcup_{\beta\in\Omega} Cl(\mathcal{S}_\beta,\Gamma)$ if $\mathfrak{S}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Definition 2.14. Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then:

1. The family $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) : \beta \in \Omega\}$ is said to be soft [21] (soft open [21], soft pre-open [8]) cover, denoted by Cover-$\mathcal{S}(\mathcal{Z},\mathfrak{A},\Gamma)$ (resp. Cover-$\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$, Cover-$\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$), of $(\mathcal{Z},\Gamma)$ if $(\mathcal{Z},\Gamma) = \bigsqcup_{\beta\in\Omega}(\mathcal{S}_\beta,\Gamma)$ where $(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma)$(resp. $(\mathcal{S}_\beta,\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$, $(\mathcal{S}_\beta,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$).
2. A soft cover $\mathfrak{T} = \{(\mathcal{T}_\beta,\Gamma) : \beta \in \Omega\}$ is soft refinement of $\mathfrak{S} = \{(\mathcal{S}_{\beta^*},\Gamma) : \beta^* \in \Omega\}$ [28] iff for each $(\mathcal{T}_\beta,\Gamma) \in \mathfrak{T}$ there is $(\mathcal{S}_{\beta^*},\Gamma) \in \mathfrak{S}$ with $(\mathcal{T}_\beta,\Gamma) \lll (\mathcal{S}_{\beta^*},\Gamma)$. If $(\mathcal{T}_\beta,\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ (resp. $(\mathcal{T}_\beta,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$), then $\mathfrak{T}$ is said to be soft open (resp. soft pre-open) refinement.
3. A soft set $(\mathcal{H},\Gamma)$ is said to be ω_0-Soft-Compact if each countable Cover-$\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ of $(\mathcal{H},\Gamma)$ has a finite soft subcover of $(\mathcal{H},\Gamma)$[34].
4. A soft set $(\mathcal{H},\Gamma)$ is said to be soft pre-compact [8], denoted by soft$_{(pre)}$-Compact, if each Cover-$\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ of $(\mathcal{H},\Gamma)$ has a finite soft subcover of $(\mathcal{H},\Gamma)$. If each countable Cover-$\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ of $(\mathcal{H},\Gamma)$ has a finite soft subcover of $(\mathcal{H},\Gamma)$, then $(\mathcal{H},\Gamma)$ is said to be ω_0-soft$_{(pre)}$-Compact.

Definition 2.15. ([27]) Let $(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ and $(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma^*)$ be a soft-$\mathcal{TS}$ with $\mathfrak{g} : \mathcal{Z} \to \mathcal{Z}^*$ and $\mathfrak{f} : \Gamma \to \Gamma^*$. Then for each $(\mathcal{H},\Gamma) \in (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ and $(\mathcal{H}^*,\Gamma^*) \in (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma^*)$ define a soft function $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma^*)$ by:

1. $(\mathfrak{g},\mathfrak{f})(\mathcal{H},\Gamma)(\gamma^*) = ((\mathfrak{g},\mathfrak{f})(\mathcal{H}),\Gamma^*) = \mathfrak{g}(\bigcup_{\gamma\in\mathfrak{f}^{-1}(\gamma^*)}\mathcal{H}(\gamma))$, where $\gamma^* \in \Gamma^*$.
2. $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{H},\Gamma^*)(\gamma) = ((\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{H}),\Gamma) = \mathfrak{g}^{-1}((\mathcal{H},\Gamma^*)(\mathfrak{f}(\gamma)))$, where $\gamma \in \Gamma$.

Definition 2.16. Let $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ be a soft function. Then $(\mathfrak{g},\mathfrak{f})$ is said to be:

1. soft surjective if $\mathfrak{g}$ and $\mathfrak{f}$ are surjective [45].

2. soft continuous if $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U},\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ for each $(\mathcal{U},\Gamma) \in \mathcal{SO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ [45].
3. soft closed if $(\mathfrak{g},\mathfrak{f})(\mathcal{U},\Gamma) \in \mathcal{SC}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ for each $(\mathcal{U},\Gamma) \in \mathcal{SC}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ [32].
4. $\text{soft}_{(pre)}$-Irresoulte iff $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ for each $(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ [8].

Definition 2.17. Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. A soft set $(\mathcal{H},\Gamma) \lll (\mathcal{Z},\Gamma)$ is said to be:

1. soft nowhere dense if $Int(Cl(\mathcal{H},\Gamma)) = \Phi$[41].
2. soft nodec if $(\mathcal{H},\Gamma) \in \mathcal{SC}(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever $(\mathcal{H},\Gamma)$ is soft nowhere dense.

Definition 2.18. ([26]) A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is said to be soft submaximal, denoted by soft-$\mathcal{SM}$, if $(\mathcal{D},\Gamma) \in \mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$.

Lemma 2.19. ([26]) *A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{SM}$ iff $\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma) = \mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Definition 2.20. ([14]) Let $\mathcal{Z} = \bigcup_{\alpha\in\Omega} \mathcal{Z}_\alpha$ and a family $\{(\mathcal{Z}_\alpha,\mathfrak{A}_\alpha,\Gamma) : \alpha \in \Omega\}$ be pairwise disjoint soft-$\mathcal{TS}$. Then, the family $\mathfrak{A} = \{(\mathcal{S},\Gamma)$ over $\mathcal{Z} : (\mathcal{S},\Gamma) \sqcap (\mathcal{Z}_\alpha,\Gamma) \in \mathcal{SO}(\mathcal{Z}_\alpha,\mathfrak{A}_\alpha,\Gamma)$ for each $\alpha \in \Omega$ defines a soft topology on $\mathcal{Z}$ with a fixed set of parameters Γ, denoted by $(\bigoplus_{\alpha\in\Omega} \mathcal{Z}_\alpha,\mathfrak{A},\Gamma)$, and $(\bigoplus_{\alpha\in\Omega} \mathcal{Z}_\alpha,\mathfrak{A},\Gamma)$ is called the sum of soft-$\mathcal{TS}$.

3. Soft quasi-submaximal

In this section, we introduce the notion of soft quasi-submaximal soft-$\mathcal{TS}$ which is a generalization of soft submaximal. Then, we give some characterizations of this notion with other some main properties.

Definition 3.1. A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is said to be soft quasi-submaximal, denoted by soft-$\mathcal{QSM}$, if $Bd(\mathcal{D},\Gamma) = Cl(\mathcal{D},\Gamma) - Int(\mathcal{D},\Gamma)$ is soft nowhere dense whenever $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$.

Theorem 3.2. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then the following are equivalent:*

1. *A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$.*
2. *For each $(\mathcal{H},\Gamma) \lll (\mathcal{Z},\Gamma)$, if $Int(\mathcal{H},\Gamma) = \Phi$ then $(\mathcal{H},\Gamma)$ is soft nowhere dense.*
3. *The soft set $Int(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$.*
4. *For each $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$, the soft set $(\mathcal{D},\Gamma)^c$ is soft nowhere dense.*
5. *For each $(\mathcal{H},\Gamma) \lll (\mathcal{Z},\Gamma)$, the soft set $Bd(\mathcal{H},\Gamma)$ has soft nowhere dense.*

Proof. $(3 \to 4)$ and $(5 \to 1)$ are trivial.

$(1 \to 2)$ Assume that $Int(\mathcal{H},\Gamma) = \Phi$, then $(\mathcal{H},\Gamma)^c \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$ and hence $\Phi = Int(Cl(Cl(\mathcal{H},\Gamma)^c - Int(\mathcal{H},\Gamma)^c)) = Int(Cl(\mathcal{H},\Gamma))$.

$(2 \to 3)$ Assume that $(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$, then $Int(\mathcal{D},\Gamma)^c = \Phi$ and hence $(\mathcal{D},\Gamma)^c$ is soft nowhere dense. Therefore $Int(\mathcal{D},\Gamma) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$.

$(4 \to 5)$ Assume that $(\mathcal{H},\Gamma) \lll (\mathcal{Z},\Gamma)$, then $(\mathcal{H},\Gamma) \cup Int((\mathcal{H},\Gamma)^c) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$. By (4), $Int((\mathcal{H},\Gamma) \cup Int((\mathcal{H},\Gamma)^c)) \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$ and since $(Bd(\mathcal{H},\Gamma))^c = Int((\mathcal{H},\Gamma) \cup Int(\mathcal{H},\Gamma)^c) \cap Int((\mathcal{H},\Gamma)^c \cup Int(\mathcal{H},\Gamma))$, then $(Bd(\mathcal{H},\Gamma))^c \in \mathcal{SD}(\mathcal{Z},\mathfrak{A},\Gamma)$. Therefore, $Bd(\mathcal{H},\Gamma)$ has soft nowhere dense. □

Theorem 3.3. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be soft-$\mathcal{QSM}$. Then:*

1. *for every soft dense subset $(\mathcal{H},\Gamma)$ of $(\mathcal{Z},\Gamma)$, $Int(\mathcal{H},\Gamma)$ is soft dense, and*
2. *if $(\mathcal{H}_1,\Gamma)$ and $(\mathcal{H}_2,\Gamma)$ are soft dense sets, then $(\mathcal{H}_1,\Gamma) \cap (\mathcal{H}_2,\Gamma)$ is non-null. That is, there are no disjoint soft dense subsets of $(\mathcal{Z},\Gamma)$.*

Proof. 1. Assume that $(\mathcal{H},\Gamma)$ is soft dense. By assumption, $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$, we obtain $Bd(\mathcal{H},\Gamma) = Cl(\mathcal{H},\Gamma) - Int(\mathcal{H},\Gamma) = (\mathcal{Z},\Gamma) - Int(\mathcal{H},\Gamma)$ is soft nowhere dense. Hence, $(\mathcal{Z},\Gamma) - Cl[(\mathcal{Z},\Gamma) - Int(\mathcal{H},\Gamma)] = (\mathcal{Z},\Gamma) - [(\mathcal{Z},\Gamma) - Int(\mathcal{H},\Gamma)] = Int(\mathcal{H},\Gamma)$ is soft dense.

2. Suppose that $(\mathcal{H}_1,\Gamma)$ and $(\mathcal{H}_2,\Gamma)$ are disjoint soft dense sets. Since $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$, $Bd(\mathcal{H}_1,\Gamma) = Cl(\mathcal{H}_1,\Gamma) - Int(\mathcal{H}_1,\Gamma)$ is soft nowhere dense. But $Bd(\mathcal{H}_1,\Gamma) = Cl(\mathcal{H}_1,\Gamma) \cap Cl(\mathcal{H}_1,\Gamma)^c \ggg Cl(\mathcal{H}_1,\Gamma) \cap Cl(\mathcal{H}_2,\Gamma) = (\mathcal{Z},\Gamma)$ is not soft nowhere dense. This is a contradiction. □

Corollary 3.4. *Every soft-$\mathcal{SM}$ is soft-$\mathcal{QSM}$.*

Proof. Immediately from 3 of Theorem 3.2. □

The converse is not true as the following example shows.

Example 3.5. Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ over $\mathcal{Z} = \{z_1, z_2, z_3\}$ with $\Gamma = \{a_1, a_2\}$ and $\mathfrak{A} = \{\Phi, (\mathcal{Z}, \Gamma), \{(\gamma_1, \{z_1\}), (\gamma_2, \emptyset)\}\}$. Then $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$ but not soft-$\mathcal{SM}$.

Theorem 3.6. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft nodec soft-$\mathcal{TS}$. If $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$, then it is soft-$\mathcal{SM}$.*

Proof. Assume that $(\mathcal{D}, \Gamma) \in \mathcal{SD}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, then by Theorem 3.2, $(\mathcal{D}, \Gamma)^c$ is soft nowhere dense. By assumption, $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is a soft nodec, we obtain $(\mathcal{D}, \Gamma) \in \mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. □

In the following theorem, we will give another characterization of soft-$\mathcal{QSM}$ by using $\mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, $\mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $\mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.

Theorem 3.7. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. Then the following are equivalent:*

1. *A soft-$\mathcal{TS}$ $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$.*
2. *$\mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma) = \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.*
3. *$\mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma) \subseteq \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.*

Proof. $(1 \to 2)$ Assume that $(\mathcal{U}, \Gamma) \in \mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, since $\Phi = Int(Bd(\mathcal{U}, \Gamma)) = Int(Cl(\mathcal{U}, \Gamma)) - Cl(Int(\mathcal{U}, \Gamma))$, then $Int(Cl(\mathcal{U}, \Gamma)) \lll Cl(Int(\mathcal{U}, \Gamma))$ and hence $(\mathcal{U}, \Gamma) \lll Cl(Int(Cl(\mathcal{U}, \Gamma))) \lll Cl(Int(\mathcal{U}, \Gamma))$. Since $\mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma) \subseteq \mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ then $\mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma) = \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.

$(2 \to 3)$ is trivial.

$(3 \to 1)$ Assume that $(\mathcal{D}, \Gamma) \in \mathcal{SD}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, since $(\mathcal{D}, \Gamma) \lll Int(Cl(\mathcal{D}, \Gamma))$ then by (2) $(\mathcal{D}, \Gamma) \in \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. Therefore, $Int(\mathcal{D}, \Gamma) \in \mathcal{SD}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and hence by Theorem 3.2, $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$. □

Lemma 3.8. ([26]) *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ and $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$. Then $Cl_{(pre)}(\mathcal{H}, \Gamma) = (\mathcal{H}, \Gamma) \cup Cl(Int((\mathcal{H}, \Gamma)))$*

Corollary 3.9. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. Then the following are equivalent:*

1. *A soft-$\mathcal{TS}$ $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$.*
2. *$Cl_{(pre)}(\mathcal{H}, \Gamma) = Cl(\mathcal{H}, \Gamma)$ for each $(\mathcal{H}, \Gamma) \in \mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.*
3. *$Cl_{(pre)}(\mathcal{H}, \Gamma) = Cl(\mathcal{H}, \Gamma)$ for each $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.*

Proof. $(1 \to 2)$ Assume that $(\mathcal{H}, \Gamma) \in \mathcal{SSPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. By Theorem 3.7, $(\mathcal{H}, \Gamma) \in \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. Hence by Lemma 3.8, $Cl_{(pre)}(\mathcal{H}, \Gamma) = (\mathcal{H}, \Gamma) \cup Cl(Int((\mathcal{H}, \Gamma))) = Cl(\mathcal{H}, \Gamma)$.

$(2 \to 3)$ is trivial.

$(3 \to 1)$ Assume that $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. By (3), $(\mathcal{H}, \Gamma) \lll Int(Cl((\mathcal{H}, \Gamma))) = Int(Cl_{(pre)}(\mathcal{H}, \Gamma)) = Int(Cl(Int(\mathcal{H}, \Gamma))) \lll Cl(Int(\mathcal{H}, \Gamma))$. Thus $(\mathcal{H}, \Gamma) \in \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. □

Proposition 3.10. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. If $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-$\mathcal{QSM}$, then there are no disjoint soft dense.*

Proof. Assume that $(\mathcal{D}_1, \Gamma), (\mathcal{D}_2, \Gamma) \in \mathcal{SD}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ with $(\mathcal{D}_1, \Gamma) \sqcap (\mathcal{D}_2, \Gamma) = \Phi$. Since $(\mathcal{Z}, \Gamma) = Cl(\mathcal{D}_1, \Gamma) \sqcap Cl(\mathcal{D}_2, \Gamma) \lll Cl(\mathcal{D}_1, \Gamma) \sqcap Cl(\mathcal{D}_1, \Gamma)^c = (Int(\mathcal{D}_1, \Gamma))^c$ which means $(Int(\mathcal{D}_1, \Gamma))^c$ is not soft nowhere dense, which is a contradiction. □

4. Soft$_{(pre)}$-expandable

In this section, we introduce a new generalization for soft expandable space [37], namely soft$_{(pre)}$-expandable. Then, we examine the relationships between this soft topological space with some known soft topological spaces.

Definition 4.1. Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then a family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ is said to be:

1. soft pre-locally finite in $(\mathcal{Z},\mathfrak{A},\Gamma)$, denoted by Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$, if for each $\mathcal{P}^z_\gamma\in(\mathcal{Z},\Gamma)$, there is $(\mathcal{U},\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ containing $\mathcal{P}^z_\gamma$ and $(\mathcal{U},\Gamma)$ meets at most finite members of $\mathfrak{S}$.
2. soft strongly-locally finite in $(\mathcal{Z},\mathfrak{A},\Gamma)$, denoted by Soft$_{(reg)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$, if for each $\mathcal{P}^z_\gamma\in(\mathcal{Z},\Gamma)$, there is $(\mathcal{U},\Gamma)\in\mathcal{SRO}(\mathcal{Z},\mathfrak{A},\Gamma)$ containing $\mathcal{P}^z_\gamma$ and $(\mathcal{U},\Gamma)$ meets at most finite members of $\mathfrak{S}$.

In the whole paper, the notion of Soft-$\mathcal{LF}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$ (resp., Soft$_{(pre)}$-$\mathcal{LF}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$ will be used to describe $\mathfrak{S}$ if it is soft locally finite (resp, soft pre-locally finite) family in $((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$, where $(\mathcal{H},\Gamma)\lll(\mathcal{Z},\Gamma)$.

Note that each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ but the following example shows the converse is not always true.

Example 4.2. It is well known that by taking Γ as a singleton set, the soft topology coincides with general topology. Also, $(\mathcal{Z},\mathfrak{A},\Gamma)$ is a soft subspace of itself, so Example 2.2 of [5] elaborates on that the converse is false, in general.

Theorem 4.3. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with the family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$. If $\mathfrak{S}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$, then:*

1. *A family $\mathfrak{T}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ for each $\mathfrak{T}\subseteq\mathfrak{S}$.*
2. *$Cl_{(pre)}(\bigsqcup_{\beta\in\Omega}(\mathcal{S}_\beta,\Gamma))=\bigsqcup_{\beta\in\Omega}Cl_{(pre)}(\mathcal{S}_\beta,\Gamma)$ whenever $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$.*

Proof. 1. It is immediately from Definition 4.1.

2. Since $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$, then $\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)=\alpha\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$. if $(\mathcal{U},\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ then $(\mathcal{U},\Gamma)\in\alpha\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever the soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$. As it was proved the family of $\alpha\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ forms a soft topology; hence, we obtain the desired result. □

Proposition 4.4. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. The family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ iff one of the following holds:*

1. *$Cl_{(pre)}(\mathfrak{S})=\{Cl_{(pre)}(\mathcal{S}_\beta,\Gamma):\beta\in\Omega\}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family.*
2. *The family $\mathfrak{S}$ is Soft$_{(reg)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ provided that $(\mathcal{S}_\beta,\Gamma)\in\mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Proof. (1) For each $(\mathcal{U},\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $\beta\in\Omega$, $(\mathcal{U},\Gamma)\sqcap(\mathcal{S}_\beta,\Gamma)\neq\Phi$ iff $(\mathcal{U},\Gamma)\sqcap Cl_{(pre)}(\mathcal{S}_\beta,\Gamma)\neq\Phi$ and hence the result follows.

(2) Sufficiency: It is immediately from the Theorem 2.8.

Necessity: Let $\mathcal{P}^z_\gamma\in(\mathcal{Z},\Gamma)$. Then there is $(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $\mathcal{P}^z_\gamma\in(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma)$ and $(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma)$ meets at most finite members of $\mathfrak{S}$, say $\{(\mathcal{S}_{\beta_1},\Gamma),(\mathcal{S}_{\beta_2},\Gamma),...(\mathcal{S}_{\beta_n},\Gamma)\}$. Define $(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)=Int(Cl(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma))$, then:

- The soft point $\mathcal{P}^z_\gamma\in(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)$ and $(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)\in\mathcal{SRO}(\mathcal{Z},\mathfrak{A},\Gamma)$.
- For each $\beta\in\Omega$, pick $(\mathcal{U}_\beta,\Gamma)\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $(\mathcal{U}_\beta,\Gamma)\lll(\mathcal{S}_\beta,\Gamma)\lll Cl(\mathcal{U}_\beta,\Gamma)$. If $(\mathcal{S}_\beta,\Gamma)\sqcap(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)\neq\Phi$, then $Cl(\mathcal{U}_\beta,\Gamma)\sqcap(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)\neq\Phi$ and hence $(\mathcal{U}_\beta,\Gamma)\sqcap Int(Cl(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma))\neq\Phi$ which implies that $\Phi\neq(\mathcal{U}_\beta,\Gamma)\sqcap(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma)\lll(\mathcal{S}_\beta,\Gamma)\sqcap(\mathcal{U}_{\mathcal{P}^z_\gamma},\Gamma)$. Thus $(\mathcal{S}_\beta,\Gamma)\sqcap(\mathcal{W}_{\mathcal{P}^z_\gamma},\Gamma)=\Phi$ for each $\beta\in\Omega-\{\beta_1,\beta_2,...\beta_n\}$.

Therefore, the family $\mathfrak{S}$ is Soft$_{(reg)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$. □

Corollary 4.5. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ where $(\mathcal{S}_\beta,\Gamma)\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$(resp. $\mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)$). If $\mathfrak{S}$ is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family, then it is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Proof. It is immediately from Theorem 2.8, Proposition 4.4 and the fact $\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)\cup\mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)\subseteq\mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$. □

Theorem 4.6. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{QSM}$ soft-$\mathcal{TS}$ with $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ where $(\mathcal{S}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$. If $\mathfrak{S}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family, then it is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Proof. Assume that $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-$\mathcal{QSM}$. Then by Corollary 3.9, $Cl_{(pre)}(\mathcal{S}_\beta,\Gamma)=Cl(\mathcal{S}_\beta,\Gamma)$ for each $\beta\in\Omega$ and hence, by Proposition 4.4, the family $\{Cl(\mathcal{S}_\beta,\Gamma):\beta\in\Omega\}$ is soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$. By Theorem 2.8, $Cl(\mathcal{S}_\beta,\Gamma)\in\mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)$ and hence by Theorem 2.13 and Corollary 4.5, the family $\mathfrak{S}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$. □

Lemma 4.7. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H},\Gamma)\in\mathcal{SC}(\mathcal{Z},\mathfrak{A},\Gamma)$. If $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{H},\Gamma):\beta\in\Omega\}$ is a Soft-$\mathcal{LF}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$ family, then it is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Proof. Let $\mathcal{P}^z_\gamma\in(\mathcal{Z},\Gamma)$. If $\mathcal{P}^z_\gamma\in(\mathcal{H},\Gamma)$, then there is $(\mathcal{U},\Gamma)\in\mathcal{SO}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$ with $\mathcal{P}^z_\gamma\in(\mathcal{U},\Gamma)$ and $(\mathcal{U},\Gamma)$ meets at most finite members of $\mathfrak{S}$. Now there is $(\mathcal{U}^*,\Gamma)\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $(\mathcal{U},\Gamma)=(\mathcal{U}^*,\Gamma)\sqcap(\mathcal{H},\Gamma)$ and $(\mathcal{U}^*,\Gamma)$ meets at most finite members of $\mathfrak{S}$. Now if $\mathcal{P}^z_\gamma\in(\mathcal{H},\Gamma)^c$, then $(\mathcal{H},\Gamma)^c\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and meets no members of $\mathfrak{S}$. □

In the next theorem, we will employ the $\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ to give several characterizations for soft-expandable space and from which we deduce the appropriate definition of soft$_{(pre)}$-expandable.

Theorem 4.8. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$. Then the following are equivalent:*

1. *A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-expandable.*
2. *For each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.*
3. *For each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.*
4. *For each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{S}\alpha\mathcal{O}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.*

Proof. $(1\to2\to3\to4)$ These implication follow from Theorem 2.8 and Corollary 4.5.

$(4\to1)$ Assume that $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$. Then, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{S}\alpha\mathcal{O}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$. By Proposition 4.4, the family $\mathfrak{T}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$. Therefore, $\{\mathrm{Int}(Cl(\mathrm{Int}(\mathcal{T}_\beta,\Gamma))):\beta\in\Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ with $\mathrm{Int}(Cl(\mathrm{Int}(\mathcal{T}_\beta,\Gamma)))\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll\mathrm{Int}(Cl(\mathrm{Int}(\mathcal{T}_\beta,\Gamma)))$ for each $\beta\in\Omega$. Hence, $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft-expandable. □

Definition 4.9. A soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is said to be:

1. soft$_{(pre)}$-expandable if for each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$. If $|\Omega|\leqslant\omega_0$, then a soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is called ω_0-soft$_{(pre)}$-expandable space.
2. soft$_{(pre)_I}$-expandable if for each Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.
3. soft$_{(pre)_{II}}$-expandable if for each Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.

From Theorem 2.13, we note that a soft-$\mathcal{TS}$ $(\mathcal{Z},\mathfrak{A},\Gamma)$ is soft$_{(pre)}$-expandable iff for each Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{S}_\beta,\Gamma)\in\mathcal{SC}(\mathcal{Z},\mathfrak{A},\Gamma)$, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$.

The following diagram follows immediately from the definitions in which none of these implications is reversible.

$$\begin{array}{ccc} \text{soft-expandable} & \rightarrow & \text{soft}_{(pre)}\text{-expandable} \\ \uparrow & & \uparrow \\ \text{soft}_{(pre)_I}\text{-expandable} & \rightarrow & \text{soft}_{(pre)_{II}}\text{-expandable} \end{array}$$

Example 4.10. Let $(\mathbb{R}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $\Gamma = \{a_1, a_2\}$ and $\mathfrak{A} = \{\mathbb{R}, (\mathcal{U}, \Gamma) : (\mathcal{U}, \Gamma) \sqsubseteq \widetilde{\mathbb{Q}}\}$. Note that each Soft-$\mathcal{LF}(\mathbb{R}, \mathfrak{A}, \Gamma)$ family is finite. Hence, $(\mathbb{R}, \mathfrak{T}, \Gamma)$ is soft-expandable and so soft$_{(pre)}$-expandable. But, it is not soft$_{(pre)_{II}}$-expandable because the family of soft points constructs soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family which has not a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family satisfying the condition of soft$_{(pre)_{II}}$-expandable.

Example 4.11. Let $(\mathbb{N}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $\Gamma = \{a_1, a_2\}$ and $\mathfrak{A} = \{\Phi, (\mathcal{U}, \Gamma) \sqsubseteq \widetilde{\mathbb{N}} : 1 \in (\mathcal{U}, \Gamma)\}$. Then $(\mathbb{N}, \mathfrak{A}, \Gamma)$ is neither soft-expandable nor soft$_{(pre)_I}$-expandable.

Example 4.12. Let $(\mathbb{R}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $\Gamma = \{a_1, a_2\}$ and $\mathfrak{A} = \{\Phi, (\mathbb{R}, \Gamma), (\mathbb{Q}, \Gamma), (\mathbb{Q}, \Gamma)^c\}$. Let $(\mathcal{H}, \Gamma)$ be a soft subset of $(\mathbb{R}, \mathfrak{A}, \Gamma)$. Now, we have the following three cases:

- $(\mathcal{H}, \Gamma) \lll (\mathbb{Q}, \Gamma)$. Then, $Cl(\mathcal{H}, \Gamma) = (\mathbb{Q}, \Gamma)$, so $\text{Int}(Cl(\mathcal{H}, \Gamma)) = (\mathbb{Q}, \Gamma)$ as well, which means that $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathbb{R}, \mathfrak{A}, \Gamma)$.
- $(\mathcal{H}, \Gamma) \lll (\mathbb{Q}, \Gamma)^c$. Then, $Cl(\mathcal{H}, \Gamma) = (\mathbb{Q}, \Gamma)^c$, so $\text{Int}(Cl(\mathcal{H}, \Gamma)^c) = (\mathbb{Q}, \Gamma)^c$ as well, which means that $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathbb{R}, \mathfrak{A}, \Gamma)$.
- Neither $(\mathcal{H}, \Gamma) \lll (\mathbb{Q}, \Gamma)$ nor $(\mathcal{H}, \Gamma) \lll (\mathbb{Q}, \Gamma)^c$ hold true. Then, $Cl(\mathcal{H}, \Gamma) = (\mathbb{R}, \Gamma)$, so $\text{Int}(Cl(\mathcal{H}, \Gamma)) = (\mathbb{R}, \Gamma)$ as well, which means that $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathbb{R}, \mathfrak{A}, \Gamma)$.

Thus, $\mathcal{SPO}(\mathbb{R}, \mathfrak{A}, \Gamma)$ is the soft discrete topology. So, $(\mathbb{R}, \mathfrak{A}, \Gamma)$ is soft$_{(pre)_{II}}$-expandable. In contrast, it can be checked that $(\mathbb{R}, \mathfrak{A}, \Gamma)$ is not soft$_{(pre)_I}$-expandable.

Note that the Example 4.2 and Example 4.12 show that soft-expandable and soft$_{(pre)}$-expandable soft-$\mathcal{TS}$ are independent notions.

Theorem 4.13. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. Then the following are equivalent:*

1. *$(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft$_{(pre)_I}$-expandable.*
2. *Every Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$.*
3. *Every Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ has a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$.*

Proof. (1→2→3) Immediately from Theorem 2.8 and Corollary 4.5.

(3→1) Let $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ be a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$. Then, $\mathfrak{S}$ has a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. Since $\{\text{Int}(Cl(\mathcal{T}_\beta, \Gamma)) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ with $\text{Int}(Cl(\mathcal{T}_\beta, \Gamma)) \in \mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll \text{Int}(Cl(\mathcal{T}_\beta, \Gamma))$ for each $\beta \in \Omega$. Therefore, $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft$_{(pre)_I}$-expandable. □

Proposition 4.14. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a Soft-$\mathcal{TS}$. If $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is:*

1. *A soft-$\mathcal{QSM}$, then $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-expandable iff it is soft$_{(pre)}$-expandable.*
2. *A soft-$\mathcal{SM}$, then $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft-expandable iff it is soft$_{(pre)_{II}}$-expandable.*

Proof. Immediately from Lemma 2.19 and Theorem 4.6. □

By Example 4.2 and Example 4.10, we observe that the conditions in Proposition 4.14 are essential.

Theorem 4.15. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. Then $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft$_{(pre)}$-expandable if each Cover-$\mathcal{SO}(\mathcal{Z}, \mathfrak{T}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$ has Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family soft pre-open refinement.*

Proof. Assume that a family $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ with $(\mathcal{S}_\beta, \Gamma) \in \mathcal{SC}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $\Omega^* = \{\mathcal{F} \subseteq \Omega : \mathcal{F} \text{ is finite}\}$. Define $\mathfrak{T} = \{(\mathcal{T}_{\beta^*}, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta^* \in \Omega^*\}$ where $(\mathcal{T}_{\beta^*}, \Gamma) = (\mathcal{Z}, \Gamma) - \sqcup\{(\mathcal{S}_\beta, \Gamma) : \beta \notin \beta^*\}$ for each $\beta^* \in \Omega^*$. Then:

- The family $\mathfrak{T}$ is Cover-$\mathcal{SO}(\mathcal{Z}, \mathfrak{T}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$.
- The family $\mathfrak{T}$ has Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ soft pre-open refinement, say $\mathfrak{W} = \{(\mathcal{W}_\alpha, \Gamma) \lll (\mathcal{Z}, \Gamma) : \alpha \in \Omega\}$.
- Define $(\mathcal{M}_\beta, \Gamma) = \sqcup\{(\mathcal{W}_\alpha, \Gamma) \in \mathfrak{W} : (\mathcal{W}_\alpha, \Gamma) \sqcap (\mathcal{S}_\beta, \Gamma) \neq \Phi\}$ for each $\beta \in \Omega$. Then, $(\mathcal{M}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ with $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{M}_\beta, \Gamma)$ for each $\beta \in \Omega$.
- The family $\{(\mathcal{M}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$, since for each $\mathcal{P}_\gamma^z \in (\mathcal{Z}, \Gamma)$ there is $(\mathcal{M}_{\mathcal{P}_\gamma^z}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ containing $\mathcal{P}_\gamma^z$ and $(\mathcal{M}_{\mathcal{P}_\gamma^z}, \Gamma)$ meets at most finite members of $\mathfrak{W}$. Note that, $(\mathcal{M}_{\mathcal{P}_\gamma^z}, \Gamma) \sqcap (\mathcal{M}_\beta, \Gamma) \neq \Phi$ iff $(\mathcal{M}_{\mathcal{P}_\gamma^z}, \Gamma) \sqcap (\mathcal{W}_\alpha, \Gamma) \neq \Phi$ and $(\mathcal{W}_\alpha, \Gamma) \sqcap (\mathcal{S}_\beta, \Gamma) \neq \Phi$ for some $\alpha \in \Omega$. Since $(\mathcal{T}_{\beta^*}, \Gamma)$ meets at most finite members of $\mathfrak{S}$ and $\mathfrak{W}$ is refinement of $\mathfrak{T}$, then $(\mathcal{W}_\alpha, \Gamma)$ meets at most finite members of $\mathfrak{S}$ for each $\alpha \in \Omega$.

Therefore, $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is soft$_{(pre)}$-expandable. □

Theorem 4.16. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$. Then $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is ω_0-soft$_{(pre)}$-expandable iff each countable Cover-$\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$ has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family soft pre-open refinement.*

Proof. Sufficiency. The proof is similar of Theorem 4.15.

Necessity. Assume that $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \mathbb{N}\}$ is a Cover-$\mathcal{SO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$. For each $\beta \in \mathbb{N}$, set $(\mathcal{T}_\beta, \Gamma) = \sqcup\{(\mathcal{S}_\delta, \Gamma) : \delta \leq \beta\}$ and define

$$(\mathcal{W}_\beta, \Gamma) = \left\{ \begin{array}{ll} (\mathcal{T}_1, \Gamma) & : \text{if } \beta = 1 \\ (\mathcal{T}_\beta, \Gamma) - (\mathcal{T}_{\beta-1}, \Gamma) & : \text{if } \beta \geqslant 2 \end{array} \right\}$$

Then $(\mathcal{W}_\beta, \Gamma) \lll (\mathcal{S}_\beta, \Gamma)$ for each $\beta \in \mathbb{N}$. Now for $\mathcal{P}_\gamma^z \in (\mathcal{Z}, \Gamma)$, let $\beta(\mathcal{P}_\gamma^z) = \min\{\beta \in \mathbb{N} : \mathcal{P}_\gamma^z \in (\mathcal{S}_\beta, \Gamma)\}$ and hence $\mathcal{P}_\gamma^z \in (\mathcal{W}_{\beta(\mathcal{P}_\gamma^z)}, \Gamma)$. Put $\mathfrak{W} = \{(\mathcal{W}_\beta, \Gamma) : \beta \in \mathbb{N}\}$. Then:

- The family $\mathfrak{W}$ is a refinement of $\mathfrak{S}$ and it is Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family since $(\mathcal{S}_\beta, \Gamma) \sqcap (\mathcal{W}_\beta, \Gamma) = \Phi$ for $\delta > \beta$ and for each $\beta \in \mathbb{N}$, $(\mathcal{W}_\beta, \Gamma) \lll (\mathcal{S}_\beta, \Gamma)$. Hence, there is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\{(\mathcal{M}_\beta, \Gamma) : \beta \in \mathbb{N}\}$ with $(\mathcal{M}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{W}_\beta, \Gamma) \lll (\mathcal{M}_\beta, \Gamma)$ for each $\beta \in \mathbb{N}$.
- Define $\mathfrak{T}^* = \{(\mathcal{S}_\beta, \Gamma) \sqcap (\mathcal{M}_\beta, \Gamma) : \beta \in \mathbb{N}\}$. Then the family $\mathfrak{T}^*$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family since $\{(\mathcal{M}_\beta, \Gamma) : \beta \in \mathbb{N}\}$ is soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and, by Lemma 2.11, $(\mathcal{S}_\beta, \Gamma) \sqcap (\mathcal{M}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ for each $\beta \in \mathbb{N}$.
- Each element of $\mathfrak{T}^*$ is a soft subset of some element of $\mathfrak{S}$ and $\mathfrak{T}^*$ is a Cover-$\mathcal{S}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$ since $\mathfrak{W}$ is a Cover-$\mathcal{S}(\mathcal{Z}, \mathfrak{T}, \Gamma)$ of $(\mathcal{Z}, \Gamma)$ and for each $\beta \in \mathbb{N}$, $(\mathcal{W}_\beta, \Gamma) \lll (\mathcal{S}_\beta, \Gamma) \sqcap (\mathcal{M}_\beta, \Gamma)$. Hence, $\mathfrak{T}^*$ is refinement of $\mathfrak{S}$.
  Therefore, $\mathfrak{S}$ has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family soft pre-open refinement.

□

Definition 4.17. Let $(\mathfrak{g}, \mathfrak{f}) : (\mathcal{Z}, \mathfrak{A}_{\mathcal{Z}}, \Gamma) \to (\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ be a soft function. Then $(\mathfrak{g}, \mathfrak{f})$ is said to be:

1. soft strongly$_{(pre)}$-open if $(\mathfrak{g}, \mathfrak{f})(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ for each $(\mathcal{H}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}_{\mathcal{Z}}, \Gamma)$.
2. soft strongly$_{(pre)}$-closed if $(\mathfrak{g}, \mathfrak{f})(\mathcal{H}, \Gamma) \in \mathcal{SPC}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ for each $(\mathcal{H}, \Gamma) \in \mathcal{SPC}(\mathcal{Z}, \mathfrak{A}_{\mathcal{Z}}, \Gamma)$.

Proposition 4.18. *A soft surjective function $(\mathfrak{g}, \mathfrak{f}) : (\mathcal{Z}, \mathfrak{A}_{\mathcal{Z}}, \Gamma) \to (\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ is soft strongly$_{(pre)}$-closed iff for each $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*, \Gamma)$ and each $(\mathcal{U}, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}_{\mathcal{Z}}, \Gamma)$ with $(\mathfrak{g}, \mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*}) \lll (\mathcal{U}, \Gamma)$ there is $(\mathcal{U}^*, \Gamma) \in \mathcal{SPO}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ with $\mathcal{P}_\gamma^{z^*} \in (\mathcal{U}^*, \Gamma)$ and $(\mathfrak{g}, \mathfrak{f})^{-1}(\mathcal{U}^*, \Gamma) \lll (\mathcal{U}, \Gamma)$.*

Proof. Necessity: Assume that $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma)$ and $(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ with $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*}) \lll (\mathcal{U},\Gamma)$. Take $(\mathcal{U}^*,\Gamma) = (\mathcal{Z}^*,\Gamma) - (\mathfrak{g},\mathfrak{f})((\mathcal{Z},\Gamma) - (\mathcal{U},\Gamma))$. Then, $(\mathcal{U}^*,\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $\mathcal{P}_\gamma^{z^*} \in (\mathcal{U}^*,\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U}^*,\Gamma) \lll (\mathcal{U},\Gamma)$.

Sufficiency: Assume that $(\mathcal{G},\Gamma) \in \mathcal{SPC}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$. For each $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma) - (\mathfrak{g},\mathfrak{f})(\mathcal{G},\Gamma)$ we have $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*}) \lll (\mathcal{Z},\Gamma) - (\mathcal{G},\Gamma) = (\mathcal{U},\Gamma)$. Then, there is $(\mathcal{U}^*_{\mathcal{P}_\gamma^{z^*}},\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $\mathcal{P}_\gamma^{z^*} \in (\mathcal{U}^*_{\mathcal{P}_\gamma^{z^*}},\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U}^*_{\mathcal{P}_\gamma^{z^*}},\Gamma) \lll (\mathcal{U},\Gamma)$. Set $(\mathcal{U}^*,\Gamma) = \sqcup\{(\mathcal{U}^*_{\mathcal{P}_\gamma^{z^*}},\Gamma) : \mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma) - (\mathfrak{g},\mathfrak{f})(\mathcal{G},\Gamma)\} \lll (\mathcal{Z}^*,\Gamma) - (\mathfrak{g},\mathfrak{f})(\mathcal{G},\Gamma)$. Then $(\mathcal{U}^*,\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $\mathcal{P}_\gamma^{z^*} \in (\mathcal{U}^*,\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U}^*,\Gamma) \lll (\mathcal{U},\Gamma)$. Therefore, $(\mathfrak{g},\mathfrak{f})(\mathcal{G},\Gamma) \in \mathcal{SPC}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$. □

Lemma 4.19. ([37]) *Let* $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *be a soft surjective function. Then:*

1. *A family* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathfrak{S}) = \{(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ *is Soft-*$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ *whenever* $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ *is Soft-*$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *and* $(\mathfrak{g},\mathfrak{f})$ *is a soft continuous.*
2. *A family* $(\mathfrak{g},\mathfrak{f})(\mathfrak{S}) = \{(\mathfrak{g},\mathfrak{f})(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ *is Soft-*$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *whenever* $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ *is Soft-*$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ *and* $(\mathfrak{g},\mathfrak{f})$ *is soft closed with* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$ *is soft compact for each* $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma)$.

Lemma 4.20. *Let a soft surjective function* $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *be soft*$_{(pre)}$*-Irresolute. If a family* $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ *is a Soft*$_{(pre)}$*-*$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$*, then* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathfrak{S}) = \{(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ *is a Soft*$_{(pre)}$*-*$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$.

Proof. Let $\mathcal{P}_\gamma^{z} \in (\mathcal{Z},\Gamma)$. Then there is $(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $(\mathfrak{g},\mathfrak{f})(\mathcal{P}_\gamma^{z}) \in (\mathcal{U},\Gamma)$ and $(\mathcal{U},\Gamma)$ meets at most finite members of $\mathfrak{S}$. Since $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ with $\mathcal{P}_\gamma^{z} \in (\mathfrak{g},\mathfrak{f})^{-1}((\mathfrak{g},\mathfrak{f})(\mathcal{P}_\gamma^{z})) \lll (\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U},\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U},\Gamma)$ meets at most finite members of $(\mathfrak{g},\mathfrak{f})^{-1}(\mathfrak{S})$. Therefore, $(\mathfrak{g},\mathfrak{f})^{-1}(\mathfrak{S})$ is a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$. □

Lemma 4.21. *If a soft surjective function* $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *is soft strongly*$_{(pre)}$*-closed with* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$ *is soft*$_{(pre)}$*-Compact for each* $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma)$*, then* $(\mathfrak{g},\mathfrak{f})(\mathfrak{S}) = \{(\mathfrak{g},\mathfrak{f})(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ *is a Soft*$_{(pre)}$*-*$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *family whenever* $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ *is a Soft*$_{(pre)}$*-*$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$.

Proof. Let $\mathcal{P}_\gamma^{z} \in (\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$. Then $\mathcal{P}_\gamma^{z} \in (\mathcal{U}_{\mathcal{P}_\gamma^{z}},\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ and $(\mathcal{U}_{\mathcal{P}_\gamma^{z}},\Gamma)$ meets at most finite members of $\mathfrak{S}$. Therefore, $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*}) \lll \bigsqcup_{\mathcal{P}_\gamma^{z} \in (\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})} (\mathcal{U}_{\mathcal{P}_\gamma^{z}},\Gamma)$ and hence $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*}) \lll \bigsqcup_{k=1}^{n} (\mathcal{U}_{\mathcal{P}_\gamma^{z_k}},\Gamma) = (\mathcal{U},\Gamma)$ where $\{\mathcal{P}_\gamma^{z_1}, \mathcal{P}_\gamma^{z_2}, \dots \mathcal{P}_\gamma^{z_n}\} \lll (\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$. By Proposition 4.18, there is $(\mathcal{U}^*,\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $\mathcal{P}_\gamma^{z^*} \in (\mathcal{U}^*,\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{U}^*,\Gamma) \lll (\mathcal{U},\Gamma)$. Since $(\mathcal{U}^*,\Gamma)$ meets at most finite members of $(\mathfrak{g},\mathfrak{f})(\mathfrak{S})$, then $(\mathfrak{g},\mathfrak{f})(\mathfrak{S})$ is Soft-$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$. □

Theorem 4.22. *Let a soft surjective function* $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *be soft continuous, soft strongly*$_{(pre)}$*-closed and soft strongly*$_{(pre)}$*-open with* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$ *is soft*$_{(pre)}$*-Compact for each* $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma)$*. If* $(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ *is soft*$_{(pre)}$*-expandable then* $(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *is soft*$_{(pre)}$*-expandable.*

Proof. Assume that $\mathfrak{S} = \{(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ family, then by Lemma 4.19, $(\mathfrak{g},\mathfrak{f})^{-1}(\mathfrak{S}) = \{(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ family and hence it has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta,\Gamma) \lll (\mathcal{Z},\Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta,\Gamma) \in \mathcal{SPO}(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ and $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{S}_\beta,\Gamma) \lll (\mathcal{T}_\beta,\Gamma)$ for each $\beta \in \Omega$. Since $(\mathfrak{g},\mathfrak{f})$ is soft strongly$_{(pre)}$-open and by Lemma 4.21, then $(\mathfrak{g},\mathfrak{f})(\mathfrak{T}) = \{(\mathfrak{g},\mathfrak{f})(\mathcal{T}_\beta,\Gamma) \lll (\mathcal{Z}^*,\Gamma) : \beta \in \Omega\}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ with $(\mathfrak{g},\mathfrak{f})(\mathcal{T}_\beta,\Gamma) \in \mathcal{SPO}(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma) \lll (\mathfrak{g},\mathfrak{f})(\mathcal{T}_\beta,\Gamma)$ for each $\beta \in \Omega$. □

Theorem 4.23. *Let a soft surjective function* $(\mathfrak{g},\mathfrak{f}) : (\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma) \to (\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *be a soft closed and soft*$_{(pre)}$*- Irresolute. If* $(\mathcal{Z}^*,\mathfrak{A}_{\mathcal{Z}^*},\Gamma)$ *is soft*$_{(pre)}$*-expandable then* $(\mathcal{Z},\mathfrak{A}_{\mathcal{Z}},\Gamma)$ *is soft*$_{(pre)}$*-expandable provided that* $(\mathfrak{g},\mathfrak{f})^{-1}(\mathcal{P}_\gamma^{z^*})$ *is soft*$_{(pre)}$*-Compact for each* $\mathcal{P}_\gamma^{z^*} \in (\mathcal{Z}^*,\Gamma)$.

Proof. Assume that $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}_\mathcal{Z}, \Gamma)$ family. By Lemma 4.19, $(\mathfrak{g}, \mathfrak{f})(\mathfrak{S}) = \{(\mathfrak{g}, \mathfrak{f})(\mathcal{S}_\beta, \Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ family and hence it has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}^*, \mathfrak{A}_{\mathcal{Z}^*}, \Gamma)$ and $(\mathfrak{g}, \mathfrak{f})(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. Since $(\mathfrak{g}, \mathfrak{f})$ is soft$_{(pre)}$- Irresolute function and by Theorem 4.20, then $(\mathfrak{g}, \mathfrak{f})^{-1}(\mathfrak{T}) = \{(\mathfrak{g}, \mathfrak{f})^{-1}(\mathcal{T}_\beta, \Gamma) : \beta \in \Omega\}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}_\mathcal{Z}, \Gamma)$ with $(\mathfrak{g}, \mathfrak{f})^{-1}(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}_\mathcal{Z}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathfrak{g}, \mathfrak{f})^{-1}((\mathfrak{g}, \mathfrak{f})(\mathcal{S}_\beta, \Gamma)) \lll (\mathfrak{g}, \mathfrak{f})^{-1}(\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. □

5. Soft $_{(pre)}$-expandable sets

In this section, we define when the soft set $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$ is a soft$_{(pre)_\alpha}$-expandable set and when it is soft$_{(pre)_\beta}$-expandable set. To clarify the results and relationships investigated in this part, we display some illustrative examples.

Definition 5.1. Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H}, \Gamma) \lll (\mathcal{Z}, \Gamma)$. Then $(\mathcal{H}, \Gamma)$ is said to be:

1. soft$_{(pre)_\beta}$-expandable set in $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ if $((\mathcal{H}, \Gamma), \mathfrak{A}_{(\mathcal{H},\Gamma)}, \Gamma)$ is soft$_{(pre)}$-expandable.
2. soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ if each family $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{H}, \Gamma) : \beta \in \Omega\}$ which is Soft-$\mathcal{LF}((\mathcal{H}, \Gamma), \mathfrak{A}_{(\mathcal{H},\Gamma)}, \Gamma)$ has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$.

Note that soft$_{(pre)_\beta}$-expandable and soft$_{(pre)_\alpha}$-expandable sets are linearly independent. To see that we give the following examples.

Example 5.2. Let $(\mathbb{N}, \mathfrak{A}, \Gamma)$ be as soft-$\mathcal{TS}$ given in Example 4.11. One can not that the family of $\mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ is $\mathfrak{A}$. Obviously, the soft set $(\mathcal{H}, \Gamma) = (\mathbb{N}, \Gamma) - \{(\gamma_1, \{1\}), (\gamma_2, \{1\})\}$ is not soft pre-open and $\mathfrak{A}_{(\mathcal{H},\Gamma)}$ is the soft discrete topology. This implies that $(\mathcal{H}, \Gamma)$ is a soft$_{(pre)_\beta}$-expandable set but not a soft$_{(pre)_\alpha}$-expandable set.

Theorem 5.3. *Let $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H}_1, \Gamma), (\mathcal{H}_2, \Gamma) \lll (\mathcal{Z}, \Gamma)$ and $(\mathcal{H}_1, \Gamma) \lll (\mathcal{H}_2, \Gamma)$. Then $(\mathcal{H}_1, \Gamma)$ is:*

1. *soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z}, \mathfrak{A}, \Gamma)$ if $(\mathcal{H}_2, \Gamma) \in \mathcal{SPCO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{H}_1, \Gamma)$ is soft$_{(pre)_\alpha}$-expandable set in $((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$.*
2. *soft$_{(pre)_\alpha}$-expandable set in $((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$ if $(\mathcal{H}_2, \Gamma) \in \mathcal{SSO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{H}_1, \Gamma)$ is soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z}, \mathfrak{A}, \Gamma)$.*

Proof. 1) Assume that $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{H}_1, \Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}((\mathcal{H}_1, \Gamma), \mathfrak{A}_{(\mathcal{H}_1,\Gamma)}, \Gamma)$ family. Then $\mathfrak{S}$ has a Soft$_{(pre)}$-$\mathcal{LF}((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{H}_2, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. Then:

- By Lemma 2.11, $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$.
- The family $\mathfrak{T}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ since if $\mathcal{P}^z_\gamma \in (\mathcal{Z}, \Gamma)$, then either $\mathcal{P}^z_\gamma \notin (\mathcal{H}_2, \Gamma)$ or $\mathcal{P}^z_\gamma \in (\mathcal{H}_2, \Gamma)$.When $\mathcal{P}^z_\gamma \in (\mathcal{Z}, \Gamma) - (\mathcal{H}_2, \Gamma)$ then $(\mathcal{Z}, \Gamma) - (\mathcal{H}_2, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ which meets no member of $\mathfrak{T}$. Now, if $\mathcal{P}^z_\gamma \in (\mathcal{H}_2, \Gamma)$, then there is $(\mathcal{T}^*, \Gamma) \in \mathcal{SPO}((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$ with $\mathcal{P}^z_\gamma \in (\mathcal{T}^*, \Gamma)$ and $(\mathcal{T}^*, \Gamma)$ meets at most finite members of $\mathfrak{T}$.
- By Lemma 2.11, $(\mathcal{T}^*, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and hence $\mathfrak{T}$ is Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family.

Therefore, $(\mathcal{H}_1, \Gamma)$ is soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z}, \mathfrak{A}, \Gamma)$.

2) Assume that $\mathfrak{S} = \{(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{H}_1, \Gamma) : \beta \in \Omega\}$ is a Soft-$\mathcal{LF}((\mathcal{H}_1, \Gamma), \mathfrak{A}_{(\mathcal{H}_1,\Gamma)}, \Gamma)$ family. Then it has a soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ family $\mathfrak{T} = \{(\mathcal{T}_\beta, \Gamma) \lll (\mathcal{Z}, \Gamma) : \beta \in \Omega\}$ with $(\mathcal{T}_\beta, \Gamma) \in \mathcal{SPO}(\mathcal{Z}, \mathfrak{A}, \Gamma)$ and $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma)$ for each $\beta \in \Omega$. Set $\mathfrak{T}^* = \{(\mathcal{T}_\beta, \Gamma) \sqcap (\mathcal{H}_2, \Gamma) : \beta \in \Omega\}$. Then by Lemma 2.11 we have:

- The family $\mathfrak{T}^*$ is Soft$_{(pre)}$-$\mathcal{LF}((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$.
- For each $\beta \in \Omega$, $(\mathcal{S}_\beta, \Gamma) \lll (\mathcal{T}_\beta, \Gamma) \sqcap (\mathcal{H}_2, \Gamma)$ and $(\mathcal{T}_\beta, \Gamma) \sqcap (\mathcal{H}_2, \Gamma) \in \mathcal{SPO}((\mathcal{H}_2, \Gamma), \mathfrak{A}_{(\mathcal{H}_2,\Gamma)}, \Gamma)$.

Therefore, $(\mathcal{H}_1,\Gamma)$ is soft$_{(pre)_\alpha}$-expandable set in $((\mathcal{H}_2,\Gamma),\mathfrak{A}_{(\mathcal{H}_2,\Gamma)},\Gamma)$. □

Corollary 5.4. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft-$\mathcal{TS}$ with $(\mathcal{H},\Gamma)\lll(\mathcal{Z},\Gamma)$. Then $(\mathcal{H},\Gamma)$ is:*

1. *soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$ if $(\mathcal{H},\Gamma)\in\mathcal{SPCO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and it is soft$_{(pre)_\beta}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$.*
2. *soft$_{(pre)_\beta}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$ if $(\mathcal{H},\Gamma)\in\mathcal{SSO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and it is soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$.*

By Example 4.11, we remark that the assumption $(\mathcal{H},\Gamma)\in\mathcal{SPCO}(\mathcal{Z},\mathfrak{A},\Gamma)$ in Corollary 5.4 can not be replaced by $(\mathcal{H},\Gamma)\in\mathcal{SPC}(\mathcal{Z},\mathfrak{A},\Gamma)$.

Proposition 5.5. *Let $(\mathcal{Z},\mathfrak{A},\Gamma)$ be a soft$_{(pre)}$-expandable soft-$\mathcal{TS}$ with $(\mathcal{H},\Gamma)\lll(\mathcal{Z},\Gamma)$. Then $(\mathcal{H},\Gamma)$ is:*

1. *soft$_{(pre)_\beta}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever $(\mathcal{H},\Gamma)\in\mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)$.*
2. *soft$_{(pre)_\alpha}$-expandable set in $(\mathcal{Z},\mathfrak{A},\Gamma)$ whenever $(\mathcal{H},\Gamma)\in\mathcal{SC}(\mathcal{Z},\mathfrak{A},\Gamma)$.*

Proof. 1) Assume that $(\mathcal{H},\Gamma)\in\mathcal{SRC}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{H},\Gamma):\beta\in\Omega\}$ is a Soft-$\mathcal{LF}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$ family. By Lemma 4.7, $\mathfrak{S}$ is Soft-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ and hence it has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z},\mathfrak{A},\Gamma)$ family $\mathfrak{T}=\{(\mathcal{T}_\beta,\Gamma)\lll(\mathcal{Z},\Gamma):\beta\in\Omega\}$ with $(\mathcal{T}_\beta,\Gamma)\in\mathcal{SPO}(\mathcal{Z},\mathfrak{A},\Gamma)$ and $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)$ for each $\beta\in\Omega$. Set $\mathfrak{T}^*=\{(\mathcal{T}_\beta,\Gamma)\sqcap(\mathcal{H},\Gamma):\beta\in\Omega\}$. Then by Lemma 2.11 we have:

- The family $\mathfrak{T}^*$ is Soft$_{(pre)}$-$\mathcal{LF}((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$.
- For each $\beta\in\Omega$, $(\mathcal{S}_\beta,\Gamma)\lll(\mathcal{T}_\beta,\Gamma)\sqcap(\mathcal{H},\Gamma)$ and $(\mathcal{T}_\beta,\Gamma)\sqcap(\mathcal{H}_2,\Gamma)\in\mathcal{SPO}((\mathcal{H}_2,\Gamma),\mathfrak{A}_{(\mathcal{H}_2,\Gamma)},\Gamma)$.

Therefore, $(\mathcal{H},\Gamma)$ is soft$_{(pre)_\beta}$-expandable set in $((\mathcal{H},\Gamma),\mathfrak{A}_{(\mathcal{H},\Gamma)},\Gamma)$.
2) Immediately from Lemma 4.7. □

Theorem 5.6. *Let $(\mathcal{Z}_\gamma,\mathfrak{A}_\gamma,\Gamma)$ be a soft-$\mathcal{TS}$ for each $\gamma\in\Omega$. Then $(\mathcal{Z}_\gamma,\mathfrak{A}_\gamma,\Gamma)$ is soft$_{(pre)}$-expandable iff the sum of soft-$\mathcal{TS}$ $(\bigoplus_{\gamma\in\Gamma}\mathcal{Z}_\gamma,\mathfrak{A},\Gamma)$ is soft$_{(pre)}$-expandable.*

Proof. Sufficiency: Immediately from Proposition 5.5.

Necessity: Assume that $\mathfrak{S}=\{(\mathcal{S}_\beta,\Gamma):\beta\in\Omega\}$ is a soft-$\mathcal{LF}(\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma,\mathfrak{A},\Gamma)$ family. Then the family $\mathfrak{S}_\gamma=\{(\mathcal{S}_\beta,\Gamma)\sqcap(\mathcal{Z}_\gamma,\Gamma):(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}\}$ is a Soft-$\mathcal{LF}(\mathcal{Z}_\gamma,\mathfrak{A}_\gamma,\Gamma)$ for each $\gamma\in\Omega$ and hence it has a Soft$_{(pre)}$-$\mathcal{LF}(\mathcal{Z}_\gamma,\mathfrak{A}_\gamma,\Gamma)$ family $\mathfrak{T}_\gamma=\{(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)_\gamma},\Gamma):(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}\}$ with $(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)_\gamma},\Gamma)\in\mathcal{SPO}(\mathcal{Z}_\gamma,\mathfrak{A}_\gamma,\Gamma)$ and for each $\gamma\in\Omega$, $(\mathcal{S}_\beta,\Gamma)\sqcap(\mathcal{Z}_\gamma,\Gamma)\lll(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)_\gamma},\Gamma)$ for each $(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}$. Define $\mathfrak{T}=\{(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)},\Gamma):(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}\}$ where $(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)},\Gamma)=\bigsqcup_{\gamma\in\Omega}(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)_\gamma},\Gamma)$. Then:

- By Lemma 2.11, $(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)},\Gamma)\in\mathcal{SPO}(\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma,\mathfrak{A},\Gamma)$ for each $(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}$.
- The family $\mathfrak{T}$ is Soft$_{(pre)}$-$\mathcal{LF}(\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma,\mathfrak{A},\Gamma)$, since for each $\mathcal{P}_\gamma^z\in(\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma,\Gamma)$ there is $\gamma^*\in\Omega$ with $\mathcal{P}_\gamma^z\in(\mathcal{Z}_{\gamma^*},\Gamma)$ and hence there is $(\mathcal{W}_{\gamma^*},\Gamma)\in\mathcal{SPO}(\mathcal{Z}_{\gamma^*},\mathfrak{A}_{\gamma^*},\Gamma)$ that meets at most finite members of $\mathfrak{T}_{\gamma^*}$, say $(\mathcal{T}_{(\mathcal{S}_1,\Gamma)_{\gamma_*}},\Gamma),(\mathcal{T}_{(\mathcal{S}_2,\Gamma)_{\gamma_*}},\Gamma),\ldots(\mathcal{T}_{(\mathcal{S}_n,\Gamma)_{\gamma_*}},\Gamma)$. Since $(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)_\delta},\Gamma)\sqcap(\mathcal{W}_{\gamma^*},\Gamma)=\Phi$ for each $(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}$, then $(\mathcal{W}_{\alpha^*},\Gamma)\sqcap(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)},\Gamma)=\Phi$ for each $(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}-\{(\mathcal{S}_1,\Gamma),(\mathcal{S}_2,\Gamma),\ldots.(\mathcal{S}_n,\Gamma)\}$.
- For each $(\mathcal{S}_\beta,\Gamma)\in\mathfrak{S}$, $(\mathcal{S}_\beta,\Gamma)=(\mathcal{S}_\beta,\Gamma)\sqcap\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma\lll(\mathcal{T}_{(\mathcal{S}_\beta,\Gamma)},\Gamma)$.

Therefore, $(\bigoplus_{\gamma\in\Omega}\mathcal{Z}_\gamma,\mathfrak{A},\Gamma)$ is soft$_{(pre)}$-expandable. □

6. Conclusion

Researchers have paid a lot of attention to soft topology recently, and there has been a lot of advancement. Soft topologies enable us to investigate more concepts and features than classical topologies because certain of its concepts and ideas, like the soft separation axioms [24] and generalizations of soft-open sets [15, 17], lack analogs in general topology. Additionally, under certain types of soft topologies, like extended topologies [11] and full topologies [16], many topological ideas behaviors can be examined using their corresponding soft topological concepts.

A new generalization of soft expandable spaces called soft$_{(pre)}$-expandable spaces has been familiarized in this work. We have initiated its master properties and displayed some of its extensions. We have demonstrated that soft$_{(pre)}$-expandable and soft expandable spaces are equivalent if the soft topological space is either soft-$\mathcal{QSM}$ or ω_0-soft$_{(pre)}$-compact. To demonstrate how they relate to one another and other soft areas, we have included some elucidative examples. The characteristics of these ideas have been established and discussed lucidly.

In the following studies, we plan to examine the ideas and findings utilizing various soft structures such as supra and infra soft topologies and weak soft structures. We also look at the interrelations between the current concepts (as part of covering properties) and the different types of soft separation axioms defined with respect to partial and total belonging relations and distinct ordinary points. Finally, we anticipate that our effort will support the investigation of soft topology and enable the production of new findings.

References

[1] M. Akdag, A. Ozkan, *Soft α-open sets and soft α-continuous functions*, Abstr. Appl. Anal. **2014** (2014), Art. ID 891341.
[2] M. Akdag, A. Ozkan, *On soft preopen sets and soft pre separation axioms*, Gazi Univ. J. Sci. **27** (2014), 1077–1083.
[3] M. I. Ali, F. Feng, X. Liu, M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. **57** (2009), 1547–1553.
[4] H. H. Al-jarrah, A. Rawshdeh, T. M. Al-shami, *On soft compact and soft Lindelöf spaces via soft regular closed sets*, Afr. Mat. **33** (2022), Art. 23.
[5] H. H. Al-jarrah, K. Y. Al-zoubi, *P-expandable spaces*, Ital. J. Pure Appl. Math. **41** (2019), 497–507.
[6] T. M. Al-shami, *Soft somewhere dense sets on soft topological spaces*, Commun. Korean Math. Soc. **33** (2018), 1341–1356.
[7] T. M. Al-shami, *Homeomorphism and quotient mappings in infra soft topological spaces*, J. Math. **2021** (2021), Art. ID 3388288.
[8] T.M. Al-shami, M. E. El-shafei, *On soft compact and soft Lindelöf spaces via soft pre-open sets*, Ann. Fuzzy Math. Inform. **17** (2019), 79–100.
[9] T. M. Al-shami, M. E. El-shafei, *T-soft equality relation*, Turk. J. Math. **44** (2020), 1427–1441.
[10] T. M. Al-shami, M. E. El-shafei, M. Abo-Elhamayel, *Almost soft compact and approximately soft Lindelöf spaces*, J. Taibah Univ. Sci. **12** (2018), 620–630.
[11] T. M. Al-shami, Lj. D. R. Kočinac, *The equivalence between the enriched and extended soft topologies*, Appl. Comput. Math. **18** (2019), 149–162.
[12] T. M. Al-shami, Lj. D. R. Kočinac, *Nearly soft Menger spaces*, J. Math. **2020** (2020), Art. ID 3807418.
[13] T. M. Al-shami, L. D. R. Kočinac, *Almost soft Menger and weakly soft Menger spaces*, Appl. Comput. Math. **21** (2022), 35–51.
[14] T. M. Al-shami, Lj. D. R. Kočinac, B. A. Asaad, *Sum of soft topological spaces*, Mathematics **8** (2020), 990.
[15] T. M. Al-shami, A. Mhemdi, *On soft parametric somewhat-open sets and applications via soft topologies*, Heliyon **9** (2023), e21472.
[16] T. M. Al-shami, A. Mhemdi, R. Abu-Gdairi, M.E. El-shafei, *Compactness and connectedness via the class of soft somewhat open sets*, AIMS Math. **8** (2023), 815–840.
[17] T. M. Al-shami, A. Mhemdi, A. Rawshdeh, H. H. Al-jarrah, *On weakly soft somewhat open sets*, Rocky Mountain J. Math. **54** (2024), 13–30.
[18] T. M. Al-shami, A. Mhemdi, A. Rawshdeh, H. H. Al-jarrah, *Soft version of compact and Lindelöf spaces using soft somewhere dense set*, AIMS Math. **6** (2021), 8064–8077.
[19] I. Arockiarani, A. A. Lancy, *Generalized soft gβ- closed sets and soft gsβ-closed sets in soft topological spaces*, Internat. J. Math. Arch. **4** (2013), 1–7.
[20] B. A. Asaad, *Results on soft extremally disconnectedness of soft topological spaces*, J. Math. Computer Sci. **17** (2017), 448–464.
[21] A. Aygünoğlu, H. Aygün, *Some notes on soft topological spaces*, Neural Comput. Applic. **21** (2012), 113–119.
[22] B. Chen, *Soft semi-open sets and related properties in soft topological spaces*, Appl. Math. Inf. Sci. **7** (2013), 287–294.
[23] S. Das, S. K. Samanta, *Soft metric*, Ann. Fuzzy Math. Inform. **6** (2013), 77–94.
[24] M. E. El-shafei, M. Abo-Elhamayel, T. M. Al-shami, *Partial soft separation axioms and soft compact spaces*, Filomat **32** (2018), 4755–4771.
[25] S. Hussain, B. Ahmad, *Some properties of soft topological spaces*, Comput. Math. Appl. **62** (2011), 4058–4067.
[26] G. Ilango, M. Ravindran, *On soft preopen sets in soft topological spaces*, Int. J. Math. Res. **5** (2013), 399–409.
[27] A. Kharal, B. Ahmad, *Mappings on soft classes*, New Math. Nat. Comput. **7** (2011), 471–481.

[28] F. Lin, *Soft connected and soft paracompact spaces*, Int. J. Math. Comput. Sci. Eng. **7** (2013), 37–43.
[29] P. K. Maji, R. Biswas, R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003), 555–562.
[30] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl. **62** (2011), 3524–3528.
[31] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl. **37** (1999), 19–31.
[32] Sk. Nazmul, S. K. Samanta, *Neighbourhood properties of soft topological spaces*, Ann. Fuzzy Math. Inform. **6** (2013), 1-15.
[33] D. Pei, D. Miao, *From soft sets to information systems*, in Proceedings of the IEEE International Conference on Granular Computing, **2** (2005) 617–621.
[34] E. Peyghan , B. Samadi , A. Tayebi, *Some results related to soft topological spaces*, Facta Univ. Ser. Math. Inform. **29** (2014), 325–336.
[35] M. Ravindran, G. Manju, *Some results on soft pre-continuity*, Malaya J. Mat. **1** (2015), 10–17.
[36] M. Ravindran, Manju, G. Ilango, *A note on soft pre-open sets*, Int. J. Pure Appl. Math. **106** (2016), 63–78.
[37] A. Rawshdeh, H. H. Al-jarrah, T. M. Al-shami, *Soft expandable spaces*, Filomat **37** (2023), 2845–2858.
[38] A. Rawshdeh, H. H. Al-jarrah, S. Tiwari, A. Tallafha, *Soft semi-linear uniform spaces and their perceptual application*, J. Intell. Fuzzy Syst. **44** (2023), 4175–4184.
[39] M. Shabir, M. Naz, *On soft topological spaces*, Comput. Math. Appl. **61** (2011), 1786–1799.
[40] D. Sivaraj, V. E. Sasikala, *A study on soft α-open sets*, IOSR Journal of Mathematics **12** (2016), 70–74.
[41] J. Subhashinin, C. Sekar, *Related properties of soft dense and soft pre open sets in a soft topological spaces*, IOSR Journal of mathematics, **2** (2014) 34–48.
[42] E. Turanli, I. Demirb , O. B. Özbakir, *Some types of soft paracompactness via soft ideals*, Int. J. Nonlinear Anal. Appl. **10** (2019), 197–211.
[43] Ş. Yüksel, N. Tozlu, Z.G. Ergül, *Soft regular generalized closed sets in soft topological spaces*, Int. J. Math. Anal. **8** (2014), 355–367.
[44] Y. Yumak , A. K. Kaymakci, *Soft β-open sets and their applications*, J. New Theory **4** (2015), 80-89.
[45] I. Zorlutuna, M. Akdag, W. K. Min, S. Atmaca, *Remarks on soft topological spaces*, Ann. Fuzzy Math. Inform. **2** (2012), 171–185.
[46] I. Zorlutuna, H. Çakir, *On continuity of soft mappings*, Appl. Math. Inf. Sci. **9** (2015), 403–409.