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Spectral extremal problems for nearly k-uniform hypergraphs

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Abstract. A hypergraph $G = (V, E)$ is called R-graph if $R = \{|e| : e \in E\}$. The spectral radius of G is the maximum modulus of eigenvalues of its adjacency tensor. Let $\mathcal{G}_{n,r}$ be the class of connected $\{k, k-1\}$-graphs of n vertices with r pendent vertices. In this paper, we characterize the hypergraphs with the maximum spectral radius in $\mathcal{G}_{n,r}$ for $n - r \geq k$, $n - r = 2, k - 1$, respectively.

1. Introduction

It is well known that hypergraphs are generalizations of graphs. At present, hypergraphs have a wide range of applications, such as obtaining multidimensional relationships [11] and constructing relational networks (protein-protein interaction, coauthorship, film actor/actress) [9]. In recent years, research on spectral theory of hypergraphs has attracted extensive attention. There are many results on uniform hypergraphs, see [2, 5–7, 10, 12, 13]. However, there are only few results on general hypergraphs, such as [4, 14]. The purpose of this paper is to study the spectral extremal problems for a class of general hypergraphs.

Let $G = (V, E)$ be a hypergraph with $V = [n] = \{1, 2, \ldots, n\}$ and $E \subseteq P(V)$, where $P(V)$ is the power set of V. The rank $r(G) = \max\{|e| : e \in E\}$. For each edge $e \in E$, we name an ordered sequence $\mu = (i_1, i_2, \ldots, i_k)$ as an k-expanded edge from e (e-expanded edge), denoted by $e \prec \mu$, if the set of distinct vertices in μ is e. Let $S(e) = \{\mu : e \prec \mu\}$ and $S(G) = \cup_{e \in E} S(e)$. Furthermore, let $S_i(e) = \{\mu \in S(e) : i$ be the first element of ordered sequence $\mu\}$ and $S_i(G) = \cup_{e \in E_i} S_i(e)$, where $E_i = \{e : i \in e \in E\}$. If $|E_i| = 1$, then vertex i is called pendent vertex. For each edge $e \in E$ satisfying $i \in e$ and $|e| = s$, we have $|S(e)| = s|S_i(e)|$ and

$$|S(e)| = \sum_{k_1, \ldots, k_s \geq 1; k_1 + \cdots + k_s = k} \frac{k!}{k_1! k_2! \cdots k_s!}.$$

The adjacency tensor $\mathcal{A}_G = (a_{i_1 i_2 \cdots i_k})$ of G is defined as follows

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{|e|}{|S(e)|} := a(e), & \text{if } e \prec (i_1, \ldots, i_k) \text{ for some } e \in E, \\ 0, & \text{otherwise.} \end{cases}$$

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For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ and an k-expanded edge $\mu = (i_1, i_2, \ldots, i_k)$, we write $a_{i_1 i_2 \cdots i_k} = a_\mu$, $x(\mu) = x_{i_1} + x_{i_2} + \cdots + x_{i_k}$, $x^\mu = x_{i_1} x_{i_2} \cdots x_{i_k}$ and $x^{\mu - i_m} = x_{i_1} \cdots x_{i_{m-1}} x_{i_{m+1}} \cdots x_{i_k}$. Then

$$(\mathcal{A}_G x)_i = \sum_{i_2, \cdots, i_k = 1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} = \sum_{\mu \in S_i(G)} a_\mu x^{\mu - i} = \sum_{e \in E_i} a(e) \sum_{\mu \in S_i(e)} x^{\mu - i}. \tag{1.1}$$

If $\mathcal{A}_G \mathbf{x} = \lambda \mathbf{x}^{[k-1]}$ and $\mathbf{x} \neq 0$, then λ is called an eigenvalue of $\mathcal{A}_G$ and $\mathbf{x}$ is its corresponding eigenvector, where $\mathbf{x}^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^{\mathrm{T}}$. The spectral radius of $\mathcal{A}_G$ is the largest modulus of the eigenvalues of $\mathcal{A}_G$. If G is connected, then $\mathcal{A}_G$ is weakly irreducible [14]. Further by the Perron-Frobenious theorem for weakly irreducible tensor [3], there is a unique eigenvector $\mathbf{x}$ satisfying $\|\mathbf{x}\|_k = 1$ associated with $\rho(\mathcal{A}_G)$, is called Perron vector of $\mathcal{A}_G$. The maximum and minimum entries of $\mathbf{x}$ are denoted by $x_{\max}$ and $x_{\min}$, respectively. We call $\gamma := \frac{x_{\max}}{x_{\min}}$ Perron ratio of $\mathcal{A}_G$.

A hypergraph $G = (V, E)$ is called R-graph if $R = \{|e| : e \in E\}$. For a set S and integer i, let $\binom{S}{i}$ be the family of all i-subsets of S. A R-graph G with vertex set $[n]$ and edge set $\bigcup_{i \in R} \binom{[n]}{i}$ is called complete R-graph. In particular, if $R = \{k\}$, then G is k-uniform hypergraph (k-graph). A hypergraph is non-uniform if $|R| \geq 2$. For a vertex i, let $R(i) = \{|e| : e \in E_i\}$ [4]. In 1986, Brualdi and Solheid [1] posed the following problem:

Problem 1.1. *Maximizing the spectral radius and determining the extremal* 2*-graph for a given class of* 2*-graphs.*

Generally, we may ask a similar problem for R-graphs as Problem 1.1.

Problem 1.2. *Maximizing the spectral radius and determining the extremal R-graph for a given class of R-graphs.*

For $R = \{k\}$, that is the case for uniform hypergraphs. For $|R| \geq 2$, Problem 1.2 becomes more difficult because of more complex structure of general hypergraphs. In this paper, we will study the spectral extremal problems of $\{k, k-1\}$-graphs. Let $\mathcal{G}_{n,r}$ be the class of connected $\{k, k-1\}$-graphs of n vertices with r pendent vertices.

2. Preliminaries

In this section, we present some notations and lemmas which will be used in our proof.

Lemma 2.1. *For a connected general hypergraph $G = (V, E)$ with rank k, let $\mathbf{x}$ be its Perron vector and $u, v \in V(G)$. If $i \in e$ implies $j \in e$ for each $e \in E$, then $x_j \geq x_i$. Moreover, if there is an edge e_0 such that $j \in e_0$ but $i \notin e_0$, then $x_j > x_i$.*

Proof. Since G is connected, $\rho(G) > 0$ and $\mathbf{x} \in \mathbb{R}^n_{++}$. Let $A_1 = \{e \in E : i, j \in e\}$ and $A_2 = \{e \in E : j \in e, i \notin e\}$, then $E_j = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and $A_1 = E_i$. By (1.1), we have

$$\begin{aligned}
(\mathcal{A}_G x)_i &= \sum_{e \in E_i} a(e) \sum_{\mu \in S_i(e)} x^{\mu - i} = \rho(G) x_i^{k-1}, \\
(\mathcal{A}_G x)_j &= \sum_{e \in E_i} a(e) \sum_{\mu \in S_j(e)} x^{\mu - j} + \sum_{e \in A_2} a(e) \sum_{\mu \in S_j(e)} x^{\mu - j} = \rho(G) x_j^{k-1}.
\end{aligned}$$

Then

$$\rho(G)(x_j^k - x_i^k) = \sum_{e \in A_2} a(e) \sum_{\mu \in S_j(e)} x^\mu \geq 0,$$

so $x_j \geq x_i$. If there is an edge $e_0 \in A_2$, we have $\rho(G)(x_j^k - x_i^k) > 0$ and $x_j > x_i$. □

For a general hypergraph G, its weighted incidence matrix $M = (M(u, e'))_{|V| \times |S(G)|}$ is defined as following:

$$M(u, e') \begin{cases} > 0, & \text{for } u \in e \text{ and } e\text{-expanded edge } e', \\ = 0, & \text{otherwise.} \end{cases}$$

Definition 1. *[14] A general hypergraph $G = (V, E)$ with rank k is called β-normal, if it has a weighted incidence matrix M such that the following conditions hold.*

(1) $\sum_{e' \in S_i(G)} a(e)M(v, e') = 1$*, for any $i \in V$ and any e-expanded edge e'.*
(2) $\prod_{v \in e'} M(v, e') = \beta$*, for any e-expanded edge e'.*
(3) $M(u, e'_1) = M(u, e'_2)$*, if e'_1 is deferent from e'_2 only their order.*

Furthermore, M is referred as consistent if for any cycle $u_0e_1u_1e_2 \cdots u_l(u_l = u_0)$ and any e_i-expanded edge e'_i,

$$\prod_{i=1}^{l} \frac{M(u_i, e'_i)}{M(u_{i-1}, e'_i)} = 1.$$

In this situation, G is named consistently β-normal.

Lemma 2.2. *[14] The spectral radius of a general hypergraph $G = (V, E)$ with rank k is $\rho(G)$ if and only if G is consistently $\rho(G)^{-k}$-normal.*

Definition 2. *A general hypergraph $G = (V, E)$ with rank k is called β-subnormal, if it has a weighted incidence matrix M such that the following conditions hold.*

(1) $\sum_{e' \in S_i(G)} a(e)M(v, e') \leq 1$*, for any $i \in V$ and any e-expanded edge e'.*
(2) $\prod_{v \in e'} M(v, e') \geq \beta$*, for any e-expanded edge e'.*
(3) $M(u, e'_1) = M(u, e'_2)$*, if e'_1 is deferent from e'_2 only their order.*

Furthermore, β-subnormal hypergraph G is referred as strictly if it isn't β-normal.

Lemma 2.3. *For a general hypergraph $G = (V, E)$ with rank k, if it is β-subnormal, then $\rho(G) \leq \beta^{-\frac{1}{k}}$. Furthermore, for strict β-subnormal hypergraph G, $\rho(G) < \beta^{-\frac{1}{k}}$.*

Proof. Assume that M be a weighted incidence matrix satisfying the conditions in Definition 2. Then for any unit positive vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, we have

$$\begin{aligned}
\mathcal{A}_G\mathbf{x}^k &= \sum_{e \in E} \sum_{e' \in S(e)} a(e) \prod_{v \in e'} x_v \\
&\leq \frac{1}{\beta^{1/k}} \sum_{e' \in S(G)} a(e) \prod_{v \in e'} (M^{\frac{1}{k}}(v, e')x_v) \\
&\leq \frac{1}{\beta^{1/k}} \sum_{e' \in S(G)} \frac{\sum_{v \in e'} a(e)(M(v, e')x_v^k)}{k} \\
&= \frac{1}{\beta^{1/k}} \frac{k \sum_v \sum_{e' \in S_v(G)} a(e)(M(v, e')x_v^k)}{k} \\
&\leq \frac{\sum_v x_v^k}{\beta^{1/k}} = \frac{1}{\beta^{1/k}}.
\end{aligned}$$

Then $\rho(G) \leq \beta^{-\frac{1}{k}}$, and if β-subnormal hypergraph G is strictly, $\rho(G) < \beta^{-\frac{1}{k}}$. □

Lemma 2.4. *[14] If G is a subgraph of H with $r(G) = r(H)$, then $\rho(G) \leq \rho(H)$.*

Lemma 2.5. *[14] Suppose that H is a connected general hypergraph with $r(H) = k$ and H' is the hypergraph obtained from H by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u, where $v_i \in e_i$, $u \notin e_i$ and H' contains no multiple edges. If $\mathbf{x} \in \mathbb{R}^n$ is the Perron eigenvector of H and $x_u \geq \max_{1 \leq i \leq r}\{x_{v_i}\}$, then $\rho(H') > \rho(H)$.*

Lemma 2.6. *[14] If H is the hypergraph with the maximum spectral radius among connected general hypergraphs with fixed number of edges, then H contains a vertex adjacent to all the other vertices.*

3. $\{k, k-1\}$-graphs with the maximum spectral radius

Denote the complete $\{k,k-1\}$-graph with order n by $K_n(k,k-1)$. If $n-r \geq k$, let $A_n^r(k,k-1)$ be the general hypergraph obtained from $K_{n-r}(k,k-1)$ by adding r new edges and r new pendent vertices, each of new edge contains exactly the same $k-1$ distinct vertices in $V(K_{n-r}(k,k-1))$ and a new pendent vertex. See Figure 1. Obviously, $A_n^0(k,k-1) \cong K_n(k,k-1)$.

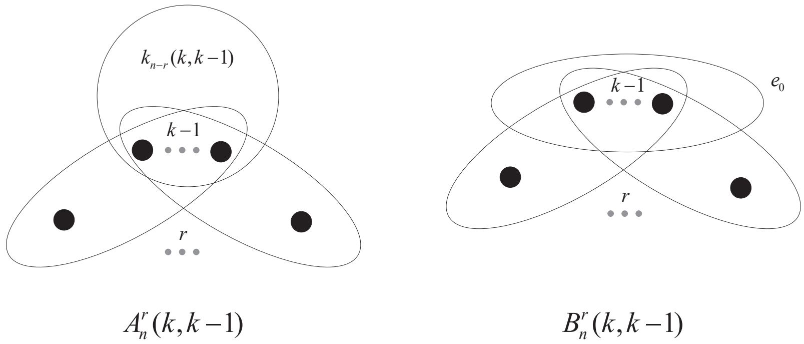


Figure 1: The hypergraph $A_n^r(k,k-1)$ and $B_n^r(k,k-1)$

Theorem 3.1. *If $H \in \mathcal{G}_{n,r}$ and $n-r \geq k$, then $\rho(H) \leq \rho(A_n^r(k,k-1))$ with equality if and only if $H \cong A_n^r(k,k-1)$.*

Proof. Let $G = (V(G), E(G))$ be the $\{k,k-1\}$-graph with maximum spectral radius in $\mathcal{G}_{n,r}$, and $\overline{V}$ be the set of pendent vertices in G. According to Lemma 2.4, we claim that $G[V(G)\setminus\overline{V}]$ is a complete $\{k,k-1\}$-hypergraph. Let

$$\overline{E} = \{e \in E(G) : e \cap \overline{V} \neq \emptyset\} = \{e_1, \ldots, e_s\},$$

and $V_i = e_i \cap \overline{V}$ for $i \in [s]$. Obviously, $s \leq r$ and $\overline{V} = V_1 \cup V_2 \cup \cdots \cup V_s$. Suppose that $|e_1| - |V_1| \geq |e_2| - |V_2| \geq \ldots \geq |e_s| - |V_s|$. Let $F(V_i) = e_i \setminus V_i$, then $|F(V_i)| = |e_i| - |V_i|$ for $i \in [s]$ and

$$|F(V_1)| \geq |F(V_2)| \geq \cdots \geq |F(V_s)| \tag{3.1}$$

Let $\mathbf{x}$ be the Perron vector of $\mathcal{A}_G$, and $\gamma = \frac{x_{\max}}{x_{\min}}$ be Perron ratio of $\mathcal{A}_G$. Then

$$\rho(G) \quad = \quad \sum_{e \in E(G)} a(e) \sum_{\mu \in S(e)} x^{\mu}.$$

Fact 1. $F(V_s) \subseteq \cdots \subseteq F(V_2) \subseteq F(V_1)$.

If there have two vertices v_i, v_j satisfy that $v_i \in F(V_i), v_j \in F(V_j)$ and $v_i \notin F(V_j), v_j \notin F(V_i)$. Without loss of generality, we assume that $x_{v_i} \geq x_{v_j}$. Let H_1 be the hypergrph obtained from $G - e_j$ by adding the edge $(e_j - v_j) \cup \{v_i\}$. Then by Lemma 2.5, $\rho(H_1) > \rho(G)$, a contradiction. Thus $F(V_i) \supseteq F(V_j)$ or $F(V_i) \subseteq F(V_j)$. Further by (3.1), we have Fact 1.

Let $V_0 = V(G) - (\overline{V} \cup F(V_1))$, $\overline{F}(V_i) = F(V_i) \setminus F(V_{i+1})$ for $i = 1, 2, \ldots, s-1$ and $\overline{F}(V_s) = F(V_s)$. Obviously, $|V_0| + |F(V_1)| = n - r \geq k$ and $|V_1| + |F(V_1)| \leq k$, then $|V_0| \geq |V_1|$.

By Lemma 2.1, we have $x_{u_1} = x_{u_2}$ if $u_1, u_2 \in V_i, \overline{F}(V_i)$ $(i = 1, \ldots, s)$ or V_0. For $i = 1, 2, \ldots, s$, let $x_u := x_i$ for any $u \in V_i$, $x_u := \overline{x}_i$ for any $u \in \overline{F}(V_i)$, and $x_u := x_0$ for $u \in V_0$.

Fact 2. If there exists some $i \in [s-1]$ such that $\overline{F}(V_i)$ is not empty, then $x_i < \overline{x}_i < x_{i+1}$ and $\overline{x}_s > x_s$.

It is easy to see that $\overline{x}_i > x_i$ for $i \in [s]$ by Lemma 2.1.

Assume that $\overline{x}_i \geq x_{i+1}$. Let $v \in V_{i+1}$, $u \in \overline{F}(V_i)$, $e_{i+1}^* = (e_{i+1} - v) \cup \{u\}$ and e_0 be the edge containing v and any $k-1$ vertices of $V(G) - \overline{V}$. Obviously, $|e_{i+1}| = |e_{i+1}^*|$ and $|S(e_{i+1})| = |S(e_{i+1}^*)|$. It is easy to see that there exists a bijection $\varphi : S(e_{i+1}) \to S(e^*_{i+1})$, for any $\mu \in S(e_{i+1})$, $\varphi(\mu) = \mu' \in S(e^*_{i+1})$ obtained from μ by replacing v by u and keeping its number of times unchanged. Then $x^\mu \le x^{\mu'}$.

Now let H_2 be the hypergraph obtained from G by deleting e_{i+1} and adding edges e^*_{i+1} and e_0. Obviously, $H_2 \in \mathcal{G}_{n,r}$, and $E(H_2) = (E(G) - e_{i+1}) \cup \{e^*_{i+1}, e_0\}$. Furthermore

$$\begin{aligned}
\rho(H_2) &\ge \sum_{e\in E(H_2)} a(e) \sum_{\mu\in S(e)} x^\mu \\
&= \sum_{e\in E(G)-e_{i+1}} a(e) \sum_{\mu\in S(e)} x^\mu + a(e^*_{i+1}) \sum_{\mu\in S(e^*_{i+1})} x^\mu + a(e_0) \sum_{\mu\in S(e_0)} x^\mu \\
&\ge \sum_{e\in E(G)-e_{i+1}} a(e) \sum_{\mu\in S(e)} x^\mu + a(e_{i+1}) \sum_{\mu\in S(e_{i+1})} x^\mu + a(e_0) \sum_{\mu\in S(e_0)} x^\mu \\
&> \sum_{e\in E(G)-e_{i+1}} a(e) \sum_{\mu\in S(e)} x^\mu + a(e_{i+1}) \sum_{\mu\in S(e_{i+1})} x^\mu = \rho(G),
\end{aligned}$$

a contradiction.

Fact 3. $|V_i| = 1$ for any $i \in [s]$.

Assume that $|V_i| > 1$ for some $i \in [s]$. Note that $n - r \ge k > |e_i| - 1$. Set $v_0 \in V_i$, $V(G)\backslash\overline{V} = \{v_1, v_2, \dots, v_{n-r}\}$, and $x_{v_1} \ge x_{v_2} \ge \cdots \ge x_{v_{n-r}}$. For any $u \in V(G)\backslash\{\overline{V} \cup F(V_i)\}$, let $e_i^1 = (e_i\backslash\{v_0\}) \cup \{u\}$, e_i^2 contain v_0 and the first $|e_i| - 1$ vertices of $\{v_1, v_2, \dots, v_{n-r}\}$. Obviously, $|e_i| = |e_i^2|$ and $|S(e_i)| = |S(e_i^2)|$. Similar to Fact 2, there exists a bijection $\varphi_1 : S(e_i) \to S(e_i^2)$, for any $\mu \in S(e_i)$, $\varphi_1(\mu) = \mu' \in S(e_i^2)$ is obtained from μ by replacing $e_i\backslash\{v_0\}$ by the first $|e_i| - 1$ vertices of $\{v_1, v_2, \dots, v_{n-r}\}$ and keeping its number of times unchanged. Then $x^\mu \le x^{\mu'}$.

Now let H_3 be the hypergraph obtained from G by deleting e_i and adding edges e_i^1 and e_i^2. Obviously, $H_3 \in \mathcal{G}_{n,r}$, and $E(H_3) = (E(G) - e_i) \cup \{e_i^1, e_i^2\}$. Furthermore

$$\begin{aligned}
\rho(H_3) &\ge \sum_{e\in E(H_3)} a(e) \sum_{\mu\in S(e)} x^\mu \\
&= \sum_{e\in E(G)-e_i} a(e) \sum_{\mu\in S(e)} x^\mu + a(e_i^1) \sum_{\mu\in S(e_i^1)} x^\mu + a(e_i^2) \sum_{\mu\in S(e_i^2)} x^\mu \\
&\ge \sum_{e\in E(G)-e_i} a(e) \sum_{\mu\in S(e)} x^\mu + a(e_i) \sum_{\mu\in S(e_i)} x^\mu + a(e_i^1) \sum_{\mu\in S(e_i^1)} x^\mu \\
&> \sum_{e\in E(G)-e_i} a(e) \sum_{\mu\in S(e)} x^\mu + a(e_i) \sum_{\mu\in S(e_i)} x^\mu = \rho(G),
\end{aligned}$$

a contradiction.

By Fact 3, we have $s = r$ and $k - 2 \le |F(V_i)| \le k - 1$.

Fact 4. $|e_i| = k$ for any $i \in [s]$.

By Fact 1 and Fact 3, we have $k \ge |e_1| \ge |e_2| \ge \cdots \ge |e_s| \ge k - 1$. Without loss of generality, we assume that $|e_1| = \cdots = |e_{s_1}| = k$ and $|e_{s_1+1}| = \cdots = |e_s| = k - 1$, where $1 \le s_1 \le s$. Let the spectral radius $\rho(G) = \rho$ of G. Note that $\mathbf{x}$ be the Perron vector of G. Define a weighted incidence matrix M as follows:

$$M(u, e') = \begin{cases} \frac{\prod_{v\in e'} x_v}{\rho x_u^k}, & \text{for } u \in e', \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, if e'_1 is deferent from e'_2 only their order, then $M(u, e'_1) = M(u, e'_2)$.

Then for any $e' \in S(G)$ have

$$\prod_{u\in e'} M(u, e') = \prod_{u\in e'} \frac{\prod_{v\in e'} x_v}{\rho x_u^k} = \rho^{-k} = \beta,$$

for any $u \in V(G)$, according to the eigenequation, we have

$$\sum_{e' \in S_u(G)} a(e)M(u,e') = \sum_{e' \in S_u(G)} \frac{a(e)\prod_{v \in e'} x_v}{\rho x_u^k} = 1,$$

and for any cycle $u_0e_1u_1e_2 \cdots u_l(u_l = u_0)$ and any k-expanded edge e_i' have

$$\prod_{i=1}^{l} \frac{M(u_i, e_i')}{M(u_{i-1}, e_i')} = \prod_{i=1}^{l} \frac{x_{u_{i-1}}^k}{x_{u_i}^k} = 1.$$

So M satisfies Definition 1.

Since $a(e) = \frac{|e|}{|S(e)|}$, we have

$$a(e) = \begin{cases} \frac{1}{(k-1)!}, & \text{for } |e| = k, \\ \frac{2(k-1)}{k!}, & \text{for } |e| = k-1. \end{cases}$$

Next, we analyze the edges as follows:

(i) For an $(k-1)$-edge e that contains v, we can extend it into $k-1$ different k-edge if we don't consider the order of the vertices, for convenience, denoted them as $e_{(1)}\{v\}, e_{(2)}\{v\}, \ldots, e_{(k-1)}\{v\}$, where $e_{(k-1)}\{v\}$ contains two v. For each of $e_{(i)}\{v\}, i \in [k-1]$, if $e_{(i)}\{v\}$ contains only one v, there are $\frac{(k-1)!}{2}$ k-expanded edges in $S_v(e)$; if $e_{(i)}\{v\}$ contains two v, there are $(k-1)!$ k-expanded edges in $S_v(e)$.

(ii) For an k-edge e, there are $(k-1)!$ k-expanded edges in $S_v(e)$ for any $v \in e$.

Suppose $(e_{s_1} \backslash e_{s_1+1}) \cap F(V_{s_1}) = \{w\}$, $\overline{v} \in F(V_{s_1+1})$, $v_0 \in \overline{V}$, $u \in V_0$.

- Let $\{e_1^*, \ldots, e_{c_1}^*, \ldots, e_c^*\} \subseteq E(G[V(G)\backslash\overline{V}])$ such that $w \in e_i^*, i = 1, 2, \ldots, c$, and $|e_1^*| = \cdots = |e_{c_1}^*| = k, |e_{c_1+1}^*| = \cdots = |e_c^*| = k-1$;
- Let $\{e_1', \ldots, e_{c_1'}', \ldots, e_{c'}'\} \subseteq E(G[V(G)\backslash\overline{V}])$ such that $\overline{v} \in e_i', i = 1, 2, \ldots, c'$, and $|e_1'| = \cdots = |e_{c_1'}'| = k, |e_{c_1'+1}'| = \cdots = |e_{c'}'| = k-1$;
- Let $\{e_1'', \ldots, e_{c_1''}'', \ldots, e_{c''}''\} \subseteq E(G[V(G)\backslash\overline{V}])$ such that $u \in e_i'', i = 1, 2, \ldots, c''$, and $|e_1''| = \cdots = |e_{c_1''}''| = k, |e_{c_1''+1}''| = \cdots = |e_{c''}''| = k-1$.

Now we may write:

(1) $\sum_{i=1}^{s_1} M(w,e_i) + \sum_{i=1}^{c_1} M(w,e_i^*) + \sum_{i=c_1+1}^{c} [\frac{1}{k}\sum_{j=1}^{k-2} M(w,e_{i,(j)}^*\{w\}) + \frac{2}{k}M(w,e_{i,(k-1)}^*\{w\})] = 1;$

(2) $\sum_{i=1}^{s_1} M(\overline{v},e_i) + \sum_{i=s_1+1}^{s} [\frac{1}{k}\sum_{j=1}^{k-2} M(\overline{v},e_{i,(j)}\{\overline{v}\}) + \frac{2}{k}M(\overline{v},e_{i,(k-1)}\{\overline{v}\})] + \sum_{i=1}^{c_1'} M(\overline{v},e_i')+$
$\sum_{i=c_1'+1}^{c'} [\frac{1}{k}\sum_{j=1}^{k-2} M(\overline{v},e'_{i,(j)}\{\overline{v}\}) + \frac{2}{k}M(\overline{v},e'_{i,(k-1)}\{\overline{v}\})] = 1;$

(3) $\sum_{i=1}^{c_1''} M(u,e_i'') + \sum_{i=c_1''+1}^{c''} [\frac{1}{k}\sum_{j=1}^{k-2} M(u,e''_{i,(j)}\{u\}) + \frac{2}{k}M(u,e''_{i,(k-1)}\{u\})] = 1;$

(4) $M(v_0,e_i) = 1,$ for $v_0 \in e_i,\ i = 1,2,\ldots,s_1;$

(5) $\frac{1}{k}\sum_{j=1}^{k-2} M(v_0,e_{i,(j)}\{v_0\}) + \frac{2}{k}M(v_0,e_{i,(k-1)}\{v_0\}) = 1,$ for $v_0 \in e_i,\ i = s_1+1,\ldots,s;$

(6) $\prod_{v\in e_i} M(v,e_i) = \beta,$ for $i = 1,2,\ldots,s_1;$

(7) $\prod_{v\in e_{i,(j)}\{v\}} M(v,e_{i,(j)}\{v\}) = \beta,$ for any $j \in [k-1],\ i = s_1+1,\ldots,s.$

For e_{s_1}, for convenience, we set $M(w,e_{s_1}) = \frac{x_{\overline{v}}^{k-2}x_{v_0}}{\rho x_w^{k-1}} := x_0$ and $M(\overline{v},e_{s_1}) = \frac{x_w x_{v_0}}{\rho x_{\overline{v}}^2} := y_0$. Note that $M(v_0,e_{s_1}) = \frac{x_w x_{\overline{v}}^{k-2}x_{v_0}}{\rho x_{v_0}^k} = \frac{x_w x_{\overline{v}}^{k-2}}{\rho x_{v_0}^{k-1}} = 1$, and $x_{\max} = x_{\overline{v}}, x_{\min} = x_{v_0}$ by Lemma 2.1, then

$$y_0 = \frac{x_w x_{v_0}}{\rho x_{\overline{v}}^2} = (\frac{x_{v_0}}{x_{\overline{v}}})^k = (\frac{1}{\gamma})^k.$$

Let $\overline{e}_{s_1} = e_{s_1}\backslash w$ and $H_4 = G - e_{s_1} + \overline{e}_{s_1}$. Construct a weighted incidence matrix M' of H_4 as following:

$$M'(v,e') = \begin{cases} M(v,e'), & \text{for } e' \notin S(\overline{e}_{s_1}), \\ \beta^{\frac{1}{k-1}}, & e' = \overline{e}_{s_1,(j)}\{v_0\}, v = \overline{v}, \text{for } i = 1,\ldots,k-2, \\ \beta^{\frac{1}{k-2}}, & e' = \overline{e}_{s_1,(k-1)}\{v_0\}, v = \overline{v}, \\ 1, & v \in e' \in S(\overline{e}_{s_1}), d(v) = 1. \end{cases}$$

where $v_0 \in \overline{e}_{s_1}$. For some pendent vertex $v_0' \in e \in \{e_1,\ldots,e_s\}$, we have

$$\rho x_{v_0'}^{k-1} = a(e)\sum_{\mu\in S_{v_0'}(e)} x^{\mu - v_0'},$$

that is

$$\rho = a(e)\sum_{\mu\in S_{v_0'}(e)} \frac{x^{\mu-v_0'}}{x_{v_0'}^{k-1}} \le k\gamma^{k-1}.$$

So, we have

$$\begin{aligned} \frac{k-2}{k}\beta^{\frac{1}{k-1}} + \frac{2}{k}\beta^{\frac{1}{k-2}} &= \frac{k-2}{k}\rho^{-\frac{k}{k-1}} + \frac{2}{k}\rho^{-\frac{k}{k-2}} \\ &\le \frac{k-2}{k}(k\gamma^{k-1})^{-\frac{k}{k-1}} + \frac{2}{k}(k\gamma^{k-1})^{-\frac{k}{k-2}} \end{aligned}$$

$$\begin{aligned}
&= (k-2)k^{-\frac{2k-1}{k-1}}\gamma^{-k} + 2k^{-\frac{2k-2}{k-2}}\gamma^{-\frac{k(k-1)}{k-2}}\\
&\leq \frac{k-2}{k^{\frac{2k-1}{k-1}}}\gamma^{-k} + \frac{2}{k^{\frac{2k-2}{k-2}}}\gamma^{-k}\\
&\leq \gamma^{-k} = y_0.
\end{aligned}$$

Now for $\bar{e}_{s_1}$, it has

(1) $\sum_{i=1}^{s_1-1} M'(w,e_i) + \sum_{i=1}^{c_1} M'(w,e_i^*) + \sum_{i=c_1+1}^{c} [\frac{1}{k}\sum_{j=1}^{k-2} M'(w,e^*_{i,(j)}\{w\}) + \frac{2}{k}M'(w,e^*_{i,(k-1)}\{w\})]$
$= 1 - M(w,e_{s_1}) < 1$;

(2) $\frac{1}{k}\sum_{j=1}^{k-2} M'(\bar{v},\bar{e}_{s_1,(j)}\{\bar{v}\}) + \frac{2}{k}M'(\bar{v},\bar{e}_{s_1,(k-1)}\{\bar{v}\}) = \frac{k-2}{k}\beta^{\frac{1}{k-1}} + \frac{2}{k}\beta^{\frac{1}{k-2}} \leq y_0 = M(\bar{v},e_{s_1})$;

(3) $\frac{1}{k}\sum_{j=1}^{k-2} M'(v_0,\bar{e}_{s_1,(j)}\{v_0\}) + \frac{2}{k}M'(v_0,\bar{e}_{s_1,(k-1)}\{v_0\}) = 1$, for $v_0 \in \bar{e}_{s_1}$;

(4) $\prod_{v\in\bar{e}_{s_1,(j)}\{v_0\}} M'(v,\bar{e}_{s_1,(j)}\{v_0\}) = \beta$, for $v_0 \in \bar{e}_{s_1}$, $1 \leq j \leq k-2$;

(5) $\prod_{v\in\bar{e}_{s_1,(k-1)}\{v_0\}} M'(v,\bar{e}_{s_1,(k-1)}\{v_0\}) = \beta$, for $v_0 \in \bar{e}_{s_1}$.

So H_4 is strictly β-subnormal. By Lemma 2.3, $\rho(H_4) < \beta^{-\frac{1}{k}} = \rho(G)$. Then $|e_i| = k, i \in [s]$ and $F(v_1) = F(v_2) = \cdots = F(v_s)$. So $G \cong A_n^r(k,k-1)$. □

Let $n-r = k-1$ and $r \geq 2$, let $B_n^r(k,k-1)$ be the general hypergraph obtained from edge $e_0 = \{u_1,u_2,\ldots,u_{k-1}\}$ by adding r new pendent vertices and r new edges, each of new edges consists of all vertices in e_0 and a new pendent vertex. See Figure 1.

Theorem 3.2. *If $H \in \mathcal{G}_{n,r}$, $n-r = k-1$ and $r \geq 2$, then $\rho(H) \leq \rho(B_n^r(k,k-1))$ with equality if and only if $H \cong B_n^r(k,k-1)$.*

Proof. Let $G = (V(G),E(G))$ be the $\{k.k-1\}$-graph with maximum spectral radius in $\mathcal{G}_{n,r}$, and $\overline{V}$ be the set of pendent vertices in G. Let $V_0 = V(G)\backslash\overline{V} = \{u_1,u_2,\ldots,u_{k-1}\}$ and $\overline{E} = \{e \in E : e\cap\overline{V} \neq \emptyset\} = \{e_1,\ldots,e_s\}$. Let $V_i = e_i \cap \overline{V}$, then $\overline{V} = V_1\cup V_2\cup\cdots\cup V_s$. Obviously, we have $e_0 = \{u_1,u_2,\ldots,u_{k-1}\} \in E(G)$ and $E(G) = \{e_0\}\cup\overline{E}$. Without loss of generality, suppose that

$$|e_1| - |V_1| \geq |e_2| - |V_2| \geq \cdots \geq |e_s| - |V_s|.$$

Let $\mathbf{x}$ be the Perron vector of G. Similar to the proof of Facts 1-3 in Theorem 3.1, we have $|V_i| = 1$ for any $i \in [s]$ and $s = r$.

Fact 5. $|e_i| = k$ for any $i \in [r]$.

Noting that $k = |e_1| \geq |e_2| \geq \cdots \geq |e_r| \geq k-1$, without loss of generality, we assume that $|e_1| = \cdots = |e_{s_2}| = k$, $|e_{s_2+1}| = \cdots = |e_r| = k-1$, then $F(V_1) = \cdots = F(V_{s_2}) = V_0$, where $1 \leq s_2 \leq r$. Let $\rho(G) = \rho = \beta^{-\frac{1}{k}}$. Define a weighted incidence matrix M_1 as follows:

$$M_1(u,e') = \begin{cases} \frac{\prod_{v\in e'} x_v}{\rho x_u^k}, & \text{for } u \in e',\\ 0, & \text{otherwise.}\end{cases}$$

Then M_1 satisfies Definition 1.

Suppose $e_{s_2}\backslash e_{s_2+1} = w, \overline{v} \in F(V_{s_2+1})$, $v_0 \in \overline{V}$. Now we may write:

(1) $\sum_{i=1}^{s_2} M_1(w, e_i) + \frac{1}{k}\sum_{j=1}^{k-2} M_1(w, e_{0,(j)}\{w\}) + \frac{2}{k} M_1(w, e_{0,(k-1)}\{w\}) = 1;$

(2) $\sum_{i=1}^{s_2} M_1(\overline{v}, e_i) + \sum_{i=s_2+1}^{s} [\frac{1}{k}\sum_{j=1}^{k-2} M_1(\overline{v}, e_{i,(j)}\{\overline{v}\}) + \frac{2}{k} M_1(\overline{v}, e_{i,(k-1)}\{\overline{v}\})]$

$+ \frac{1}{k}\sum_{j=1}^{k-2} M_1(\overline{v}, e_{0,(j)}\{\overline{v}\}) + \frac{2}{k} M_1(\overline{v}, e_{0,(k-1)}\{\overline{v}\}) = 1;$

(3) $M_1(v_0, e_i) = 1,$ for $v_0 \in e_i,\ i = 1, 2, \ldots, s_2;$

(4) $\frac{1}{k}\sum_{j=1}^{k-2} M_1(v_0, e_{i,(j)}\{v_0\}) + \frac{2}{k} M_1(v_0, e_{i,(k-1)}\{v_0\}) = 1,$ for $v_0 \in e_i,\ i = s_2 + 1, \ldots, s;$

(5) $\prod_{v \in e_i} M_1(v, e_i) = \beta,$ for $i = 1, 2, \ldots, s_2;$

(6) $\prod_{v \in e_{i,(j)}\{v\}} M_1(v, e_{i,(j)}\{v\}) = \beta,$ for any $j \in [k-1],\ i = s_2 + 1, \ldots, s;$

(7) $\prod_{v \in e_{0,(j)}\{v\}} M_1(v, e_{0,(j)}\{v\}) = \beta.$

For e_{s_2}, for convenience, we set $M_1(w, e_{s_2}) = \frac{x_{\overline{v}}^{k-2} x_{v_0}}{\rho x_w^{k-1}} := x_0$, $M_1(\overline{v}, e_{s_2}) = \frac{x_w x_{v_0}}{\rho x_{\overline{v}}^2} := y_0$. Note that $M_1(v_0, e_{s_2}) = \frac{x_w x_{\overline{v}}^{k-2} x_{v_0}}{\rho x_{v_0}^k} = \frac{x_w x_{\overline{v}}^{k-2}}{\rho x_{v_0}^{k-1}} = 1$, and $x_{\max} = x_{\overline{v}}, x_{\min} = x_{v_0}$ by Lemma 2.1, then

$$y_0 = \frac{x_w x_{v_0}}{\rho x_{\overline{v}}^2} = (\frac{x_{v_0}}{x_{\overline{v}}})^k = (\frac{1}{\gamma})^k.$$

Let $H_5 = G - e_{s_2} + \overline{e}_{s_2}$, where $\overline{e}_{s_2} = e_{s_2}\backslash w$. Construct a weighted incidence matrix M_1' for H_5 as following:

$$M_1'(v, e') = \begin{cases} M_1(v, e'), & e' \notin S(\overline{e}_{s_2}), \\ \beta^{\frac{1}{k-1}}, & e' = \overline{e}_{s_2,(j)}\{v_0\}, v = \overline{v}, \text{for } i = 1, \ldots, k-2, \\ \beta^{\frac{1}{k-2}}, & e' = \overline{e}_{s_2,(k-1)}\{v_0\}, v = \overline{v}, \\ 1, & v \in e' \in S(\overline{e}_{s_2}), d(v) = 1. \end{cases}$$

where $v_0 \in \overline{e}_{s_2}$. Similar to Theorem 3.1, we have

$$\frac{k-2}{k}\beta^{\frac{1}{k-1}} + \frac{2}{k}\beta^{\frac{1}{k-2}} \le \gamma^{-k} = y_0.$$

Now for $\bar{e}_{s_2}$, it has

(1) $\sum_{i=1}^{s_2-1} M_1'(w, e_i) + \frac{1}{k}\sum_{j=1}^{k-2} M_1'(w, e_{0,(j)}\{w\}) + \frac{2}{k} M_1'(w, e_{0,(k-1)}\{w\}) = 1 - M_1(w, e_{s_2}) < 1$;

(2) $\frac{1}{k}\sum_{j=1}^{k-2} M_1'(\bar{v}, \bar{e}_{s_2,(j)}\{\bar{v}\}) + \frac{2}{k} M_1'(\bar{v}, \bar{e}_{s_2,(k-1)}\{\bar{v}\}) = \frac{k-2}{k}\beta^{\frac{1}{k-1}} + \frac{2}{k}\beta^{\frac{1}{k-2}} \leq y_0 = M_1(\bar{v}, e_{s_2})$;

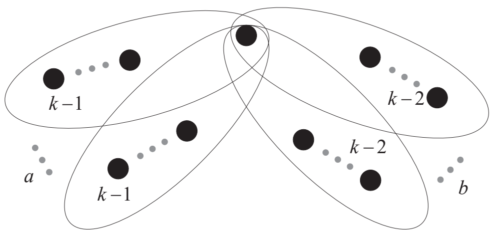
(3) $\frac{1}{k}\sum_{j=1}^{k-2} M_1'(v_0, \bar{e}_{s_2,(j)}\{v_0\}) + \frac{2}{k} M_1'(v_0, \bar{e}_{s_2,(k-1)}\{v_0\}) = 1$, for $v_0 \in \bar{e}_{s_2}$;

(4) $\prod_{v \in \bar{e}_{s_2,(j)}\{v_0\}} M_1'(v, \bar{e}_{s_2,(j)}\{v_0\}) = \beta$, for $v_0 \in \bar{e}_{s_2}$, $1 \leq j \leq k-2$;

(5) $\prod_{v \in \bar{e}_{s_2,(k-1)}\{v_0\}} M_1'(v, \bar{e}_{s_2,(k-1)}\{v_0\}) = \beta$, for $v_0 \in \bar{e}_{s_2}$.

So M_1' is strictly β-subnormal. Also by Lemma 2.3, $\rho(H_5) < \beta^{-\frac{1}{k}} = \rho(G)$. Then $|e_i| = k, i = 1, 2, \ldots, r$ and $F(v_1) = F(v_2) = \cdots = F(v_r) = V_0$. Thus we get that $G \cong B_n^r(k, k-1)$. □

Let $n - r = 1$, $C_{n,n-1}^{a,b}(k, k-1)$ be the hypergraph in $\mathcal{G}_{n,r}$ with a k-edges and b $(k-1)$-edges. See Figure 2. Obviously, each of k-edges contains $k-1$ pendent vertices, each of $(k-1)$-edges contains $k-2$ pendent vertices, then $a(k-1) + b(k-2) = r$. According to Theorem 4.3 in [2], we know that $C_{n,n-1}^{a_1,b_1}(k, k-1)$ have the maximum spectral radius in $\mathcal{G}_{n,n-1}$, where b_1 is the maximum solution of congruence $(n - b_1(k-2) - 1) \equiv 0(\text{mod } k-1)$.



$C_{n,n-1}^{a,b}(k, k-1)$

Figure 2: The hypergraph $C_{n,n-1}^{a,b}(k, k-1)$

For $n - r = 2$, let

- k^1-edge be an edge consisting of two non-pendent vertices and $k-2$ pendent vertices;
- k^2-edge be an edge consisting of a non-pendent vertex and $k-1$ pendent vertices;
- $(k-1)^1$-edge be an edge consisting of two non-pendent vertices and $k-3$ pendent vertices;
- $(k-1)^2$-edge be an edge consisting of a non-pendent vertex and $k-2$ pendent vertices.

Let $D_{n,n-2}^{a,b,c,d}(k, k-1)$ be a $\{k, k-1\}$-graph in $\mathcal{G}_{n,r}$ with a k^1-edges, b $(k-1)^1$-edges, c k^2-edges, d $(k-1)^2$-edges, and $a(k-2) + b(k-3) + c(k-1) + d(k-2) = n-2$. See Figure 3. For convenience, let

- E_1 be the set of k^1-edges in $D_{n,n-2}^{a,b,c,d}(k, k-1)$;
- E_2 be the set of k^2-edges in $D_{n,n-2}^{a,b,c,d}(k, k-1)$;
- E_3 be the set of $(k-1)^1$-edges in $D_{n,n-2}^{a,b,c,d}(k, k-1)$;
- E_4 be the set of $(k-1)^2$-edges in $D_{n,n-2}^{a,b,c,d}(k, k-1)$.

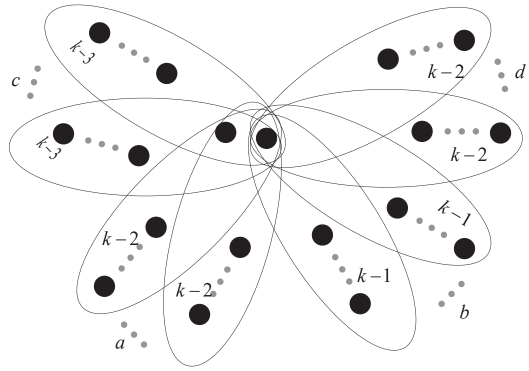


Figure 3: The hypergraph $D^{a,b,c,d}_{n,n-2}(k,k-1)$

Then $E(D^{a,b,c,d}_{n,n-2}(k,k-1)) = E_1 \cup E_2 \cup E_3 \cup E_4$.

By Lemma 2.6, we may assume that k^2-edges and $(k-1)^2$-edges of $D^{a,b,c,d}_{n,n-2}(k,k-1)$ have a non-pendent vertex in common.

Lemma 3.3. *Suppose that $D^{a_1,b_1,c,d}_{n,n-2}(k,k-1)$ and $D^{a_2,b_2,c,d}_{n,n-2}(k,k-1)$ are two $\{k,k-1\}$-graphs with $a_1(k-2)+b_1(k-3) = a_2(k-2)+b_2(k-3)$ and $b_1 < b_2$. Then $\rho(D^{a_1,b_1,c,d}_{n,n-2}(k,k-1)) < \rho(D^{a_2,b_2,c,d}_{n,n-2}(k,k-1))$.*

Proof. Let u_1, u_2 be the two non-pendent vertices in $V(D^{a_2,b_2,c,d}_{n,n-2}(k,k-1))$ and $E(D^{a_2,b_2,c,d}_{n,n-2}(k,k-1)) = E_1 \cup E_2 \cup E_3 \cup E_4$. According to the definition of $D^{a,b,c,d}_{n,n-2}(k,k-1)$, without loss of generality, we set $u_1 \in e \in E_3 \cup E_4$. Clearly, $|E_1| = a_2, |E_2| = b_2, |E_3| = c, |E_4| = d$.

Let $G := D^{a_2,b_2,c,d}_{n,n-2}(k,k-1)$ and $\rho(D^{a_2,b_2,c,d}_{n,n-2}(k,k-1)) = \beta^{-\frac{1}{k}}$, by Lemma 2.2, there is a weighted incidence matrix M_2 which satisfies the following conditions:

$$\begin{cases} \sum_{e'\in S_v(G)} a(e)M_2(v,e') = 1, & \forall\, v \in V(G) \text{ and any } e\text{-expanded edge } e', \\ \prod_{v\in e'} M_2(v,e') = \beta, & \forall\, e' \in S(G), \\ M_2(v,e'_1) = M_2(v,e'_2), & e'_1 \text{ is deferent from } e'_2 \text{ only their order.} \end{cases} \tag{3.2}$$

For an $(k-1)^1$-edge e, we may extend it into $k-1$ different k-edge if we don't consider the order of the vertices, denoted by $e_{(1)}\{u_1\}, e_{(2)}\{u_1\}, \ldots, e_{(k-2)}\{u_1\}, e_{(k-1)}\{u_1\}$. We may suppose that $e_{(k-2)}\{u_1\}$ contains a u_1 and two u_2, and $e_{(k-1)}\{u_1\}$ contains two u_1 and a u_2. Now we may write:

$$\begin{aligned}
&(1)\ \sum_{e\in E_1} M_2(u_1,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_2(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_2(u_1,e_{(k-1)}\{u_1\})]\\
&\quad + \sum_{e\in E_3} M_2(u_1,e) + \sum_{e\in E_4}[\frac{1}{k}\sum_{i=1}^{k-2} M_2(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_2(u_1,e_{(k-1)}\{u_1\})] = 1;\\
&(2)\ \sum_{e\in E_1} M_2(u_2,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_2(u_2,e_{(i)}\{u_1\}) + \frac{2}{k}M_2(u_2,e_{(k-1)}\{u_1\})] = 1;\\
&(3)\ \frac{1}{k}\sum_{i=1}^{k-2} M_2(v,e_{(i)}\{v\}) + \frac{2}{k}M_2(v,e_{(k-1)}\{v\}) = 1,\ \text{for } v\in e\in E_2\cup E_4,\ d(v)=1;\\
&(4)\ M_2(v,e) = 1,\ \text{for } v\in e\in E_1\cup E_3,\ d(v)=1;\\
&(5)\ \prod_{v\in e} M_2(v,e) = M_2(u_1,e)M_2(u_2,e) = \beta,\ \text{for } e\in E_1;\\
&(6)\ \prod_{v\in e_{(i)}\{u_1\}} M_2(v,e_{(i)}\{u_1\}) = \beta,\ \text{for } e\in E_2\cup E_4;\\
&(7)\ M_2(u_1,e) = \beta,\ \text{for } e\in E_3.
\end{aligned} \tag{3.3}$$

For any $e\in E_2$, for convenience, we may write $M_2(u_1,e_{(i)}\{u_1\}) := x$, $M_2(u_1,e_{(k-2)}\{u_1\}) := x_1$, $M_2(u_1,e_{(k-1)}\{u_1\}) := x_2$, $M_2(u_2,e_{(i)}\{u_1\}) := y$, $M_2(u_2,e_{(k-2)}\{u_1\}) := y_1$, $M_2(u_2,e_{(k-1)}\{u_1\}) := y_2$, where $i = 1,\ldots,k-3$. Then according to (6) of (3.3), we have

$$\begin{cases} xy = \beta,\\ x_1y_1^2 = \beta,\\ x_2^2y_2 = \beta. \end{cases}$$

Note that $x > \beta, y > \beta, x_1 > \beta, y_2 > \beta$.

Choose $b_2 - b_1$ edges in E_2, and let E_2' be the set containing all these edges. Further let E_1' be a set having $\frac{(b_2-b_1)(k-3)}{k-2} = a_1 - a_2$ edges, each edge in E_1' consists of u_1, u_2 and $(k-2)$ pendent vertices. Then $D_{n,n-2}^{a_1,b_1,c,d}(k,k-1)$ may be obtained from $D_{n,n-2}^{a_2,b_2,c,d}(k,k-1)$ by deleting the edges in E_2' and adding the edges in E_1'.

Define a weighted incidence matrix M_2' for $D_{n,n-2}^{a_1,b_1,c,d}(k,k-1)$:

$$M_2'(v,e') = \begin{cases} M_2(v,e'), & \text{for } e'\notin S(E_1'),\\ x_0, & v\in e'\in S(E_1'), v = u_1,\\ y_0, & v\in e'\in S(E_1'), v = u_2,\\ 1, & v\in e'\in S(E_1'), d(v) = 1. \end{cases}$$

where $0 < x_0, y_0 < 1$ and x_0, y_0 satisfy

$$\begin{cases} x_0 < \frac{1}{a_1-a_2}\sum_{e\in E_2'}[\frac{k-3}{k}x + \frac{1}{k}x_1 + \frac{2}{k}x_2],\\ y_0 \le \frac{1}{a_1-a_2}\sum_{e\in E_2'}[\frac{k-3}{k}y + \frac{1}{k}y_1 + \frac{2}{k}y_2],\\ x_0y_0 \ge \beta. \end{cases}$$

where x_0, y_0 may be taken because

$$\begin{aligned}
&\frac{(\frac{1}{a_1-a_2}\sum_{e\in E_2'}[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2])(\frac{1}{a_1-a_2}\sum_{e\in E_2'}[\frac{k-3}{k}y+\frac{1}{k}y_1+\frac{2}{k}y_2])}{\beta}\\
&=\frac{(b_2-b_1)^2[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2][\frac{k-3}{k}y+\frac{1}{k}y_1+\frac{2}{k}y_2]}{(a_1-a_2)^2\beta}\\
&>\frac{[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2][\frac{k-3}{k}y+\frac{1}{k}y_1+\frac{2}{k}y_2]}{\beta}\\
&>[\frac{k-3}{k}+\frac{1}{k}+\frac{2}{k}\frac{x_2}{\beta}][\frac{k-3}{k}+\frac{1}{k}\frac{y_1}{\beta}+\frac{2}{k}]\\
&=[\frac{k-3}{k}+\frac{1}{k}+\frac{2}{k}\frac{(\frac{\beta}{y_2})^{\frac{1}{2}}}{\beta}][\frac{k-3}{k}+\frac{1}{k}\frac{(\frac{\beta}{x_1})^{\frac{1}{2}}}{\beta}+\frac{2}{k}]\\
&=[\frac{k-2}{k}+\frac{2}{k}(\frac{1}{\beta y_2})^{\frac{1}{2}}][\frac{k-1}{k}+\frac{1}{k}(\frac{1}{\beta x_1})^{\frac{1}{2}}]\\
&>1.
\end{aligned}$$

For each edge in E_1', it has

(1) $\sum_{e\in E_1'} M_2'(u_1,e)=(a_1-a_2)x_0<\sum_{e\in E_2'}[\frac{1}{k}\sum_{i=1}^{k-2}M_2(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_2(u_1,e_{(k-1)}\{u_1\})]$;

(2) $\sum_{e\in E_1'} M_2'(u_2,e)=(a_1-a_2)y_0\le\sum_{e\in E_2'}[\frac{1}{k}\sum_{i=1}^{k-2}M_2(u_2,e_{(i)}\{u_1\})+\frac{2}{k}M_2(u_2,e_{(k-1)}\{u_1\})]$;

(3) $\prod_{v\in e} M_2'(v,e)=M_2'(u_1,e)M_2'(u_2,e)=x_0y_0\ge\beta$, for $e\in E_1'$;

(4) $M_2'(v,e)=1$, for $v\in e\in E_1'$, $d(v)=1$.

So M_2' is strictly β-subnormal. By Lemma 2.3, $\rho(D_{n,n-2}^{a_1,b_1,c,d}(k,k-1))<\rho(D_{n,n-2}^{a_2,b_2,c,d}(k,k-1))$. □

Lemma 3.4. *Suppose that $D_{n,n-2}^{a,b,c_1,d_1}(k,k-1)$ and $D_{n,n-2}^{a,b,c_2,d_2}(k,k-1)$ are two $\{k,k-1\}$-graphs with $c_1(k-1)+d_1(k-2)=c_2(k-1)+d_2(k-2)$ and $d_1<d_2$. Then $\rho(D_{n,n-2}^{a,b,c_1,d_1}(k,k-1))<\rho(D_{n,n-2}^{a,b,c_2,d_2}(k,k-1))$.*

Proof. Let u_1, u_2 be the two non-pendent vertices in $V(D_{n,n-2}^{a,b,c_2,d_2}(k,k-1))$ and $E(D_{n,n-2}^{a,b,c_2,d_2}(k,k-1))=E_1\cup E_2\cup E_3\cup E_4$. According to the definition of $D_{n,n-2}^{a,b,c,d}(k,k-1)$, without loss of generality, we set $u_1\in e\in E_3\cup E_4$. Clearly, $|E_1|=a, |E_2|=b, |E_3|=c_2, |E_4|=d_2$.

Let $\rho(D_{n,n-2}^{a,b,c_2,d_2}(k,k-1))=\beta^{-\frac{1}{k}}$, by Lemma 2.2, there is a weighted incidence matrix M_3 which satisfies the following conditions:

$$\begin{cases}\sum_{e'\in S_v(G)}a(e)M_3(v,e')=1, & \forall\, v\in V(G) \text{ and any } e\text{-expanded edge } e',\\ \prod_{v\in e'}M_3(v,e')=\beta, & \forall\, e'\in S(G),\\ M_3(v,e_1')=M_3(v,e_2'), & e_1' \text{ is deferent from } e_2' \text{ only their order.}\end{cases}\tag{3.4}$$

Now we may write:

$$\begin{aligned}
&(1)\ \sum_{e\in E_1} M_3(u_1,e)+\sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_3(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_3(u_1,e_{(k-1)}\{u_1\})]\\
&\quad +\sum_{e\in E_3} M_3(u_1,e)+\sum_{e\in E_4}[\frac{1}{k}\sum_{i=1}^{k-2} M_3(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_3(u_1,e_{(k-1)}\{u_1\})]=1;\\
&(2)\ \sum_{e\in E_1} M_3(u_2,e)+\sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_3(u_2,e_{(i)}\{u_1\})+\frac{2}{k}M_3(u_2,e_{(k-1)}\{u_1\})]=1;\\
&(3)\ \frac{1}{k}\sum_{i=1}^{k-2} M_3(v,e_{(i)}\{v\})+\frac{2}{k}M_3(v,e_{(k-1)}\{v\})=1,\ \text{for } v\in e\in E_2\cup E_4,\ d(v)=1;\\
&(4)\ M_3(v,e)=1,\ \text{for } v\in e\in E_1\cup E_3,\ d(v)=1;\\
&(5)\ \prod_{v\in e} M_3(v,e)=M_3(u_1,e)M_3(u_2,e)=\beta,\ \text{for } e\in E_1;\\
&(6)\ \prod_{v\in e_{(i)}\{u_1\}} M_3(v,e_{(i)}\{u_1\})=\beta,\ \text{for } e\in E_2\cup E_4;\\
&(7)\ M_3(u_1,e)=\beta,\ \text{for } e\in E_3.
\end{aligned} \tag{3.5}$$

Choose d_2-d_1 edges in E_4, and let E_4' be the set containing all these edges. Further let E_3' be a set having $\frac{(d_2-d_1)(k-2)}{k-1}=c_1-c_2$ edges, each edge in E_3' contains $(k-1)$ pendent vertices and u_1. Then $D^{a,b,c_1,d_1}_{n,n-2}(k,k-1)$ may be obtained from $D^{a,b,c_2,d_2}_{n,n-2}(k,k-1)$ by deleting the edges in E_4' and adding edges in E_3'.

Define a weighted incidence matrix M_3' for $D^{a,b,c_1,d_1}_{n,n-2}(k,k-1)$:

$$M_3'(v,e')=\begin{cases} M_3(v,e'), & \text{for } e'\notin S(E_3'),\\ \beta, & v\in e'\in S(E_3'), v=u_1,\\ 1, & v\in e'\in S(E_3'), d(v)=1.\end{cases}$$

For each edge in E_3', it has

$$\begin{aligned}
&(1)\ \sum_{e\in E_3'} M_3'(u_1,e)=(c_1-c_2)\beta<\sum_{e\in E_4'}[\frac{1}{k}\sum_{i=1}^{k-2}\beta+\frac{2}{k}\beta]<\sum_{e\in E_4'}[\frac{1}{k}\sum_{i=1}^{k-2}\beta+\frac{2}{k}\beta^{\frac{1}{2}}]\\
&\quad =\sum_{e\in E_4'}[\frac{1}{k}\sum_{i=1}^{k-2} M_3(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_3(u_1,e_{(k-1)}\{u_1\})];\\
&(2)\ \prod_{v\in e} M_3'(v,e)=M_3'(u_1,e)=\beta,\ \text{for } e\in E_3';\\
&(3)\ M_3'(v,e)=1,\ \text{for } v\in e\in E_3',\ d(v)=1.
\end{aligned}$$

So M_3' is strictly β-subnormal. By Lemma 2.3, $\rho(D^{a,b,c_1,d_1}_{n,n-2}(k,k-1))<\rho(D^{a,b,c_2,d_2}_{n,n-2}(k,k-1))$. □

Lemma 3.5. *Suppose that $D^{a,b_3,c,d_3}_{n,n-2}(k,k-1)$ and $D^{a,b_4,c,d_4}_{n,n-2}(k,k-1)$ are two $\{k,k-1\}$-graphs with $b_3(k-3)+d_3(k-2)=b_4(k-3)+d_4(k-2)$ and $b_3>b_4$. Then $\rho(D^{a,b_3,c,d_3}_{n,n-2}(k,k-1))>\rho(D^{a,b_4,c,d_4}_{n,n-2}(k,k-1))$.*

Proof. Let u_1,u_2 be the two non-pendent vertices in $D^{a,b_3,c,d_3}_{n,n-2}(k,k-1)$ and $E(D^{a,b_3,c,d_3}_{n,n-2}(k,k-1))=E_1\cup E_2\cup E_3\cup E_4$. According to the definition of $D^{a,b,c,d}_{n,n-2}(k,k-1)$, without loss of generality, we set $u_1\in e\in E_3\cup E_4$. Clearly, $|E_1|=a,|E_2|=b_3,|E_3|=c,|E_4|=d_3$.

Let $\rho(D^{a,b_3,c,d_3}_{n,n-2}(k,k-1)) = \beta^{-\frac{1}{k}}$, by Lemma 2.2, there is a weighted incidence matrix M_4 which satisfies the following conditions:

$$\begin{cases} \sum_{e'\in S_v(G)} a(e)M_4(v,e') = 1, & \forall\, v \in V(G) \text{ and any } e\text{-expanded edge } e', \\ \prod_{v\in e'} M_4(v_{l_i},e') = \beta, & \forall\, e' \in S(G), \\ M_4(v,e'_1) = M_4(v,e'_2), & e'_1 \text{ is deferent from } e'_2 \text{ only their order.} \end{cases} \tag{3.6}$$

Now we may write:

$$\begin{aligned}
&(1)\ \sum_{e\in E_1} M_4(u_1,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_4(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_4(u_1,e_{(k-1)}\{u_1\})] \\
&\quad + \sum_{e\in E_3} M_4(u_1,e) + \sum_{e\in E_4}[\frac{1}{k}\sum_{i=1}^{k-2} M_4(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_4(u_1,e_{(k-1)}\{u_1\})] = 1; \\
&(2)\ \sum_{e\in E_1} M_4(u_2,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_4(u_2,e_{(i)}\{u_1\}) + \frac{2}{k}M_4(u_2,e_{(k-1)}\{u_1\})] = 1; \\
&(3)\ \frac{1}{k}\sum_{i=1}^{k-2} M_4(v,e_{(i)}\{v\}) + \frac{2}{k}M_4(v,e_{(k-1)}\{v\}) = 1, \text{ for } v \in e \in E_2 \cup E_4,\ d(v) = 1; \\
&(4)\ M_4(v,e) = 1, \text{ for } v \in e \in E_1 \cup E_3,\ d(v) = 1; \\
&(5)\ \prod_{v\in e} M_4(v,e) = M_4(u_1,e)M_4(u_2,e) = \beta, \text{ for } e \in E_1; \\
&(6)\ \prod_{v\in e_{(i)}\{u_1\}} M_4(v,e_{(i)}\{u_1\}) = \beta, \text{ for } e \in E_2 \cup E_4; \\
&(7)\ M_4(u_1,e) = \beta, \text{ for } e \in E_3.
\end{aligned} \tag{3.7}$$

Similar to Lemma 3.3, for any $e \in E_2$, we may write $M_4(u_1,e_{(i)}\{u_1\}) := x$, $M_4(u_1,e_{(k-2)}\{u_1\}) := x_1$, $M_4(u_1,e_{(k-1)}\{u_1\}) := x_2$, $M_4(u_2,e_{(i)}\{u_1\}) := y$, $M_4(u_2,e_{(k-2)}\{u_1\}) := y_1$, $M_4(u_2,e_{(k-1)}\{u_1\}) := y_2$, where $i = 1,\ldots,k-3$. Then according to (6) of (3.7), we have

$$\begin{cases} xy = \beta, \\ x_1y_1^2 = \beta, \\ x_2^2y_2 = \beta. \end{cases}$$

Note that $x > \beta, y > \beta, x_1 > \beta, y_2 > \beta$.

Choose $b_3 - b_4$ edges in E_2, and let E'_2 be the set containing all these edges. Further let E'_4 be a set having $\frac{(b_3-b_4)(k-3)}{k-2} = d_4 - d_3$ edges, each edge in E'_4 contains $(k-2)$ pendent vertices and $\{u_1\}$. Then $D^{a,b_4,c,d_4}_{n,n-2}(k,k-1)$ may be obtained from $D^{a,b_3,c,d_3}_{n,n-2}(k,k-1)$ by deleting the edges in E'_2 and adding edges in E'_4.

Define a weighted incidence matrix M'_4 for $D^{a,b_4,c,d_4}_{n,n-2}(k,k-1)$:

$$M'_4(v,e') = \begin{cases} M_4(v,e'), & \text{for } e' \notin S(E'_4), \\ \beta, & e' = e_{(i)}\{u_1\}, e \in E'_4, v = u_1, \text{for } i = 1,\ldots,k-2, \\ \beta^{\frac{1}{2}}, & e' = e_{(k-1)}\{u_1\}, e \in E'_4, v = u_1, \\ 1, & v \in e' \in S(E'_4), d(v) = 1. \end{cases}$$

We can get

$$\begin{aligned}
&\sum_{e\in E_2'}[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2]-(d_4-d_3)[\frac{k-2}{k}\beta+\frac{2}{k}\beta^{\frac{1}{2}}]\\
&=(b_3-b_4)[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2]-(d_4-d_3)[\frac{k-2}{k}\beta+\frac{2}{k}\beta^{\frac{1}{2}}]\\
&>[\frac{k-3}{k}x+\frac{1}{k}x_1+\frac{2}{k}x_2]-[\frac{k-2}{k}\beta+\frac{2}{k}\beta^{\frac{1}{2}}]\\
&>\frac{2}{k}(x_2-\beta^{\frac{1}{2}})=\frac{2}{k}((\frac{\beta}{y_2})^{\frac{1}{2}}-\beta^{\frac{1}{2}})\\
&>0.
\end{aligned}$$

For each edge in E_4', it has

$$\begin{aligned}
(1)\ &\sum_{e\in E_4'}[\frac{1}{k}\sum_{i=1}^{k-2}M_4'(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_4'(u_1,e_{(k-1)}\{u_1\})]\\
&\quad<\sum_{e\in E_2'}[\frac{1}{k}\sum_{i=1}^{k-2}M_4(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_4(u_1,e_{(k-1)}\{u_1\})];\\
(2)\ &\sum_{e\in E_1}M_4'(u_2,e)+\sum_{e\in E_2\setminus E_2'}[\frac{1}{k}\sum_{i=1}^{k-2}M_4'(u_2,e_{(i)}\{u_1\})+\frac{2}{k}M_4'(u_2,e_{(k-1)}\{u_1\})]<1;\\
(3)\ &\frac{1}{k}\sum_{i=1}^{k-2}M_4'(v,e_{(i)}\{v\})+\frac{2}{k}M_4'(v,e_{(k-1)}\{v\})=1,\ \text{for } v\in e\in E_4',\ d(v)=1;\\
(4)\ &\prod_{v\in e_{(i)}\{u_1\}}M_4'(v,e_{(i)}\{u_1\})=M_4'(u_1,e_{(i)}\{u_1\})=\beta,\ \text{for } e\in E_4',\ i=1,\dots,k-2;\\
(6)\ &\prod_{v\in e_{(k-1)}\{u_1\}}M_4'(v,e_{(k-1)}\{u_1\})=(M_4'(u_1,e_{(k-1)}\{u_1\}))^2=\beta.
\end{aligned}$$

So M_4' is strictly β-subnormal. By Lemma 2.3, $\rho(D_{n,n-2}^{a,b_4,c,d_4}(k,k-1))<\rho(D_{n,n-2}^{a,b_3,c,d_3}(k,k-1))$. □

Lemma 3.6. *Suppose that $D_{n,n-2}^{a_5,b,c_5,d}(k,k-1)$ and $D_{n,n-2}^{a_6,b,c_6,d}(k,k-1)$ are two $\{k,k-1\}$-graphs with $a_5(k-2)+c_5(k-1)=a_6(k-2)+c_6(k-1)$ and $a_5>a_6$. Then $\rho(D_{n,n-2}^{a_5,b,c_5,d}(k,k-1))>\rho(D_{n,n-2}^{a_6,b,c_6,d}(k,k-1))$.*

Proof. Let u_1,u_2 be the two non-pendent vertices of $D_{n,n-2}^{a_5,b,c_5,d}(k,k-1)$ and $E(D_{n,n-2}^{a_5,b,c_5,d}(k,k-1))=E_1\cup E_2\cup E_3\cup E_4$. According to the definition of $D_{n,n-2}^{a,b,c,d}(k,k-1)$, without loss of generality, we set $u_1\in e\in E_3\cup E_4$. Clearly, $|E_1|=a_5,|E_2|=b,|E_3|=c_5,|E_4|=d$.

Let $\rho(D_{n,n-2}^{a_5,b,c_5,d}(k,k-1))=\beta^{-\frac{1}{k}}$, by Lemma 2.2, there is a weighted incidence matrix M_5 which satisfies the following conditions:

$$\begin{cases}\sum_{e'\in S_v(G)}a(e)M_5(v,e')=1, & \forall\ v\in V(G) \text{ and any } e\text{-expanded edge } e',\\ \prod_{v\in e'}M_5(v,e')=\beta, & \forall\ e'\in S(G),\\ M_5(v,e_1')=M_5(v,e_2'), & e_1' \text{ is deferent from } e_2' \text{ only their order.}\end{cases}\tag{3.8}$$

Now we may write:

$$(1)\ \sum_{e\in E_1} M_5(u_1,e)+\sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_5(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_5(u_1,e_{(k-1)}\{u_1\})]$$
$$+\sum_{e\in E_3} M_5(u_1,e)+\sum_{e\in E_4}[\frac{1}{k}\sum_{i=1}^{k-2} M_5(u_1,e_{(i)}\{u_1\})+\frac{2}{k}M_5(u_1,e_{(k-1)}\{u_1\})]=1;$$

$$(2)\ \sum_{e\in E_1} M_5(u_2,e)+\sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_5(u_2,e_{(i)}\{u_1\})+\frac{2}{k}M_5(u_2,e_{(k-1)}\{u_1\})]=1;$$

$$(3)\ \frac{1}{k}\sum_{i=1}^{k-2} M_5(v,e_{(i)}\{v\})+\frac{2}{k}M_5(v,e_{(k-1)}\{v\})=1,\ \text{for } v\in e\in E_2\cup E_4,\ d(v)=1;$$

$$(4)\ M_5(v,e)=1,\ \text{for } v\in e\in E_1\cup E_3,\ d(v)=1;$$

$$(5)\ \prod_{v\in e} M_5(v,e)=M_5(u_1,e)M_5(u_2,e)=\beta,\ \text{for } e\in E_1;$$

$$(6)\ \prod_{v\in e_{(i)}\{u_1\}} M_5(v,e_{(i)}\{u_1\})=\beta,\ \text{for } e\in E_2\cup E_4;$$

$$(7)\ M_5(u_1,e)=\beta,\ \text{for } e\in E_3.$$

Choose a_5-a_6 edges in E_1, and let E_1' be the set containing all these edges. Further let E_3' be a set having $\frac{(a_5-a_6)(k-2)}{k-1}=c_6-c_5$ edges, each edge in E_3' contains $(k-1)$ pendent vertices and $\{u_1\}$. Then $D_{n,n-2}^{a_6,b,c_6,d}(k,k-1)$ may be obtained from $D_{n,n-2}^{a_5,b,c_5,d}(k,k-1)$ by deleting the edges in E_1' and adding edges in E_3'.

Define a weighted incidence matrix M_5' for $D_{n,n-2}^{a_5,b,c_5,d}(k,k-1)$:

$$M_5'(v,e')=\begin{cases} M_5(v,e'), & \text{for } e'\notin S(E_3'),\\ M_5(u_1,e_0), & v\in e'\in S(E_3'), v=u_1,\\ 1, & v\in e'\in S(E_3'), d(v)=1.\end{cases}$$

where $e_0\in E_1$. For each edge in E_3', it has

$$(1)\ \sum_{e\in E_3'} M_5'(u_1,e)=(c_6-c_5)M_5(u_1,e_0)<\sum_{e\in E_1'} M_5(u_1,e)=(a_5-a_6)M_5(u_1,e_0);$$

$$(2)\ \sum_{e\in E_1\setminus E_1'} M_5'(u_2,e)+\sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_5'(u_2,e_{(i)}\{u_1\})+\frac{2}{k}M_5'(u_2,e_{(k-1)}\{u_1\})]<1;$$

$$(3)\ \prod_{v\in e} M_5'(v,e)=M_5'(u_1,e)=M_5(u_1,e_0)>\beta,\ \text{for } e\in E_3';$$

$$(4)\ M_5'(v,e)=1,\ \text{for } v\in e\in E_3',\ d(v)=1.$$

So M_5' is strictly β-subnormal. By Lemma 2.3, $\rho(D_{n,n-2}^{a_6,b,c_6,d}(k,k-1))<\rho(D_{n,n-2}^{a_5,b,c_5,d}(k,k-1))$. □

Lemma 3.7. *Suppose that $D_{n,n-2}^{a_7,b,c,d_7}(k,k-1)$ is a $\{k,k-1\}$-hypergraph. Then $\rho(D_{n,n-2}^{a_7,b,c,d_7}(k,k-1))>\rho(D_{n,n-2}^{a_7-1,b,c,d_7+1}(k,k-1))$.*

Proof. Let u_1,u_2 be the two non-pendent vertices of $D_{n,n-2}^{a_7,b,c,d_7}(k,k-1)$ and $E(D_{n,n-2}^{a_7,b,c,d_7}(k,k-1))=E_1\cup E_2\cup E_3\cup E_4$. According to the definition of $D_{n,n-2}^{a,b,c,d}(k,k-1)$, without loss of generality, we set $u_1\in e\in E_3\cup E_4$. Clearly, $|E_1|=a_7, |E_2|=b, |E_3|=c, |E_4|=d_7$.

Let $\rho(D^{a_7,b,c,d_7}_{n,n-2}(k,k-1)) = \rho = \beta^{-\frac{1}{k}}$ and $\mathbf{x}$ be the Perron vector of $D^{a_7,b,c,d_7}_{n,n-2}(k,k-1)$. Define a weighted incidence matrix M_6 as follows:

$$M_6(u,e') = \begin{cases} \frac{\prod_{v\in e'} x_v}{\rho x_u^k}, & \text{for } u \in e', \\ 0, & \text{otherwise.} \end{cases}$$

Then M_6 satisfies Definition 1.

Now we may write:

$$(1)\ \sum_{e\in E_1} M_6(u_1,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_6(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_6(u_1,e_{(k-1)}\{u_1\})]$$

$$+\sum_{e\in E_3} M_6(u_1,e) + \sum_{e\in E_4}[\frac{1}{k}\sum_{i=1}^{k-2} M_6(u_1,e_{(i)}\{u_1\}) + \frac{2}{k}M_6(u_1,e_{(k-1)}\{u_1\})] = 1;$$

$$(2)\ \sum_{e\in E_1} M_6(u_2,e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_6(u_2,e_{(i)}\{u_1\}) + \frac{2}{k}M_6(u_2,e_{(k-1)}\{u_1\})] = 1;$$

$$(3)\ \frac{1}{k}\sum_{i=1}^{k-2} M_6(v,e_{(i)}\{v\}) + \frac{2}{k}M_6(v,e_{(k-1)}\{v\}) = 1, \text{ for } v \in e \in E_2 \cup E_4,\ d(v) = 1;$$

$$(4)\ M_6(v,e) = 1, \text{ for } v \in e \in E_1 \cup E_3,\ d(v) = 1;$$

$$(5)\ \prod_{v\in e} M_6(v,e) = M_6(u_1,e)M_6(u_2,e) = \beta, \text{ for } e \in E_1;$$

$$(6)\ \prod_{v\in e_{(i)}\{u_1\}} M_6(v,e_{(i)}\{u_1\}) = \beta, \text{ for } e \in E_2 \cup E_4;$$

$$(7)\ M_6(u_1,e) = \beta, \text{ for } e \in E_3.$$

For $e_1 \in E_1$, let $v_0 \in e_1$ and $d(v_0) = 1$. For convenience, we set $M_6(u_1,e_1) = \frac{x_{u_2}x_{v_0}^{k-2}}{\rho x_{u_1}^{k-1}} := y_0$. Note that $M_6(v_0,e_1) = \frac{x_{u_1}x_{u_2}x_{v_0}^{k-2}}{\rho x_{v_0}^k} = \frac{x_{u_1}x_{u_2}}{\rho x_{v_0}^2} = 1$, and $x_{\max} = x_{u_1}, x_{\min} = x_{v_0}$ by Lemma 2.1, then

$$y_0 = \frac{x_{u_2}x_{v_0}^{k-2}}{\rho x_{u_1}^{k-1}} = (\frac{x_{v_0}}{x_{u_1}})^k = (\frac{1}{\gamma})^k.$$

Let $e_1' = e_1 \backslash u_2$ and $D^{a_7-1,b,c,d_7+1}_{n,n-2}(k,k-1) = D^{a_7,b,c,d_7}_{n,n-2}(k,k-1) - e_1 + e_1'$. Construct a weighted incidence matrix M_6' of $D^{a_7-1,b,c,d_7+1}_{n,n-2}(k,k-1)$ as following:

$$M_6'(v,e') = \begin{cases} M_6(v,e'), & \text{for } e' \notin S(e_1'), \\ \beta, & e' = e'_{1,(j)}\{u_1\}, v = u_1, \text{for } i = 1,\dots,k-2, \\ \beta^{\frac{1}{2}}, & e' = e'_{1,(k-1)}\{u_1\}, v = u_1, \\ 1, & v \in e' \in S(e_1'), d(v) = 1. \end{cases}$$

Similar to Theorem 3.1, we have

$$\begin{aligned} \frac{k-2}{k}\beta + \frac{2}{k}\beta^{\frac{1}{2}} &= \frac{k-2}{k}\rho^{-k} + \frac{2}{k}\rho^{-\frac{k}{2}} \\ &\leq \frac{k-2}{k}(k\gamma^{k-1})^{-k} + \frac{2}{k}(k\gamma^{k-1})^{-\frac{k}{2}} \\ &= (k-2)k^{-k-1}\gamma^{-k(k-1)} + 2k^{-\frac{k}{2}-1}\gamma^{-\frac{k(k-1)}{2}} \end{aligned}$$

$$\leq \frac{k-2}{k^{k+1}}\gamma^{-k} + \frac{2}{k^{\frac{k}{2}+1}}\gamma^{-k}$$
$$\leq \gamma^{-k} = y_0.$$

Now for e_1', it has

(1) $\frac{1}{k}\sum_{i=1}^{k-2} M_6'(u_1, e'_{1,(i)}\{u_1\}) + \frac{2}{k}M_6'(u_1, e'_{1,(k-1)}\{u_1\}) = \frac{k-2}{k}\beta + \frac{2}{k}\beta^{\frac{1}{2}} \leq y_0 = M_6(u_1, e_1);$

(2) $\sum_{e\in E_1\setminus e_1'} M_6'(u_2, e) + \sum_{e\in E_2}[\frac{1}{k}\sum_{i=1}^{k-2} M_6'(u_2, e_{(i)}\{u_1\}) + \frac{2}{k}M_6'(u_2, e_{(k-1)}\{u_1\})] < 1;$

(3) $\prod_{v\in e'_{1,(j)}\{u_1\}} M_6'(v, e'_{1,(j)}\{u_1\}) = M_6'(u_1, e'_{1,(i)}\{u_1\}) = \beta,$ for $1 \leq j \leq k-2;$

(4) $\prod_{v\in e'_{1,(k-1)}\{u_1\}} M_6'(v, e'_{1,(k-1)}\{u_1\}) = (M_6'(u_1, e'_{1,(k-1)}\{u_1\}))^2 = \beta.$

So M_6' is strictly β-subnormal. By Lemma 2.3, $\rho(D_{n,n-2}^{a_7-1,b,c,d_7+1}(k,k-1)) < \rho(D_{n,n-2}^{a_7,b,c,d_7}(k,k-1))$. □

Theorem 3.8. *Among all $\{k,k-1\}$-graphs in $\mathcal{G}_{n,n-2}$. The hypergraph $D_{n,n-2}^{a_0,b_0,c_0,0}(k,k-1)$ has uniquely the maximum spectral radius, where $c_0 = \min\{c \mid a(k-2)+b(k-3)+c(k-1) = n-2,\ a,b \geq 1\}$ and b_0 is the maximum solution of congruence $\Big((n-c_0(k-1)) - b_0(k-3) - 2\Big) \equiv 0 \pmod{k-2}$.*

Proof. Let H be the $\{k,k-1\}$-hypergraph with maximum spectral radius in $\mathcal{G}_{n,n-2}$. By Lemma 3.6, the more k^1-edge, the bigger the spectral radius of H; By Lemma 3.5, the more $(k-1)^1$-edge, the bigger the spectral radius of H. Thus H has as many k^1-edge and $(k-1)^1$-edge as possible.

By Lemmas 3.4 and 3.6, we get H has as few k^2-edge as possible. By Lemmas 3.3 and 3.7, we have $H \cong D_{n,n-2}^{a_0,b_0,c_0,0}(k,k-1)$, where $c_0 = \min\{c \mid a(k-2)+b(k-3)+c(k-1) = n-2,\ a,b \geq 1\}$ and b_0 is the maximum solution of congruence $\Big((n-c_0(k-1)) - b_0(k-3) - 2\Big) \equiv 0 \pmod{k-2}$. □

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