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On parabolic problems involving fractional p-Laplacian via topological degree

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Abstract. The main objective of this paper is to establish the existence result of weak solutions for the initial boundary value parabolic problems involving p-Laplacian operator. The main tool used here is the topological degree method combined with the theory of fractional Sobolev spaces.

1. Introduction

In this research paper, we focus on the parabolic problems, which is represented by the following equation

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = f(x,t) & \text{in } \mathcal{K}_T = \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Here, Ω denotes a bounded open domain of $\mathbb{R}^n (n \geqslant 2)$, T a positive real number, f is a given function. The operator $(-\Delta)_p^s u$ known as the fractional p-Laplacian is defined in the following manner:

$$(-\Delta)_p^s u(x,t) = P.V \int_{\mathbb{R}^n} \frac{|U(x,y,t)|^{p-2} U(x,y,t)}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n,$$

where $U(x,y,t) = u(x,t) - u(y,t)$ and $P.V$, which stands for "in the principal value sense," is a frequently used abbreviation, with $1 < p < \infty$. For more information on this operator see [10].

Many mathematicians, physicists, economics, biologists, and other scientists, have recently become interested in studying problems involving fractional and nonlocal operators. Find more details at [3, 5, 7, 9, 11, 14, 16–18, 22, 24].

For the nonlocal fractional p-Laplacian operator, there are a large number of references in the literature studied th problem (1). Among all of them, [2] proved the existence results when $(f, u_0) \in L^1(\mathcal{K}_T) \times L^1(\Omega)$. See also [25] for more results.

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When $p = 2$, problem (1) reduces to the fractional Laplacian problem

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^s u = f(x,t) & \text{in } \mathcal{K}_T = \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{2}$$

The existence, uniqueness and summa-bility of the solutions to the problem (2), was proved in [20], We refer also to [15] for more details and results.

On the other hand, the topological degree theory is developed in [4] for operators of type $P + Q + C$, where P is maximal monotone, Q is bounded and of type (S_+) and C compact such that $D(P) \subset D(C)$. Authors in [19] studied nonlinear equations for compact identity perturbations in the framework of infinite dimensional Banach spaces, where the concept of topological degree was first established. We suggest to the readers to consult [1, 6, 12], which has been applied to some elliptic and parabolic problems.

Based on the work of Asfaw [4], and from the works cited above. We investigate the existence result of weak solutions to the problem (1) involving nonlocal fractional p-Laplacian operator by using the tool of topological degree. To our knowledge, the problem (1) has never been studied by the topological degree theory.

2. Preliminaries

2.1. Fractional Sobolev spaces

In this section, we present some definitions, notations, and properties of functional spaces which will be used in the sequel.

Let s, p be two real numbers such that $0 < s < 1$ and $1 < p < \infty$. We define p_s^* the fractional critical exponent as follows

$$p_s^* = \begin{cases} \infty \text{ if } ps \geq n, \\ np/(n-ps) \text{ if } ps < n. \end{cases}$$

Let Ω be an open subset of $\mathbb{R}^n$, $\mathcal{C}\Omega = \mathbb{R}^n \backslash \Omega$ and $\mathcal{K}_\Omega = (\mathbb{R}^n \times \mathbb{R}^n) \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. It is clear that $\Omega \times \Omega$ is strictly contained in $\mathcal{K}_\Omega$. Let W be a linear space of measurable Lebesgue functions defined from $\mathbb{R}^n$ to $\mathbb{R}$ which satisfy two conditions: their restriction on Ω belongs to $L^p(\Omega)$, and they satisfy the following inequality

$$\iint_{\mathcal{K}_\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} dydx < \infty.$$

In this context, W has a norm defined as follows

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\iint_{\mathcal{K}_\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} dydx \right)^{\frac{1}{p}}.$$

Moreover, there exists a closed linear subspace of W denoted W_0, consisting of the elements u of W that satisfy the following condition

$$u = 0 \text{ almost everywhere in } \mathcal{C}\Omega.$$

In the subspace W_0, a norm can also be defined as follows

$$\|u\|_{W_0} = \left(\iint_{\mathcal{K}_\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

It is well known that the pair $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space, see [26]. The dual space of $(W_0, \|\cdot\|_{W_0})$ is indecated by $\left(W_0^*, \|\cdot\|_{W_0^*}\right)$.

Lemma 2.1. *[13] The following embedding $W_0 \hookrightarrow L^{\theta}(\Omega)$ is compact for each $1 \leq \theta < p_s^*$, and continuous for each $1 \leq \theta \leq p_s^*$.*

Lemma 2.2. *[23] For any $\xi, \eta \in \mathbb{R}^n$, we get*

$$\begin{cases} |\xi-\eta|^p \leq c_p \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta); & p \geq 2 \\ |\xi-\eta|^p \leq C_p \left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi-\eta)\right]^{\frac{p}{2}} \left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{2}}; & 1 < p < 2, \end{cases} \tag{3}$$

where $c_p = \left(\frac{1}{2}\right)^{-p}$ and $C_p = \frac{1}{p-1}$.

In the sequel, let $ps < n$ and denote the following functional space

$$\Gamma := L^p(0,T;W_0), \quad T > 0.$$

This space is a separable and reflexive Banach space, equipped with the following norm

$$\|u\|_\Gamma = \left(\int_0^T \|u\|_{W_0}^p dt\right)^{\frac{1}{p}}.$$

2.2. Topological degree theory

Consider a real, separable, and reflexive Banach space X, its dual space represented by X^* with a continuous pairing $\langle \cdot, \cdot \rangle$. In this context, we use $\|\cdot\|$ to denote the norm of the space X and its dual X^*. We will indicate strong (weak) convergence by the symbol $\to$ ($\rightharpoonup$).

Let $\mathcal{F} : X \to 2^{X^*}$ be a multi-valued mapping, where the values are subsets of X^*. The graph of $\mathcal{F}$ is defined by

$$G(\mathcal{F}) = \left\{(u,w) \in X \times X^* : w \in \mathcal{F}(u)\right\}.$$

Definition 2.3. *Under the above notation, we say that $\mathcal{F}$ is monotone, if for each $(u_1, w_1), (u_2, w_2)$ in $G(\mathcal{F})$ we have*

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq 0.$$

Definition 2.4. *We say that the map $\mathcal{F}$ is maximal monotone if*

i. *$\mathcal{F}$ is monotone,*

ii. *for any $(u_0, w_0) \in X \times X^*$ such that $\langle w_0 - w, u_0 - u \rangle \geq 0$, for each $(u,w) \in G(\mathcal{F})$, then $(u_0, w_0) \in G(\mathcal{F})$.*

Definition 2.5. *The map $\mathcal{F} : D(\mathcal{F}) \subset X \to Y$ is demicontinuous if for each $(u_k)_k \subset \Omega$ such that $u_k \to u$, implies that $\mathcal{F}(u_k) \rightharpoonup \mathcal{F}(u)$.*

Definition 2.6. *The map $\mathcal{F}$ is called of type (S_+), if for every $(u_k) \subset D(\mathcal{F})$ such that u_k converges weakly to u and $\limsup_{k\to+\infty} \langle \mathcal{F} u_k, u_k - u \rangle \leq 0$, it implies that u_k converges strongly to u.*

Let $P : D(P) \subset X \to X^*$ be a linear maximal monotone mapping such that $\overline{D(P)} = X$ and E be an open bounded subset of X. We define the following classes of operators

$$\mathcal{N}_E := \Big\{P + Q : \overline{E} \cap D(P) \to X^* | \quad Q \text{ is bounded, demicontinuous map of type } (S_+) \text{ with respect to } D(P) \text{ from } \overline{E} \text{ to } X^*\Big\},$$

$$\mathcal{H}_E := \Big\{P + Q(t) : \overline{E} \cap D(P) \to X^* | \quad Q(t) \text{ is a bounded homotopy of type } (S_+) \text{ with respect to } D(P) \text{ from } \overline{E} \text{ to } X^*\Big\}.$$

It is clear that the set $\mathcal{H}_E$ (set of admissible homotopies) includes all affine homotopies $P + (1-t)Q_1 + tQ_2$ with $(P + Q_i) \in \mathcal{N}_E, i = 1, 2$.

Theorem 2.7. *Let $P : D(P) \subset X \to X^*$ be a maximal monotone densely linear mapping. There exists a function which is called the topological degree function*

$$d : \Big\{(N, E, h) : N \in \mathcal{N}_E, \text{ } E \text{ an open bounded subset of } X, \text{ } h \notin N(\partial E \cap D(P))\Big\} \to \mathbb{Z}$$

satisfying the following properties:

1. ***(Existence)*** *If $d(N, E, h) \neq 0$, then there is a solution to the equation $Nu = h$ in the set $E \cap D(P)$.*
2. ***(Additivity)*** *For any E_1 and E_2 two disjoint open subsets of E with $h \notin N[(\overline{E}\backslash(E_1 \cup E_2)) \cap D(P)]$, then we get*

$$d(N, E, h) = d(N, E_1, h) + d(N, E_2, h).$$

3. ***(Invariance under homotopies)*** *If $N(t) \in \mathcal{H}_E$ and $h(t) \notin N(t)(\partial E \cap D(P))$ for each t in the interval $[0, 1]$, with $h(t)$ is a continuous curve in X^*, then the following can be concluded*

$$d(N(t), E, h(t)) \text{ remains constant for all } t \text{ in } [0, 1].$$

4. ***(Normalization)*** *Let $P + M$ be a normalising map such that $M : X \to X^*$ is a duality mapping. Then, we have*

$$d(P + M, E, h) = 1, \text{ for every } h \in (P + M)(E \cap D(P)).$$

Theorem 2.8. *Let $P + Q \in \mathcal{N}_X$ with $0 \in P(0)$ and $h \in X^*$. Suppose that there exists a strictly positive real number R such that*

$$\langle Pu + Qu - h, u\rangle > 0, \tag{4}$$

for every $u \in \partial B_R(0) \cap D(P)$. Then, we get

$$(P + Q)(D(P)) = X^*,$$

provided that $P + Q$ is coercive.

Proof. Let $\nu > 0$ and $t \in [0, 1]$. We define de following operator

$$H_\nu(t, u) = Pu + (1-t)Mu + t(Qu + \nu Mu - h).$$

Since $0 \in P(0)$ and according to the boundary condition (4), we can obtain

$$\begin{aligned}\langle H_\nu(t, u), u\rangle &= \langle t(Pu + Qu - h, u\rangle + \langle(1-t)Pu + (1-t+\nu)Mu, u\rangle \\ &\geq \langle(1-t)Pu + (1-t+\nu)Mu, u\rangle \\ &= (1-t)\langle Pu, u\rangle + (1-t+\nu)\langle Mu, u\rangle \\ &\geq (1-t+\nu)\|u\|^2 = (1-t+\nu)R^2 > 0.\end{aligned}$$

This means that 0 is not an element of $H_\nu(t, u)$. Moreover, since M and $Q + \nu M$ are bounded, continuous and of type (S_+), then $\{H_\nu(t, \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Consequently, from the normalisation and invariance under homotopy in Theorem 2.7, we can conclude that

$$\begin{aligned}d(H_\nu(t, \cdot), B_R(0), 0) &= d(P + M, B_R(0), 0) \\ &= d(\nu M, B_R(0), 0) \\ &= 1,\end{aligned}$$

that is $d(P+Q+\nu M, B_R(0), f) = 1$. From Theorem 2.7, we can deduce that

$$f^* \in (P+Q+\nu M)(D(P)\cap B_R(0)).$$

Therefore, for every $\epsilon_n \to 0_+$, there exist $y_n \in D(P)\cap B_R(0)$, $v_n^* \in Py_n$, and $w_n^* \in Qy_n$ such that

$$v_n^* + w_n^* + \epsilon_n M y_n = f^* \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Since $(y_n)_n$ is bounded, it follows that $\epsilon_n M y_n \to 0$ as n tends to infinity. By using (5) and letting limit as $n \to +\infty$, we get

$$f^* \in \overline{(P+Q+\nu M)(D(P)\cap B_R(0))}.$$

If $P+Q$ is coercive, then for any $f^* \in X^*$, there exists a positive constant $R = R(f^*) > 0$ such that the boundary condition is satisfied. As $f^* \in X^*$ is chosen arbitrarily, we can deduce that

$$(P+Q)(D(P)) = X^*.$$

The proof is complete. □

Throughout the paper, we denote $U_k(x,y,t)$ and $V(x,y,t)$ by $u_k(x,t)-u_k(y,t)$ and $v(x,t)-v(y,t)$ respectively.

3. Main result

In this section, we demonstrate our main results. The first result is the following lemma.

Lemma 3.1. *Let $Q : \Gamma \to \Gamma^*$ be an operator defined by*

$$\langle Qu, \phi\rangle = \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt,$$

for all u, $\phi \in \Gamma$. Then, Q is
i) *bounded,*
ii) *continuous,*
iii) *of type (S_+).*

Proof. **i)** We will prove that Q is bounded. For all u, $\phi \in \Gamma$, we have

$$\begin{aligned}
|\langle Qu, \phi\rangle| &= \left|\int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{n+ps}}(\phi(x,t)-\phi(y,t))dxdydt\right| \\
&\le \left|\int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} \frac{(\phi(x,t)-\phi(y,t))}{|x-y|^{\frac{n+ps}{p}}}dxdydt\right| \\
&\le \|u\|_\Gamma^{p-1}\|\phi\|_\Gamma \\
&\le C\|\phi\|_\Gamma.
\end{aligned}$$

ii) We are going to prove that the operator Q is continuous. Let $u_k \to u$ in Γ and $\phi \in \Gamma$, we have

$$\begin{aligned}
&\langle Q(u_k), \phi\rangle - \langle Q(u), \phi\rangle \\
&\quad = \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U_k(x,y,t)|^{p-2}U_k(x,y,t)(\phi(x,t)-\phi(y,t))}{|x-y|^{n+ps}}dxdydt \\
&\quad - \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2}U(x,y,t)(\phi(x,t)-\phi(y,t))}{|x-y|^{n+ps}}dxdydt.
\end{aligned}$$

Then,

$$\begin{aligned}
&\langle Q(u_k),\phi\rangle - \langle Q(u),\phi\rangle \\
&= \int_0^T \iint_{\mathcal{K}_\Omega} \left[\frac{|U_k(x,y,t)|^{p-2}U_k(x,y,t) - |U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{n+ps}} (\phi(x,t)-\phi(y,t)) \right] dxdydt \\
&= \int_0^T \iint_{\mathcal{K}_\Omega} \left[\left(\frac{|U_k(x,y,t)|^{p-2}U_k(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} - \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} \right) \frac{(\phi(x,t)-\phi(y,t))}{|x-y|^{\frac{n+ps}{p}}} \right] dxdydt.
\end{aligned}$$

Let $p' = \frac{p}{p-1}$ and denote

$$A_k = \frac{|U_k(x,y,t)|^{p-2}U_k(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} \in L^{p'}(\mathcal{K}_\Omega \times (0,T)),$$

$$A = \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} \in L^{p'}(\mathcal{K}_\Omega \times (0,T)),$$

$$\phi(x,y,t) = \frac{(\phi(x,t)-\phi(y,t))}{|x-y|^{\frac{n+ps}{p}}} \in L^{p}(\mathcal{K}_\Omega \times (0,T)).$$

Then, we have by Hölder inequality that

$$\begin{aligned}
&|\langle Q(u_k),\phi\rangle - \langle Q(u),\phi\rangle| \\
&= \left| \int_0^T \iint_{\mathcal{K}_\Omega} \left[\left(\frac{|U_k(x,y,t)|^{p-2}U_k(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} - \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{(n+ps)\frac{p-1}{p}}} \right) \frac{(\phi(x,t)-\phi(y,t))}{|x-y|^{\frac{n+ps}{p}}} \right] dxdydt \right| \\
&\leq \|A_k - A\|_{L^{p'}(\mathcal{K}_\Omega\times(0,T))} \left\|\phi\right\|_{L^{p}(\mathcal{K}_\Omega\times(0,T))}.
\end{aligned}$$

Now, we denote

$$B_k = \frac{U_k(x,y,t)}{|x-y|^{\frac{n+ps}{p}}} \in L^{p}(\mathcal{K}_\Omega \times (0,T)),$$

$$B = \frac{U(x,y,t)}{|x-y|^{\frac{n+ps}{p}}} \in L^{p}(\mathcal{K}_\Omega \times (0,T)).$$

Since $u_k \to u$ in Γ, then

$$B_k \to B \text{ in } L^{p}(\mathcal{K}_\Omega \times (0,T)).$$

As a result, there exists a subsequence (still denoted by $(B_k)_k$) of B_k such that

$$B_k \to B \text{ a.e. in } \mathcal{K}_\Omega \times (0,T),$$

and there exists a function $b(x,y,t)$ such that $|B_k| \leq b(x,y,t)$.
Hence

$$A_k \to A \text{ a.e. in } \mathcal{K}_\Omega \times (0,T),$$

and

$$A_k = |B_k|^{p-1} \leq |b|^{p-1}.$$

Then, by the Dominated convergence theorem, we deduce that

$$A_k \to A \text{ in } L^{p'}(\mathcal{K}_\Omega \times (0,T)).$$

This implies that the operator Q is continuous on Γ.

iii) Now, we will show that Q is of type (S_+). Let $(u_k)_k \in D(Q)$ be a sequence with

$$u_k \rightharpoonup u \text{ in } \Gamma \text{ and } \limsup_{k\to+\infty} \langle Q(u_k) - Q(u), u_k - u\rangle \leq 0. \tag{6}$$

Firstly, we prove the strict monotonicity of Q. For any u, $v \in \Gamma$, we have

$$\begin{aligned}
&\langle Q(u) - Q(v), u - v\rangle \\
&\quad= \langle Q(u), u - v\rangle - \langle Q(v), u - v\rangle \\
&\quad= \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2}U(x,y,t)}{|x-y|^{n+ps}}((u-v)(x,t) - (u-v)(y,t))dxdydt \\
&\quad- \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|V(x,y,t)|^{p-2}V(x,y,t)}{|x-y|^{n+ps}}((u-v)(x,t) - (u-v)(y,t))dxdydt \\
&\quad= \|u\|_\Gamma^p + \|v\|_\Gamma^p - \langle Q(u), v\rangle - \langle Q(v), u\rangle \\
&\quad\geqslant \|u\|_\Gamma^p + \|v\|_\Gamma^p - \|u\|_\Gamma^{p-1}\|v\|_\Gamma - \|v\|_\Gamma^{p-1}\|u\|_\Gamma \\
&\quad= \left(\|u\|_\Gamma^{p-1} - \|v\|_\Gamma^{p-1}\right)(\|u\|_\Gamma - \|v\|_\Gamma).
\end{aligned} \tag{7}$$

By Lemma 2.2. If $p \geqslant 2$, then from (7) we have

$$\begin{aligned}
c_p\langle Q(u) - Q(v), u - v\rangle &\geqslant c_p\left(\|u\|_\Gamma^{p-1} - \|v\|_\Gamma^{p-1}\right)(\|u\|_\Gamma - \|v\|_\Gamma) \\
&\geqslant c_p\left(\|u\|_\Gamma^{p-2}\|u\|_\Gamma - \|v\|_\Gamma^{p-2}\|v\|_\Gamma\right)(\|u\|_\Gamma - \|v\|_\Gamma) \\
&\geqslant \left|\|u\|_\Gamma - \|v\|_\Gamma\right|^p > 0.
\end{aligned}$$

If $1 < p < 2$, using (7) we get

$$C_p\left(\langle Q(u) - Q(v), u - v\rangle\right)^{\frac{p}{2}}\left(\|u\|_\Gamma^p - \|v\|_\Gamma^p\right)^{\frac{2-p}{2}} \geqslant \left|\|u\|_\Gamma - \|v\|_\Gamma\right|^p.$$

For $u \neq v$, Q is strictly monotone. Therefore, using (6), we can deduce that

$$\lim_{k\to+\infty} \langle Q(u_k) - Q(u), u_k - u\rangle = 0. \tag{8}$$

According to [21, Theorem 5.1] and Lemma 2.1, we can get

$$u_k \to u \text{ a.e. in } \mathcal{K}_T. \tag{9}$$

This along with Fatou's lemma yield

$$\liminf_{k\to+\infty} \int_0^T \iint_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^p}{|x-y|^{n+ps}}dxdydt \geq \int_0^T \iint_{\mathcal{K}_\Omega} \frac{\left|U(x,y,t)\right|^p}{|x-y|^{n+ps}}dxdydt. \tag{10}$$

On the other hand, we know that

$$\lim_{k\to+\infty} \langle Q(u_k), u_k - u\rangle = \lim_{k\to+\infty} \langle Q(u_k) - Q(u), u_k - u\rangle = 0. \tag{11}$$

Now, by applying Young's inequality, we can find a positive constant C such that

$$\begin{aligned}
&\langle Q(u_k), u_k - u\rangle \\
&\quad = \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt \\
&\quad - \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^{p-2} (U_k(x,y,t))\, U(x,y,t)}{|x-y|^{n+ps}} dxdydt \\
&\quad \geq \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt \\
&\quad - \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^{p-1} U(x,y,t)}{|x-y|^{n+ps}} dxdydt \\
&\quad \geq C\left(\int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt - \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt\right).
\end{aligned} \tag{12}$$

As a consequence of (10), (11) and (12), we get

$$\lim_{k\to+\infty} \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U_k(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt = \int_0^T \int_{\mathcal{K}_\Omega} \frac{\left|U(x,y,t)\right|^p}{|x-y|^{n+ps}} dxdydt. \tag{13}$$

By combining (9) and (13) with [8, Brezis-Lieb Lemma], we have established our result.
□

Next, we prove the existence of weak solutions to problem (1) using the topological degree theory introduced in the above section.

Theorem 3.2. *Let $f \in \Gamma^*$ and $u_0 \in L^2(\Omega)$. Then, the problem (1) has at least one weak solution $u \in D(P)$ in the following sense*

$$\begin{aligned}
&\int_{\mathcal{K}_T} \frac{\partial u}{\partial t}\phi dxdt + \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^{p-2} U(x,y,t)}{|x-y|^{n+ps}} (\phi(x,t) - \phi(y,t)) dxdydt \\
&\qquad = \int_{\mathcal{K}_T} f(x,t)\phi dxdt
\end{aligned}$$

for all $\phi \in \Gamma$.

Proof. Let $P : D(P) \subset \Gamma \to \Gamma^*$ be an operator defined by

$$\langle Pu, v\rangle = \int_{\mathcal{K}_T} \frac{\partial u}{\partial t} v dxdt, \qquad \text{for every } u \in D(P),\ v \in \Gamma,$$

where

$$D(P) = \left\{ v \in \Gamma : \frac{\partial v}{\partial t} \in \Gamma^*, v(0) = 0 \right\}.$$

As in [27], it can be shown that P is a densely defined maximal monotone operator. According to the monotonicity of P (i.e. $\langle Pu, u\rangle \geq 0$ for every $u \in D(P)$), we get

$$\begin{aligned}
\langle Pu + Qu, u\rangle &\geq \langle Qu, u\rangle \\
&= \int_0^T \iint_{\mathcal{K}_\Omega} \frac{|U(x,y,t)|^p}{|x-y|^{n+ps}} dxdydt \\
&= \|u\|_\Gamma^p
\end{aligned}$$

for all $u \in \Gamma$. Clearly, the right-hand side of the above inequality grows towards infinity as $\|u\|_\Gamma$ approaches infinity. This implies that for any $f \in \Gamma^*$ we can find a constant $R = R(f)$ such that $\langle Pu + Qu - f, u\rangle > 0$ for every $u \in B_R(0) \cap D(P)$. By using Theorem 2.8, we can deduce that the equation $Pu + Qu = f$ has a solution in the domain $D(P)$. This demonstrates that the problem (1) possesses at least one weak solution. □

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