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Global existence theorem for the 3-D generalized micropolar fluid system in critical Fourier-Besov-Morrey spaces with variable exponent

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Abstract. In this paper, we consider the 3-D generalized micropolar fluid system in critical Fourier-Besov-Morrey spaces with variable exponent. Using the Littlewood-Paley theory and Banach fixed point theorem we establish the global existence result with the small initial data belonging to $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}(\mathbb{R}^3)$.

1. Introduction and statement of main results

In this work we are interested in looking for the global well-posedness and the Gevrey class regurality for the following three-dimensional generalized incompressible micropolar system :

$$\begin{cases} \partial_t u + (\chi+\nu)(-\Delta)^{\alpha_1} u + u\cdot\nabla u + \nabla\pi - 2\chi\nabla\times w = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \partial_t w + \mu(-\Delta)^{\alpha_2} w + u\cdot\nabla b + 4\chi w - \kappa\nabla \operatorname{div} w - 2\chi\nabla\times u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ (u,w)|_{t=0} = (u_0, w_0) & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(x,t), w = w(x,t)$ and $\pi = \pi(x,t)$ are unknown functions representing the linear velocity field, the micro-rotation velocity field, the pressure of the fluid particle passing at the point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ respectively. u_0 and w_0 represent the initial velocities and we assume that $\operatorname{div} u_0 = 0$. κ, μ, ν and χ are nonnegative constants reflecting various viscosity of the fluid. Throughout this paper we only consider the situation with $\kappa = \mu = 1$ and $\chi = \nu = 1/2$, and we denote $\alpha = min(\alpha_1, \alpha_2)$.

Eringen [9] introduced the micropolar fluid system in 1960, which is a significant step toward generalizing the Navier-Stokes equations to better describe the motion of various real fluids consisting of rigid but randomly oriented particles (e.g., blood) by taking into account the influence of micro-rotation of the particles suspended in the fluid. It can describe many phenomena that occur in a large number of complex fluids, such as suspensions and liquid crystals. For additional applications, see [8, 20] and the references therein.

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There is a rich literature about global well-posedness for the system (1) and its related systems with singular data in different spaces, where the smallness conditions are taken in the weak-norms of the critical spaces. For instance, the existence theorem of the classical micropolar fluid system ($\alpha_1 = \alpha_2 = 1$) with sufficiently regular initial data has been obtained by Lukaszewicz [21]. Inoue et al.[14] established similar result for the magneto-micropolar fluid system. Chen and Miao [6] obtained the global existence for the classical micropolar fluid system with small initial data in Besov spaces $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p \in [1, 6)$ and $q = \infty$. Zhao and Zhu [33] established the local existence of the classical micropolar fluid system and the global existence in the Fourier-Besov spaces $\mathcal{F}\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $1 < p \leq \infty$ and $1 \leq q < \infty$ when the initial data are small. In the recent years, Weipeng Zhu [34] obtained the local existence for large initial data in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $1 \leq q \leq 2$, and proved that the solution is global in these spaces when the initial data are small. In addition, Zhu obtained the ill-posedness of the classical micropolar system in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $2 < q \leq \infty$. Chen and Miao [15] proved the global existence for the micropolar system with small initial data in Besov spaces $\dot{\mathcal{B}}_{p,q}^{-1+\frac{3}{p}}$ when $p \in [1, 6)$ and $q = \infty$. Recently, Weipeng Zhu [34] considered a critical case $p = 1$ and proved that the classical system of micropolar fluid is locally well-posed for large initial data in $\mathcal{F}\dot{B}_{1,q}^{-1}$ for $1 \leq q \leq 2$, and globally well-posed in these spaces with small initial data. However, the ill-posedness of same problem has been showed by Zhu in $\mathcal{F}\dot{B}_{1,q}^{-1}$ where $2 < q \leq \infty$, which implies the optimality of the space $\mathcal{F}\dot{B}_{1,q}^{-1}$, and this was previously observed by Iwabuchi and Takada [15] for the Navier-Stokes-Coriolis system(NSC). Recently, Ferreira et al. [17] showed global existence of solutions for (NSC) when the small initial data belongs to the Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}_{1,\lambda,\infty}^{\lambda-1}$ (larger than Fourier-Besov spaces) for $1 < \lambda < 3$. Moreover, the authors [17] shown an optimality with respect to parameter $\lambda > 0$ when the initial data $u_0 \in \mathcal{F}\dot{\mathcal{N}}_{1,\lambda,q}^{\lambda-1}$ for $2 < q \leq \infty$. The well-posedness of the problem (1) was proved by Ferreira and Villamizar-Roa [11] in pseudo-measure space PM^c (c is a given nonnegative parameter), where the pseudo-measure space PM^c is defined by

$$PM^c = \left\{ f \in S'(\mathbb{R}^3) : \ \hat{f} \in L^1_{loc}(\mathbb{R}^3), \ \|f\|_{PM^c} = ess\ sup_{x \in \mathbb{R}^3} |\xi|^c |\hat{f}(\xi)| < \infty \right\}.$$

Zhu and Zhao [33] established the global existence and asymptotic stability of solutions for the initial value problem of system (1) in Fourier-Besov spaces $\mathcal{F}\dot{B}_{p,q}^{4-\frac{3}{p}-2\alpha}$ for $1 < p \leq \infty$, and $1 \leq q \leq \infty$. Moreover, they showed the spatial analyticity and the temporal decay of global solutions. For more studies in this direction see [26, 27, 30]. In the last years, function spaces with variable exponent have been attracted much attention not only for theoretical reasons, but also because of the special role played in some applications, such as image processing [7], partial differential equations [10] and the fluid dynamics [29]. The aim of this paper is to establish the global well-posedness of the problem (1) in variable exponent Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}_{p(\cdot),h(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}$, and this work is a complement of our above work in Fourier-Besov spaces [27]. It is clear that the variable exponent Fourier-Besov-Morrey spaces is larger than variable exponent Fourier-Besov spaces and variable exponent Lebesgue spaces $L^{p(\cdot)}$ which is discovered by Orlicz [25] and was further investigated by Nakano [22]; however, the current development began with the papers of Kovaik and Rakosnik [17]. We refer the reader to see [3, 4, 13, 23, 24] and the references therein for a detailed study of Besov spaces with variable exponent and Besov-Morrey spaces with variable exponent. Notice that the variable exponent Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}_{p(\cdot),h(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}$ is invariant under the scaling of the system (1). In fact if (u, w) solves system (1) with the initial data (u_0, w_0) then $(u_\lambda, w_\lambda) := (\lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \lambda^{2\alpha-1} w_0(\lambda x))$ also solves the system (1) with the initial data

$$(u_{0,\lambda}, w_{0,\lambda}) := (\lambda^{2\alpha-1} u_0(\lambda x), \lambda^{2\alpha-1} w_0(\lambda x)), \tag{2}$$

and

$$\|(u_0, w_0)\|_{\mathcal{F}\dot{\mathcal{N}}_{p(\cdot),h(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}} = \|(u_{0,\lambda}, w_{0,\lambda})\|_{\mathcal{F}\dot{\mathcal{N}}_{p(\cdot),h(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}},$$

for any $\lambda > 0$.

Definition 1.1. *Let $E \in S'(\mathbb{R}^n)$ be a Banach space. The space E is a critical space for the initial data of the system (1) if and only if whose norm is invariant under the scaling (2) for all $\lambda > 0$, i.e*

$$\|(u_{0,\lambda}(x), w_{0,\lambda}(x))\|_E \approx \|(u_0(x), w_0(x))\|_E.$$

We recall an existence and uniquness result for an abstract operator equation in a Banach space, which will be used to show the main result.

Lemma 1.2. *([5]) Let X be a Banach space with norm $\|\cdot\|$ and $B : X \longrightarrow X$ a bilinear operator, such that for any $x_1, x_2 \in X$, $\|B(x_1, x_2)\| \le \eta\|x_1\|\|x_2\|$, then for any $y \in X$ such that $\|y\| < \dfrac{1}{4\eta}$ the equation $x = y + B(x, x)$ has a solution $x \in X$. In particular, the solution is such that $\|x\| \le 2\|y\|$ and it is the only one such that $\|x\| < \dfrac{1}{2\eta}$.*

Our main theorems is stated as below.

Theorem 1.3. *Let $\frac{1}{2} < \alpha \le 1$, $\alpha_1 \le \alpha_2$, $p(\cdot), h(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ such that $2 \le p(\cdot) \le \frac{6}{5-4\alpha}$, $p(\cdot) \le h(\cdot) < \infty$, $1 \le \rho < \infty$, $1 \le q < \frac{3}{2\alpha-1}$ and there exists a small δ_0 such that for any $(u_0, w_0) \in \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}$ satisfying $\nabla \cdot u_0 = 0$ with $\|(u_0, w_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}} < \delta_0$. Then, the problem (1) admits a unique global mild solution (u, w) in*

$$\mathcal{L}^{\rho}([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),h(\cdot),q}) \cap \mathcal{L}^{\rho}([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha}_{2,h(\cdot),q}) \cap \mathcal{L}^{\infty}([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q}).$$

Such that

$$\|(u,w)\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),h(\cdot),q})\cap\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha}_{2,h(\cdot),q})\cap\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})} \le \|(u_0,w_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}}.$$

Presented below are certain notations which will be employed throughout this document.
- The symbol C represents a positive constant, the specific value of which may vary in different contexts.
- The notation $x \lesssim y$ implies the existence of a positive constant C such that $x \le Cy$.
- In the context of a Banach space E, we use $(x, y) \in E$ to denote $(x, y) \in E \times E$.
- We denote $\|(x, y)\|_E$ as $\|(x, y)\|_{E\times E}$.
- For two spaces X and Y, we define the norm $\|\cdot\|_{X\cap Y}$ as the sum of the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.
- The symbol $\mathcal{S}(\mathbb{R}^n)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing complex-valued functions on $\mathbb{R}^n$.

By $\hat{\varphi}$ we denote the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in the version

$$\hat{\varphi}(x) := \mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

and we define its inverse Fourier transform by

$$\check{\varphi}(\xi) = \mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(x)dx.$$

2. Preliminaries

We first introduce some important harmonic analysis related to the variable exponent function spaces.

Definition 2.1. *([4]) Let $\mathcal{P}_0(\mathbb{R}^n)$ denotes the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that*

$$0 < p_- = ess\inf_{x\in\mathbb{R}^n} p(x), \; ess\sup_{x\in\mathbb{R}^n} p(x) = p_+ < \infty.$$

The Lebesgue space with variable exponent is defined by

$L^{p(\cdot)}(\mathbb{R}^n)=\left\{f:\mathbb{R}^n\to\mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^n} f(x)|^{p(x)}dx<\infty\right\},$

with Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}(\frac{|f(x)|}{\lambda})^{p(x)}dx\le 1\right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{L^{p(\cdot)}}$ is a Banach space.
Since the $L^{p(\cdot)}$ does not have the same desired properties as L^p. So, we propose the following standard conditions to ensure that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

Definition 2.2. *([4]) Let $p:\mathbb{R}^n\to\mathbb{R}$.*

i) *We say that p is locally log-Hölder continuous, $p\in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists a constant $c_{\log}>0$ with*

$$|p(x)-p(y)|\le\frac{c_{\log}}{\log\left(e+\frac{1}{|x-y|}\right)}\quad \textit{for all}\quad x,y\in\mathbb{R}^n \textit{and } x\ne y.$$

ii) *We say that p is globally log-Hölder continuous, $p\in C^{\log}(\mathbb{R}^n)$, if $p\in C^{\log}_{loc}(\mathbb{R}^n)$ and there exists a $p_\infty\in\mathbb{R}$ and a constant $c_\infty>0$ with*

$$\left|p(x)-p_\infty\right|\le\frac{c_\infty}{\log(e+|x|)}\quad for\ \ all\ \ x\in\mathbb{R}^n.$$

iii) *We write $p\in\mathcal{P}^{\log}_0(\mathbb{R}^n)$ if $0<p^-\le p(x)\le p^+\le\infty$ with $1/p\in C^{\log}(\mathbb{R}^n)$.*

We recall the littlewood-Paley decomposition. Let $\varphi\in S(\mathbb{R}^n)$ be a radial positive function such that $0\le\varphi\le 1,\ \ \text{supp}(\varphi)\subset\left\{\xi\in\mathbb{R}^n:\frac{3}{4}\le|\xi|\le\frac{8}{3}\right\}$
and

$$\sum_{j\in\mathbb{Z}}\varphi\left(2^{-j}\xi\right)=1,\quad \text{for all } \xi\ne 0.$$

We denote

$$\varphi_j(\xi)=\varphi\left(2^{-j}\xi\right),\quad \psi_j(\xi)=\sum_{k\le j-1}\varphi_k(\xi),$$

and

$$l(x)=\mathcal{F}^{-1}\varphi(x),\quad g(x)=\mathcal{F}^{-1}\psi(x).$$

Now, we present some frequency localization operators:

$$\Delta_j f:=\mathcal{F}^{-1}\left(\varphi_j\mathcal{F}(f)\right)=2^{nj}\int_{\mathbb{R}^n}l\left(2^jy\right)f(x-y)dy,$$
$$S_jf:=\sum_{k\le j-1}\Delta_kf=\mathcal{F}^{-1}\left(\psi_j\mathcal{F}(f)\right)=2^{nj}\int_{\mathbb{R}^n}g\left(2^jy\right)f(x-y)dy,$$

where $\Delta_j=S_j-S_{j-1}$ is a frequency projection to the annulus $\{|\xi|\sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi|\lesssim 2^j\}$.
By using the definition of Δ_j and S_j, we easily check that

$$\Delta_j\Delta_kf=0,\quad \text{if } |j-k|\ge 2$$
$$\Delta_j\left(S_{k-1}f\Delta_kf\right)=0,\quad \text{if } |j-k|\ge 5.$$

The following Bony para-product decomposition will be applied around the paper:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

where $\dot{T}_u v = \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v$, $\dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \Delta_j u\tilde{\Delta}_j v$ *and* $\tilde{\Delta}_j v = \sum_{|j'-j|\leq 1} \Delta_{j'} v$.

2.1. Variable exponent Morrey spaces, Fourier-Besov spaces and Fourier-Besov-Morrey spaces

We now define the Morrey spaces with variable exponent.

Definition 2.3. *([3]) Let $p(\cdot)$, $h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < p_- \leq p(x) \leq h(x) \leq \infty$, the Morrey space with variable exponent $\mathcal{M}^{h(\cdot)}_{p(\cdot)} := \mathcal{M}^{h(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions on $\mathbb{R}^n$ such that*

$$\|f\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}} := \sup_{x_0\in\mathbb{R}^n, r>0} \|r^{\frac{n}{h(x)}-\frac{n}{p(x)}} f\chi_{B(x_0,r)}\|_{L^{p(\cdot)}} < \infty.$$

According to the definition of the $L^{p(\cdot)}$-norm, $\|f\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}}$ also has the following form

$$\|f\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}} := \sup_{x_0\in\mathbb{R}^n, r>0} \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(r^{\frac{n}{h(x)}-\frac{n}{p(x)}} \frac{f}{\lambda}\chi_{B(x_0,r)}) \leq 1\right\}.$$

Then, we define the homogeneous Fourier-Besov space with variable exponent $\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 2.4. *([2]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \leq p(\cdot) \leq \infty$. The homogeneous Fourier-Besov space with variable exponent $\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j \hat{f}\}^{\infty}_{-\infty}\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^n) = \{f \in S(\mathbb{R}^n) : (D^\alpha f)(0) = 0, \forall\alpha \text{ multi-index}\}.$$

We also need to give the semimodular of mixed Morrey-sequence spaces $l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})$.

Definition 2.5. *([3]) Let $p(\cdot)$, $q(\cdot)$, $h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $p(\cdot) \leq h(\cdot)$, the mixed Morrey-sequence space $l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})$ includes all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that*
$\rho_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})}(\lambda\{f_j\}_{j\in\mathbb{Z}}) < \infty$ for some $\lambda > 0$. For $\{f_j\}_j \in l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})$, we define

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})} := \inf\left\{\lambda > 0,\ \rho_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})}\left(\{\frac{f_j}{\lambda}\}_{j\in\mathbb{Z}}\right) \leq 1\right\} < \infty,$$

where

$$\rho_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \inf\left\{\gamma > 0,\ \int_{\mathbb{R}^n}\left(\frac{|r^{\frac{n}{h(x)}-\frac{n}{p(x)}} f_j\chi_{B(x_0,r)}|}{\gamma^{\frac{1}{q(x)}}}\right)^{p(x)} dx \leq 1\right\}.$$

Notice that if $q_+ < \infty$ or $q_+ < \infty$ and $p(x) \geq q(x)$, then

$$\rho_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})}(\{f_i\}_{i\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} \sup_{x_0\in\mathbb{R}^n, r>0} \|(|r^{\frac{n}{h(x)}-\frac{n}{p(x)}} f_i|\chi_{B(x_0,r)})^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Definition 2.6. *([3])(Homogeneous Besov-Morrey space with variable exponent)*
Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot)$, $q(\cdot)$, $h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \leq p(x) \leq h(x) \leq \infty$. The homogeneous Besov-Morrey space with variable exponent $\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})} < \infty.$$

Definition 2.7. *([1])(Homogeneous Fourier-Besov-Morrey space with variable exponent) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $p(\cdot),\ q(\cdot),\ h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ *with* $0 < p_- \le p(\cdot) \le h(\cdot) \le \infty$. *The homogeneous Fourier-Besov-Morrey space with variable exponent* $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q(\cdot)}$ *is defined by the set of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j \hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(\mathcal{M}^{h(\cdot)}_{p(\cdot)})} < \infty.$$

Proposition 2.8. *([1]) For Morrey spaces with variable exponent, the following inclusions are established.*

(1) *(Hölder inequality)([1]) Let* $p(\cdot),\ p_1(\cdot),\ p_2(\cdot),\ h(\cdot),\ h_1(\cdot),\ h_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, *such that* $p(x) \le h(x),\ p_1(x) \le h_1(x),\ p_2(x) \le h_2(x),\ \dfrac{1}{p(x)} = \dfrac{1}{p_1(x)} + \dfrac{1}{p_2(x)}$ *and* $\dfrac{1}{h(x)} = \dfrac{1}{h_1(x)} + \dfrac{1}{h_2(x)}$, *then there exists a constant C depending only on* p_- *and* p_+ *such that*

$$\|fg\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}} \le C\|f\|_{\mathcal{M}^{h_1(\cdot)}_{p_1(\cdot)}}\|g\|_{\mathcal{M}^{h_2(\cdot)}_{p_2(\cdot)}},$$

holds for every $f \in \mathcal{M}^{h_1(\cdot)}_{p_1(\cdot)}$ *and* $g \in \mathcal{M}^{h_2(\cdot)}_{p_2(\cdot)}$.

(2) *([1]) Let* $p_0(\cdot),\ p_1(\cdot),\ h_0(\cdot),\ h_1(\cdot),\ q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, *and* $s_0(\cdot),\ s_1(\cdot) \in L^\infty \cap C^{log}(\mathbb{R}^n)$ *with* $s_0(\cdot) \ge s_1(\cdot)$. *If* $\dfrac{1}{q}$ *and* $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$ *are locally log-Hölder continuous, then*

$$\mathcal{N}^{s_0(\cdot)}_{p_0(\cdot),h_0(\cdot),q} \hookrightarrow \mathcal{N}^{s_1(\cdot)}_{p_1(\cdot),h_1(\cdot),q}.$$

(3) *([3]) For* $p(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $\psi \in L^1(\mathbb{R}^n)$, *assume* $\Psi(x) = \sup_{y \notin B(0,|x|)} |\psi(y)|$ *is integrable, then*

$$\|f * \psi_\epsilon\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f \in \mathcal{M}^{h(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, *where* $\psi_\epsilon = \dfrac{1}{\epsilon^n}\psi(\dfrac{1}{\epsilon})$ *and C depends only on n.*

To establish the global well-posedness of (1), we need to introduce the Chemin-Lerner type homogeneous Fourier-Besov-Morrey spaces with variable exponent.

Definition 2.9. *([1]) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot),\ q(\cdot),\ h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$, $T \in [0,\infty)$ *and* $1 \le q,\ \rho \le \infty$. *We define the Chemin-Lerner type homogeneous Fourier-Besov-Morrey space with variable exponent* $\mathcal{L}^\rho([0,T); \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q})$ *by*

$$\mathcal{L}^\rho\left([0,T); \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q}\right) = \left\{f \in \mathcal{Z}'(\mathbb{R}^n);\ \|f\|_{\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q})} < \infty\right\},$$

with the norm

$$\|f\|_{\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),h(\cdot),q})} = \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j \hat{f}\|^q_{L^\rho([0,T);\mathcal{M}^{h(\cdot)}_{p(\cdot)})}\right)^{\frac{1}{q}}.$$

We will use the following proposition to prove our main theorem.

Proposition 2.10. *([1]) Let* $I = (0,T],\ s > 0,\ 1 \le \gamma,\ \rho,\ \rho_1,\ \rho_2,\ q \le \infty,\ p(\cdot),\ h(\cdot),\ r(\cdot) \in C^{\log} \cap \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{h(\cdot)} = \frac{1}{h_1(\cdot)} + \frac{1}{h_2(\cdot)}$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ *and* $\frac{1}{\gamma} = \frac{1}{r(\cdot)} + \frac{1}{p(\cdot)}$. *Then, we have*

$$\begin{aligned} |ab|_{\mathcal{L}^\rho(I,\dot{\mathcal{N}}^s_{\gamma,h(\cdot),q})} \lesssim & |a|_{\mathcal{L}^{\rho_1}(I,\mathcal{M}^{h_1(\cdot)}_{r(\cdot)})}|b|_{\mathcal{L}^{\rho_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),h_2(\cdot),q})} \\ & + |b|_{\mathcal{L}^{\rho_1}(I,\mathcal{M}^{h_1(\cdot)}_{r(\cdot)})}|a|_{\mathcal{L}^{\rho_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),h_2(\cdot),q})}. \end{aligned}$$

2.2. Fractional micropolar semigroup and mild solutions

The corresponding linear system of (1) is as follows:

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha_1} u - \nabla \times w = 0 \\ \partial_t w + (-\Delta)^{\alpha_2} w + 2w - \nabla \operatorname{div} w - \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ (u,w)|_{t=0} = (u_0, w_0)\,. \end{cases} \tag{3}$$

The solution operator of the previous problem is denoted by the notation $G(t)$, i.e., for specified initial data (u_0, w_0) in suitable function space, if we denote $(u,w)^T = G(t)\,(u_0,w_0)^T$ the unique solution of the problem (3), then

$$(\widehat{G(t)f})(\xi) = e^{-\mathcal{A}(\xi)t}\hat{f}(\xi) \quad \text{for } f(x) = (f_1(x), f_2(x))^T\,,$$

where

$$\mathcal{A}(\xi) = \begin{bmatrix} |\xi|^{2\alpha_1} I & \mathcal{B}(\xi) \\ \mathcal{B}(\xi) & \left(|\xi|^{2\alpha_2}+2\right)I + C(\xi) \end{bmatrix},$$

with

$$\mathcal{B}(\xi) = i\begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \text{ and } C(\xi) = \begin{bmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & {\xi_3}^2 \end{bmatrix}.$$

Moreover, by applying the Leray projection $\mathbf{P}$ to both sides of the first equations of (1), we can eliminate the pressure π and we obtain

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha_1} u + \mathbf{P}(u\cdot\nabla u) - \nabla \times w = 0 \\ \partial_t w + (-\Delta)^{\alpha_2} w + u\cdot\nabla w + 2w - \nabla \operatorname{div} w - \nabla \times u = 0 \\ \operatorname{div} u = 0 \\ (u,w)|_{t=0} = (u_0, w_0)\,, \end{cases} \tag{4}$$

where $\mathbf{P} = I + \nabla(-\Delta)^{-1} \operatorname{div}$ is the 3×3 matrix pseudo-differential operator in $\mathbb{R}^3$ with the symbol $\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)_{i,j=1}^3$. We denote

$$U(x,t) = \begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix}, \quad U_i(x,t) = \begin{pmatrix} u_i(x,t) \\ w_i(x,t) \end{pmatrix}, \quad i = 1,2$$

and

$$U_1 \tilde{\otimes} U_2 = \begin{pmatrix} u_1 \otimes u_2 \\ u_1 \otimes w_2 \end{pmatrix}, \quad \widetilde{\mathbf{P}}\nabla\cdot(U_1\tilde{\otimes}U_2) = \begin{pmatrix} \mathbf{P}\nabla\cdot(u_1\otimes u_2) \\ \nabla\cdot(u_1\otimes w_2) \end{pmatrix}.$$

Solving system (4) can be reduced to finding a solution U to the following integral equation:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\otimes U)(\tau)d\tau. \tag{5}$$

A solution of (5) **is called a mild solution of** (1).
Now, we will look at a property of the semigroup $G(\cdot)$.

Lemma 2.11. *[11] Let $\frac{1}{2} < \alpha = min(\alpha_1, \alpha_2) \leq 1$. Then for $t \geq 0$ and $|\xi| \neq 0$, there exists $C = C(\alpha_1,\alpha_2) > 0$ (independent of ξ) such that*

$$|e^{-t\mathcal{A}(\xi)}| \leq \begin{cases} e^{-|\xi|^{\alpha_1}t} & \text{if } |\xi| \leq 1, \\ e^{-C|\xi|^{\alpha}t} & \text{if } |\xi| > 1. \end{cases} \tag{6}$$

In particular, if $\alpha = \alpha_1$, then it holds

$$|e^{-t\mathcal{A}(\xi)}| \leq e^{-|\xi|^{2\alpha}t} \text{ for all } |\xi| > 0. \tag{7}$$

3. A priori estimates

The key to the proof of Theorem 1.3 is to establish the linear and bilinear estimates for (1), in order to use Lemma 1.2. We start by proving the linear estimate for Equation (5) in the lemma given below.

Proposition 3.1 (Linear estimate). *Let* $\frac{1}{2} < \alpha = \alpha_1 \leq 1$, $1 \leq \rho, q \leq +\infty$, $p(\cdot), p_1(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $2 \leq p_1(\cdot) \leq p(\cdot) \leq h(\cdot) < \infty$. *Assume that* $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}$, *then the following inequality holds*

$$\|G(t)U_0\|_{\mathcal{L}^\rho\left([0;\infty);\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho}}_{p_1(\cdot),h(\cdot),q}\right)} \lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}}.$$

Proof. According to Hôlder's inequality, Lemma 2.11 and the hypothesis $p_1(\cdot) \leq p(\cdot)$, we get

$$\begin{aligned}
\Big\|G(t)U_0\Big\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)+3-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho}1}_{p_1(\cdot),h(\cdot),q})} &\leq \Big\|\|2^{j(4-2\alpha+\frac{2\alpha}{\rho}-\frac{3}{p_1(\cdot)})}\varphi_j e^{-t|\xi|^{2\alpha}}\hat{U}_0\|_{\mathcal{L}^\rho\left([0,\infty),\mathcal{M}^{h(\cdot)}_{p_1(\cdot)}\right)}\Big\|_{\ell^q} \\
&\lesssim \Big\|\sum_{k=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\hat{U}_0\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}} \\
&\qquad \|r^{\frac{-3(p(\cdot)-p_1(\cdot))}{p(\cdot)p_1(\cdot)}}2^{j(\frac{2\alpha}{\rho}+\frac{3}{p_1'(\cdot)}-\frac{3}{p'(\cdot)})}\varphi_{j+k}e^{-t2^{2\alpha(j+k)}}\|_{L^\rho\left([0,\infty),L^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}}\right)}\Big\|_{\ell^q} \\
&\lesssim \Big\|\sum_{k=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\hat{U}_0\|_{\mathcal{M}^{h(\cdot)}_{p(\cdot)}}\Big\|_{\ell^q} \\
&\lesssim \Big\|U_0\Big\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}},
\end{aligned}$$

where

$$\begin{aligned}
&\|r^{\frac{-3(p(\cdot)-p_1(\cdot))}{p(\cdot)p_1(\cdot)}}2^{j(\frac{2\alpha}{\rho}+\frac{3}{p_1'(\cdot)}-\frac{3}{p'(\cdot)})}\varphi_{j+k}e^{-t2^{2\alpha(j+k)}}\|_{L^\rho\left([0,\infty),L^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}}\right)} \\
&= \|r^{\frac{-3(p(\cdot)-p_1(\cdot))}{p(\cdot)p_1(\cdot)}}2^{j\frac{2\alpha}{\rho}}e^{-t2^{2\alpha(j+k)}}\|_{L^\rho([0,\infty))}\inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}2^{j(\frac{3}{p(x)}-\frac{3}{p_1(x)})}}{\lambda}|^{\frac{p(x)p_1(x)}{p(x)-p_1(x)}}dx\leq 1\right\} \\
&\lesssim \inf\left\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+k}}{\lambda}|^{\frac{p(x)p_1(x)}{p(x)-p_1(x)}}2^{-3j}dx\leq 1\right\} \\
&\leq C.
\end{aligned}$$

Consequently, one obtains

$$\|G(t)U_0\|_{\mathcal{L}^\rho\left([0;\infty);\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho}}_{p_1(\cdot),h(\cdot),q}\right)} \lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}}. \tag{8}$$

□

We have the following lemma for the bilinear estimate.

Lemma 3.2 (Bilinear estimate). *Let* $\frac{1}{2} < \alpha = \alpha_1 \leq 1$, $p_1(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ *such that* $p_1(\cdot) \leq h(\cdot)$ *and* $2 \leq p_1(\cdot) \leq \frac{6}{5-4\alpha}$, $1 \leq \rho \leq \infty$, $1 \leq q < \frac{3}{2\alpha-1}$, *and* $\rho_1 \in [\rho,\infty]$. *Then, we get*

$$\begin{aligned}
&\|\int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\otimes U_2)(\tau)d\tau\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),h(\cdot),q})} \\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})} + \|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}.
\end{aligned}$$

Proof. Let $\frac{1}{2} < \alpha \le 1,\ 1 \le q < \frac{3}{2\alpha-1}$. From Propositions 2.8 and 2.10 and Hausdorf-Young's inequality, we have

$$
\begin{aligned}
&\|\int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\otimes U_2)(\tau)d\tau\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),h(\cdot),q})}\\
&\lesssim \Big\|\|\int_0^t 2^{j\left(4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\varphi_j e^{-(t-\tau)\mathcal{A}(\xi)}\operatorname{div}(\widehat{U_1\otimes U_2})\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty),\mathcal{M}^{h(\cdot)}_{p_1(\cdot)})}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|\int_0^t 2^{j\left(4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\varphi_j e^{-(t-\tau)|\cdot|^{2\alpha}}\operatorname{div}(\widehat{U_1\otimes U_2})\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty),\mathcal{M}^{h(\cdot)}_{p_1(\cdot)})}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|\int_0^t \|r^{-3(\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)})}2^{j\left(5-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\varphi_j e^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\|\Delta_j(U_1\otimes U_2)\|_{\mathcal{M}^{h(\cdot)}_{\frac{6}{4\alpha+1}}}\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty))}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|\int_0^t 2^{j\left(\frac{5}{2}+\frac{2\alpha}{\rho_1}\right)}\|r^{-3(\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)})}\varphi_j e^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\|\Delta_j(U_1\otimes U_2)\|_{\mathcal{M}^{h(\cdot)}_{\frac{6}{4\alpha+1}}}\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty))}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|\int_0^t 2^{j\left(\frac{2\alpha}{\rho_1}+\frac{5}{2}\right)}e^{-(t-\tau)2^{2\alpha j}}\|r^{-3(\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)})}2^{-3j\cdot\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)}}\varphi_j\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\|\Delta_j(U_1\otimes U_2)\|_{\mathcal{M}^{h(\cdot)}_{\frac{6}{4\alpha+1}}}\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty))}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|2^{j\left(\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha\right)}\|\Delta_j(U_1\otimes U_2)\|_{\mathcal{M}^{h(\cdot)}_{\frac{6}{5-4\alpha}}}\|_{L^\rho([0,\infty))}\|e^{-t2^{2\alpha j}}2^{2\alpha j\left(1+\frac{1}{\rho_1}-\frac{1}{\rho}\right)}\|_{\mathcal{L}^{\left(1+\frac{1}{\rho_1}-\frac{1}{\rho}\right)^{-1}}([0,\infty))}\Big\|_{\ell^q}\\
&\lesssim \Big\|\|2^{j\left(\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha\right)}\|\Delta_j(U_1\otimes U_2)\|_{\mathcal{M}^{h(\cdot)}_{\frac{6}{5-4\alpha}}}\|_{L^\rho([0,\infty))}\Big\|_{\ell^q}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),L^{\frac{3}{2\alpha-1}})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),L^{\frac{3}{2\alpha-1}})}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\dot{N}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\dot{N}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}.
\end{aligned}
$$

where we used the following inequality:

$$
\begin{aligned}
&\|r^{-3(\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)})}2^{-3j\cdot\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)}}\varphi_j\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\\
&\le \inf\left\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{2^{-3j\cdot\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)}}\varphi_j}{\lambda}\right|^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}\mathrm{d}x<1\right\}\\
&<\ \text{Constant}.
\end{aligned}
$$

Then, we obtain the result. □

4. Proof of Theorem 1.3

We consider the Banach space $\mathcal{H}$ such that

$$
\mathcal{H}=\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),h(\cdot),q}\left(\mathbb{R}^3\right))\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q}\left(\mathbb{R}^3\right))\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q}\left(\mathbb{R}^3\right)),
$$

and define mappings as follows,

$$\Phi(U)=G(t)U_0+B(U,U).$$

Using Proposition 3.1, we get

$$\|G(t)U_0\|_{\mathcal{H}} \le C_1\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),h(\cdot),q}}. \tag{9}$$

By Lemma 3.2, we obtain

$$\begin{aligned}
&\|B(U_1\otimes U_2)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),h(\cdot),q})}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})},\\
&\|B(U_1\otimes U_2)\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})},
\end{aligned}$$

and

$$\begin{aligned}
&\|B(U_1\otimes U_2)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\\
&\lesssim \|U_1\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_2\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}+\|U_2\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,h(\cdot),q})}\|U_1\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,h(\cdot),q})}.
\end{aligned}$$

Consequently,

$$\|B(U_1\otimes U_2)\|_{\mathcal{H}} \le C_2\|U_1\|_{\mathcal{H}}\|U_2\|_{\mathcal{H}}. \tag{10}$$

Then, by (9) and (10), we get

$$\begin{aligned}
\|\Phi(U)\|_{\mathcal{H}} &\le \|G(t)U_0\|_{\mathcal{H}} + \|\int_0^t G(t-\tau)\tilde{\mathbf{P}}\tilde{\nabla}\cdot(U\otimes U)d\tau\|_{\mathcal{H}}\\
&\le \|G(t)U_0\|_{\mathcal{H}} + \|B(U\otimes U)\|_{\mathcal{H}}\\
&\le C_1\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{p(\cdot)}-2\alpha}_{p(\cdot),h(\cdot),q}} + C_2\|U_1\|_{\mathcal{H}}\|U_2\|_{\mathcal{H}}.
\end{aligned}$$

Put $\delta_0 < \frac{1}{4C_2}$ for any $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{p(\cdot)}-2\alpha}_{p(\cdot),h(\cdot),q}$ with $\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-\frac{3}{p(\cdot)}-2\alpha}_{p(\cdot),h(\cdot),q}} < \frac{\delta_0}{C_1}$. Then, by Lemma 1.2, we conclude that the problem (1) admits a unique global mild solution $U \in \mathcal{H}$.

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