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Positive solutions for double multi-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator

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Abstract. This study delves into the investigation of positive solutions for a specific class of such problems, namely double multi-point boundary value problems. In this research, we employ a monotone iterative approach combined with the theory of the fixed point index within a cone to establish the existence and multiplicity of positive solutions for double multi-point boundary value problems associated with nonlinear fractional differential equations involving the p-Laplacian operator. These findings not only advance the theoretical understanding of fractional differential equations but also hold promise for applications in diverse scientific and engineering disciplines. Additionally, we present straightforward and illustrative examples that reinforce the core findings of this study.

1. Introduction

Fractional differential equations, with derivatives of non-integer order, have emerged as a powerful mathematical framework for modeling intricate phenomena with applications spanning across various scientific and engineering disciplines. These equations provide a more accurate representation of systems that exhibit memory effects, non-local interactions, and anomalous diffusion. Some examples are seen in biology, economics, control theory, chemistry, physics and biophysics, just to mention but a few [11]-[14]. In particular, the incorporation of the p-Laplacian operator adds an extra layer of complexity and relevance to the study of such equations. Double multi-point boundary conditions are of particular interest as they arise in many real-world applications. Understanding the existence of positive solutions in the context of double multi-point boundary value problems for such equations is a fundamental endeavor with broad implications. This trend comes about due to advances in fractional calculus theories with its widespread applications as evidently seen in [1],[15]-[20].

Moreover, the use of fractional differential equations has extended to hereditary properties of diverse materials and processes. This has caught the attention of a lot of researchers who have over the years concentrated on the study of existence and uniqueness of solutions for boundary value problems (BVP) of fractional differential equations involving nonlocal boundary conditions through means of techniques

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of nonlinear such as the upper and lower solution method, fixed point theorems and the Leray-Schauder theory among others [2]-[5], [7], [8], [10].

In [6], the following fractional differential systems

$$\begin{cases} D^{\alpha}_{0^+}y(x) + \mu_1 f(x, v(x)) + \mu_2 g(x, v(x)) = 0, & x \in (0,1), \\ D^{\alpha}_{0^+}v(x) + \mu_1 f(x, y(x)) + \mu_2 g(x, y(x)) = 0, & x \in (0,1), \end{cases}$$

and

$$\begin{cases} D^{\alpha}_{0^+}y(x) + \mu_1 f(x, y(x)) + \mu_2 g(x, v(x)) = 0, & x \in (0,1), \\ D^{\alpha}_{0^+}v(x) + \mu_1 f(x, v(x)) + \mu_2 g(x, y(x)) = 0, & x \in (0,1), \end{cases}$$

with the multipoint boundary conditions

$$\begin{cases} D^{\beta}_{0^+}y(1) = \sum_{k=1}^{m-2} \xi_k D^{\beta}_{0^+}y(\eta_k),\ y(0) = 0, \\ D^{\beta}_{0^+}v(1) = \sum_{k=1}^{m-2} \xi_k D^{\beta}_{0^+}v(\eta_k),\ v(0) = 0, \end{cases}$$

were considered, where $D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ are the Riemann-Liouville fractional derivatives, $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous, $1 < \alpha \le 2$, $0 \le \beta \le 1$ with $1 \le \alpha - \beta$, $0 < \xi_k, \eta_k < 1$, $k = 1, 2, \cdots, m-2$, such that $\sum_{k=1}^{m-2} \xi_k \eta_k^{\alpha-\beta-1} < 1$ and $\mu_1, \mu_2 \in (0, +\infty)$ such that $\mu_1 \ge \mu_2$. The application of fixed-point theorems concerning γ concave and $(-\gamma)$ convex operators played a pivotal role in verifying the existence of positive solutions in fractional differential systems subjected to multipoint boundary conditions.

In [9], the following BVP was investigated

$$\begin{cases} D^{\gamma}(\varphi_p(D^{\alpha}y(x))) = f(x, y(x)),\ 0 < x < 1, \\ y(0) = D^{\alpha}y(0) = 0,\ D^{\beta}y(1) = aD^{\beta}y(\xi),\ D^{\alpha}y(1) = bD^{\alpha}y(\eta), \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $1 < \alpha, \gamma \le 2$, $\beta > 0$ such that $1 + \beta \le \alpha$, ξ, $\eta \in (0,1)$, a, $b \in [0, +\infty)$ such that $1 - a\xi^{\alpha-\beta-1} > 0$, $1 - b^{p-1}\eta^{\gamma-1} > 0$, $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $\varphi_q = (\varphi_p)^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, D^{α}, D^{β} and D^{γ} are the Riemann-Liouville derivatives and $f : [0,1] \times [0,\infty) \to [0,\infty)$. The authors applied a monotone iterative method to obtain some existence results for positive solutions of the BVP of nonlinear fractional differential equations with the p-Laplacian operator.

As far as the authors are aware, there is a limited body of work that delves into the investigation of the Riemann-Liouville fractional derivative involving a p-Laplacian operator and double multi-point boundary value conditions. The inclusion of double multi-point boundary conditions introduces additional layers of complexity and mathematical richness, making this problem both challenging and captivating. Motivated by the literature mentioned, in this study we concentrate on the existence results of the following BVP of fractional differential equations

$$\begin{cases} D^{\beta}(\varphi_p(D^{\alpha}y(x))) + \lambda h(x) f(x, y(x)) = 0, & x \in (0,1), \\ y(0) = 0, \quad \varphi_p(D^{\alpha}y(0)) = 0, \\ D^{\gamma}y(1) = \sum_{k=1}^{m-2} a_k D^{\gamma}y(\eta_k), \quad \varphi_p(D^{\alpha}y(1)) = \sum_{k=1}^{m-2} b_k \varphi_p(D^{\alpha}y(\xi_k)), \end{cases} \tag{1}$$

where $1 < \alpha, \beta \le 2$, $0 < \gamma \le 1$ such that $1 \le \alpha - \gamma$, $0 \le a_k, b_k, \eta_k, \xi_k \le 1$, for $k = 1, 2, \cdots, m-2$ such that $\sum_{k=1}^{m-2} a_k \eta_k^{\alpha-\gamma-1} < 1$, $\sum_{k=1}^{m-2} b_k \xi_k^{\beta-1} < 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$, $h : [0,1] \to [0,+\infty))$, $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, with D^{α}, D^{β} and D^{γ} are the standard Riemann-Liouville fractional derivatives.

The primary objective of this study is to explore the existence of positive solutions for this class of nonlinear fractional differential equations. This pursuit holds significant value not only from a theoretical perspective but also for its potential implications in real-world problem-solving. Positive solutions often represent physical quantities of interest, and their existence or non-existence can profoundly affect the outcome of mathematical models.

In Section 2 of this paper, we present essential definitions and lemmas that serve as the foundational elements for our main findings. Additionally, for the reader's convenience, we introduce a fixed point theorem. In Section 3, we consider the nonlinear BVP (1) and give the existence results of this problem. In this section, we also present multiplicity results for the BVP (1). In addition, some comprehensive examples to illustrate our main results are provided herein. Finally, Section 4 concludes the paper, emphasizing the significance of our results and suggesting avenues for future research.

In summary, this study contributes to the understanding of fractional differential equations with the p-Laplacian operator, particularly in the context of double multi-point boundary value problems. The exploration of positive solutions not only advances the theoretical understanding but also holds promise for applications in diverse scientific and engineering disciplines.

2. Basic Definitions and Preliminaries

In this section, we commence by presenting crucial definitions and lemmas. These auxiliary lemmas are indispensable for demonstrating the existence of solutions for the problem (1).

Definition 2.1. *(see [19], [20]) The integral*

$$I_a^{\beta} g(x) = \int_a^x \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} g(t)dt,$$

the fractional order integral operation onto the function $g \in L^1([a,b), R_+)$ with order $\beta \in R_+$ and lower limit a, where Γ represents the gamma function.

Definition 2.2. *(see [19], [20]) The definition of the Riemann-Liouville fractional-order derivative of a function g at order β denoted as,*

$$D_{0^+}^{\beta} g(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_0^1 (x-t)^{n-\beta-1} g(t)dt,$$

where $\beta \in R_+$

Lemma 2.3. *(see [1]) Suppose that $g \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$ that belongs to domain $C(0,1) \cap L(0,1)$. Then*

$$I^{\beta} D^{\beta} g(x) = g(x) + c_1 x^{\beta-1} + c_2 x^{\beta-2} + \cdots + c_N x^{\beta-N},$$

for some $c_k \in \mathbb{R}$, $k = 1, 2, \cdots, N$, where the number N is the smallest integer greater than or equal to β.

Lemma 2.4. *(see [6]) Let $e \in C[0,1]$. Then the linear fractional BVP*

$$D^{\alpha} y(x) + e(x) = 0$$

$$y(0) = 0, \qquad D^{\gamma} y(1) = \sum_{k=1}^{m-2} a_k D^{\gamma} y(\eta_k)$$

has a unique solution which is given by

$$y(x) = \int_0^1 G(x,t) e(t)dt,$$

where

$$G(x,t) = G_1(x,t) + G_2(x,t),$$

in which

$$G_1(x,t) = \begin{cases} \frac{x^{\alpha-1}(1-t)^{\alpha-\gamma-1}-(x-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le x \le 1, \\ \frac{x^{\alpha-1}(1-t)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le x \le t \le 1, \end{cases}$$

$$G_2(x,t) = \begin{cases} \frac{\sum_{0\le t\le \eta_i}[a_i\eta_i^{\alpha-\gamma-1}x^{\alpha-1}(1-t)^{\alpha-\gamma-1}-a_i x^{\alpha-1}(\eta_i-t)^{\alpha-\gamma-1}]}{A\Gamma(\alpha)}, & x\in[0,1], \\ \frac{\sum_{\eta_i\le t\le 1} a_i\eta_i^{\alpha-\gamma-1}x^{\alpha-1}(1-t)^{\alpha-\gamma-1}}{A\Gamma(\alpha)}, & x\in[0,1], \end{cases} \quad (2)$$

where
$A = 1-\sum_{k=1}^{m-2} a_k\eta_k^{\alpha-\gamma-1}$.

Lemma 2.5. *(see* [4]*) Let $e \in C[0,1]$. Then the linear fractional BVP*

$$D^\beta(\varphi_p(D^\alpha y(x))) + e(x) = 0, \qquad x\in[0,1],$$

$$y(0)=0, \quad \varphi_p(D^\alpha y(0)) = 0, \quad \varphi_p(D^\alpha y(1)) = \sum_{k=1}^{m-2} b_k\varphi_p(D^\alpha y(\xi_k)), \quad D^\gamma y(1) = \sum_{k=1}^{m-2} a_k D^\gamma y(\eta_k)$$

admits a unique solution expressed as

$$y(x) = \int_0^1 G(x,t)\varphi_q(\rho(t))dt,$$

where
$G(x,t)$ is given in Lemma 2.2 and

$$\rho(x) = \int_0^1 H(x,t)e(t)ds + \frac{x^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k\int_0^1 H(\xi_k,t)e(t)dt,$$

in which

$$B = 1-\sum_{k=1}^{m-2} b_k\xi_k^{\beta-1},$$

$$H(x,t) = \begin{cases} \frac{x^{\beta-1}(1-t)^{\beta-1}-(x-t)^{\beta-1}}{\Gamma(\beta)}, & 0\le t\le x\le 1, \\ \frac{x^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)}, & 0\le x\le t\le 1. \end{cases} \quad (3)$$

Lemma 2.6. *(see*[4], [6]*) The functions $G(x,s)$ and $H(x,s)$ given by* (2) *and* (3) *respectively meet the following conditions:*

1. *For $t,x\in[0,1], G(x,t)\ge 0, H(x,t)\ge 0, H(x,t)\le H(t,t)$ and if $\sum_{k=1}^{m-2} a_k\eta_k^{\alpha-\gamma-1} < 1$,*
   $G(x,t) \le \overline{G}(x,t)$,
   where

   $$\overline{G}(x,t) = \overline{G}_1(x,t) + \overline{G}_2(x,t),$$

   in which

   $$\overline{G}_1(x,t) = \frac{x^{\alpha-1}(1-t)^{\alpha-\gamma-1}}{\Gamma(\alpha)}$$

   and

   $$\overline{G}_2(x,t) = \frac{\sum_{k=1}^{m-2} a_k\eta_k^{\alpha-\gamma-1}x^{\alpha-1}(1-t)^{\alpha-\gamma-1}}{A\Gamma(\alpha)},$$

2. *$G(x,t) \geq x^{\alpha-1}G(1,t)$ for all $t, x \in [0,1]$ and there exists a positive function $g_2 \in C(0,1)$ such that*

$$\min_{\vartheta \leq x \leq \delta} H(x,t) \geq g_2(t)H(t,t) \text{ for } t \in (0,1),$$

where

$$g_2(t) = \begin{cases} \frac{\delta^{\beta-1}(1-t)^{\beta-1}-(\delta-t)^{\beta-1}}{x^{\beta-1}(1-t)^{\beta-1}}, & \text{if } t \in [0, m_1], \\ (\frac{\vartheta}{t})^{\beta-1}, & \text{if } t \in [m_1, 1] \end{cases}$$

for $0 \leq \vartheta < m_1 < \delta \leq 1$.

3. *For $t, x \in [0,1]$, $\max_{0\leq x\leq 1} \int_0^1 H(x,t)ds = \frac{\Gamma(\beta)}{\Gamma(2\beta)}$ and*

$$G_*(t,t) = \max_{x\in[0,1]} \overline{G}_1(x,t) + \max_{x\in[0,1]} \overline{G}_2(x,t),$$

where

$$\max_{x\in[0,1]} \overline{G}_1(x,t) = \frac{(1-t)^{\alpha-\gamma-1}}{\Gamma(\alpha)}$$

and

$$\max_{x\in[0,1]} \overline{G}_2(x,t) = \frac{\sum_{k=1}^{m-2} a_k \eta_k^{\alpha-\gamma-1}(1-t)^{\alpha-\gamma-1}}{A\Gamma(\alpha)}.$$

Moreover, we leverage the following fixed point theorems to establish the existence of results.

Theorem 2.7. *(Schauder-Tychonov Fixed Point Theorem)(see* [21]*) Let E be a Banach space and P be a bounded, closed, convex subset of E. If $\Theta : P \to P$ is compact, then Θ has a fixed point in P.*

Theorem 2.8. *(see* [5]*) Let E be a Banach space, $D \subset E$ be a cone and $\Omega_R = \{v \in D : \|v\| \leq R\}$. Let the operator $\Theta : D \cap \Omega_R \to D$ be completely continuous that satisfying $\Theta w \neq w$, $\forall w \in \partial\Omega_R$.*
Then

1. *If $\|\Theta w\| \leq \|w\|$, $\forall w \in \partial\Omega_R$, then $i(\Theta, \Omega_R, D) = 1$,*
2. *If $\|\Theta w\| \geq \|w\|$, $\forall w \in \partial\Omega_R$, then $i(\Theta, \Omega_R, D) = 0$.*

3. Existence of Solutions for BVP (1)

We examine the Banach space $E = C([0,1], \mathbb{R})$ equipped with the norm $\|y\| = \sup_{0\leq x\leq 1} |y(x)|$. Let we define the set $D = \{y \in E : y(x) \geq 0\}$, then D is a cone within E.

Define the operator $\Theta : D \to D$ as follows:

$$(\Theta y)(x) = \int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau)f(\tau, y(\tau))d\tau + \frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k \int_0^1 H(\xi_k, \tau)\lambda h(\tau)f(\tau, y(\tau))d\tau\right)dt.$$

Then, the operator Θ has a fixed point if and only if the BVP (1) has a solution.

Lemma 3.1. *Suppose $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ and $h \in C([0,1], [0,+\infty))$, then the operator $\Theta : D \to D$ is a completely continuous operator.*

Proof. From the non-negativeness and continuity of $G(x,t)$, $H(x,t)$, $f(x,y(x))$ and $h(x)$, we see that the operator $\Theta : D \to D$ is continuous.

Let $\Omega \subset D$ be bounded. So, for all $x \in [0,1]$ and $y \in \Omega$, there exists a positive constant M such that $|f(x,y(x))| \leq M$. Then, we get

$$\begin{aligned}
|(\Theta y)(x)| =& \left| \int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau) f(\tau,y(\tau))d\tau \right.\right.\\
&\left.\left. + \frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k \int_0^1 H(\xi_k,\tau)\lambda h(\tau) f(\tau,y(\tau))d\tau \right) dt \right| \\
\leq& (\lambda\|h\|M)^{q-1} \int_0^1 G(x,t)\varphi_q\left(\int_0^1 \frac{\tau^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau \right.\\
&\left. + \frac{1}{B}\sum_{k=1}^{m-2} b_k \int_0^1 \frac{\tau^{\beta-1}(1-\tau)^{\beta-1}}{B\Gamma(\beta)} d\tau \right) dt \\
\leq& (\lambda\|h\|M)^{q-1} \int_0^1 G_*(t,t)\varphi_q\left(\int_0^1 H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right) d\tau\right) dt \\
\leq& L\left[\frac{\lambda\|h\|M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]^{q-1},
\end{aligned}$$

where

$$L = \int_0^1 G_*(t,t)dt,$$

which shows that $\Theta(\Omega)$ is uniformly bounded.

Also, by the continuity of $G(x,t)$ on $[0,1]\times[0,1]$, we know that this function is uniformly continuous on $[0,1]\times[0,1]$. Therefore, for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that $x_1, x_2 \in [0,1]$ and $|x_1 - x_2| < \delta$,

$$|G(x_1,t) - G(x_2,t)| < \varphi_p\left[\frac{\lambda\|h\|M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]\varepsilon,$$

for fixed $t \in [0,1]$.
Thus, for all $y \in \Omega$, we get

$$\begin{aligned}
|(\Theta y)(x_2) - (\Theta y)(x_1)| &\leq \int_0^1 |G(x_2,t) - G(x_1,t)|\varphi_q(\rho(t))\,dt \\
&\leq \int_0^1 |G(x_2,t) - G(x_1,t)| \left[\frac{\lambda\|h\|M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]^{q-1} dt \\
&= \varphi_q\left[\frac{\lambda\|h\|M\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right] \int_0^1 |G(x_2,t) - G(x_1,t)|dt \\
&\leq \varepsilon,
\end{aligned}$$

which gives that the operator Θ is an equicontinuous operator. Thus, using the Arzella-Ascoli theorem, we get $\Theta : D \to D$ is completely continuous operator. □

Definition 3.2. *Let $p(x)$ be a solution of the BVP* (1). *The $p(x)$ is said to be a maximal solution of the BVP* (1), *if every solution $y(x)$ of the BVP* (1) *satisfies $y(x) < p(x)$ for $x \in [0,1]$. A minimal solution $q(x)$ can be defined by similar way by reversing the above inequality, i.e $y(x) > q(x)$ for $x \in [0,1]$.*

Theorem 3.3. *Suppose $f:[0,1]\times[0,\infty)\to[0,\infty)$ and $h:[0,1]\to[0,+\infty)$ are continuous and f is non-decreasing in the second variable, then BVP* (1) *has a maximal positive solution $\overline{w}$ and a minimal positive solution $\overline{v}$ such that $w_n(x)\to\overline{w}(x)$ and $v_n(x)\to\overline{v}(x)$ as $n\to\infty$ uniformly on* $[0,1]$*, where*

$$v_n(x)=\int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau)f(\tau,v_{n-1}(\tau))d\tau + \frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2}b_k\int_0^1 H(\xi_k,\tau)\lambda h(\tau)f(\tau,v_{n-1}(\tau))d\tau\right)dt, \tag{4}$$

and

$$w_n(x)=\int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau)f(\tau,w_{n-1}(\tau))d\tau + \frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2}b_k\int_0^1 H(\xi_k,\tau)\lambda h(\tau)f(\tau,w_{n-1}(\tau))d\tau\right)dt. \tag{5}$$

Proof. Let

$$B_r=\{y\in D:\|y\|\le r\},$$

where

$$r\ge L\left[\frac{\lambda\|h\|Q_1\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)\right]^{q-1}.$$

Step 1: Firstly, let show BVP (1) has at least one solution.
For $y\in B_r$, there exists a positive constant Q_1 such that $|f(x,y(x))|\le Q_1$,

$$\begin{aligned}|(\Theta y)(x)| &= \left|\int_0^1 G(x,t)\varphi_q(\rho(t))\,dt\right| \\ &\le L\left[\frac{\lambda\|h\|Q_1\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)\right]^{q-1}.\end{aligned}$$

Therefore,

$$\Theta:B_r\to B_r.$$

By Lemma 3.1, we can easily say that the operator $\Theta:B_r\to B_r$ is completely continuous. Hence, by using the Schauder fixed point theorem, it is clear that the operator Θ has at least one fixed point in B_r and so BVP (1) has at least one solution in the set B_r.

Step 2: BVP (1) has a positive solution in the set B_r, that is a minimal positive solution. Let $v_0(x)=0$, $x\in[0,1]$ and $v_1(x)=\Theta v_0(x)$. From $\Theta:B_r\to B_r$, we have $v_1\in B_r$. From (4), it is obvious that

$$v_n(x)=(\Theta v_{n-1})(x),\ x\in[0,1],\ for\ n=1,2,3,\ldots. \tag{6}$$

Also, since $f(x,y)$ is non-decreasing in the second variable y, we have

$$0=v_0(x)\le v_1(x)\le\cdots v_n(x)\le\cdots,\ x\in[0,1],$$

so we get $\{v_n\}$ is a sequentially compact set since Θ is compact. Consequently, there exists a $\overline{v}\in B_r$ such that $v_n\to\overline{v}$ as $n\to\infty$.
Let $y(x)$ be any positive solution of BVP (1) in the set B_r. It is easily seen that

$$0=v_0(x)\le y(x)=(\Theta y)(x).$$

Therefore,

$$v_n(x) \leq y(x) \; for \; n = 0, 1, 2, \cdots. \tag{7}$$

If we take a limit as $n \to \infty$ in (7), we have $\overline{v}(x) \leq y(x)$ for $x \in [0, 1]$.

Step 3: BVP (1) has a positive solution in B_r, that is a maximal positive solution. Let $w_0(x) = r, \; x \in [0, 1]$ and $w_1(x) = Tw_0(x)$. From $\Theta : B_r \to B_r$, we get $w_1 \in B_r$. So we have

$$0 \leq w_1(x) \leq r = w_0(x).$$

Moreover, since $f(x, y)$ in non-decreasing in y, we obtain

$$\cdots \leq w_n(x) \leq \cdots \leq w_1(x) \leq w_0(x) = r, \; x \in [0, 1].$$

Following the same steps used in Step 2, we see that

$$w_n(x) \to \overline{w}(x) \; as \; n \to \infty$$

and

$$\overline{w}(x) = \int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau) f(\tau, \overline{w}(\tau))d\tau + \frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k \int_0^1 H(\xi_k, \tau)\lambda h(\tau) f(\tau, \overline{w}(\tau))d\tau\right)dt.$$

Let $y(x)$ be any positive solution of BVP (1) in B_r.
Trivially,

$$y(x) \leq w_0(x).$$

Thus,

$$y(x) \leq w_n(x). \tag{8}$$

If we take a limit as $n \to \infty$ in (8), we have $y(x) \leq \overline{w}(x)$ for $x \in [0, 1]$. □

Let we define the numbers $f^0, f_0, f^\infty, f_\infty$ as followings:

$$f^0 = \lim_{y\to 0^+} \sup_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_1\|y\|)}, \qquad f_0 = \lim_{y\to 0^+} \inf_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_2\|y\|)},$$
$$f^\infty = \lim_{y\to +\infty} \sup_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_3\|y\|)}, \qquad f_\infty = \lim_{y\to +\infty} \inf_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_4\|y\|)},$$

where r_1, r_2, r_3, r_4 are any positive numbers.

Let

$$A_1 = L\left[\frac{\|h\|\Gamma(\beta)}{\Gamma(2\beta)}\left(1 + \frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]^{q-1}$$

and

$$A_2 = \int_\vartheta^\delta G(1,t)\varphi_q\left(\int_\vartheta^\delta g_2(\tau)H(\tau,\tau)\left(1 + \frac{1}{B}\sum_{k=1}^{m-2} b_k\right)h(\tau)d\tau\right)dt,$$

where ϑ and δ are given in Lemma 2.4.

Theorem 3.4. *Assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $h : [0,1] \to [0,+\infty)$ are continuous and the following conditions hold:*

(C_1) $f_0 = f_\infty = +\infty$,

(C_2) *There exists a constant $\rho_1 > 0$ such that $f(t,y) \leq \varphi_p(r_5\|y\|)$ for $t \in [0,1]$ with $r_5 > 0$ and $y \in [0,\rho_1]$.*

Then, BVP (1) *has at least two positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < \rho_1 < \|y_2\|,$$

for

$$\lambda^{q-1} \in \left(\frac{1}{r_2A_2}, \frac{1}{r_5A_1}\right) \cap \left(\frac{1}{r_4A_2}, \frac{1}{r_5A_1}\right), \tag{9}$$

where $r_2A_2 > r_5A_1$ and $r_4A_2 > r_5A_1$.

Proof. Since

$$f_0 = \lim_{y\to 0^+} \inf_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_2\|y\|)} = +\infty,$$

there is $\rho_0 \in (0,\rho_1)$ such that

$$f(x,y) \geq \varphi_p(r_2\|y\|) \text{ for } x \in [0,1],\ y \in [0,\rho_0].$$

Let

$$\Omega_{\rho_0} = \{y \in D : \|y\| \leq \rho_0\}.$$

Then, for any $y \in \partial\Omega_{\rho_0}$, it follows from Lemma 2.6 that

$$\begin{aligned}
(\Theta y)(x) &= \int_0^1 G(x,t)\varphi_q\left(\rho(t)\right)dt \\
&\geq \min_{\vartheta\leq x\leq\delta}\left\{\int_0^1 G(x,t)\varphi_q\left(\rho(t)\right)dt\right\} \\
&\geq r_2\lambda^{q-1}\|y\|\int_0^1 x^{\alpha-1}G(1,t)\varphi_q\left(\int_0^1 g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)h(\tau)d\tau\right)dt \\
&\geq r_2\lambda^{q-1}\|y\|\int_\vartheta^\delta G(1,t)\varphi_q\left(\int_\vartheta^\delta g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)h(\tau)d\tau\right)dt.
\end{aligned}$$

Therefore,

$$\|\Theta y\| \geq \lambda^{q-1}r_2A_2\|y\|.$$

Considering also (9), we obtain

$$\|\Theta y\| \geq \|y\|, \text{ for all } y \in \partial\Omega_{\rho_0}.$$

By Lemma 2.8, we have

$$i(\Theta,\Omega_{\rho_0},D) = 0. \tag{10}$$

Also, since

$$f_\infty = \lim_{y\to\infty} \inf_{x\in[0,1]} \frac{f(x,y)}{\varphi_p(r_4\|y\|)} = +\infty,$$

there is a number ρ_0^* with $\rho_0^* > \rho_1$, such that

$$f(x,y) \geq \varphi_p(r_4\|y\|) \text{ for } x \in [0,1],\ y \in [\rho_0^*, +\infty).$$

Let

$$\Omega_{\rho_0^*} = \{y \in D : \|y\| \leq \rho_0^*\}.$$

Then, for any $y \in \partial\Omega_{\rho_0^*}$, it follows from Lemma 2.6 that

$$\begin{aligned}
(\Theta y)(x) &= \int_0^1 G(x,t)\varphi_q\left(\rho(s)\right)dt \\
&\geq \min_{\vartheta\leq t\leq\delta}\left\{\int_0^1 G(x,t)\varphi_q\left(\rho(t)\right)dt\right\} \\
&\geq r_4\lambda^{q-1}\|y\|\int_0^1 x^{\alpha-1}G(1,t)\varphi_q\left(\int_0^1 g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)h(\tau)d\tau\right)ds \\
&\geq r_4\lambda^{q-1}\int_\vartheta^\delta G(1,t)\varphi_q\left(\int_\vartheta^\delta g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)h(\tau)d\tau\right)dt.
\end{aligned}$$

Therefore,

$$\|\Theta y\| \geq \lambda^{q-1}r_4A_2\|y\|.$$

Considering also (9), we get

$$\|\Theta y\| \geq \|y\|, \text{ for all } y \in \partial\Omega_{\rho_0^*}.$$

By Lemma 2.8, we get

$$i(\Theta, \Omega_{\rho_0^*}, D) = 0. \tag{11}$$

Finally, let $\Omega_{\rho_1} = \{y \in D : \|y\| \leq \rho_1\}$ for any $y \in \partial\Omega_{\rho_1}$, it follows from Lemma 2.6 and (C_2) that

$$\begin{aligned}
(\Theta y)(x) &= \int_0^1 G(x,t)\varphi_q\left(\rho(t)\right)dt \\
&\leq L\left[\frac{\lambda\|h\|\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]^{q-1}.
\end{aligned}$$

Therefore,

$$\|\Theta y\| \leq \lambda^{q-1}r_5A_1\|y\|.$$

Considering also (9), we have

$$\|\Theta y\| \leq \|y\|, \text{ for all } y \in \partial\Omega_{\rho_1}.$$

By Lemma 2.8, we get

$$i(\Theta, \Omega_{\rho_1}, D) = 1. \tag{12}$$

From (10)-(12) and $\rho_0 < \rho_1 < \rho_0^*$, we get

$$i(\Theta, \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}, D) = -1, \qquad i(\Theta, \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}, D) = 1.$$

Thus, Θ has a fixed point $y_1 \in \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}$ and a fixed point $y_2 \in \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}$. Trivially, y_1, y_2 are both positive solutions of BVP (1) and $0 < \|y_1\| < \rho_1 < \|y_2\|$. □

Similarly, we can get the following results;

Corollary 3.5. *Assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $h : [0,1] \to [0,+\infty)$ are continuous and the following conditions hold:*

(C_3) $f^0 = f^\infty = 0$.
(C_4) *There exists a constant $\rho_2 > 0$ such that $f(x,y) \geq \varphi_p(r_6\|y\|)$ for $x \in [0,1]$ with $r_6 > 0$ and $y \in [0,\rho_2]$.*

Then BVP (1) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \rho_2 < \|y_2\|$$

for

$$\lambda^{q-1} \in \left(\frac{1}{r_6A_2}, \frac{1}{r_3A_1}\right) \cap \left(\frac{1}{r_6A_2}, \frac{1}{r_1A_1}\right),$$

where $r_6A_2 > r_3A_1$ and $r_6A_2 > r_1A_1$.

Example 3.6. *Consider the following boundary value problem:*

$$\begin{cases} D^{\frac{4}{3}}(\varphi_2(D^{\frac{3}{2}}y(x))) + \lambda h(x)f(t,y(x)) = 0, \quad x \in [0,1], \\ y(0) = 0, \quad \varphi_2(D^{\frac{4}{3}}y(0)) = 0, \\ D^{\frac{1}{2}}y(1) = \sum_{k=1}^{m-2} b_k D^{\frac{1}{2}}y(\eta_k), \quad \varphi_2(D^{\frac{3}{2}}y(1)) = \sum_{k=1}^{m-2} a_k\varphi_2(D^{\frac{3}{2}}y(\xi_k)), \end{cases} \tag{13}$$

where

$$h(x) = \frac{x}{1+x}, \qquad f(x,y(x)) = 3|\cos(x+y(x))|,$$

$\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $\gamma = \frac{1}{2}$, $p = q = 2$, $m = 4$, $a_1 = b_1 = \frac{1}{2}$, $a_2 = b_2 = \frac{1}{4}$, $\xi_1 = \eta_1 = \frac{1}{8}$, $\xi_2 = \eta_2 = \frac{1}{3}$, $\lambda \in (0,+\infty)$ *and* $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.
Then,

$$M = |f(x,y(x))| = 3,$$
$$A = 1 - \sum_{k=1}^{2} a_i\eta_k^{\alpha-\gamma-1} = \frac{1}{4},$$
$$B = 1 - \sum_{k=1}^{2} b_k\xi_k^{\beta-1} = 1 - (b_1\xi_1^{\beta-1} + b_2\xi_2^{\beta-1}) = 0.57666.$$

By computation we see that

$$L = \int_0^1 G_*(t,t)dt = \frac{1}{A\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-\gamma-1}dt = \frac{8}{\sqrt{\pi}},$$

$$r \geq L\left[\frac{\lambda\|h\|M_1\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)\right]^{q-1} = 9.2450\lambda^{q-1}.$$

Hence, according the Theorem 3.3, we can say that BVP (13) has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{w}$ in B_r.

Example 3.7. *Consider the following fractional boundary value problem:*

$$\begin{cases} D^{\frac{4}{3}}(\varphi_2(D^{\frac{3}{2}}y(x))) + \lambda h(x)f(t,y(x)) = 0, \quad x\in[0,1],\\ y(0)=0, \quad \varphi_2(D^{\frac{4}{3}}y(0)) = 0,\\ D^{\frac{1}{2}}y(1) = \sum_{k=1}^{m-2} b_k D^{\frac{1}{2}}y(\eta_k), \quad \varphi_2(D^{\frac{3}{2}}y(1)) = \sum_{k=1}^{m-2} a_k\varphi_2(D^{\frac{3}{2}}y(\xi_k)), \end{cases} \tag{14}$$

where

$$h(x) = 2\sin\left(\frac{x}{\pi}\right), \qquad f(x,y(x)) = 3|y(x)|^{\frac{1}{2}} + \|y\|,$$

$\alpha=\frac{3}{2},\ \beta=\frac{4}{3}, \gamma=\frac{1}{2},\ p=q=2,\ m=4,\ a_1=b_1=\frac{1}{2}, a_2=b_2=\frac{1}{4},\ \xi_1=\eta_1=\frac{1}{8}, \xi_2=\eta_2=\frac{1}{3},$
$\lambda\in(0,+\infty)$ *and* $f\in C([0,1]\times[0,+\infty),\ [0,+\infty))$.

We set $\vartheta=\frac{1}{3}$ *and* $\delta=\frac{2}{3}$.

By computation, we find $A=\frac{1}{4},\ B=0.57666,$

$$A_1 = L\left[\frac{\|h\|\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)\right]^{q-1} = 12.327$$

and

$$A_2 = \int_{\vartheta}^{\delta} G(1,t)\varphi_q\left(\int_{\vartheta}^{\delta} g_2(\tau)H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)h(\tau)d\tau\right)dt = 0.22150.$$

Taking $\rho_1=9,\ r_5=2,$ *we get*

$$f(x,y)\leq 3(3)+9 = 18 = \varphi_p(r_5\|y\|) = \varphi_2(2\times 9),\ \textit{for } x\in[0,1],\ y\in[0,\rho_1].$$

Therefore, condition (C_2) *is satisfied. It can be easily seen that condition* (C_1) *holds.*
Also, let $r_2=200$ *and* $r_4=150$, *we have* $r_2A_2 > r_5A_1$ *and* $r_4A_2 > r_5A_1$.
Hence, by Theorem 3.4, for

$$\lambda\in\left(\frac{1}{r_2A_2},\frac{1}{r_5A_1}\right)\cap\left(\frac{1}{r_4A_2},\frac{1}{r_5A_1}\right) = \left(\frac{1}{r_4A_2},\frac{1}{r_5A_1}\right) = (0.0300, 0.0406),$$

BVP (14) has at least two solutions y_1 and y_2 which satisfy the inequality $0<\|y_1\|<9<\|y_2\|$ *for the given values of* $r_5,\ r_2$ *and* r_4.

Theorem 3.8. *Assume that* $f:[0,1]\times[0,\infty)\to[0,\infty)$ *and* $h:[0,1]\to[0,+\infty)$ *are continuous and there exist constants* $\gamma_1,\gamma_2>0$ *such that:*

(C_5) $f(x,y_1)\leq f(x,y_2)$ *for any* $0\leq x\leq 1$ *and* $0\leq y_1\leq y_2\leq\gamma_2$,
(C_6) $\max_{0\leq x\leq 1} f(x,\gamma_2)\leq\varphi_p(\gamma_2\gamma_1)$,
(C_7) $f(x,0)\neq 0$ *for* $0\leq x\leq 1$.

Then BVP (1) *has at least two positive solutions u^* and v^* such that*

$$0 < \|u^*\| \le \gamma_2 \text{ and } \lim_{n\to\infty} \Theta^n u_0 = u^*, \text{ where } u_0(x) = \gamma_2,$$
$$0 < \|v^*\| \le \gamma_2 \text{ and } \lim_{n\to\infty} \Theta^n v_0 = v^*, \text{ where } v_0(x) = 0.$$

for

$$\lambda^{q-1} \in \left(0, \frac{1}{\gamma_1 A_1}\right)$$

where $\gamma_1 A_1 > 0$.

Proof. We define $B_{\gamma_2} = \{u \in D : \|u\| \le \gamma_2\}$. We proceed to prove that $\Theta B_{\gamma_2} \subseteq B_{\gamma_2}$. Let $u \in B_{\gamma_2}$, then $0 \le u(x) \le \|u\| \le \gamma_2$.
By assumptions C_5 and C_6, we obtain $0 \le f(x, u(x)) \le f(x, \gamma_2) \le \varphi_p(\gamma_2\gamma_1)$.
For any $u \in B_{\gamma_2}$, by Lemma 3.1, $\Theta u \in D$, then

$$\begin{aligned}
\|\Theta u\| = &\max_{0\le x\le 1}\left\{\int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau) f(\tau, y(\tau))d\tau \right.\right.\\
&\left.+\frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k \int_0^1 H(\xi_k,\tau)\lambda h(\tau) f(\tau, y(\tau))d\tau\right)dt\\
\le&\gamma_2\gamma_1(\lambda\|h\|)^{q-1}\int_0^1 G_*(t,t)\varphi_q\left(\int_0^1 H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)d\tau\right)dt\\
=&\gamma_2\lambda^{q-1}\gamma_1 L\left[\frac{\|h\|\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2} b_k\right)\right]^{q-1} = \gamma_2\lambda^{q-1}\gamma_1 A_1\\
\le&\gamma_2.
\end{aligned}$$

Therefore, $\Theta u \in B_{\gamma_2}$. Hence, we obtain $\Theta B_{\gamma_2} \subseteq B_{\gamma_2}$. Let $u_0(x) = \gamma_2,\ 0 \le x \le 1$, then $\|u_0\| = \gamma_2$ and $u_0 \in B_{\gamma_2}$. Let $u_1(x) = \Theta u_0(x)$, thus $u_1 \in B_{\gamma_2}$.

We define

$$u_{n+1} = \Theta u_n = \Theta^{n+1} u_0,\ n = 0, 1, 2, \cdots .$$

Since $\Theta B_{\gamma_1} \subseteq B_{\gamma_1},\ u_n \in B_{\gamma_1}$ $(n = 0, 1, 2, \cdots)$ by using Lemma 3.1, Θ is compact, we see that $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and there exists $u_* \in B_{\gamma_2}$ such that $u_{n_k} \to u^*$. By the definition of Θ and (C_5), for any $x \in [0, 1]$, we get

$$\begin{aligned}
u_1(x) =&(\Theta u_0)(x)\\
=&\int_0^1 G(x,t)\varphi_q\left(\int_0^1 H(t,\tau)\lambda h(\tau) f(\tau, u_0(\tau))d\tau\right.\\
&\left.+\frac{t^{\beta-1}}{B}\sum_{k=1}^{m-2} b_k \int_0^1 H(\xi_k,\tau)\lambda h(\tau) f(\tau, u_0(\tau))d\tau\right)dt\\
\le&\gamma_2\gamma_1(\lambda\|h\|)^{q-1}\int_0^1 G(x,t)\varphi_q\left(\int_0^1 \frac{\tau^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right.
\end{aligned}$$

$$+\frac{1}{B}\sum_{k=1}^{m-2}b_k\int_0^1\frac{\tau^{\beta-1}(1-\tau)^{\beta-1}}{B\Gamma(\beta)}d\tau\Bigg)dt$$
$$\leq\gamma_2\gamma_1(\lambda\|h\|)^{q-1}\int_0^1 G_*(t,t)\varphi_q\left(\int_0^1 H(\tau,\tau)\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)d\tau\right)dt$$
$$=\gamma_2\lambda^{q-1}\gamma_1 L\left[\frac{\|h\|\Gamma(\beta)}{\Gamma(2\beta)}\left(1+\frac{1}{B}\sum_{k=1}^{m-2}b_k\right)\right]^{q-1}$$
$$=\gamma_2\lambda^{q-1}\gamma_1 A_1$$
$$\leq\gamma_2=u_0(x).$$

Therefore, we get

$$u_2(x)=\Theta u_1(x)\leq\Theta u_0(x)=u_1(x),\quad 0\leq x\leq 1.$$

Thus, by induction method, we obtain

$$u_{n+1}\leq u_n,\quad n-0,1,2,\cdots.$$

for $0\leq x\leq 1$.
Therefore, there exists a $u^*\in B_{\gamma_2}$ such that $u_n\to u_*$. By the continuity of the operator Θ and $u_{n+1}=\Theta u_n$, we have $\Theta u^*=u^*$.
We let $v_0=0,\ 0\leq x\leq 1$, then $v_0\in B_{\gamma_2}$. Let $v_1=\Theta v_0$, then $v_1\in B_{\gamma_2}$.
We define

$$v_{n+1}=\Theta v_n=\Theta^{n+1}v_0,\ n=0,1,2,\cdots.$$

Since $\Theta:B_{\gamma_2}\to B_{\gamma_2}$, we obtain $v_n\subseteq B_{\gamma_2},\ n=0,1,2,\cdots$. Since also Θ is completely continuous, we see that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Also, $v_1(x)=\Theta v_0(x)=(\Theta 0)(x)\geq 0,\ 0\leq x\leq 1$, it implies that

$$v_2(x)=\Theta v_1(x)\geq(\Theta 0)(x)=v_1(x),\ 0\leq x\leq 1.$$

Thus, by induction method, we obtain

$$v_{n+1}\geq v_n,\ 0\leq x\leq 1,\ n=0,1,2,\cdots.$$

Therefore, there exists a $v^*\in B_{\gamma_1}$ such that $v_n\to v^*$. By the continuity of the operator Θ and $v_{n+1}=\Theta v_n$, we obtain $\Theta v^*=v^*$.

It is evident that every fixed point of the operator Θ in D is also a solution of BVP (1). In addition, if $f(x,0)\neq 0$ for $x\in[0,1]$, then the zero function is not the solution of BVP (1). Thus, we get $\|u^*\|>0,\ \|v^*\|>0$. Then u^* and v^* are two positive solutions of BVP (1). So, the proof is completed. □

Applying Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Suppose $f:[0,1]\times[0,\infty)\to[0,\infty)$ and $h:[0,1]\to[0,+\infty)$ are continuous, also C_7 and the following conditions hold:*

(C_8) $f(x,y_1)\leq f(x,y_2)$ *for any* $0\leq x\leq 1$ *and* $0\leq y_1\leq y_2$;
(C_9) $\lim_{y\to\infty}\max_{0\leq x\leq 1}\frac{f(x,y)}{y^{p-1}}\leq\varphi_p(\gamma_1)$, *in particular* $\lim_{y\to\infty}\max_{0\leq x\leq 1}\frac{f(x,y)}{y^{p-1}}=0$.

Then BVP (1) *has two positive solutions u^* and v^*, for*

$$\lambda^{q-1} \in \left(0, \frac{1}{\gamma_1 A_1}\right)$$

where $\gamma_1 A_1 > 0$.

Example 3.10. *We consider the following BVP:*

$$\begin{cases} D^{\frac{4}{3}}(\varphi_2(D^{\frac{3}{2}}y(x))) + \lambda h(x)f(x,y(x)) = 0, \quad x \in [0,1], \\ y(0) = 0, \quad \varphi_2(D^{\frac{4}{3}}y(0)) = 0, \\ D^{\frac{1}{2}}y(1) = \sum_{i=1}^{m-2} b_i D^{\frac{1}{2}}y(\eta_i), \quad \varphi_2(D^{\frac{3}{2}}y(1)) = \sum_{i=1}^{m-2} a_i\varphi_2(D^{\frac{3}{2}}y(\xi_i)), \end{cases} \tag{15}$$

where

$$h(x) = 2\sin\left(\frac{x}{\pi}\right), \qquad f(x,y(x)) = e^x + 3|y(x)|x^{\frac{1}{2}} + \|y\|,$$

$\alpha = \frac{3}{2},\ \beta = \frac{4}{3},\ \gamma = \frac{1}{2}\ p = q = 2,\ m = 4,\ a_1 = b_1 = \frac{1}{2}, a_2 = b_2 = \frac{1}{4},\ \xi_1 = \eta_1 = \frac{1}{8}, \xi_2 = \eta_2 = \frac{1}{3},$
$\lambda \in (0,+\infty)$ *and* $f \in C([0,1]\times[0,+\infty),\ [0,+\infty))$.

We set $\vartheta = \frac{1}{3}$ and $\delta = \frac{2}{3}$. By computation, $A = \frac{1}{4}, B = 0.57666$ *and* $A_1 = 12.327$. *Taking* $\gamma_2 = 9$ *and* $\gamma_1 = 2.5$. *Thus, $f(x,y)$ satisfies*

1. $f(x,y_1) \le f(x,y_2)$ *for any* $0 \le x \le 1,\ 0 \le y_1 \le y_2 \le 9$;
2. $\max_{0\le x\le 1} f(x,\gamma_2) = f(1,9) \approx 20.718 < \varphi_2(\gamma_2\gamma_1) \approx 22.5$;
3. $f(x,0) \ne 0$ *for* $0 \le x \le 1$.

Then by Theorem 3.8, BVP (15) has two positive solutions u^* and v^* such that

$$0 < \|u^*\| \le 9 \quad and \quad \lim_{n\to\infty}\Theta^n u_0 = u^*, \quad where \quad u_0(x) = 9,$$
$$0 < \|v^*\| \le 9 \quad and \quad \lim_{n\to\infty}\Theta^n v_0 = v^*, \quad where \quad v_0(x) = 0,$$

for

$$\lambda^{q-1} \in \left(0, \frac{1}{\gamma_1 A_1}\right)$$

or

$$\lambda \in (0, 0.00361).$$

4. Conclusion

In this study, we have explored the existence of positive solutions for a class of nonlinear fractional differential equations subjected to the influence of the p-Laplacian operator within double multi-point boundary value problems. These equations hold significant importance in modeling complex phenomena across various applications. Our research aspires to produce results that aid in both the theoretical understanding and practical applications of fractional differential equations. Our findings lay a foundation that may benefit mathematicians and engineers seeking to investigate such equations.

Our results demonstrate the existence of positive solutions in double multi-point boundary value problems, a key finding of relevance to researchers and engineers tackling problems across diverse application domains. Furthermore, it is anticipated that our results will make a meaningful contribution to the existing body of literature in the field of fractional differential equations.

Future studies may extend the scope to encompass a broader class of equations or investigate more complex boundary value problems involving different operators. Additionally, exploring the implications of our theoretical results in practical applications through numerical solution techniques and stability analysis could offer valuable insights.

In summary, this study offers a valuable contribution to the field of fractional differential equations by giving the existence results for positive solutions within double multi-point boundary value problems. This derived results hold promise for advancing both theoretical and practical understanding in this intricate domain, serving as a foundation for further research and exploration.

Competing interest: The authors declare that they have no conflict of interest.

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