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Geodesic and F-geodesic exploration with a vertical generalized Berger-type deformed Sasaki metric on the tangent and φ-unit tangent bundles

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Abstract. In this research paper, we delve into a comprehensive exploration of geodesics and F-geodesics inquiries. Our investigation centers around a vertical generalized Berger-type deformed Sasaki metric, which is applied to both the tangent bundle TM and the φ-unit tangent bundle $T_1^{\varphi}M$. These bundles are situated over an anti-paraKähler manifold (M^{2m}, φ, g).

1. Introduction

The study of the differential geometry of the tangent bundle has opened up a rich domain in the field of differential geometry, presenting various new challenges and problems to explore. Since the middle of the last century, geometric structures on bundles have been a subject of extensive study. The natural extensions of a Riemannian metric g from a Riemannian manifold (M, g) to its tangent or cotangent bundles create new (pseudo) Riemannian structures, each possessing interesting geometric properties. One of the most well-known Riemannian metrics on the tangent bundle over a Riemannian manifold (M, g) is the Sasaki metric, denoted as g_S, introduced by Sasaki in [22]. Over time, the geometric properties of the Sasaki metric garnered significant attention from researchers. However, in many instances, their investigations led to the conclusion that the base manifold was flat, as exemplified in [12, 16]. Consequently, this realization prompted many researchers to explore various deformations of the Sasaki metric. In addition to the Sasaki metric, there is another Riemannian metric defined on the tangent bundle TM by Musso and Tricerri [16], known as the Cheeger-Gromoll metric g_{CG}. Although originally introduced by Cheeger and Gromoll in [5], Musso and Tricerri later provided a more comprehensible expression for it and gave it its name. In this context, Abbassi and Sarih [1] introduced natural metrics on both the tangent bundle and the unit tangent bundles. These metrics encompass the Sasaki metric, the Cheeger-Gromoll metric and all other well-known g-natural metrics in the literature.

Inspired by the Berger deformation of metrics on a unit sphere, Yampolsky [26] proposed an alternative approach to deform the Sasaki metric on slashed and unit tangent bundles over Kählerian manifolds, utilizing an almost complex structure denoted as J. This deformed metric, referred to as a Berger type

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deformed Sasaki metric, was examined for its geodesic properties. In a subsequent work published as [2], Altunbas, Simsek, and Gezer introduced the Berger type deformed Sasaki metric on the tangent bundle over an anti-paraKähler manifold. They conducted a comprehensive analysis of the Riemannian curvature tensors associated with this metric and presented various geometric results. Additionally, they defined certain almost anti-paraHermitian structures on the tangent bundle and established conditions under which these structures could be classified as anti-paraKähler or quasi-anti-paraKähler.

Geodesics have applications in various fields, including physics, geometry, and computer graphics, where they are used to find optimal paths on curved surfaces. Many authors have extensively investigated geodesy on the tangent bundle, focusing on oblique geodesics, non-vertical geodesics, and their projections onto the base manifold (see [9, 20, 26, 28]). Sasaki [23] and Sato [24] provided a comprehensive description of the curves and the associated vector fields that generate non-vertical geodesics on the tangent bundle and the unit tangent bundle, respectively. Their works demonstrated that the projected curves exhibit constant geodesic curvatures (Frenet curvatures). Nagy [18] extended these findings to the scenario of locally symmetric base manifolds, further enriching our understanding of geodesics in such contexts. Yampolsky [26] pursued similar studies on the tangent bundle and the unit tangent bundle, utilizing the Berger-type deformed Sasaki metric over Kählerian manifolds. This research extended to both locally symmetric base manifolds and base manifolds with constant holomorphic curvature.

The concept of F-planar curves serves as a generalization encompassing magnetic curves and, by extension, geodesics, as detailed in references [11] and [17]. It is worth noting that the notion of F-geodesics, introduced in [3], presents a variation that slightly differs from that of F-planar curves. In recent mathematical literature, there has been a series of papers dedicated to the exploration of magnetic curves, F-planar curves, and F-geodesics (for example, see [7], [8], and [19]). These works have contributed to a deeper understanding of these mathematical concepts and their applications.

In this paper, we begin with an introduction and provide preliminary information. In Section 3, we introduce and analyze the vertical generalized Berger-type deformed Sasaki metric on both the tangent bundle TM and the φ-unit tangent bundle $T_1^{\varphi}M$ over an anti-paraKähler manifold (M^{2m}, φ, g). We also delve into the Levi-Civita connection associated with this metric, as demonstrated in Theorem 3.3. Moving on to Section 4, we explore various aspects of geodesics and F-geodesics related to the vertical generalized Berger-type deformed Sasaki metric. Firstly, we investigate geodesics on the tangent bundle, establishing both necessary and sufficient conditions for a curve to be a geodesic with respect to this metric, as described in Theorems 4.1 and 4.7. Secondly, we study the geodesics on the φ-unit tangent bundle concerning the vertical generalized Berger-type deformed Sasaki metric. In this context, we once again provide necessary and sufficient conditions for a curve to qualify as a geodesic under this metric, as elucidated in Theorem 4.9. Furthermore, we delve into the Frenet curvatures of the projected non-vertical geodesics, as discussed in Theorems 4.11, 4.13, 4.15 and 4.17. In the third part of Section 4, our focus shifts to F-geodesics and F-planar curves on the tangent bundle. We examine these concepts in relation to the Levi-Civita connection of the vertical generalized Berger-type deformed Sasaki metric, providing relevant conditions and results, including Theorems 5.1, 5.3 and 5.5. Finally, we extend our exploration to the φ-unit tangent bundle in the same section. Here, we study F-geodesics and F-planar curves with respect to the Levi-Civita connection of this metric. The results and conditions for these cases are given in Theorems 5.9, 5.11, 5.13 and 5.15.

2. Preliminaries

Consider the tangent bundle TM over an m-dimensional Riemannian manifold (M^m, g), with the natural projection $\pi : TM \to M^m$. If you have a local chart $(U, x^i)_{i=\overline{1,m}}$ for M^m, it induces a local chart $(\pi^{-1}(U), x^i, \xi^i)_{i=\overline{1,m}}$ for TM. Let Γ_{ij}^k represent the Christoffel symbols of g, and ∇ be the Levi-Civita connection of g.

The Levi-Civita connection ∇ provides a direct sum decomposition of the tangent bundle of TM at any point (x, ξ) in TM into the vertical subspace

$$V_{(x,\xi)}TM = Ker(d\pi_{(x,\xi)}) = \{a^i \frac{\partial}{\partial \xi^i}|_{(x,\xi)},\ a^i \in \mathbb{R}\}$$

and the horizontal subspace

$$H_{(x,\xi)}TM = \{a^i \frac{\partial}{\partial x^i}|_{(x,\xi)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}|_{(x,\xi)},\ a^i \in \mathbb{R}\}.$$

Let us consider $X = X^i \frac{\partial}{\partial x^i}$ as a local vector field on the manifold M^m. We define the vertical and horizontal lifts of X as follows [27]

$$\text{Vertical lift} \quad : \quad {}^V X = X^i \frac{\partial}{\partial \xi^i},$$

$$\text{Horizontal lift} \quad : \quad {}^H X = X^i (\frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}).$$

It is worth noting that ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial \xi^i}$. Consequently, the pair $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ for $i = 1$ to m forms a locally adapted frame on the tangent bundle TM.

Additionally, we can define the vertical spray ${}^V\xi$ on TM as

$${}^V\xi = \xi^i\, {}^V(\frac{\partial}{\partial x^i}) = \xi^i \frac{\partial}{\partial \xi^i}.$$

${}^V\xi$ is also known as the canonical or Liouville vector field on TM [27].

3. A vertical generalized Berger-type deformed Sasaki metric

An almost product structure φ on a manifold M is a $(1,1)$-tensor field on M that satisfies the condition: $\varphi^2 = id_M$, where id_M represents the identity tensor field of type $(1,1)$ on M and φ is distinct from $\pm id_M$. The pair (M, φ) is denoted as an almost product manifold. An almost para-complex manifold is essentially an almost product manifold (M, φ) with the additional requirement that the two eigenbundles, TM^+ and TM^-, corresponding to the eigenvalues $+1$ and -1 of φ, must have the same rank. It is important to note that the dimension of an almost para-complex manifold is always even, as noted by [6]. An almost para-complex structure φ is considered integrable when the Nijenhuis tensor N_φ, defined as

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X, Y]$$

vanishes entirely on the manifold M^{2m}. Moreover, an almost para-complex structure is integrable if and only if it is possible to introduce a torsion-free linear connection ∇ such that $\nabla\varphi = 0$, as indicated by [21].

A Riemannian metric g is identified as an anti-paraHermitian metric when it satisfies the condition

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently, the purity condition, often referred to as a B-metric

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for all vector fields X and Y on the manifold M^{2m} [10, 13–15, 21].

If (M^{2m}, φ) is an almost para-complex manifold with an anti-paraHermitian metric g, the triple (M^{2m}, φ, g) is recognized as an almost anti-paraHermitian manifold, also referred to as an almost B-manifold [10, 13–15, 21]. Additionally, (M^{2m}, φ, g) is labeled as an anti-paraKähler manifold (B-manifold) if the almost para-complex structure φ is parallel with respect to the Levi-Civita connection ∇ of the metric g, meaning $\nabla\varphi = 0$. It is a well-known fact that in the case of an anti-paraKähler manifold (M^{2m}, φ, g), the Riemannian curvature tensor is pure, as mentioned by [21].

Definition 3.1. *Suppose we have an almost anti-paraHermitian manifold (M^{2m},φ,g) with its tangent bundle denoted as TM. We define a fiber-wise vertical generalized Berger-type deformation of the Sasaki metric on TM as follows [4]*

$$\begin{aligned}\tilde{g}({}^HX,{}^HY) &= g(X,Y),\\ \tilde{g}({}^VX,{}^HY) &= \tilde{g}({}^HX,{}^VY)=0,\\ \tilde{g}({}^VX,{}^VY) &= g(X,Y)+fg(X,\varphi\xi)g(Y,\varphi\xi)\end{aligned}$$

for all vector fields X and Y on M^{2m}, where $f: M^{2m}\rightarrow]0,+\infty[$ is a strictly positive smooth function.

Subsequently, we consider $\lambda = 1 + fr^2$, where $r^2 = g(\xi,\xi) = |\xi|^2$ and $|.|$ denotes the norm with respect to g.

The Levi-Civita connection $\widetilde{\nabla}$ associated with the vertical generalized Berger-type deformed Sasaki metric on TM is described by the Koszul formula, which can be expressed as

$$\begin{aligned}2\tilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}\left(\tilde{g}(\widetilde{Y},\widetilde{Z})\right)+\widetilde{Y}\left((\tilde{g}(\widetilde{Z},\widetilde{X})\right)-\widetilde{Z}\left(\tilde{g}(\widetilde{X},\widetilde{Y})\right)\\ &\quad+\tilde{g}(\widetilde{Z},[\widetilde{X},\widetilde{Y}])+\tilde{g}(\widetilde{Y},[\widetilde{Z},\widetilde{X}])-\tilde{g}(\widetilde{X},[\widetilde{Y},\widetilde{Z}]),\end{aligned}$$

where $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ are vector fields defined on TM. Through standard direct calculations, we arrive at the following outcome.

Theorem 3.2. *[4] In the context of an anti-paraKähler manifold (M^{2m},φ,g), when we consider its tangent bundle $(TM,\tilde{g})$ endowed with the vertical generalized Berger-type deformed Sasaki metric, the following formulas can be established*

$$\begin{aligned}1)\ \widetilde{\nabla}_{^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)\xi),\\ 2)\ \widetilde{\nabla}_{^HX}{}^VY &= \frac{1}{2}{}^H(R(\xi,Y)X)+{}^V(\nabla_XY)+\frac{1}{2\lambda}X(f)g(Y,\varphi\xi){}^V(\varphi\xi),\\ 3)\ \widetilde{\nabla}_{^VX}{}^HY &= \frac{1}{2}{}^H(R(\xi,X)Y)+\frac{1}{2\lambda}Y(f)g(X,\varphi\xi){}^V(\varphi\xi),\\ 4)\ \widetilde{\nabla}_{^VX}{}^VY &= -\frac{1}{2}g(X,\varphi\xi)g(Y,\varphi\xi){}^H(grad f)+\frac{f}{\lambda}g(X,\varphi Y){}^V(\varphi\xi)\end{aligned}$$

for all vector fields X and Y on M^{2m}, where ∇ is the Levi-Civita connection of (M^{2m},φ,g) and R is its Riemannian curvature tensor.

The φ-unit tangent (sphere) bundle over an anti-paraKähler manifold (M^{2m},φ,g) is a hypersurface defined as

$$T_1^{\varphi}M=\left\{(x,\xi)\in TM,\ g(\xi,\varphi\xi)=1\right\}.$$

The unit normal vector field to $T_1^{\varphi}M$ is expressed as

$$\mathcal{N}=\sqrt{\frac{f}{\lambda(\lambda-1)}}\,{}^V(\varphi\xi),$$

where $\lambda = 1 + fg(\xi,\xi)$.

The tangential lift, denoted as ${}^TX$, of a vector $X\in T_xM$ at point x on the manifold M^{2m} to the point $(x,\xi)\in T_1^{\varphi}M$ is obtained by projecting the vertical lift of X to the point (x,u) with respect to the unit normal vector $\mathcal{N}$. This is expressed as

$${}^TX={}^VX-\tilde{g}_{(x,\xi)}({}^VX,\mathcal{N}_{(x,\xi)})\mathcal{N}_{(x,\xi)}={}^VX-\frac{f}{\lambda-1}g_x(X,\varphi\xi){}^V(\varphi\xi)_{(x,\xi)}.$$

The tangent space $T_{(x,\xi)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at $(x,\xi) \in T_1^{\varphi}M$ is defined as

$$T_{(x,\xi)}T_1^{\varphi}M = \{{}^{H}X + {}^{T}Y \,/\, X \in T_xM, Y \in (\varphi\xi)^{\perp} \subset T_xM\},$$

where $(\varphi\xi)^{\perp} = \Big\{Y \in T_xM,\ g(Y,\varphi\xi) = 0\Big\}$. Given a vector field X on M^{2m}, the tangential lift ${}^{T}X$ of X is given by

$${}^{T}X_{(x,\xi)} = \Big({}^{V}X - \tilde{g}({}^{V}X, \mathcal{N})\mathcal{N}\Big)_{(x,\xi)} = {}^{V}X_{(x,\xi)} - \frac{f}{\lambda - 1} g_x(X_x, \varphi\xi)^{V}(\varphi\xi)_{(x,\xi)}.$$

For the sake of clarity in notation, we can express $\bar{X}$ as $\bar{X} = X - \dfrac{f}{\lambda - 1} g(X,\varphi\xi)\varphi\xi,$, and in this form, ${}^{T}X$ is equivalent to ${}^{V}\bar{X}$.

The Levi-Civita connection $\widehat{\nabla}$ on $T_1^{\varphi}M$ with respect to the vertical generalized Berger-type deformed Sasaki metric is characterized by the Gauss formula

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widetilde{\nabla}_{\widehat{X}}\widehat{Y} - \tilde{g}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y}, \mathcal{N})\mathcal{N}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$. Using usual direct calculations, we find the following result.

Theorem 3.3. *[4] In the context of an anti-paraKähler manifold (M^{2m}, φ, g) and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, the Levi-Civita connection $\widehat{\nabla}$ of this metric on $T_1^{\varphi}M$ gives rise to the following formulas*

$$\begin{aligned}
1)\ \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X,Y)\xi),\\
2)\ \widehat{\nabla}_{{}^{H}X}{}^{T}Y &= \frac{1}{2}{}^{H}(R(\xi,Y)X) + {}^{T}(\nabla_X Y),\\
3)\ \widehat{\nabla}_{{}^{T}X}{}^{H}Y &= \frac{1}{2}{}^{H}(R(\xi,X)Y),\\
4)\ \widehat{\nabla}_{{}^{T}X}{}^{T}Y &= \frac{f^2}{(\lambda-1)^2} g(X,\varphi\xi)g(Y,\varphi\xi)^{T}\xi - \frac{f}{\lambda-1} g(Y,\varphi\xi)^{T}(\varphi X)
\end{aligned}$$

for all vector fields X, Y on M^{2m}.

4. Geodesics

4.1. Geodesics on the tangent bundle with the vertical generalized Berger-type deformed Sasaki metric

Consider a curve $\Gamma = (\gamma(t), \xi(t))$ naturally parameterized on the tangent bundle TM, where t serves as an arc length parameter along Γ. In this parameterization, γ represents a curve on the manifold M, and ξ is a vector field along this curve. We introduce the following notations: $\gamma'_t = \frac{d\gamma}{dt}$, $\gamma''_t = \nabla_{\gamma'_t}\gamma'_t$, $\xi'_t = \nabla_{\gamma'_t}\xi$, $\xi''_t = \nabla_{\gamma'_t}\xi'_t$ and $\Gamma'_t = \frac{d\Gamma}{dt}$. With these notations in place, the relationship can be expressed as

$$\Gamma'_t = {}^{H}\gamma'_t + {}^{V}\xi'_t. \tag{1}$$

Theorem 4.1. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), where $(TM, \tilde{g})$ represents its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric, a curve $\Gamma = (\gamma(t), \xi(t))$ on TM is a geodesic if and only if the following conditions hold*

$$\left\{\begin{aligned}
\gamma''_t &= R(\xi'_t, \xi)\gamma'_t + \frac{1}{2} g(\xi'_t, \varphi\xi)^2\, grad f,\\
\xi''_t &= -\frac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi.
\end{aligned}\right. \tag{2}$$

Proof. From 1 and Theorem 3.2, we find

$$\begin{aligned}
\widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \widetilde{\nabla}_{({}^H\gamma'_t + {}^V\xi'_t)}({}^H\gamma'_t + {}^V\xi'_t) \\
&= \widetilde{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^H\gamma'_t}{}^V\xi'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^V\xi'_t \\
&= {}^H\gamma''_t + {}^H(R(\xi,\xi'_t)\gamma'_t) + {}^V\xi''_t + \frac{1}{\lambda}\gamma'_t(f)g(\xi'_t,\varphi\xi)^V(\varphi\xi) \\
&\quad -\frac{1}{2}g(\xi'_t,\varphi\xi)^{2H}(grad f) + \frac{f}{\lambda}g(\xi'_t,\varphi\xi'_t)^V(\varphi\xi) \\
&= {}^H\Big(\gamma''_t + R(\xi,\xi'_t)\gamma'_t - \frac{1}{2}g(\xi'_t,\varphi\xi)^2\, grad f\Big) \\
&\quad +{}^V\Big(\xi''_t + \frac{1}{\lambda}\Big(g(\gamma'_t, grad f)g(\xi'_t,\varphi\xi) + f\, g(\xi'_t,\varphi\xi'_t)\Big)\varphi\xi\Big).
\end{aligned}$$

If we put $\widetilde{\nabla}_{\Gamma'_t}\Gamma'_t$ equal to zero, we find (2). □

Corollary 4.2. *In the context of an anti-paraKähler manifold (M^{2m},φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the vertical generalized Berger-type deformed Sasaki metric, when considering a curve $\Gamma = (\gamma(t),\xi(t))$ on TM, if the function f is a constant, then Γ is a geodesic if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t,\xi)\gamma'_t \\ \xi''_t = -\dfrac{f}{\lambda}g(\xi'_t,\varphi\xi'_t)\varphi\xi. \end{cases}$$

If γ is a curve on the manifold M^{2m}, then the curve $\Gamma = (\gamma(t),\gamma'_t(t))$ is referred to as the natural lift of the curve γ [27]. Thus, we have the following result.

Corollary 4.3. *In the context of an anti-paraKähler manifold (M^{2m},φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the vertical generalized Berger-type deformed Sasaki metric, it is noteworthy that the natural lift $\Gamma = (\gamma(t),\gamma'_t(t))$ of any geodesic curve γ is itself a geodesic on $(TM,\tilde{g})$.*

When discussing a curve $\Gamma = (\gamma(t),\xi(t))$ on TM, it is termed a horizontal lift of the curve $\gamma(t)$ on M^{2m} if and only if the condition $\xi'_t = 0$ holds [27]. Hence, we have the following.

Corollary 4.4. *In the context of an anti-paraKähler manifold (M^{2m},φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the vertical generalized Berger-type deformed Sasaki metric, it is important to note that the horizontal lift $\Gamma = (\gamma(t),\xi(t))$ of any geodesic curve γ is itself a geodesic on $(TM,\tilde{g})$.*

Remark 4.5. *As a reminder, note that locally we have*

$$\gamma''_t = \sum_{l=1}^{2m}\Big(\frac{d^2\gamma^l}{dt^2} + \sum_{i,j=1}^{2m}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^l_{ij}\Big)\frac{\partial}{\partial x^l}, \tag{3}$$

and

$$\xi'_t = \sum_{l=1}^{2m}\Big(\frac{d\xi^l}{dt} + \sum_{i,j=1}^{2m}\frac{d\gamma^j}{dt}\xi^i\Gamma^l_{ij}\Big)\frac{\partial}{\partial x^l}. \tag{4}$$

Example 4.6. *Let $\big(]0,+\infty[^2, g, \varphi\big)$ be an anti-paraKähler manifold such that*

$$g = x^2dx^2 + y^2dy^2$$

and

$$\varphi\frac{\partial}{\partial x} = \frac{x}{y}\frac{\partial}{\partial y} \quad , \quad \varphi\frac{\partial}{\partial y} = \frac{y}{x}\frac{\partial}{\partial x}.$$

The non-null Christoffel symbols of the Riemannian connection are

$$\Gamma_{11}^{1}=\frac{1}{x}\,,\ \ \Gamma_{22}^{2}=\frac{1}{y}.$$

1) *Let γ be a curve such that $\gamma(t)=(x(t),y(t))$, from* (3)*, the geodesic γ such that $\gamma(0)=(a,b)\in]0,+\infty[^2$ and $\gamma'_t(0)=(\mu,\eta)\in]0,+\infty[^2$ satisfies the system of differential equations*

$$\gamma''_t=0\Leftrightarrow \frac{d^2\gamma^l}{dt^2}+\sum_{i,j=1}^{2}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^{l}=0\quad\Leftrightarrow\quad \begin{cases}\dfrac{d^2x}{dt^2}+\dfrac{(\frac{dx}{dt})^2}{x}=0\\ \dfrac{d^2y}{dt^2}+\dfrac{(\frac{dy}{dt})^2}{y}=0\end{cases}\Leftrightarrow\begin{cases}x(t)=\sqrt{2a\mu t+a^2}\\ y(t)=\sqrt{2b\eta t+b^2}\end{cases}.$$

Hence $\gamma'_t(t)=\dfrac{a\mu}{\sqrt{2a\mu t+a^2}}\partial_x+\dfrac{b\eta}{\sqrt{2b\eta t+a^2}}\partial_y$ *and* $\gamma(t)=(\sqrt{2a\mu t+a^2},\sqrt{2b\eta t+b^2})$.
From Corollary 4.3*, the curve $\Gamma_1=(\gamma(t),\gamma'_t(t))$ is a geodesic on $T]0,+\infty[^2$.*
2) *If $\Gamma_2=(\gamma(t),\xi(t))$ is the horizontal lift of γ, such that $\xi(t)=(u(t),v(t))$, i.e., $\xi'_t=0$, from* (4)*, we have*

$$\xi'_t=0\Leftrightarrow\frac{d\xi^l}{dt}+\sum_{i,j=1}^{2}\frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^{l}=0\Leftrightarrow\begin{cases}\dfrac{du}{dt}+\dfrac{dx}{dt}\dfrac{u}{x}=0,\\ \dfrac{dv}{dt}+\dfrac{dy}{dt}\dfrac{v}{y}=0,\end{cases}\Leftrightarrow\begin{cases}u(t)=\dfrac{k_1}{\sqrt{2a\mu t+a^2}},\\ v(t)=\dfrac{k_2}{\sqrt{2b\eta t+b^2}}.\end{cases}$$

Hence $\xi(t)=\dfrac{k_1}{\sqrt{2a\mu t+a^2}}\partial_x+\dfrac{k_2}{\sqrt{2b\eta t+b^2}}\partial_y$*, where* $k_1,k_2\in\mathbb{R}$.
From Corollary 4.4*, the curve $\Gamma_2=(\gamma(t),\xi(t))$ is a geodesic on $T]0,+\infty[^2$.*

Theorem 4.7. *In the context of an anti-paraKähler manifold (M^{2m},φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the vertical generalized Berger-type deformed Sasaki metric, if we have a geodesic γ on M^{2m} and $\Gamma=(\gamma(t),\xi(t))$ is a geodesic on TM with the condition that $g(\xi,\varphi\xi)$ is not constant, then it follows that the function f is constant along the curve γ.*

Proof. Let γ be a geodesic on M^{2m}, then $\gamma''_t=0$. Using the first equation of the formula (2), we obtain

$$\begin{aligned} g(\gamma''_t,\gamma'_t)=0 \quad&\Rightarrow\quad g(R(\xi'_t,\xi)\gamma'_t,\gamma'_t)+\frac{1}{2}g(\xi'_t,\varphi\xi)^2 g(grad f,\gamma'_t)=0\\ &\Rightarrow\quad \frac{1}{2}g(\xi'_t,\varphi\xi)^2 g(grad f,\gamma'_t)=0,\end{aligned}$$

from which we have

$$\begin{aligned} g(\xi,\varphi\xi)\neq const \quad&\Rightarrow\quad \gamma'_t g(\xi,\varphi\xi)\neq 0\\ &\Rightarrow\quad g(\xi'_t,\varphi\xi)\neq 0.\end{aligned}$$

Hence, $g(grad f,\gamma'_t)=0\Rightarrow\gamma'_t(f)=0$. □

4.2. Geodesics on the φ-unit tangent bundle with the vertical generalized Berger-type deformed Sasaki metric

Lemma 4.8. *In the context of an anti-paraKähler manifold (M^{2m},φ,g) and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, when considering a curve $\Gamma=(\gamma(t),\xi(t))$ on $T_1^{\varphi}M$, we can state the following*

$$\Gamma'_t={}^{H}\gamma'_t+{}^{T}\xi'_t. \tag{5}$$

Proof. Utilizing equation (1), we can express

$$\Gamma'_t = {}^H\gamma'_t + {}^V\xi'_t = {}^H\gamma'_t + {}^T\xi'_t + \frac{f}{\lambda - 1} g(\xi'_t, \varphi\xi)^V(\varphi\xi).$$

Since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$, we have $g(\xi, \varphi\xi) = 1$. Additionally, we observe

$$0 = \gamma'_t g(\xi, \varphi\xi) = 2g(\xi'_t, \varphi\xi),$$

which implies

$$g(\xi'_t, \varphi\xi) = 0. \tag{6}$$

Thus, we have successfully completed the proof of the lemma. □

Subsequently, considering t as an arc length parameter on C, based on equation (5), we can state the following

$$1 = |\gamma'_t|^2 + |\xi'_t|^2. \tag{7}$$

Theorem 4.9. *In the scenario of an anti-paraKähler manifold $(M^{2m}, \varphi, g))$ and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, the curve $\Gamma = (\gamma(t), \xi(t))$ qualifies as a geodesic on $T_1^{\varphi}M$ if and only if*

$$\begin{cases} \gamma''_t &= R(\xi'_t, \xi)\gamma'_t, \\ \xi''_t &= 0. \end{cases} \tag{8}$$

Moreover

$$\begin{cases} |\xi'_t| &= \kappa, \\ |\gamma'_t| &= \sqrt{1-\kappa^2}, \end{cases} \tag{9}$$

where $\kappa = const.$ and $0 \leq \kappa \leq 1$.

Proof. By employing equation (5) and Theorem 3.3, we can compute the derivative $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$ as follows

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \widehat{\nabla}_{({}^H\gamma'_t + {}^T\xi'_t)}({}^H\gamma'_t + {}^T\xi'_t) \\ &= \widehat{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^H\gamma'_t}{}^T\xi'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^T\xi'_t \\ &= {}^H\gamma''_t + {}^H(R(u, \xi'_t)\gamma'_t) + {}^T\xi''_t \\ &= {}^H\!\left(\gamma''_t - R(\xi'_t, \xi)\gamma'_t\right) + {}^T\xi''_t. \end{aligned}$$

If we set $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$ equal to zero, we obtain equation (8). On the other hand, using the second equation from equation (8), we derive

$$\gamma'_t|\xi'_t|^2 = \gamma'_t g(\xi'_t, \xi'_t) = 2g(\xi''_t, \xi'_t) = 0,$$

which leads to $|\xi'_t| = \kappa = const$. From equation (7), we deduce $0 \leq \kappa \leq 1$ and $|\gamma'_t| = \sqrt{1-\kappa^2}$. □

Remark 4.10. *Based on equation* (9)*, the geodesics $\Gamma = (\gamma(t), \xi(t))$ of $T_1^{\varphi}M$ can be naturally categorized into three distinct classes, as follows:*

(1) Horizontal geodesics: These geodesics occur when $\kappa = 0$, which is determined by equation (9)*. They are characterized by $|\gamma'_t| = 1$. Equation* (7) *further reveals that $\xi'_t = 0$, meaning that these geodesics are generated by parallel vector fields ξ along the geodesics γ on the base manifold.*

(2) Vertical geodesics: When $\kappa = 1$, in accordance with equation (9)*, we observe that $|\gamma'_t| = 0$. Consequently, $\gamma(t)$ becomes a constant, and Γ represents a geodesic in Euclidean space, specifically on a fixed fiber.*

(3) Umbilical (oblique) geodesics: These geodesics correspond to $0 < \kappa < 1$, as indicated in equation (9)*. In such cases, Γ can be interpreted as a non-zero vector field ξ along the curve γ (see also [25]).*

When we have a curve Γ on TM, we use the term "projection" to describe the curve $\gamma = \pi \circ \Gamma$, which represents the curve Γ projected onto M^{2m}.

Theorem 4.11. *In the context of a locally symmetric anti-paraKähler manifold (M^{2m}, φ, g) and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, when considering Γ as a non-vertical geodesic on $T_1^{\varphi}M$, it can be concluded that all the Frenet curvatures of the projected curve $\gamma = \pi \circ \Gamma$ are constant.*

Proof. By utilizing the first equation in (8), we have $\gamma_t'' = R(\xi_t', \xi)\gamma_t'$. It is straightforward to observe that

$$\gamma_t' g(\gamma_t', \gamma_t') = 2g(\gamma_t'', \gamma_t') = 2g(R(\xi_t', \xi)\gamma_t', \gamma_t') = 0,$$

which leads to the conclusion that $|\gamma_t'| = const$. Calculating the third derivative, we obtain

$$\begin{aligned} \gamma_t''' &= (\nabla_{\gamma_t'} R)(\xi_t', \xi)\gamma_t' + R(\xi_t'', \xi)\gamma_t' + R(\xi_t', \xi_t')\gamma_t' + R(\xi_t', \xi)\gamma_t'' \\ &= R(\xi_t', \xi)\gamma_t''. \end{aligned}$$

Since

$$\gamma_t' g(\gamma_t'', \gamma_t'') = 2g(\gamma_t''', \gamma_t'') = 2g(R(\xi_t', \varphi\xi)\gamma_t'', \gamma_t'') = 0,$$

we deduce that $|\gamma_t''| = const$. Continuing this process, we arrive at

$$\gamma_t^{(p+1)} = R(\xi_t', \xi)\gamma_t^{(p)}, \quad p \geq 1$$

and

$$\gamma_t' g(\gamma_t^{(p)}, \gamma_t^{(p)}) = 2g(\gamma_t^{(p+1)}, \gamma_t^{(p)}) = 2g(R(\xi_t', \xi)\gamma_t^{(p)}, \gamma_t^{(p)}) = 0.$$

Thus, we establish that

$$|\gamma_t^{(p)}| = const, \quad p \geq 1. \tag{10}$$

Denoting s as an arc length parameter on γ, i.e., ($|x_s'| = 1$), we have $\gamma_t' = \gamma_s' \dfrac{ds}{dt}$. Using (9), we find

$$\frac{ds}{dt} = \sqrt{1 - \kappa^2} = const. \tag{11}$$

Let $\nu_1 = \gamma_s', \nu_2, \ldots, \nu_{2m-1}$ represent the Frenet frame along γ and $k_1, \ldots, k_{2m-1}$ denote the Frenet curvatures of γ. Then the Frenet formulas hold

$$\begin{cases} (\nu_1)_s' &= k_1 \nu_2 \\ (\nu_i)_s' &= -k_{i-1}\nu_{i-1} + k_i \nu_{i+1}, \quad 2 \leq i \leq 2m-2 \\ (\nu_{2m-1})_s' &= -k_{2m-2}\nu_{2m-2}. \end{cases}$$

From (11), we have

$$\gamma_t' = \gamma_s' \frac{ds}{dt} = \sqrt{1 - \kappa^2}\, \nu_1.$$

By applying the Frenet formulas, we deduce

$$\gamma_t'' = \sqrt{1 - \kappa^2}\,(\nu_1)_t' = \sqrt{1 - \kappa^2}\,(\nu_1)_s' \frac{ds}{dt} = (1 - \kappa^2)k_1 \nu_2. \tag{12}$$

Now, (10) implies that $k_1 = const$. Similarly, we find

$$\begin{aligned} \gamma_t''' &= (1 - \kappa^2)k_1(\nu_2)_t' = (1 - \kappa^2)k_1(\nu_2)_s' \frac{ds}{dt} \\ &= (1 - \kappa^2)^{\frac{3}{2}} k_1(-k_1\nu_1 + k_2\nu_3). \end{aligned} \tag{13}$$

and again (10) reveals that k_2 is also a constant. This process continues, and the proof is completed. □

Lemma 4.12. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g) and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, if $\Gamma = (\gamma(t), \xi(t))$ is a curve on $T_1^{\varphi}M$, then we have*

(1) *If $\Gamma = (\gamma(t), \xi(t))$ is a curve on $T_1^{\varphi}M$, then $\Upsilon = (\gamma(t), \varphi\xi(t))$ is also a curve on $T_1^{\varphi}M$.*

(2) *Υ is a geodesic on $T_1^{\varphi}M$ if and only if Γ is a geodesic on $T_1^{\varphi}M$.*

Proof. (1) We put $\mu(t) = \varphi\xi(t)$. Since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$, then $g(\xi, \varphi\xi) = 1$. On the other hand, $g(\mu, \varphi\mu) = g(\varphi\xi, \varphi(\varphi\xi)) = g(\varphi\xi, \xi) = 1$, i.e.,

$$\Upsilon(t) = (\gamma(t), \mu(t)) \in T_1^{\varphi}M.$$

(2) In a similar manner as the proof of (8), and by considering $\mu'_t = \varphi\xi'_t$ and $\mu''_t = \varphi\xi''_t$, we have

$$\begin{aligned}\widehat{\nabla}_{\Upsilon'_t}\Upsilon'_t &= {}^{H}\Big(\gamma''_t + R(\mu, \mu'_t)\gamma'_t\Big) + {}^{T}\mu''_t \\ &= {}^{H}\Big(\gamma''_t + R(\varphi\xi, \varphi\xi'_t)\gamma'_t\Big) + {}^{T}(\varphi\xi''_t).\end{aligned}$$

As the Riemannian curvature tensor is pure, we can express it as follows

$$\widehat{\nabla}_{\Upsilon'_t}\Upsilon'_t = {}^{H}\Big(\gamma''_t + R(\xi, \xi'_t)\gamma'_t\Big) + {}^{T}(\varphi\xi''_t),$$

which leads to

$$\begin{aligned}\widehat{\nabla}_{\Upsilon'_t}\Upsilon'_t = 0 \quad &\Leftrightarrow \quad \left\{\begin{array}{lcl}\gamma''_t &=& -R(\xi, \xi'_t)\gamma'_t \\ \varphi\xi''_t &=& 0\end{array}\right. \\ &\Leftrightarrow \quad \left\{\begin{array}{lcl}\gamma''_t &=& R(\xi'_t, \xi)\gamma'_t \\ \xi''_t &=& 0\end{array}\right. \Leftrightarrow \widehat{\nabla}_{\Gamma'_t}\Gamma'_t = 0.\end{aligned}$$

□

From Theorem 4.11 and Lemma 4.12, we have the following theorem.

Theorem 4.13. *Suppose we have a locally symmetric anti-paraKähler manifold (M^{2m}, φ, g) and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric. Let $\Gamma = (\gamma(t), \xi(t))$ be a non-vertical geodesic on $T_1^{\varphi}M$. Then, all the Frenet curvatures of the projected curve $\pi \circ \Upsilon$, where $\Upsilon = (\gamma(t), \varphi\xi(t))$, are constants.*

Now, we will investigate geodesics on the φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric over an anti-paraKähler manifold with constant sectional curvature. According to Theorem 4.9, we can give the following result.

Corollary 4.14. *Suppose we have an anti-paraKähler manifold (M^{2m}, φ, g) with constant sectional curvature $c \neq 0$, and let $T_1^{\varphi}M$ be the φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. Consider a curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$. We can establish that Γ is a geodesic on $T_1^{\varphi}M$ if and only if the following conditions hold*

$$\left\{\begin{array}{lcl}\gamma''_t &=& cg(\xi, \gamma'_t)\xi'_t - cg(\xi'_t, \gamma'_t)\xi, \\ \xi''_t &=& 0.\end{array}\right.$$

Theorem 4.15. *In the anti-paraKähler real Euclidean space $(\mathbb{R}^{2m}, \varphi, <,>)$, with $T_1^{\varphi}\mathbb{R}^{2m}$ being its φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric, any oblique geodesic $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}\mathbb{R}^{2m}$ has the following parametric form*

$$\left\{\begin{array}{lcl}\gamma^i(t) &=& a^i t + b^i, \\ \xi^j(t) &=& c^j t + d^j,\end{array}\right.$$

where $\gamma(t) = (\gamma^i(t))_{i=\overline{1,2m}}$, $\xi(t) = (\xi^j(t))_{j=\overline{1,2m-1}}$ and a^i, b^i, c^i, d^i are real constants. This provides a concise representation of the oblique geodesics in the given space.

The power of the curvature operator $R^p(X, Y)$ is defined recursively as

$$R^p(X, Y)Z = R^{p-1}(X, Y)R(X, Y)Z,$$

for any vector fields X and Y, where $p \geq 2$.

Lemma 4.16. *[25] Let (M, g) be a Riemannian manifold of constant sectional curvature c, then we have*

$$R^p(X, Y) = \begin{cases} (-b^2c^2)^{h-1}R(X, Y), & for \;\; p = 2h-1 \\ (-b^2c^2)^{h-1}R^2(X, Y), & for \;\; p = 2h \end{cases}$$

for any vector fields X and Y on M, where $h \geq 2$ and $b^2 = |X|^2|Y|^2 - g(X, Y)^2$.

Theorem 4.17. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g) with constant sectional curvature $c \neq 0$ and its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, if Γ is a non-vertical geodesic on $T_1^{\varphi}M$ and $k_1, \ldots, k_{2m-1}$ are the Frenet curvatures of the projected curve $\gamma = \pi \circ \Gamma$, then it follows that if $k_1 \neq 0$ and $k_2 \neq 0$, then $k_3 = 0$, but it cannot be confirmed that k_i are equal to zero for $i > 3$.*

Proof. From the proof of Theorem 4.11, we can establish a recurrence relation for the derivatives of $\gamma(t)$ as follows

$$\gamma_t'' = R(\xi_t', \xi)\gamma_t'$$

and

$$\gamma_t''' = R(\xi_t', \xi)\gamma_t'' = R(\xi_t', \xi)R(\xi_t', \xi)\gamma_t' = R^2(\xi_t', \xi)\gamma_t'.$$

Continuing this process, we find that for $p \geq 1$

$$\gamma_t^{(p+1)} = R(\xi_t', \xi)\gamma_t^{(p)} = R^p(\xi_t', \xi)\gamma_t'. \tag{14}$$

Using (13) we have

$$\gamma_t^{(4)} = -(1-\kappa^2)^2 k_1(k_1^2 + k_2^2)\nu_2 + (1-\kappa^2)^2 k_1 k_2 k_3 \nu_4. \tag{15}$$

On the other hand, from Lemma 4.16, (12) and (14) we have

$$\begin{aligned} \gamma_t^{(4)} &= R^3(\xi_t', \xi)\gamma_t' \\ &= -b^2c^2 R(\xi_t', \xi)\gamma_t' \\ &= -b^2c^2\gamma_t'' \\ &= -b^2c^2(1-\kappa^2)k_1\nu_2. \end{aligned} \tag{16}$$

If $k_1 \neq 0$ and $k_2 \neq 0$, and from (15) and (16) we get

$$(b^2c^2 - (1-\kappa^2)(k_1^2 + k_2^2))\nu_2 + (1-\kappa^2)k_2k_3\nu_4 = 0.$$

Therefore, we have $k_3 = 0$, and $b^2c^2 = (1-\kappa^2)(k_1^2 + k_2^2)$, i.e., $b^2 = const$. Continuing the process, it becomes evident that we cannot confirm that k_i are equal to zero for $i > 3$. □

5. *F*-geodesics

In the context of a Riemannian manifold (M, g) and a $(1, 1)$-tensor field F on this manifold, a curve γ in M is considered F-planar if its velocity vector, when parallel transported along γ, remains within the span of the vectors γ'_t and $F\gamma'_t$ along the curve γ. Mathematically, this concept can be described by the condition

$$\gamma''_t = \varrho_1(t)\gamma'_t + \varrho_2 F\gamma'_t,$$

where ϱ_1 and ϱ_2 are some functions of the parameter t [11, 17]. It is worth noting that F-planar curves encompass and extend the concept of magnetic curves, and they also include geodesics.

An F-geodesic is a specific type of F-planar curve that satisfies the condition

$$\gamma''_t = F\gamma'_t.$$

In this context, it is important to recognize that while every F-geodesic is also an F-planar curve, not all F-planar curves qualify as F-geodesics [3].

5.1. *F-geodesics on the tangent bundle with the vertical generalized Berger-type deformed Sasaki metric*

In this section, we will always remember that $\widetilde{\nabla}$ denotes the Levi-Civita connection of the vertical generalized Berger-type deformed Sasaki metric on tangent bundle TM, given in the Theorem 3.2.

Theorem 5.1. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), with its tangent bundle TM equipped with the vertical generalized Berger-type deformed Sasaki metric, and given a $(1, 1)$-tensor field F on M^{2m}, a curve $\Gamma = (\gamma(t), \xi(t))$ on TM is considered to be an ^HF-planar curve with respect to $\widetilde{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \dfrac{1}{2} g(\xi'_t, \varphi\xi)^2 \, grad f + \varrho_1\gamma'_t + \varrho_2 F\gamma'_t, \\ \xi''_t = -\dfrac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi + \varrho_1\xi'_t + \varrho_2 F\xi'_t, \end{cases}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

Proof. From the proof of Theorem 4.1, we find

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^H\Big(\gamma''_t + R(\xi, \xi'_t)\gamma'_t - \frac{1}{2} g(\xi'_t, \varphi\xi)^2 \, grad f\Big) \\ &\quad + {}^V\Big(\xi''_t + \frac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi\Big). \end{aligned} \tag{17}$$

On the other hand, we compute

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1\Gamma'_t + \varrho_2 {}^HF\Gamma'_t \\ &= \varrho_1({}^H\gamma'_t + {}^V\xi'_t) + \varrho_2 {}^HF({}^H\gamma'_t + {}^V\xi'_t) \\ &= \varrho_1 {}^H\gamma'_t + \varrho_2 {}^HF^H\gamma'_t + \varrho_1 {}^V\xi'_t + \varrho_2 {}^HF^V\xi'_t \\ &= {}^H(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^V(\varrho_1\xi'_t + \varrho_2 F\xi'_t). \end{aligned} \tag{18}$$

From (17) and (18), the result immediately follows. □

Corollary 5.2. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), with its tangent bundle TM equipped with the vertical generalized Berger-type deformed Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is considered to be an $^H\varphi$-planar curve with respect to $\widetilde{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \dfrac{1}{2} g(\xi'_t, \varphi\xi)^2 \, grad f + \varrho_1\gamma'_t + \varrho_2 \varphi\gamma'_t, \\ \xi''_t = -\dfrac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi + \varrho_1\xi'_t + \varrho_2 \varphi\xi'_t. \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 5.1, we obtain the following result.

Theorem 5.3. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), with its tangent bundle TM equipped with the vertical generalized Berger-type deformed Sasaki metric, and given a $(1,1)$-tensor field F on M^{2m}, a curve $\Gamma = (\gamma(t), \xi(t))$ on TM is considered to be an ${}^HF$-geodesic with respect to $\widetilde{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \dfrac{1}{2} g(\xi'_t, \varphi\xi)^2 \, grad f + F\gamma'_t, \\ \xi''_t = -\dfrac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi + F\xi'_t. \end{cases}$$

Corollary 5.4. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), with its tangent bundle TM equipped with the vertical generalized Berger-type deformed Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is considered to be an ${}^H\varphi$-geodesic with respect to $\widetilde{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \dfrac{1}{2} g(\xi'_t, \varphi\xi)^2 \, grad f + \varphi\gamma'_t, \\ \xi''_t = -\dfrac{1}{\lambda}\Big(g(\gamma'_t, grad f) g(\xi'_t, \varphi\xi) + f\, g(\xi'_t, \varphi\xi'_t)\Big)\varphi\xi + \varphi\xi'_t. \end{cases}$$

Theorem 5.5. *If $\Gamma = (\gamma(t), \xi(t))$ represents the horizontal lift of a curve γ on the tangent bundle TM equipped with the vertical generalized Berger-type deformed Sasaki metric over an anti-paraKähler manifold, then Γ is an ${}^HF$-planar curve (or ${}^HF$-geodesic) if and only if γ is an F-planar curve (or F-geodesic).*

Proof. Let γ be a curve in a manifold M^{2m} that is an F-planar curve with respect to the connection ∇, meaning it satisfies the differential equation

$$\gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t,$$

where ϱ_1 and ϱ_2 are some functions of the parameter t. Now, suppose we have the horizontal lift $\Gamma = (\gamma(t), \xi(t))$ of a curve γ, where $\xi'_t = 0$. From equation (1), we have $\Gamma'_t = {}^H\gamma'_t$. Using equation (17), we can write

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^H\gamma''_t \\ &= {}^H(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) \\ &= \varrho_1 {}^H\gamma'_t + \varrho_2 {}^HF {}^H\gamma'_t \\ &= \varrho_1\Gamma'_t + \varrho_2 {}^HF\Gamma'_t. \end{aligned}$$

This expression implies that Γ is an ${}^HF$-planar curve with respect to the connection $\widetilde{\nabla}$. In the special case where $\varrho_1 = 0$ and $\varrho_2 = 1$, we obtain that Γ is an ${}^HF$-geodesic if and only if γ is an F-geodesic. □

Corollary 5.6. *Consider a manifold (M^{2m}, φ, g) equipped with an anti-paraKähler structure represented by φ and g. Let TM denote its tangent bundle, which is equipped with the vertical generalized Berger-type deformed Sasaki metric. If we have a curve $\Gamma = (\gamma(t), \xi(t))$ that is a horizontal lift of a curve γ, then Γ is an ${}^H\varphi$-planar curve (or ${}^H\varphi$-geodesic) if and only if the curve γ is a φ-planar curve (or φ-geodesic).*

Example 5.7. *Let $(\mathbb{R}^2, \varphi, g)$ be an anti-paraKähler manifold such that*

$$g = dx^2 + dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\Gamma = (\gamma(t), \xi(t))$ such that $\gamma(t) = (x(t), y(t))$ and $\xi(t) = (u(t), -u(t))$.
By a direct calculation, we find $g(\xi'_t, \varphi\xi) = g(\xi'_t, \varphi\xi'_t) = 0$.
1) *Using Corollary 5.4, Γ is an ${}^H\varphi$-geodesic if and only if*

$$\begin{cases} \gamma''_t = \varphi\gamma'_t, \\ \xi''_t = \varphi\xi'_t, \end{cases} \Leftrightarrow \begin{cases} x'' = x', \\ y'' = -y', \\ u'' = u', \\ u'' = -u', \end{cases} \Leftrightarrow \begin{cases} x(t) = k_1 e^t + k_2, \\ y(t) = k_3 e^{-t} + k_4, \\ u(t) = k_5, \end{cases}$$

then $\Gamma = (k_1e^t + k_2, k_3e^{-t} + k_4, k_5, -k_5)$ *is an* ${}^H\varphi$*-geodesic on* $T\mathbb{R}^2$*, where* k_i *are real constants.*
2) *Using Corollary 5.2,* Γ *is an* ${}^H\varphi$*-planar curve if and only if*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t, \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t, \end{cases} \Leftrightarrow \begin{cases} x'' = (\varrho_1+\varrho_2)x', \\ y'' = (\varrho_1-\varrho_2)y', \\ u'' = (\varrho_1+\varrho_2)u', \\ u'' = (\varrho_1-\varrho_2)u', \end{cases} \Leftrightarrow \begin{cases} x'' = (\varrho_1+\varrho_2)x', \\ y'' = (\varrho_1-\varrho_2)y', \\ \varrho_2 u' = 0. \end{cases}$$

If $\varrho_2 = 0$*, then* $\begin{cases} x(t) = \varepsilon_1\int(e^{\int \varrho_1 dt})dt, \\ y(t) = \varepsilon_2\int(e^{\int \varrho_1 dt})dt, \\ u(t) = \varepsilon_3\int(e^{\int \varrho_1 dt})dt, \end{cases}$

If $\varrho_2 \neq 0$*, then* $\begin{cases} x(t) = \varepsilon_1\int(e^{\int(\varrho_1+\varrho_2)dt})dt, \\ y(t) = \varepsilon_2\int(e^{\int(\varrho_1-\varrho_2)dt})dt, \\ u(t) = k = const., \end{cases}$

where $\varepsilon_i = \pm 1$.
For example: If $\varrho_1(t) = \dfrac{1}{t+1}$ *and* $\varrho_2(t) = 0$*, we find*

$$\begin{cases} x(t) = a_1t^2 + 2a_1t + a_2, \\ y(t) = a_3t^2 + 2a_3t + a_4, \\ u(t) = a_5t^2 + 2a_5t + a_6, \end{cases}$$

then $\Gamma = (a_1t^2 + 2a_1t + a_2, a_3t^2 + 2a_3t + a_4, a_5t^2 + 2a_5t + a_6, -a_5t^2 - 2a_5t - a_6)$ *is an* ${}^H\varphi$*-planar curve on* $T\mathbb{R}^2$*, where* a_i *are real constants.*
If $\varrho_1(t) = \dfrac{1}{t+1}$ *and* $\varrho_2(t) = \dfrac{1}{t-1}$*, we find*

$$\begin{cases} x(t) = b_1t^3 - 3b_1t + b_2, \\ y(t) = b_3\ln(t+1)^2 + b_3t + b_4, \\ u(t) = b_5, \end{cases}$$

then $\Gamma = (b_1t^3 - 3b_1t + b_2, b_3\ln(t+1)^2 + b_3t + b_4, b_5, -b_5)$ *is an* ${}^H\varphi$*-planar curve on* $T\mathbb{R}^2$*, where* b_i *are real constants.*

Example 5.8. *Let* $(\mathbb{R}^2, \varphi, g)$ *be an anti-paraKähler manifold such that*

$$g = x^2dx^2 + y^2dy^2, \quad \varphi = \begin{pmatrix} 0 & \frac{y}{x} \\ \frac{x}{y} & 0 \end{pmatrix} \quad and \quad F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ a, b \in \mathbb{R}^*.$$

The non-null Christoffel symbols of the Riemannian connection are

$$\Gamma^1_{11} = \frac{1}{x}, \ \ \Gamma^2_{22} = \frac{1}{y}.$$

Let $\Gamma = (\gamma(t), \xi(t))$ *be the horizontal lift of a curve* γ*, such that* $\gamma(t) = (x(t), y(t))$ *and* $\xi(t) = (u(t), v(t))$ *then* $\xi'_t = 0$*, from* (4) *we have*

$$\frac{d\xi^h}{dt} + \sum_{i,j=1}^{2}\frac{d\gamma^j}{dt}\xi^i\Gamma^h_{ij} = 0 \Leftrightarrow \begin{cases} u' + \dfrac{x'}{x}u = 0, \\ v' + \dfrac{y'}{y}v = 0, \end{cases} \Leftrightarrow \begin{cases} u(t) = \dfrac{k_1}{x(t)}, \\ v(t) = \dfrac{k_2}{y(t)}, \end{cases}$$

where k_1, k_2 *are real constants.*
(i) γ *is an F-geodesic if and only if* $\gamma''_t = F\gamma'_t$*, from* (3) *we have*

$$\begin{cases} x'' + \dfrac{(x')^2}{x} = ax', \\ y'' + \dfrac{(y')^2}{y} = by', \end{cases} \Leftrightarrow \begin{cases} x(t) = \pm\sqrt{c_1e^{at} + c_2}, \\ y(t) = \pm\sqrt{c_3e^{bt} + c_4}. \end{cases}$$

Using Theorem 5.3, the horizontal lift

$$\Gamma = (\pm\sqrt{c_1 e^{at} + c_2}, \pm\sqrt{c_3 e^{bt} + c_4}, \frac{c_5}{\sqrt{c_1 e^{at} + c_2}}, \frac{c_6}{\sqrt{c_3 e^{bt} + c_4}})$$

is an ${}^H F$*-geodesic on* $T\mathbb{R}^2$*, where* c_i *are real constants.*
(ii) γ *is an F-planar curve if and only if* $\gamma_t'' = \varrho_1 \gamma_t' + \varrho_2 F\gamma_t'$*, where* ϱ_1 *and* ϱ_2 *are some functions of the parameter t, hence, we have*

$$\begin{cases} x'' + \dfrac{(x')^2}{x} = (\varrho_1 + a\varrho_2)x', \\ y'' + \dfrac{(y')^2}{x} = (\varrho_1 + b\varrho_2)y', \end{cases} \Leftrightarrow \begin{cases} x(t) = \pm\sqrt{2\int(e^{\int(\varrho_1 + a\varrho_2)dt})dt}, \\ y(t) = \pm\sqrt{2\int(e^{\int(\varrho_1 + b\varrho_2)dt})dt}. \end{cases}$$

For example: If $\varrho_1(t) = \dfrac{1}{t+1}$ *and* $\varrho_2(t) = \dfrac{1}{t}$*, we find*

$$\begin{cases} x(t) = \pm\sqrt{\dfrac{\alpha_1}{a+2}t^{a+2} + \dfrac{\alpha_1}{a+1}t^{a+1} + \alpha_2}, \\ y(t) = \pm\sqrt{\dfrac{\beta_1}{b+2}t^{b+2} + \dfrac{\beta_1}{b+1}t^{b+1} + \beta_2}, \\ u(t) = \dfrac{\lambda_1}{\sqrt{\dfrac{\alpha_1}{a+2}t^{a+2} + \dfrac{\alpha_1}{a+1}t^{a+1} + \alpha_2}}, \\ v(t) = \dfrac{\lambda_2}{\sqrt{\dfrac{\beta_1}{b+2}t^{b+2} + \dfrac{\beta_1}{b+1}t^{b+1} + \beta_2}}, \end{cases}$$

then $\Gamma = (x(t), y(t), u(t), v(t))$ *is an* ${}^H\varphi$*-planar on* $T\mathbb{R}^2$*, where* $\alpha_i, \beta_i, \lambda_i$ *are real constants.*

5.2. F-geodesics on the φ-unit tangent bundle with the vertical generalized Berger-type deformed Sasaki metric

In this section $\widehat{\nabla}$ represents the Levi-Civita connection of the vertical generalized Berger-type deformed Sasaki metric on φ-unit tangent bundle $T_1^\varphi M$, given in the Theorem 3.3.

Theorem 5.9. *A curve* $\Gamma = (\gamma(t), \xi(t))$ *on the* φ*-unit tangent bundle* $T_1^\varphi M$ *of an anti-paraKähler manifold* (M^{2m}, φ, g) *equipped with the vertical generalized Berger-type deformed Sasaki metric is an* ${}^H F$*-planar curve with respect to* $\widehat{\nabla}$ *if and only if*

$$\begin{cases} \gamma_t'' = R(\xi_t', \xi)\gamma_t' + \varrho_1\gamma_t' + \varrho_2 F\gamma_t', \\ \xi_t'' = \varrho_1\xi_t' + \varrho_2 F\xi_t', \end{cases}$$

where ϱ_1 *and* ϱ_2 *are some functions of the parameter t.*

Proof. With help of the proof of Theorem 4.9, we find

$$\widehat{\nabla}_{\Gamma_t'}\Gamma_t' = {}^H\Big(\gamma_t'' - R(\xi_t', \xi)\gamma_t'\Big) + {}^T\xi_t''. \tag{19}$$

On the other hand, by (5), we get

$$\begin{aligned} \widehat{\nabla}_{\Gamma_t'}\Gamma_t' &= \varrho_1\Gamma_t' + \varrho_2 {}^H F\Gamma_t' \\ &= \varrho_1({}^H\gamma_t' + {}^T\xi_t') + \varrho_2 {}^H F({}^H\gamma_t' + {}^T\xi_t'). \end{aligned}$$

From (6), we have ${}^{T}\xi'_t = {}^{V}\xi'_t$, which leads to

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1{}^{H}\gamma'_t + \varrho_2{}^{H}F^{H}\gamma'_t \\ &+\varrho_1{}^{V}\xi'_t + \varrho_2{}^{H}F^{V}\xi'_t \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{V}(\varrho_1\xi'_t + \varrho_2 F\xi'_t) \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{T}(\varrho_1\xi'_t + \varrho_2 F\xi'_t). \end{aligned} \tag{20}$$

From (19) and (20), the result immediately follows. □

Corollary 5.10. *In the context of an anti-paraKähler manifold (M^{2m}, φ, g), with its φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the vertical generalized Berger-type deformed Sasaki metric, a curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-planar curve with respect to $\widehat{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t, \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t. \end{cases}$$

When we set $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 5.9, we derive the following result.

Theorem 5.11. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric and F be a $(1,1)$-tensor field on M^{2m}. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}F$-geodesic with respect to $\widehat{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + F\gamma'_t, \\ \xi''_t = F\xi'_t. \end{cases}$$

Corollary 5.12. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-geodesic with respect to $\widehat{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \xi)\gamma'_t + \varphi\gamma'_t, \\ \xi''_t = \varphi\xi'_t. \end{cases}$$

Theorem 5.13. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}(R(\xi'_t, \xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = 2R(\xi'_t, \xi)\gamma'_t, \\ \xi''_t = R(\xi'_t, \xi)\xi'_t. \end{cases}$$

Corollary 5.14. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold of constant sectional curvature $c \neq 0$ and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}(R(\xi'_t, \xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if*

$$\begin{cases} \gamma''_t = 2c(g(\xi, \gamma'_t)\xi'_t - g(\xi'_t, \gamma'_t)\xi), \\ \xi''_t = c(g(\xi, \xi'_t)\xi'_t - g(\xi'_t, \xi'_t)\xi). \end{cases}$$

Theorem 5.15. *Consider an anti-paraKähler manifold (M^{2m}, φ, g) and its φ-unit tangent bundle $T_1^{\varphi}M$ with the vertical generalized Berger-type deformed Sasaki metric. Let F be a $(1,1)$-tensor field on M^{2m}. If we have a curve $\Gamma = (\gamma(t), \xi(t))$ as the horizontal lift of a curve γ, and Γ belongs to $T_1^{\varphi}M$, then Γ is an ${}^{H}F$-planar curve (or ${}^{H}F$-geodesic) if and only if γ is an F-planar curve (or F-geodesic).*

Proof. Let γ be a curve in a manifold M^{2m} that is an F-planar curve with respect to the connection ∇, which means γ satisfies the following equation

$$\gamma_t'' = \varrho_1\gamma_t' + \varrho_2 F\gamma_t',$$

where ϱ_1 and ϱ_2 are some functions of the parameter t. If $\Gamma = (\gamma(t), \xi(t))$ is the horizontal lift of the curve γ, then $\xi_t' = 0$. Using equation (5), we find that $\Gamma_t' = {}^H\gamma_t'$. With equation (19), we can write

$$\begin{aligned}\widehat{\nabla}_{\Gamma_t'}\Gamma_t' &= {}^H\gamma_t'' = {}^H(\varrho_1\gamma_t' + \varrho_2 F\gamma_t')\\ &= \varrho_1{}^H\gamma_t' + \varrho_2{}^HF{}^H\gamma_t' = \varrho_1\Gamma_t' + \varrho_2{}^HF\Gamma_t'.\end{aligned}$$

In other words, Γ is an ${}^HF$-planar curve with respect to $\widehat{\nabla}$. In the specific case where $\varrho_1 = 0$ and $\varrho_2 = 1$, we conclude that Γ is an ${}^HF$-geodesic if and only γ is an F-geodesic. □

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