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On the families of numbers with respect to Orlicz functions

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Abstract. Summable families of numbers were defined by E. H. Moore ([11, 12]), who also showed that an infinite series of real or complex numbers converges unconditionally if and only if it is summable. In this paper, we introduce an extension of power series methods in the sense of summable families. As applications, we construct the spaces of families of numbers with respect to Orlicz functions and study some expansions of P-strongly convergent and P-statistically convergent series with respect to Orlicz functions. Our results are natural extensions of the sequence spaces defined by Orlicz, which are introduced in [13] and [14, 15].

1. Introduction

A family of numbers $(x_i)_{i\in I}$ is a set of real or complex numbers, x_i, which correspond in a unique way to the elements i of an index set I. The great advantage of the previous concept is that it can also be applied to uncountable families. In this paper, we study some natural extensions of sequence spaces in the context of families of numbers. Our results are followed by the same vein of the works [1, 2, 7, 8, 13, 15] and the references given therein. More precisely, in Section 3, we introduce an extension of power series methods in the context of summable families. As applications, we have obtained extensions of the P-strongly convergent and P-statistically convergent concepts. They are studied in more detail in Section 6. We note that power series methods play an important role in the theory of summable sequences (see [4–6]). In Section 4, in view of the results in [13], we study some properties of number families with respect to Orlicz functions and examine their linear structure and paranorms. In Section 5, we construct the p-Banach structure for spaces of number families. In particular, we obtain some interesting conclusions about degenerate Orlicz functions. In Section 6, as previously mentioned, we improve the result published by Şahin B. N. in [15] by using power series methods in the context of summable families.

2. Preliminaries

Let I be a nonempty set. A family of numbers $(x_i)_{i\in I}$ (in short (x_i)) is a set of real or complex numbers x_i that correspond in a unique way to the elements i of an index set I. Clearly, if I is countable, then $(x_i)_{i\in I}$

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is a sequence. If we denote the collection of all finite subsets I of by $\mathcal{F}(I)$, then for each family of numbers (x_i), the finite partial sums

$$\sigma_J = \sum_{i\in J} x_i \text{ with } J \in \mathcal{F}(I). \tag{1}$$

form a directed system, where set-theoretic inclusion $J_1 \leq J_2$ is used as the $\geq$ relation. The system $(\sigma_J)_{J\in\mathcal{F}(I)}$ is called convergent to σ if for every $\varepsilon > 0$, there exists $J_0 \in \mathcal{F}(I)$ such that

$$\Big|\sum_{i\in J} x_i - \sigma\Big| < \varepsilon \tag{2}$$

for every $J \in \mathcal{F}(I)$ with $J_0 \leq J$. By these means, we denote

$$\lim \sigma_J = \sigma.$$

In this paper, we denote that $|J|$ is the number of elements of $J \in \mathcal{F}(I)$..

Definition 2.1. *([12]) Let $\{x_i\}_{i\in I}$ be a family of numbers. The family $\{x_i\}_{i\in I}$ is said to be summable if the system $\{\sigma_J\}_{J\in\mathcal{F}(I)}$ converges to σ, and we write*

$$\sigma = \sum_{i\in I} x_i.$$

Definition 2.2. *([12]) Let $\{x_i\}_{i\in I}$ be a family of numbers.*
1) The family $\{x_i\}_{i\in I}$ is said to be convergent to 0 if for every $\varepsilon > 0$, there exists $J_0 \in \mathcal{F}(I)$ such that

$$|x_i| < \varepsilon \tag{3}$$

for every $i \in I \setminus J_0$.
2) The family $\{x_i\}_{i\in I}$ is said to be convergent to $L \in \mathbb{K}$ if for every $\varepsilon > 0$, there exists $J_0 \in \mathcal{F}(I)$ such that

$$|x_i - L| < \varepsilon \tag{4}$$

for every $i \in I \setminus J_0$.

Definition 2.3. *([12]) Let $\{x_i\}_{i\in I}$ be a family of numbers. The family $\{x_i\}_{i\in I}$ is said to be bounded if there exists $M > 0$ such that*

$$|x_i| < M \tag{5}$$

for every $i \in I$.

Definition 2.4. *1) The family $\left(\sigma_J\right)_{J\in\mathcal{F}(I)} \subset \mathbb{R}$ is said to be increasing if*

$$\sigma_{J_1} \leq \sigma_{J_2}$$

for every $J_1, J_2 \in \mathcal{F}(I)$ and $J_1 \leq J_2$.
2) The family $\left(\sigma_J\right)_{J\in\mathcal{F}(I)} \subset \mathbb{R}$ is said to be decreasing if

$$\sigma_{J_1} \geq \sigma_{J_2}$$

for every $J_1, J_2 \in \mathcal{F}(I)$ and $J_1 \leq J_2$.

The following fact may be non-original:

Lemma 2.5. *1) If the family* $\left(\sigma_J\right)_{J\in\mathcal{F}(I)} \subset R$ *is increasing and* $C = \sup_{J\in\mathcal{F}(I)} \sigma_J < +\infty$, *then*

$$\lim \sigma_J = C.$$

2) If the family $\left(\sigma_J\right)_{J\in\mathcal{F}(I)} \subset R$ *is decreasing and* $C = \inf_{J\in\mathcal{F}(I)} \sigma_J > -\infty$, *then*

$$\lim \sigma_J = C.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since $C = \sup_{J\in\mathcal{F}(I)} \sigma_J < +\infty$, we can find $J_0 \in \mathcal{F}(I)$ such that

$$C - \varepsilon < \sigma_{J_0} \leq C.$$

It follows from the increasing property of $\left(\sigma_J\right)$ that

$$C - \varepsilon < \sigma_{J_0} \leq \sigma_J \leq C.$$

for every $J \geq J_0$. This yields that

$$\lim \sigma_J = C.$$

This proves 1). By the same argument, we obtain 2). □

Lemma 2.6. *Suppose that* $x = (x_i)_{i\in I}$ *is a summable family of numbers. Then,* $x = (x_i)$ *converges to* 0.

Proof. Since $x = (x_i)_{i\in I}$ is summable, we have that $S_J = \sum_{i\in J\in\mathcal{F}(I)} x_i$ converges to $S \in \mathbb{K}$. Let $\varepsilon > 0$; then, there exists $J_0 \in \mathcal{F}(I)$ such that

$$|S_J - S| < \frac{\varepsilon}{2} \tag{6}$$

for all $J \geq J_0$. For each $i \in I \setminus J_0$, we set $J_1 = J_0 \cup \{i\}$, which implies that $J_1 \in \mathcal{F}(I)$ and $J_1 \geq J_0$. It follows from (6) that

$$|S_{J_1} - S| < \frac{\varepsilon}{2}. \tag{7}$$

Combining (6) and (7), we arrive at

$$\begin{aligned} |x_i| &= |S_{J_1} - S_J| = |S_{J_1} - S + S - S_J| \\ &\leq |S_{J_1} - S| + |S - S_J| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $i \in I \setminus J_0$. This proves that $x = (x_i)$ converges to 0. □

Throughout the paper, we let

$$l_\infty(I) = \left\{x = (x_i)_{i\in I} \subset \mathbb{K} : (x_i) \text{ is bounded } \right\};$$

$$C_0(I) = \left\{x = (x_i)_{i\in I} \subset \mathbb{K} : (x_i) \text{ converges to } 0\right\};$$

$$C(I) = \left\{x = (x_i)_{i\in I} \subset \mathbb{K} : (x_i) \text{ is convergent}\right\},$$

and

$$l_p(I) = \left\{x = (x_i)_{i\in I} \subset \mathbb{K} : \sum_{i\in I} |x_i|^p < +\infty\right\}$$

with $p > 0$. We have the following inclusions:

$$l_p(I) \subset C_0(I) \subset C(I) \subset l_\infty(I).$$

$l_\infty(I)$ is a Banach space with respect to the operations

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I} \text{ and } \lambda(x_i)_{i\in I} = (\lambda x_i)_{i\in I}$$

with the norm

$$\|x\| = \sup_{i\in I} |x_i|$$

for all $x \in l_\infty(I)$. Moreover, it is not difficult to check that $C(I), C_0(I)$ are closed subspaces of $l_\infty(I)$. If $p \geq 1$, then $l_p(I)$ is a Banach space with respect to the norm

$$\|x\| = \Big(\sum_{i\in I} |x_i|^p\Big)^{\frac{1}{p}}$$

for all $x = (x_i) \in l_p(I)$. In particular, if $p = 2$, then $l_2(I)$ is a Hilbert space with respect to the scalar product

$$(x|y) = \sum_{i\in I} x_i \overline{y}_i$$

for all $x = (x_i), y = (y_i) \in l_2(I)$.

Definition 2.7. ([7]) *An Orlicz function M is a continuous nondecreasing and convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim\limits_{t\to+\infty} M(t) = +\infty$. If $M(t) = 0$ for some $t > 0$, M is said to be a degenerate Orlicz function.*

Definition 2.8. ([7]) *An Orlicz function M is said to satisfy the Δ_2-condition at zero if $\limsup\limits_{t\to 0} \dfrac{M(2t)}{M(t)} < +\infty$.*

Remark 2.9. 1) It is easily checked that the Δ_2-condition at 0 implies that, for every positive number $q > 0$, $\limsup\limits_{t\to 0} \dfrac{M(qt)}{M(t)} < +\infty$ (this condition is sometimes called the Δ_q-condition).

2) It is easy to check that an Orlicz function M satisfies the Δ_2-condition at zero if there exists a constant $K > 0$ such that

$$M(2t) \leq KM(t)$$

for all $t \geq 0$. This condition is equivalent to the condition

$$M(qt) \leq KM(t)$$

for all $t \geq 0$ and $q \geq 1$.

Definition 2.10. ([3]) *Let $0 < p \leq 1$. A $p-$norm on a vector space E over $\mathbb{K}$ is a mapping $\|.\|$ from E to $[0, +\infty)$ satisfying*

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|^p \|x\|$, for every $\lambda \in \mathbb{K}$ and $x \in E$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in E$.

Then, $(E, \|.\|)$ is said to be a p-normed space.

It is easy to see that the 1-normed space is a normed space. A p-normed space E is called to be a p-Banach space if it is complete according to the metric

$$d(x, y) = \|x - y\|$$

for all $x, y \in E$.

Example 2.11. *It is easy to check that $l_\infty(I)$ is a p-Banach space with $p-$norm*

$$\|x\| = \sup_{i\in I} |x_i|^p$$

for all $x = (x_i) \in l_\infty(I)$, where $0 < p \leq 1$. Moreover, $C(I)$ and $C_0(I)$ are closed subspaces of $l_\infty(I)$.

The following definition is due from [10].

Definition 2.12. *Two p-norms $\|.\|_1$ and $\|.\|_2$ on a $\mathbb{K}$- vector space E are said to be equivalent if there are $C_1, C_2 > 0$ such that*

$$C_1\|x\|_2 \leq \|x\|_1 \leq C_1\|x\|_2$$

for every $x \in E$.

It is easy to see that this is equivalent to requiring that the identity map $\mathrm{id}_E : (E, \|.\|_1) \to (E, \|.\|_2)$ is an isomorphism.

Definition 2.13. *([9]) Let X be a linear space over field $\mathbb{K}$ and g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) is called a paranormed space and g is a paranorm for X, if the following axioms are satisfied for all elements $x, y \in X$*

a) $g(\theta) = 0$ if $x = \theta$ where θ is the zero element of X;

b) $g(x) \geq 0$;

c) $g(-x) = g(x)$;

d) $g(x + y) \leq g(x) + g(y)$ (triangle inequality);

e) If (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and (x_n) is a sequence in X with $g(x_n - x) \to 0$ as $n \to \infty$ then $g(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$ (continuity of multiplication by scalars).

A paranorm g is said to be total, if $g(x) = 0$ implies $x = \theta$.

3. An extension of power series methods

Let $\omega(I)$ be the set of families of complex numbers. Let $p = (p_i)_{i\in I}$ be any family of nonnegative real numbers and the map $\varphi : I \to [0, \infty)$. We define the corresponding power sum of (p_i) and φ

$$p_\varphi(t) = \sum_{i\in I} p_i t^{\varphi(i)}$$

for $t \in \mathbb{R}$. We say that $p_\varphi(t)$ is convergent at t_0 if the family $\left(p_i t_0^{\varphi(i)}\right)_{i\in I}$ is summable, that is,

$$\sum_{i\in I} p_i t_0^{\varphi(i)} < +\infty.$$

If $p_\varphi(t)$ is not convergent at t_0, then it is said to be divergent at t_0.

It is easy to see that $p_\varphi(t)$ becomes a normal power series when $I = \mathbb{N}$ and $\varphi(n) = n$ for all $n \in \mathbb{N}$. We have the following facts about $p_\varphi(t)$:

Proposition 3.1. *1) If $p_\varphi(t)$ is convergent at $t_0 > 0$, then it is convergent at $0 \leq t < t_0$.*

2) If $p_\varphi(t)$ is divergent at $t_0 > 0$, then it is divergent at $t > t_0$.

Proof. 1) Since $p_\varphi(t)$ is convergent at $t_0 > 0$, we have

$$0 \leq \sum_{i\in I} p_i t_0^{\varphi(i)} = K < +\infty.$$

Moreover,

$$S_J := \sum_{i\in J} p_i t^{\varphi(i)} \le \sum_{i\in J} p_i t_0^{\varphi(i)} \le K < +\infty$$

for every $0 \le t \le t_0$ and $J \in \mathcal{F}(I)$. Since p_i is nonnegative and $t \ge 0$, we can conclude that $(S_J)_{J\in\mathcal{F}(I)}$ is increasing. In view of Lemma 2.5, we infer that (S_J) is convergent and so

$$\sum_{i\in I} p_i t^{\varphi(i)} < +\infty.$$

2) Since $p_\varphi(t)$ is divergent at $t_0 > 0$, we have

$$\sum_{i\in I} p_i t_0^{\varphi(i)} = +\infty.$$

Hence, for each $n \in \mathbb{N}$, there exist $J_n \in \mathcal{F}(I)$ such that

$$S_{J_n} = \sum_{i\in J_n} p_i t^{\varphi(i)} \ge \sum_{i\in J_n} p_i t_0^{\varphi(i)} \ge n$$

for every $t \ge t_0$. This implies that $\sup_{J\in\mathcal{F}(I)} S_J = +\infty$. We obtain

$$\sum_{i\in I} p_i t^{\varphi(i)} = +\infty.$$

The proof is complete. □

Now, we set

$$R = \sup\{t_0 \in [0,+\infty) : p_\varphi(t) \text{ is convergent at } t_0\}. \tag{8}$$

By the previous proposition, we can conclude that R exists and that $p_\varphi(t)$ is convergent on $[0,R)$ and divergent on $(R,+\infty)$. We said that R is the radius of convergence of $p_\varphi(t)$. It is easy to see that R is the radius of convergence of the power series when $I = \mathbb{N}$ and $\varphi(n) = n$ for all $n \in \mathbb{N}$.

Next, we assume that p_φ has the radius of convergence $R > 0$. Let

$$C_{p_\varphi} := \Big\{ f : (-R,R) \to \mathbb{R} : \lim_{t\to R^-} \frac{f(t)}{p_\varphi(t)} \text{ exists} \Big\}. \tag{9}$$

and

$$C_{P_{p_\varphi}} := \Big\{ x = (x_i) : p_\varphi^x := \sum_{i\in I} p_i t^{\varphi(i)} x_i \text{ has the radius of convergence } \ge R \text{ and } p_\varphi^x \in C_{p_\varphi} \Big\}. \tag{10}$$

The functional $P_\varphi - \lim : C_{P_{p_\varphi}} \to \mathbb{R}$ defined by

$$(P_\varphi - \lim)(x) = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{i\in I} p_i t^{\varphi(i)} x_i \tag{11}$$

is called a *power-summable family method*, and the family $x = (x_i)$ is said to be $P_\varphi-$convergent.

Now, by means of the summable family method and following the ideas of Unver and Orhan ([16]), we introduce the concepts of strong summability and statistical convergence with respect to the power-summable family method. We set the following notations:

$$W_0(P_\varphi, I) = \Big\{ x \in \omega(I) : \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{i\in I} p_i t^{\varphi(i)} |x_i| = 0 \Big\} \tag{12}$$

and

$$W(P_\varphi, I) = \{x \in \omega(I) : x - Le \in W_0(P_\varphi), \text{ for some } L\}, \tag{13}$$

where $e = (e_i), e_i = 1$ for every $i \in I$. When $x \in W(P_\varphi, I)$, we say that x is P_φ-strongly convergent to L.

Definition 3.2. *A power-summable family method is called regular if*

$$(P_\varphi - \lim)x = \lim x$$

for every $x \in C(I)$.

The following proposition is derived from Boos's result.

Proposition 3.3. *A power-summable family method is called regular if only if*

$$\lim_{0<t\to R^-} \frac{p_i t^{\varphi(i)}}{p(t)} = 0 \tag{14}$$

for all $i \in I$.

Proof. Suppose that the summable family method is called regular. Clearly, if $p_i = 0$, then $\lim\limits_{0<t\to R^-} \dfrac{p_i t^{\varphi(i)}}{p(t)} = 0$. Hence, we can reduce to $p_i > 0$ for all $i \in I$. Since the summable family method is called regular, we infer that the family of numbers $\left(\frac{1}{p(t)} \sum_{i\in I} p_i t^{\varphi(i)} x_i\right)_{i\in I}$ is summable for each $t < R$. It follows from Proposition 1.1.5 in [12] that $\left(\frac{1}{p(t)} \sum_{i\in I} p_i t^{\varphi(i)} x_i\right)_{i\in I}$ contains at most countably many nonvanishing terms. This yields that the family $(x_i)_{i\in I}$ contains at most countably many nonvanishing terms. Hence, applying Theorem 3.6.6 in [4], we can conclude that

$$\lim_{0<t\to R^-} \frac{p_i t^{\varphi(i)}}{p(t)} = 0$$

for all $i \in I$. □

Definition 3.4. *The family* $x = (x_i)$ *is said to be* P_φ*-statistical convergent to* L *if* $\chi_{K(x-Le,\varepsilon)}$ *is contained in* $W_0(P_\varphi)$ *for every* $\varepsilon > 0$, *where* $\chi_{K(x,\varepsilon)}$ *is the characteristic function of the set*

$$K(x,\varepsilon) = \{i \in I : |x_i| \geq \varepsilon\}.$$

By $\mathrm{st}(P_\varphi, I)$, we denote the space of all P_φ-statistically convergent families.

Next, we introduce the concept of an P_φ-uniformly integrable family, which is a natural extension of the concept of the P-uniformly integrable sequence introduced by Unver and Orhan [16].

Definition 3.5. *Let* P_φ *be a power-summable family method and* $x = (x_i)$ *be a family of numbers. Then,* x *is called* P_φ*-uniformly integrable if there exists* $0 \leq t_0 < R$ *such that*

$$\lim_{c\to\infty} \sup_{t\in[t_0,R]} \frac{1}{p_\varphi(t)} \sum_{i\in I, |x_i|\geq c} p_i t^{\varphi(i)} |x_i| = 0. \tag{15}$$

It is easy to see that every bounded family is P_φ-uniformly integrable. In [16], the authors proved that a sequence x is $P-$strongly convergent if and only if it is $P-$ statistically convergent and P-uniformly integrable. In the same vein, we have obtained the following fact:

Theorem 3.6. *Let* P_φ *be a power-summable family method. A family* $x = (x_i)$ *is* $P_\varphi-$*strongly convergent if and only if it is* $P_\varphi-$ *statistically convergent and* P_φ*-uniformly integrable.*

4. Families of numbers with respect to Orlicz functions

Let $\omega(I)$ be the set of all families of complex numbers. Let $p = (p_i)_{i\in I}$ be any family of positive real numbers and M be an Orlicz function. According to the idea from [13], we define the following spaces:

$$l_M(p,I) = \Big\{x \in \omega(I) : \sum_{i\in I} \Big(M\frac{|x_i|}{\rho}\Big)^{p_i} < \infty, \text{ for some } \rho > 0\Big\}; \tag{16}$$

$$h_M(p,I) = \Big\{x \in \omega(I) : \sum_{i\in I} \Big(M\frac{|x_i|}{\rho}\Big)^{p_i} < \infty, \text{ for every } \rho > 0\Big\}; \tag{17}$$

$$W_0(p,M,I) = \Big\{x \in \omega(I) : y_J := \frac{1}{|J|}\sum_{i\in J} \Big(M\frac{|x_i|}{\rho}\Big)^{p_i} \to 0, \text{ for some } \rho > 0,\ J \in \mathcal{F}(I)\Big\}; \tag{18}$$

$$W(p,M,I) = \Big\{x \in \omega(I) : y_J := \frac{1}{|J|}\sum_{i\in J} \Big(M\frac{|x_i - L|}{\rho}\Big)^{p_i} \to 0, \text{ for some } \rho > 0 \text{ and } L \in \mathbb{C},\ J \in \mathcal{F}(I)\Big\}; \tag{19}$$

$$W_\infty(p,M,I) = \Big\{x \in \omega(I) : \sup_{J\in\mathcal{F}(I)} \frac{1}{|J|}\sum_{i\in J} \Big(M\frac{|x_i|}{\rho}\Big)^{p_i} < \infty \text{ for some } \rho > 0\Big\}. \tag{20}$$

When $p_i = 1$ for all $i \in I$, then $l_M(p,I)$ becomes $l_M(I)$. When $M(x) = x$, then the above sets of number families are denoted by $l(p,I), h(p,I), [C,1,p,I], [C,1,p,I]_0$ and $[C,1,p,I]_\infty$, respectively. We denote $W(p,M,I), W_0(p,M,I)$ and $W_\infty(p,M,I)$ as $W(M,I), W_0(M,I)$ and $W_\infty(M,I)$ when $p_i = 1$ for all i.

Next, we always assume that (p_i) is bounded and set $H := \sup_{i\in I} p_i$.

Theorem 4.1. *$l_M(p,I)$ is a linear subspace of $l_\infty(I)$.*

Proof. We first show that $l_M(p,I) \subset l_\infty(I)$. Suppose that $l_M(p,I) \subsetneq l_\infty(I)$. Then, we can seek $x = (x_i) \in l_M(p,I)$ such that $x = (x_i)$ is unbounded. Hence, for each $n = 1, 2, ...$ there exists $x_{i_n} \in \{x_i : i \in I\}$ such that $|x_{i_n}| > n$. Since $x \in l_M(p,I)$, we can find $\rho > 0$ such that

$$\sum_{i\in I} \bigg(M\Big(\frac{|x_i|}{\rho}\Big)\bigg)^{p_i} < \infty.$$

By Lemma 2.6, we can deduce that the family

$$\bigg[\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^{p_i}\bigg]_{i\in I} \tag{21}$$

converges to 0.

Since $M(t)$ is nondecreasing and $\lim_{t\to\infty} M(t) = \infty$, we can seek $t_0 \in [0,\infty)$ such that $M(t_0) \geq 1$. It follows from $\lim\limits_{n\to\infty} |x_{i_n}| = \infty$ that there is n_0 satisfying $\dfrac{|x_{i_{n_0}}|}{\rho} > t_0$. Hence,

$$M\Big(\frac{|x_{i_{n_0}}|}{\rho}\Big)^{p_{i_{n_0}}} > \Big(M(t_0)\Big)^{p_{i_{n_0}}} \geq 1.$$

This implies that

$$\lim_{n\to\infty} M\Big(\frac{|x_{i_n}|}{\rho}\Big)^{i_n} \neq 0.$$

We have obtained a contradiction to (21). Therefore, $l_M(p, I) \subset l_\infty(I)$.

Now, we claim that $l_M(p, I)$ is a linear subspace of $l_\infty(I)$. Let $x, y \in l_M(p, I)$ and $\alpha, \beta \in \mathbb{K}$. Therefore, there exists ρ_1, ρ_2 such that

$$\sum_{i\in I} \Big(M\Big(\frac{|x_i|}{\rho_1}\Big)\Big)^{p_i} = L_1 < +\infty$$

and

$$\sum_{i\in I} \Big(M\Big(\frac{|y_i|}{\rho_2}\Big)\Big)^{p_i} = L_2 < +\infty.$$

Hence,

$$0 \leq \sum_{i\in J} \Big(M\Big(\frac{|x_i|}{\rho_1}\Big)\Big)^{p_i} \leq L_1$$

and

$$0 < \sum_{i\in J} \Big(M\Big(\frac{|y_i|}{\rho_2}\Big)\Big)^{p_i} \leq L_2$$

for every $J \in F(I)$.

If $\alpha \neq 0$ and $\beta \neq 0$, then we set $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is nondecreasing and convex,

$$\begin{aligned}\sum_{i\in J} \Big(M\Big(\frac{|\alpha x_i + \beta y_i|}{\rho_3}\Big)\Big)^{p_i} &\leq \sum_{i\in J} \Big(M\Big(\frac{|\alpha x_i| + |\beta y_i|}{\rho_3}\Big)\Big)^{p_i} \\ &\leq \sum_{i\in J} \Big(M\Big(\frac{|\alpha x_i|}{\rho_3} + \frac{|\beta y_i|}{\rho_3}\Big)\Big)^{p_i} \leq \sum_{i\in J} \Big(M\Big(\frac{|x_i|}{2\rho_1} + \frac{|\beta y_i|}{2\rho_2}\Big)\Big)^{p_i} \\ &= \sum_{i\in J} \Big(\frac{1}{2} M\Big(\frac{|x_i|}{\rho_1}\Big) + M\Big(\frac{|\beta y_i|}{\rho_2}\Big)\Big)^{p_i} = \sum_{i\in J} \frac{1}{2^{p_i}}\Big(M\Big(\frac{|x_i|}{\rho_1}\Big) + M\Big(\frac{|\beta y_i|}{\rho_2}\Big)\Big)^{p_i} \\ &\leq C \sum_{i\in J} \Big(M\Big(\frac{|x_i|}{\rho_1}\Big)\Big)^{p_i} + C \sum_{i\in J} \Big(M\Big(\frac{|y_i|}{\rho_2}\Big)\Big)^{p_i} \leq C(L_1 + L_2) < +\infty\end{aligned}$$

for every $J \in \mathcal{F}(I)$, where $C = \max\{1, 2^{H-1}\}$. Applying Lemma 2.5, we can deduce that

$$\sum_{i\in I} \Big(M\Big(\frac{|\alpha x_i + \beta y_i|}{\rho_3}\Big)\Big)^{p_i} < +\infty.$$

This proves that $\alpha x + \beta y \in l_M(p, I)$.

By the same argument, we can claim that $\alpha x, \beta y \in\in l_M(p, I)$ for all $\alpha, \beta \in \mathbb{K}$. Hence, $l_M(p, I)$ is a linear subspace of $l_\infty(I)$.

□

Theorem 4.2. *$l_M(p, I)$ is the total paranormed space with*

$$g(x) = \inf\left\{\rho^{\frac{p_i}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\rho}\Big)^{p_i}\Big)^{\frac{1}{H}} \leq 1, i \in I\right\}, \tag{22}$$

where $H = \max\{1, \sup_{i\in I} p_i\}$.

Proof. The rest of the proof closely follows the lines from [13] with minor differences based on the technical details related to summable families. We only repeat a different statement for the reader's convenience. It is easy to see that $g(x) = g(-x)$. By using Theorem 4.1 for $\alpha = \beta = 1$, we infer that $g(x + y) \le g(x) + g(y)$ for every $x, y \in l_M(p, I)$. Moreover, $g(x) = 0$ if and only if $x = 0$.

Next, we need to claim that scalar multiplication is continuous. Since

$$g(\lambda x) = \inf\left\{\rho^{\frac{p_i}{H}} > 0 : \left(\sum_{i\in I} M\Big(\frac{|\lambda x_i|}{\rho}\Big)^{p_i}\right)^{\frac{1}{H}} \le 1,\ i \in I\right\},$$

we have that

$$g(\lambda x) = \inf\left\{\Big(\lambda r\Big)^{\frac{p_i}{H}} > 0 : \left(\sum_{i\in I} M\Big(\frac{|x_i|}{r}\Big)^{p_i}\right)^{\frac{1}{H}} \le 1, i \in I\right\},$$

where $r = \dfrac{p}{\lambda}$. It follows from $|\lambda|^{p_i} \le \max\{1, |\lambda|^H\}$ that

$$|\lambda|^{\frac{p_i}{H}} \le \Big(\max\{1, |\lambda|^H\}\Big)^{\frac{1}{H}}.$$

Hence,

$$\begin{aligned} 0 \le g(\lambda x) &\le \Big(\max\{1, |\lambda|^H\}\Big)^{\frac{1}{H}} \inf\left\{\Big(\lambda r\Big)^{\frac{p_i}{H}} > 0 : \left(\sum_{i\in I} M\Big(\frac{|x_i|}{r}\Big)^{p_i}\right)^{\frac{1}{H}} \le 1,\ i \in I\right\} \\ &= \Big(\max\{1, |\lambda|^H\}\Big)^{\frac{1}{H}} g(x). \end{aligned}$$

This implies that $g(\lambda x)$ converges to zero as $g(x)$ converges to zero. Now assume that $(\lambda_k)_{k\in\mathbb{N}}$ converges to 0 and $x \in l_M(p, I)$. For each sufficiently small $\varepsilon > 0$, we can seek $J_0 \in \mathcal{F}(I)$ such that

$$\sum_{i\in I\setminus J_0} \Big[M\Big(\frac{|x_i|}{\rho}\Big)\Big]^{p_i} < \frac{\varepsilon}{2}$$

for some $\rho > 0$. Hence,

$$\left(\sum_{i\in I\setminus J_0} \Big[M\Big(\frac{|x_i|}{\rho}\Big)\Big]^{p_i}\right)^{\frac{1}{H}} \le \frac{\varepsilon}{2}.$$

Moreover, since M is continuous on $[0, \infty)$, we infer that

$$f(t) = \sum_{i\in J_0} M\Big(\frac{|tx_i|}{\rho}\Big)$$

is right continuous at 0. Hence, there exists $0 < \delta < 1$ such that

$$|f(t)| < \frac{\varepsilon}{2}$$

for $0 < t < \delta$. Since $\lim_{k\to\infty} \lambda_k = 0$, we can find $k_0 \in \mathbb{N}$ such that

$$|\lambda_k| < \delta$$

for every $k \ge k_0$.

$$f(\lambda_k) = \sum_{i\in J_0} M\Big(\frac{|\lambda_k x_i|}{\rho}\Big) < \frac{\varepsilon}{2} \tag{23}$$

for every $k \geq k_0$. On the other hand, using the convexity of M, we can deduce that

$$\begin{aligned}\sum_{i\in I\setminus J_0}\Big[M\Big(\frac{|\lambda_k x_i|}{\rho}\Big)\Big]^{p_i} &\leq \sum_{i\in I\setminus J_0}\Big[M\Big(\frac{|\delta x_i|}{\rho}\Big)\Big]^{p_i} \\ &\leq \delta\Big[M\Big(\frac{|x_i|}{\rho}\Big)\Big]^{p_i} < \Big(\frac{\varepsilon}{2}\Big)^{H}.\end{aligned} \tag{24}$$

for every $k \geq k_0$. Hence,

$$\Big(\sum_{i\in I\setminus J_0}\Big[M\Big(\frac{|\lambda_k x_i|}{\rho}\Big)\Big]^{p_i}\Big)^{\frac{1}{H}} < \frac{\varepsilon}{2}. \tag{25}$$

for every $k \geq k_0$. Combining (23) and (25), we can deduce that

$$(g(\lambda_k x) = \Big(\sum_{i\in I}\Big[M\Big(\frac{|\lambda_k x_i|}{\rho}\Big)\Big]^{p_i}\Big)^{\frac{1}{H}} < \varepsilon$$

for $k \geq k_0$. This implies that $g(\lambda_k x)$ converges to 0 as $k \to \infty$. The theorem is proved. □

The following result is similar to that in sequence spaces (see [13]). We omit their proofs.

Theorem 4.3. *Let $p = (p_i)_{i\in I}$ be a bounded family of positive numbers. Then, $W_0(p, M, I)$, $W(p, M, I)$ and $W_\infty(p, M, I)$ are linear spaces.*

Theorem 4.4. *Let $p = (p_i)_{i\in I}$ be a bounded family of positive numbers. Then, $W_0(p, M, I)$ is a linear topological space paranormed by g' and defined as*

$$g'(x) = \inf\Big\{\rho^{\frac{p_i}{H}}\Big[\frac{1}{|J|}\sum_{i\in J}\Big(M\frac{|x_i|}{\rho}\Big)^{p_i}\Big]^{1/H} \leq 1,\ J \in \mathcal{F}(I)\Big\}, \tag{26}$$

where $H = \sup_{i\in I} p_i$.

Theorem 4.5. *Let M be an Orlicz function that satisfies the Δ_2-condition. Then, $W(I) \subset W(M, I)$, $W_0(I) \subset W_0(M, I)$ and $W_\infty(I) \subset W_\infty(M, I)$.*

Proof. Let $x \in W(I)$; then, the family $(S_J)_{J\in\mathcal{F}(I)}$ converges to 0, where

$$S_J = \frac{1}{|J|}\sum_{i\in J}|x_i - l|$$

for some $l \in \mathbb{K}$. Let $\varepsilon > 0$; then, we can seek $J_0 \in \mathcal{F}(I)$ such that

$$0 < S_J < \varepsilon \tag{27}$$

for every $J \geq J_0$. Now, choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for every $0 \leq t \leq \delta$. Write $y_i = |x_i - l|$ and consider

$$\frac{1}{|J|}\sum_{i\in J}M(y_i) = \frac{1}{|J|}\Big(\sum_{y_i\leq\delta}M(y_i) + \sum_{y_i>\delta}M(y_i)\Big). \tag{28}$$

It is easy to see that

$$\frac{1}{|J|}\sum_{y_i\leq\delta}M(y_i) \leq \frac{1}{|J|}\sum_{i\in J}\varepsilon \leq \varepsilon. \tag{29}$$

If $y_i > \delta$, then $y_i < 1 + \frac{y_i}{\delta}$. Since M is nondecreasing and convex, we obtain that

$$M(y_i) < M\Big(1 + \frac{y_i}{\delta}\Big) \le \frac{1}{2}M(2) + \frac{1}{2}M\Big(\frac{2y_i}{\delta}\Big).$$

Since M satisfies Δ_2 conditions, we can find $K > 0$ such that

$$M(2t) \le KM(t)$$

for all $t > 0$. Hence,

$$M(y_i) < \frac{K}{2}M(2) + \frac{M}{2\delta}y_i M(2) < \frac{K}{\delta}M(2)y_i.$$

Therefore,

$$\frac{1}{|J|}\sum_{y_i>\delta} M(y_i) \le \frac{K}{\delta|J|}M(2)\sum_{i\in J} y_i.$$

Combined with (27), we can deduce that

$$\frac{1}{|J|}\sum_{y_i>\delta} M(y_i) \le \frac{KM(2)}{\delta}\varepsilon \tag{30}$$

for all $J \ge J_0$. It follows from (27), (28), (29) and (30) that

$$\frac{1}{|J|}\sum_{i\in J} M(y_i) = \frac{1}{|J|}\sum_{i\in J} M\big(|x_i - l|\big) < \Big(1 + \frac{KM(2)}{\delta}\Big)\varepsilon$$

for every $J \ge J_0$. This proves that $x \in W(M, I)$.

By the same arguments, we have that $W_0(I) \subset W_0(M, I)$ and $W_\infty(I) \subset W_\infty(M, I)$. □

Due to [13], we also obtain the following statements:

Theorem 4.6. *1) If $0 < \inf_{i\in I} p_i \le p_i \le 1$ for all $i \in I$, then*

$$W(p, M, I) \subset W(M, I).$$

2) If $1 \le p_i$ for all $i \in I$, then $W(M, I) \subset W(p, M, I)$.

Proof. Let $x = (x_i) \in W(p, M, I)$. Then, there are $l \in \mathbb{K}$ and $\rho > 0$ such that the family

$$S_J = \frac{1}{|J|}\sum_{i\in J\in\mathcal{F}(I)} \Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i} \tag{31}$$

converges to 0. Let $0 < \varepsilon < 1$; then, there exists $J_0 \in \mathcal{F}(I)$ such that

$$S_J = \frac{1}{|J|}\sum_{i\in J} \Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i} < \varepsilon$$

for every $J \ge J_0$. This implies that

$$\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i} < \varepsilon < 1$$

for all $i \in J \ge J_0$. Since $0 < \inf_{i\in I} p_i < p_i \le 1$ for all $i \in I$, we infer that

$$M\Big(\frac{|x_i - l|}{\rho}\Big) \le \Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i}$$

for all $i \in J$. Hence,

$$\frac{1}{|J|}\sum_{i\in J} M\Big(\frac{|x_i - l|}{\rho}\Big) \le \sum_{i\in J}\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i} = S_J$$

for all $J \ge J_0$. It follows from (31) that the family

$$z_J := \frac{1}{|J|}\sum_{i\in J\in\mathcal{F}(I)} M\Big(\frac{|x_i - l|}{\rho}\Big)$$

converges to 0. Therefore, $x \in W(M, I)$.

2) Suppose that $1 \le p_i \le \sup_{i\in I} p_i < +\infty$ for all $i \in I$. Let $x \in W(M, I)$. Then, for each $0 < \varepsilon < 1$, we can find $J_0 \in \mathcal{F}(I)$ such that

$$\frac{1}{|J|}\sum_{i\in J} M\Big(\frac{|x_i - l|}{\rho}\Big) < \varepsilon < 1$$

for every $J \ge J_0$, with some $l \in \mathbb{K}$ and $\rho > 0$. This implies that

$$\frac{1}{|J|}\sum_{i\in J}\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i} \le \frac{1}{|J|}\sum_{i\in J} M\Big(\frac{|x_i - l|}{\rho}\Big) < \varepsilon$$

for all $J \ge J_0$. This means that the family

$$S_J = \frac{1}{|J|}\sum_{i\in J\in\mathcal{F}(I)}\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i}$$

converges to 0. Hence, $x \in W(p, M, I)$. The theorem is proved. □

Theorem 4.7. *Let $p = (p_i)_{i\in I}$ and $q = (q_i)_{i\in I}$ be families of positive real numbers. Assume that $0 < p_i < q_i$ for all $i \in I$ and $\Big(\frac{q_i}{p_i}\Big)_{i\in I}$ is bounded. Then,*

$$W(q, M, I) \subset W(p, M, I).$$

Proof. Let $x = (x_i) \in W(q, M, I)$. Then, there are $l \in \mathbb{K}$ and $\rho > 0$ such that the family

$$S_J = \frac{1}{|J|}\sum_{i\in J\in\mathcal{F}(I)}\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{q_i} \tag{32}$$

converges to 0. Write

$$t_i = \Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{q_i}$$

and $\lambda_i = \frac{q_i}{p_i}$. Take $0 < \lambda < \lambda_i$. Define

$$u_i = \begin{cases} t_i, & \text{if } t_i \ge 1\\ 0, & \text{otherwise,}\end{cases}$$

and

$$v_i = \begin{cases} 0, & \text{if } t_i \ge 1\\ t_i, & \text{otherwise.}\end{cases}$$

Hence, $t_i = v_i + u_i$ for each $i \in I$. Moreover,

$$t_i^{\lambda_i} = u_i^{\lambda_i} + v_i^{\lambda_i}$$

for every $i \in I$. It follows that

$$u_i^{\lambda_i} \leq u_i \leq t_i \text{ and } v_i^{\lambda_i} \leq v_i^{\lambda}.$$

Therefore,

$$\frac{1}{|J|}\sum_{i\in J} t_i^{\lambda_i} \leq \frac{1}{|J|}\sum t_i + \Big(\frac{1}{|J|}\sum_{i\in J} v_i\Big)^{\lambda}$$

$$\frac{1}{|J|}\sum t_i + \Big(\frac{1}{|J|}\sum_{i\in J} t_i\Big)^{\lambda}$$

for all $J \in \mathcal{F}(\mathcal{I})$. It follows from (32) that the family

$$\frac{1}{|J|}\sum_{i\in J} t_i^{\lambda_i} = \frac{1}{|J|}\sum_{i\in J}\Big(M\Big(\frac{|x_i - l|}{\rho}\Big)\Big)^{p_i}$$

converges to 0. This means that $x = (x_i) \in W(p, M, I)$. The proof is complete. □

5. p-Banach structure

In this section, we construct the p-Banach structure of $l_M(q, I)$. We also give some properties of a subspace $h_M(q, I)$ of $l_M(q, I)$. In particular, we have obtained some of their descriptions when M is a degenerate Orlicz function. As in the previous section, we always assume that $q = (q_i)_{\in I}$ is a bounded family of positive real numbers and that $H = \sup_{i\in I} q_i$.

Theorem 5.1. *Suppose that $P := \inf\{q_i : i \in I\} \geq 1$ and $p := \frac{P}{H}$. Then, $l_M(q, I)$ is a p-normed space with the p-norm*

$$\|x\| = \inf\Big\{\rho^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\rho}\Big)^P\Big)^{\frac{1}{H}} \leq 1\Big\}. \tag{33}$$

Proof. Clearly

$$\|x\| = \inf\Big\{\rho^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\rho}\Big)^P\Big)^{\frac{1}{H}} \leq 1\Big\} \geq 0$$

for each $x \in l_M(q, I)$. We claim that $x = 0$ if only if $\|x\| = 0$. Indeed, if $x = 0$, then $x_i = 0$ for every $i \in I$. Hence, $M(\frac{|x_i|}{\rho}) = M(\frac{0}{\rho}) = 0$ for every $\rho > 0$. This implies that

$$\|x\| = \inf\{\rho^{\frac{P}{H}} > 0\} = 0.$$

If $x \neq 0$, we need to show that $\|x\| \neq 0$. Suppose that $x \neq 0$ and $\|x\| = 0$. Since $x = (x_i) \neq 0$, we can seek $i_0 \in I$ such that $x_{i_0} \neq 0$. Therefore, $|x_{i_0}| > 0$. It follows from $\lim_{t\to\infty} M(t) = \infty$ that there exists $t_0 > 0$ such that $M(t_0) > 1$. Since

$$\inf\Big\{\rho^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\rho}\Big)^P\Big)^{\frac{1}{H}} \leq 1\Big\} = 0,$$

we can find

$$\rho_0 \in \Big\{\rho^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\rho}\Big)^P\Big)^{\frac{1}{H}} \leq 1\Big\}$$

such that $\dfrac{|x_{i_0}|}{\rho_0} > t_0$. Therefore,

$$\Big(M(\frac{|x_{i_0}|}{\rho_0})\Big)^P \geq \Big(M(t_0)\Big)^P > 1.$$

This yields that

$$\Big(\sum_{i\in I} M\big(\frac{|x_i|}{\rho}\big)^P\Big)^{\dfrac{1}{H}} > 1.$$

We arrive at a contradiction to

$$\rho_0 \in \Big\{\rho^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\big(\frac{|x_i|}{\rho}\big)^P\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Hence, $x \neq 0$ implies $\|x\| \neq 0$.

Now, we claim that $\|\lambda x\| = |\lambda|^p\|x\|$ for every $x \in l_M(q, I)$ and $\lambda \in \mathbb{K}$. If $\lambda = 0$ or $x = 0$, then our claim is obvious. If $\lambda \neq 0$ and $x \neq 0$ then

$$\begin{aligned}\|\lambda x\| &= \inf\Big\{\tau^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\big(\frac{|\lambda x_i|}{\tau}\big)^P\Big)^{\frac{1}{H}} \leq 1\Big\}\\ &= \inf\Big\{\tau^{\frac{P}{H}} > 0 : \Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\frac{\tau}{|\lambda|}}\Big)^P\Big)^{\frac{1}{H}} \leq 1\Big\}.\end{aligned}$$

Set $\rho = \dfrac{\tau}{|\lambda|}$. We have

$$\begin{aligned}\|\lambda x\| &= \inf\Bigg\{(\rho|\lambda)|^{\frac{P}{H}} : \Big(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^P\Big)^{\frac{1}{H}} \leq 1\Bigg\}\\ &= |\lambda|^{\frac{P}{H}} \inf\{\rho^{\frac{P}{H}} : \Big(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^P\Big)^{\frac{1}{H}} \leq 1\}\\ &= |\lambda|^{\frac{P}{H}}\|x\|.\end{aligned}$$

Next, we need to claim that

$$\sum_{i\in I} M\big(\frac{|x_i|}{\|x\|^{\frac{H}{P}}}\big) \leq 1. \tag{34}$$

for every $x \neq 0$. Indeed, for every $\varepsilon > 0$, there exists $\rho' = \rho^{\frac{P}{H}} > 0$ such that $\|x\| \leq \rho' \leq \|x\| + \varepsilon$ and

$$\Big(\sum_{i\in I} M\big(\frac{|x_i|}{\rho'^{\frac{H}{P}}}\big)^P\Big)^{\frac{1}{H}} \leq 1.$$

We have

$$\frac{|x_i|}{(\|x\| + \varepsilon)^{\frac{H}{P}}} \leq \frac{|x_i|}{\rho'^{\frac{H}{P}}}$$

for all $i \in I$. Since M is nondecreasing, we obtain

$$\Big(\sum_{i\in I}\big(M\big(\frac{|x_i|}{(\|x\| + \varepsilon)^{\frac{H}{P}}}\big)\big)^P\Big)^{\frac{1}{H}} \leq \Big(\sum_{i\in I}\big(M\big(\frac{|x_i|}{\rho'^{\frac{H}{P}}}\big)\big)^P\Big)^{\frac{1}{H}} \leq 1.$$

Letting $\varepsilon \to 0$, we arrive at

$$\left(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\|x\|^{\frac{H}{P}}}\Big)\Big)^P\right)^{\frac{1}{H}} \leq 1.$$

Now, for each $x, y \in l_M(q, I)$, we set

$$u = \|x\| = \inf\left\{\rho^{\frac{P}{H}} : \left(\sum_{i\in I}\left(M\left(\frac{|x_i|}{\rho}\right)\right)^P\right)^{\frac{1}{H}} \leq 1\right\}$$

and

$$v = \|y\| = \inf\left\{\rho^{\frac{P}{H}} : \left(\sum_{i\in I}\left(M\left(\frac{|y_i|}{\rho}\right)\right)^P\right)^{\frac{1}{H}} \leq 1\right\}.$$

(we may assume that $x, y \neq 0$). This implies that

$$\left(\sum_{i\in I}\left(M\left(\frac{|x_i|}{\|x\|^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq 1 \text{ and } \left(\sum_{i\in I}\left(M\left(\frac{|y_i|}{\|y\|^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq 1.$$

Suppose that $t, s \in \mathbb{R}$ satisfy $s > u$ and $t > v$. We obtain

$$\left(\sum_{i\in I}\left(M\left(\frac{|x_i|}{s^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq \left(\sum_{i\in I}\left(M\left(\frac{|x_i|}{\|x\|^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\sum_{i\in I}\left(M\left(\frac{|y_i|}{t^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq \left(\sum_{i\in I}\left(M\left(\frac{|y_i|}{\|y\|^{\frac{H}{P}}}\right)\right)^P\right)^{\frac{1}{H}} \leq 1$$

On the other hand, we have

$$\frac{|x_i| + |y_i|}{(t+s)^{\frac{H}{P}}} \leq \frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}} + t^{\frac{H}{P}}}\frac{|x_i|}{s^{\frac{H}{P}}} + \frac{t^{\frac{H}{P}}}{s^{\frac{H}{P}} + t^{\frac{H}{P}}}\frac{|y_i|}{t^{\frac{H}{P}}}.$$

for every $i \in I$. Hence,

$$\begin{aligned}\left(M\left(\frac{|x_i + y_i|}{(s+t)^{\frac{H}{P}}}\right)\right)^P &\leq \left(M\left(\frac{|x_i| + |y_i|}{(s+t)^{\frac{H}{P}}}\right)\right)^P \\ &\leq \frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}} + t^{\frac{H}{P}}}M\Big(\frac{|x_i|}{s^{\frac{H}{P}}}\Big) + \frac{t^{\frac{H}{P}}}{s^{\frac{H}{P}} + t^{\frac{H}{P}}}M\Big(\frac{|y_i|}{t^{\frac{H}{P}}}\Big)\end{aligned}$$

for every $i \in I$. It is easy to see that $(a+b)^x \leq a^x + b^x$ for all $x \in [0, 1]$. Using both induction and this fact, we obtain that

$$\Big(\sum_{k=1}^{n}(a_k + b_k)^P\Big)^{\frac{1}{H}} \leq \Big(\sum_{k=1}^{n} a_k^P\Big)^{\frac{1}{H}} + \Big(\sum_{k=1}^{n} k_k^P\Big)^{\frac{1}{H}}.$$

Supposing that $J \in \mathcal{F}(I)$, we have

$$\begin{aligned}\left[\sum_{i\in J}\left(M\left(\frac{|x_i+y_i|}{(s+t)^{\frac{H}{P}}}\right)\right)^P\right]^{\frac{1}{H}} &\le \left[\sum_{i\in J}\left(\frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}M\left(\frac{|x_i|}{s^{\frac{H}{P}}}\right)+\frac{t^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}M\left(\frac{|y_i|}{t^{\frac{H}{P}}}\right)\right)^P\right]^{\frac{1}{H}}\\ &\le \left[\sum_{i\in J}\left(\frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}M\left(\frac{|x_i|}{s}\right)\right)^P\right]^{\frac{1}{H}}+\left[\sum_{i\in J}\left(\frac{t^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}M\left(\frac{|y_i|}{t^{\frac{H}{P}}}\right)\right)^P\right]^{\frac{1}{H}}\\ &\le \left(\frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}\right)^{\frac{P}{H}}+\left(\frac{s^{\frac{H}{P}}}{s^{\frac{H}{P}}+t^{\frac{H}{P}}}\right)^{\frac{P}{H}}\\ &\le \frac{s}{s+t}+\frac{t}{s+t}=1.\end{aligned}$$

for every $s > \|x\| > 0$ and $t > \|y\| > 0$. This yields that

$$s+t \in \left\{\rho^{\frac{P}{H}} : \sum_{i\in I} M\left(\frac{|x_i+y_i|}{\rho}\right) \le 1\right\}.$$

and

$$\left\|x+y\right\| = \inf\left\{\rho^{\frac{P}{H}} : \sum_{i\in I} M\left(\frac{|x_i+y_i|}{\rho}\right) \le 1\right\} \le s+t. \tag{35}$$

Since (35) holds for every $s > \|x\|$ and $t > \|y\|$, we can deduce that

$$\left\|x+y\right\| \le \|x\| + \left\|y\right\|.$$

and $l_M(q, I)$ is a p-normed space.
□

Lemma 5.2. *Let $(x^k) \subset l_M(q, I)$ be a sequence. Suppose that (x^k) converges to 0 in $l_M(q, I)$. Then, $\lim_{k\to\infty} x_i^k = 0 \in \mathbb{K}$ for every $i \in I$.*

Proof. Suppose otherwise. Then, there exists $i_0 \in I$ such that $(x_{i_0}^k)$ does not converge to $0 \in \mathbb{K}$. This means that there is a sequence of positive integers (k_j) and $r > 0$ such that $|x_{i_0}^{k_j}| \ge r$. For each $j = 1, 2, ...,$ we have

$$\left\|x^{k_j}\right\| = \inf\left\{\rho^{\frac{P}{H}} > 0 : \left(\sum_{i\in I} M\left(\frac{|x_i^{k_j}|}{\rho}\right)^P\right)^{\frac{1}{H}} \le 1\right\}.$$

Therefore,

$$1 \ge \left(\sum_{i\in I} M\left(\frac{|x_i^{k_j}|}{\|x^{k_j}\|^{\frac{H}{P}}}\right)^P\right)^{\frac{1}{H}} \ge M\left(\frac{r}{\left\|x^{k_j}\right\|^{\frac{H}{P}}}\right) \tag{36}$$

for every k_j. Letting $k_j \to \infty$ and combining it with $\|x^{k_j}\| \to 0$, we arrive at $M\left(\frac{r}{|x^{k_j}|}\right) \to \infty$. This is contradictory to (36). The lemma is proved. □

We have the following fact:

Theorem 5.3. *$l_M(q, I)$ is a p-Banach space.*

Proof. According to Theorem 5.1, we have that $l_M(q, I)$ is a p-normed space with the p-norm

$$\|x\| = \inf\left\{\rho^{\frac{P}{H}} > 0 : \left(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^P\right)^{\frac{1}{H}} \le 1\right\}. \tag{37}$$

Suppose that $(x^k) \subset l_M(q, I)$ is a Cauchy sequence. We claim that (x^k) converges to $x \in l_M(q, I)$. We have

$$\|x^k - x^l\| = \inf\Big\{\rho^{\frac{P}{H}} : \Big(\sum_{i\in I}\Big(M\Big(\frac{|x_i^k - x_i^l|}{\rho}\Big)\Big)^P\Big)^{\frac{1}{H}} \le 1\Big\} \to 0 \tag{38}$$

as $k, l \to \infty$. By Lemma 5.2, we obtain that

$$|x_i^k - x_i^l| \to 0$$

as $k, l \to \infty$ for each $i \in I$. Hence, for each $i \in I$, $(x_i^k) \subset \mathbb{K}$ is a Cauchy sequence $i \in I$. Since $\mathbb{K}$ is complete, we infer that $\lim\limits_{k\to\infty} x_i^k := x_i \in \mathbb{K}$. Set $x = (x_i)_{i\in I}$. Let $\varepsilon > 0$. It follows from (38) that there exists $k_0 \in \mathbb{N}$

$$\|x^k - x^l\| = \inf\Big\{\rho^{\frac{P}{H}} : \left(\sum_{i\in I}\Big(M\Big(\frac{|x_i^k - x_i^l|}{\rho}\Big)\Big)^P\right)^{\frac{1}{H}} \le 1\Big\} < \varepsilon \tag{39}$$

for every $k, l \ge k_0$. In the above inequality, if we fix $k \ge k_0$ and let $l \to \infty$, then

$$\|x^k - x\| = \inf\Big\{\rho^{\frac{H}{P}} : \left(\sum_{i\in I}\Big(M\Big(\frac{|x_i^k - x_i|}{\rho}\Big)\Big)^P\right)^{\frac{1}{H}} \le 1\Big\} < \varepsilon. \tag{40}$$

We have obtained $\|x^k - x\| < \varepsilon$ for every $k > k_0$. This means that x^k converges to x.

Next, we show that $x \in l_M(q, I)$. It follows from (40) that

$$\sum_{i\in I}\Big(M\Big(\frac{|x_i^{k_0} - x_i|}{\rho}\Big)\Big)^{q_i} \le \sum_{i\in I}\Big(M\Big(\frac{|x_i^{k_0} - x_i|}{\rho}\Big)\Big)^P \le 1 < \infty.$$

Hence, $x^{k_0} - x \in l_M(I)$. This implies that $x = x^{k_0} - (x^{k_0} - x) \in l_M(q, I)$. We can conclude that $l_M(q, I)$ is a p-Banach space. □

We recall that

$$h_M(q, I) = \Big\{x = (x_i) \subset \mathbb{K} : \sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} < \infty \text{ for every } \rho > 0\Big\}.$$

We have the following fact:

Proposition 5.4. *$h_M(q, I)$ is a closed subspace of $l_M(q, I)$.*

Proof. First, we claim that $h_M(q, I)$ is a linear subspace of $l_M(q, I)$. Suppose that $x, y \in h_M(q, I)$ and $\alpha \in \mathbb{K}$. If $\alpha = 0$, then $\alpha x = 0 \in h_M(q, I)$.

If $\alpha \neq 0$, then $\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} < \infty$ for every $\rho > 0$. Hence, for every $\rho' > 0$ and $\rho = \frac{\rho'}{|\alpha|}$, we have

$$\sum_{i\in I}\left(M\Big(\frac{|\alpha x_i|}{\rho'}\Big)\right)^{q_i} = \sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} < \infty.$$

This yields that $\alpha x \in h_M(q,I)$, which implies that $2x, 2y \in h_M(q,I)$. We obtain $\sum_{i\in I}\Big(M\big(\frac{|2x_i|}{\rho}\big)\Big)^{q_i} < \infty$ and $\sum_{i\in I}\Big(M\big(\frac{|2y_i|}{\rho}\big)\Big)^{q_i} < \infty$ for every $\rho > 0$. Since M is a convex function, we have that

$$\begin{aligned} M\Big(\frac{|x_i+y_i|}{\rho}\Big) &\le M\Big(\frac{|x_i|+|y_i|}{\rho}\Big) \\ &= M\Big(\frac{1}{2}\frac{|2x_i|}{\rho}+\frac{1}{2}\frac{|2y_i|}{\rho}\Big) \\ &\le \frac{1}{2}M\Big(\frac{|2x_i|}{\rho}\Big)+\frac{1}{2}M\Big(\frac{|2y_i|}{\rho}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i\in I}\Big(M\Big(\frac{|x_i+y_i|}{\rho}\Big)\Big)^{q_i} &\le \Big(\frac{1}{2}\sum_{i\in I}M\Big(\frac{|2x_i|}{\rho}\Big)+\frac{1}{2}\sum_{i\in I}M\Big(\frac{|2y_i|}{\rho}\Big)\Big)^{q_i} \\ &\le C\Big[\sum_{i\in I}\Big(M\Big(\frac{|2x_i|}{\rho}\Big)\Big)^{q_i}+\sum_{i\in I}\Big(M\Big(\frac{|2y_i|}{\rho}\Big)\Big)^{q_i}\Big] < \infty, \end{aligned}$$

where $C = \max\Big\{1, 2^{\sup q_i - 1}\Big\}$. This proves that $x + y \in h_M(q,I)$.

Finally, we show that $h_M(q,I)$ is a closed subset of $l_M(q,I)$. Let (x^k) be a sequence in $h_M(q,I)$. Suppose that x^k converges to x in $l_M(q,I)$. For each $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|x^k - x\| < \varepsilon \tag{41}$$

for every $k \ge k_0$. We claim that $x^{k_0} - x \in h_M(q,I)$. Assume that $x^{k_0} - x \notin h_M(I)$. Then, we can find $\rho_0 > 0$ such that

$$\sum_{i\in I}\Big(M\Big(\frac{|x_i^{k_0} - x_i|}{\rho_0}\Big)\Big)^{q_i} = \infty.$$

Since M is nondecreasing, we have that

$$\sum_{i\in I}\Big(M\Big(\frac{|x_i^{k_0} - x_i|}{\rho}\Big)\Big)^{q_i} = \infty$$

for every $0 < \rho < \min\{\varepsilon, \rho_0\}$. This implies that

$$\inf\Big\{\rho > 0 : \sum_{i\in I}\Big(M\Big(\frac{|x_i^{k_0} - x_i|}{\rho}\Big) \le 1\Big\}\Big)^{q_i}\Big\} > \varepsilon.$$

We arrive at $\|x^{k_0} - x\| > \varepsilon$ and contraction to (41). Hence, there exists $k_0 \in \mathbb{N}$ such that $x^{k_0} - x \in h_M(q,I)$. Therefore,

$$x = x^{k_0} - (x^{k_0} - x) \in h_M(q,I).$$

□

If M is degenerate, then we obtain a description of $l_M(q,I)$ and $h_M(q,I)$

Theorem 5.5. *Suppose that M is a degenerate Orlicz function. Then,*
1) $l_M(q,I)$ is isomorphic to $l_\infty(I)$;
2) $h_M(q,I)$ is isomorphic to $C_0(I)$.

Proof. 1) Suppose that M is a degenerate Orlicz function. There exists $t_0 > 0$ such that $M(t_0) = 0$. From the continuity of M and $\lim_{t\to\infty} M(t) = \infty$, we can find T_0 such that $M(T_0) = 0$ and $M(t) > 0$ for every $t > T_0$. For each $x = (x_i) \in l_\infty(I)$ and $x \neq 0$, we set

$$k = \sup_{i\in I} |x_i|^p = \|x\|_\infty < \infty,$$

where $p = \frac{P}{H}$. If we choose $\rho = \frac{2k}{T_0}$, then

$$\frac{|x_i|}{\rho} = \frac{T_0|x_i|}{2k} \leq \frac{T_0}{2}.$$

This implies that

$$0 \leq M\Big(\frac{|x_i|}{\rho}\Big) \leq M\Big(\frac{T_0}{2}\Big) = 0$$

for every $i \in I$. We obtain $\sum_{i\in I}\left(M\big(\frac{|x_i|}{\rho}\big)\right)^{q_i} = 0$ so that $x = (x_i) \in l_M(q, I)$. Hence, $l_\infty(I) = l_M(q, I)$.

Next, we claim that the p-norms of $l_M(q, I)$ and $l_\infty(I)$ are equivalent. We recall that

$$\|x\|_\infty = \sup_{i\in I} |x_i|^p$$

is the p-norm on $l_\infty(I)$. It follows from the previous argument that

$$\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{p_i} = 0 < 1$$

with $\rho = \frac{2k}{T_0}$. Hence,

$$\|x\| = \inf\left\{\rho^{\frac{P}{H}} > 0 : \left(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^{P}\right)^{\frac{1}{H}} \leq 1\right\} \leq \frac{2k}{T_0} = \frac{2\|x\|_\infty}{T_0}.$$

This implies that

$$\|x\|_\infty \geq \frac{T_0}{2}\|x\| \tag{42}$$

for every $x \in l_M(q, I)$. It follows from

$$\|x\| = \inf\left\{\rho^{\frac{P}{H}} > 0 : \left(\sum_{i\in I}\Big(M\Big(\frac{|x_i|}{\rho}\Big)\Big)^{P}\right)^{\frac{1}{H}} \leq 1\right\}$$

that

$$\Big(\sum_{i\in I} M\Big(\frac{|x_i|}{\|x\|^{\frac{1}{p}}}\Big)^{\frac{1}{H}} = \left(\sum_{i\in I} M\left[\frac{|x_i|}{\|x\|^{\frac{H}{P}}}\right]\right)^{\frac{1}{H}} \leq 1$$

for every $x \in l_M(q, I)$ and $x \neq 0$. Since M is continuous $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$, we can seek $T_1 > 0$ such that $M(T_1) = 1$ and $M(t) > 1$ for all $t > T_1$. Because $M\Big(\frac{|x_i|}{\|x\|^{\frac{1}{p}}}\Big) \leq 1$ for every i, we can deduce that

$$\frac{|x_i|}{\|x\|^{\frac{1}{p}}} \leq T_1$$

for every i. This yields

$$\|x\|_\infty = \sup_{i\in I} |x_i|^p \le T_1\|x\| \tag{43}$$

for every $x \neq 0$. It is easy to see that the above inequality holds with $x = 0$. Combining (42) and (43), we can conclude that the p-norms of $l_\infty(I)$ and $l_M(q, I)$ are equivalent. This proves that $l_M(q, I)$ is isomorphic to $l_\infty(I)$.

2) It is easy to check that $C_0(I)$ is a closed subspace of $l_\infty(I)$. Because $h_M(q, I)$ is a closed subspace of $l_M(q, I)$ and according to 1), we only need to claim that $h_M(q, I) = C_0(I)$.

Let $x = (x_i) \in h_M(p, I)$. We have

$$\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} < \infty$$

for every $\rho > 0$. If $x \notin C_0(I)$, then the family $(|x_i|)_{i\in I}$ does not converges to 0. We can find the infinite subset J of I and $r > 0$ such that $|x_j| > r$ for every $j \in J$. Now, we fix $\rho > 0$ such that

$$\frac{|x_j|}{\rho} \ge \frac{r}{\rho} \ge T_1$$

for all $j \in J$. It follows from $\frac{|x_j|}{\rho} \ge T_1$ for every $j \in J$ that

$$\left(M\Big(\frac{|x_j|}{\rho}\Big)\right)^{q_i} \ge \Big(M(T_1)\Big)^{q_i} > 0$$

for every $j \in J$. Since J is infinite, we can deduce that

$$\sum_{j\in J}\left(M\Big(\frac{|x_j|}{\rho}\Big)\right)^{q_i} = \infty.$$

Hence, $\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} = \infty$. We arrive at a contradiction to $\sum_{i\in I}\Big(M\big(\frac{|x_i|}{\rho}\big)\Big)^{q_i} < +\infty$ for every $\rho > 0$. This implies that $h_M(q, I) \subset C_0(I)$.

Next, assume that $x = (x_i) \in C_0(I)$. We show that $x \in h_M(q, I)$. For each $\rho > 0$, since the family $(|x_i|)$ converges to 0, we can find $J_0 \in \mathcal{F}(I)$ such that $|x_i| < \rho T_0$ for every $i \in I \setminus J_0$. This implies that $\frac{|x_i|}{\rho} < T_0$ for every $i \in I \setminus J_0$. Hence, $M(\frac{|x_i|}{\rho}) = 0$ for every $i \in I \setminus J_0$. We obtain

$$\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} = \sum_{i\in J_0}\left(M\Big(\frac{|x_i|}{\rho}\Big)\right)^{q_i} < \infty,$$

for each $\rho > 0$. This proves that $x = (x_i) \in h_M(q, I)$. Therefore, $C_0(I) \subset h_M(q, I)$, so $C_0(I) = h_M(q, I)$.. □

We recall that a nondegenerate Orlicz function M is called, and it satisfies the Δ_q-condition at 0 if $\lim_{t\to 0} \frac{M(qt)}{M(t)} < \infty$ for some $q > 0$. A function M satisfies the Δ_q-condition at 0 for every $q > 0$ if and only if M satisfies the Δ_2-condition at 0 (see [7]).

The following theorem states that $h_M(p, I) = l_M(p, I)$ under the Δ_2 condition:

Theorem 5.6. *Let M be a nondegenerate Orlicz function. If M satisfies the Δ_2-condition at 0, then $l_M(p, I) = h_M(p, I)$.*

Proof. Since M satisfies the Δ_2-condition at 0, we can deduce that M satisfies the Δ_q-condition for each $q > 0$. Let $x \in l_M(p, I)$. There exists $\rho_0 > 0$ such that

$$\sum_{i\in I}\left(M\Big(\frac{|x_i|}{\rho_0}\Big)\right)^{p_i} < \infty.$$

Therefore, the family $\left(\left(M(\frac{|x_i|}{\rho_0}\right)^{p_i}\right)_{i\in I}$ converges to 0. It follows that $\left(\left(M\Big(\frac{|x_i|}{\rho_0}\Big)\right)^{P}\right)_{i\in I}$ converges to 0, where $P = \inf p_i > 0$. Since M is nondegenerate and continuous at 0, we can conclude that the family $\left(\frac{|x_i|}{\rho_0}\right)_{i\in I}$ converges to 0. Hence, there exists $J_0 \in \mathcal{F}(I)$ such that $\dfrac{|x_i|}{\rho_0} < 1$ for every $i \in I \setminus J_0$.

For each $\rho > 0$, invoking the Δ_q-condition at 0 with $q = \dfrac{\rho_0}{\rho}$, we can find $K > 0$ and $0 < \delta < 1$ such that

$$M\Big(\frac{\rho_0 t}{\rho}\Big) < KM(t)$$

for every $0 < t \leq \delta$. We have

$$\left(M(\frac{|x_i|}{\rho})\right)^{p_i} = \left(M(\frac{\rho_0}{\rho}\frac{|x_i|}{\rho_0})\right)^{p_i} \leq K^H\left(M(\frac{|x_i|}{\rho_0})\right)^{p_i}$$

for every $i \in I \setminus J_0$, where $H = \sup\{p_i : i \in I\}$. We obtain

$$\begin{aligned}\sum_{i\in I}\left(M\left(\frac{|x_i|}{\rho}\right)\right)^{p_i} &= \sum_{i\in J_0}\left(M\left(\frac{|x_i|}{\rho}\right)\right)^{p_i} + \sum_{i\in I\setminus J_0}^{\infty}\left(M\left(\frac{|x_i|}{\rho}\right)\right)^{p_i} \\ &\leq \sum_{i\in J_0}\left(M\left(\frac{|x_i|}{\rho}\right)\right)^{p_i} + K^H\sum_{i\in I\setminus J_0}\left(M\left(\frac{|x_i|}{\rho_0}\right)\right)^{p_i} < \infty.\end{aligned}$$

This proves that $\sum_{i\in I}\left(M(\dfrac{|x_i|}{\rho})\right)^{p_i} < \infty$ for every $\rho > 0$. Therefore, $x \in h_M(q, I)$. Hence, $l_M(p, I) \subset h_M(p, I)$, and thus, $l_M(p, I) = h_M(p, I)$. □

6. P-strong convergence of families with respect to Orlicz functions

Let $\omega(I)$ be the set of all complex families. Let $p = (p_i)_{i\in I}$ be any family of nonnegative real numbers and the map $\varphi : I \to [0, \infty)$. Suppose that P_φ is a regular power series method with a radius of convergence of $R > 0$. Let M be an Orlicz function. We introduce the following family spaces:

$$W_0(P_\varphi, M, I) = \left\{x \in \omega(I) : \lim_{t\to R^-}\frac{1}{p_\varphi(t)}\sum_{j\in I} p_j t^{\varphi(j)} M(|x_j|) = 0\right\} \tag{44}$$

and

$$W(P_\varphi, M, I) = \left\{x \in \omega(I) : x - Le \in W_0(P, M, I) \text{ for some } L\right\} \tag{45}$$

If $x \in W(P_\varphi, M, I)$, we say that the family x is P_φ-strongly convergent to L with respect to an Orlicz function M.

Proposition 6.1. *Let M be an Orlicz function satisfying the Δ_2-condition at 0. Then, we have the following inclusions:*

$$W_0(P_\varphi, I) \subset W_0(P_\varphi, M, I) \text{ and } W(P_\varphi, I) \subset W(P_\varphi, M, I).$$

Proof. It is sufficient to claim $W_0(P_\varphi, I) \subset W_0(P_\varphi, M, I)$. Let $x = (x_i) \in W_0(P_\varphi, I)$ and M be an Orlicz function satisfying the Δ_2-condition. According to the continuity of M at 0, for a given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $M(s) < \varepsilon$ for every $0 \le s < \delta$. Then, we have that

$$\begin{aligned}\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|) &= \frac{1}{p_\varphi(t)}\sum_{i\in I;\, |x_i|<\delta} p_i t^{\varphi(i)} M(|x_i|) + \frac{1}{p_\varphi(t)}\sum_{i\in I;\, |x_i|\ge\delta} p_i t^{\varphi(i)} M(|x_i|)\\ &\le \varepsilon + \frac{1}{p_\varphi(t)}\sum_{i\in I;\, |x_i|\ge\delta} p_i t^{\varphi(i)} M(|x_i|).\end{aligned} \tag{46}$$

Since $0 < \delta < 1$, we have that

$$|x_i| < \frac{1}{\delta}|x_i| < 1 + \frac{|x_i|}{\delta},$$

for every $i \in I$. It follows from the Δ_2-condition of M that

$$\begin{aligned} M(|x_i| &\le M(1 + \frac{|x_i|}{\delta}) \le \frac{1}{2}M(2) + \frac{1}{2}M\Big(\frac{2|x_i|}{\delta}\Big)\\ &\le \frac{M(2)}{2} + K\frac{|x_i|}{2\delta}M(2) < \frac{(1+K)M(2)}{\delta}|x_i|,\end{aligned} \tag{47}$$

for every $i \in I$, where K is a positive constant satisfying $M(2u) \le KM(u)$ for every $u \ge 0$. Combining (46) and (47), we arrive at

$$\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|) \le \varepsilon + \frac{(1+K)M(2)}{\delta}\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} |x_i|.$$

Since $x = (x_i) \in W_0(P_\varphi, I)$, we infer that

$$\lim_{0<t\to R^-}\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} |x_i| = 0.$$

This implies that

$$\lim_{0<t\to R^-}\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|).$$

The proof is finished. □

Lemma 6.2. *If M is an Orlicz function satisfying the Δ_2- condition, then $W_0(P_\varphi, M, I) \cap l_\infty(I)$ is an ideal of $l_\infty(I)$.*

Proof. Given $x \in W_0(P_\varphi, M, I)$ and $y \in l_\infty(I)$, we show that $xy \in W_0(P_\varphi, M, I)$. Since $y \in l_\infty(I)$, there exists $K_1 > 1$ such that

$$||y|| = \sup_{i\in I} |y_i| \le K_1.$$

Hence,

$$M(|x_i y_i|) \le M(K_1|x_i|) \le K(1 + K_1)M(|x_i|),$$

for every $i \in I$. Therefore,

$$\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i y_i)|) \le K(1+K_1)\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|). \tag{48}$$

It follows from $x \in W_0(P_\varphi, M, I)$ that

$$\lim_{0<t\to R^-}\frac{1}{p_\varphi(t)}\sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|) = 0. \tag{49}$$

Combining (48) and (49), we can conclude that

$$\lim_{0<t\to R^-} \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M(|x_i y_i)|) = 0.$$

This proves that $xy \in W_0(P_\varphi, M, I)$. □

The following lemma is similar to Lemma 2.2 in [14]. We omit the proof.

Lemma 6.3. *Let E be an ideal of $l_\infty(I)$ and $x \in l_\infty(I)$. Then, x belongs to the closure of E in $l_\infty(I)$ if and only if $\chi_{K(x;\varepsilon)} \in E$ for all $\varepsilon > 0$.*

The following lemma is analogous to Lemma 2.3 in [14]. The author states that the idea was first published by Freedman and Sember in [5] and omits the proof. We provide proof for the reader's convenience.

Lemma 6.4. *If P_φ is regular, then $W_0(P_\varphi, I) \cap l_\infty$ is a closed ideal of $l_\infty(I)$.*

Proof. By the same argument as that in the proof of Lemma 6.2, we can deduce that $W_0(P_\varphi, I) \cap l_\infty$ is an ideal of $l_\infty(I)$. Now, we claim that $W_0(P_\varphi, I) \cap l_\infty$ is a closed subset of $l_\infty(I)$. Let (x^k) be a sequence in $W_0(P_\varphi, I) \cap l_\infty$. Suppose that x^k converges to $x \in l_\infty(I)$. It is sufficient to prove that $x \in W_0(P_\varphi, I)$. Since $x^k = (x_i^k) \in W_0(P_\varphi, I) \cap l_\infty$, we infer that

$$\lim_{0<t\to R^-} \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M(|x_i^k|) = 0. \tag{50}$$

for each $k = 1, 2, ...$ We set

$$g_k(t) := \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M(|x_i^k|)$$

for each $k = 1, 2, ...$ For each $0 < t < R$, given $\varepsilon > 0$, there is a $J_0 \in \mathcal{F}(I)$ such that

$$|g_k(t) - S_J^k(t)| < \varepsilon \tag{51}$$

for every $J \in \mathcal{F}(I)$ with $J \geq J_0$, where

$$S_J^k(t) = \frac{1}{p_\varphi(t)} \sum_{i\in J} p_i t^{\varphi(i)} M(|x_i^k|).$$

Since

$$\|x^k - x\| = \sup_{i\in I} |x_i^k - x_i| \to 0$$

as $k \to \infty$, we have that $\lim_{k\to\infty} x_i^k = x_i$. Using the continuity of M, we can conclude that

$$\lim_{k\to\infty} S_J^k(t) = \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|) := S_J(t) \tag{52}$$

for every $J \in \mathcal{F}(I)$. For each $J \geq J_0$, there exists $k_0 = k_0(J) \in \mathbb{N}$ such that

$$|S_J^k(t) = S_J(t)| < \varepsilon \tag{53}$$

for every $k \geq k_0$. It follows from (51) and (53) that

$$|g_k(t) - S_J(t)| < 2\varepsilon$$

for every $J \geq J_0$ and $k \geq k_0$. Letting $t \to R^-$, and combining it with (50), we arrive at

$$\lim_{0<t\to R^-} S_J(t) < 2\varepsilon$$

for all $J \geq J_0$. This yields that

$$\lim_{0<t\to R^-} \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M(|x_i|) = 0. \tag{54}$$

The lemma is proved. □

Theorem 6.5. *Let M be an Orlicz function satisfying the Δ_2-condition and P_φ be a regular power-summable family method. Then,*

$$W(P_\varphi, M, I) \cap l_\infty(I) = W(P_\varphi, I) \cap l_\infty(I).$$

Proof. It is sufficient to prove that

$$W_0(P_\varphi, M, I) \cap l_\infty(I) = W_0(P_\varphi, I) \cap l_\infty(I).$$

By Proposition 6.1, we have that

$$W_0(P_\varphi, M, I) \cap l_\infty(I) \subset W_0(P_\varphi, I) \cap l_\infty(I).$$

We need to prove the opposite inclusion. To do so, we first see that

$$\frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M\Big(\chi_{K(x;\varepsilon)}(i)\Big) = M(1) \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} \chi_{K(x;\varepsilon)}(i). \tag{55}$$

Let $x = (x_i) \in W_0(P_\varphi, M, I) \cap l_\infty(I)$ and $\varepsilon > 0$. Consider a family $y = (y_i)$ defined by

$$y_i = \begin{cases} \dfrac{1}{x_i}, & \text{if } |x_i| \geq \varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to observe that $xy = \chi_{K(x;\varepsilon)}$ and $\chi_{K(x;\varepsilon)} \in W_0(P_\varphi, M, I) \cap l_\infty(I)$. This yields that

$$\lim_{0<t\to R^-} \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} M\Big(\chi_{K(x;\varepsilon)}(i)\Big) = 0.$$

It follows from (55) that

$$\lim_{0<t\to R^-} \frac{1}{p_\varphi(t)} \sum_{i\in I} p_i t^{\varphi(i)} \chi_{K(x;\varepsilon)}(i) = 0.$$

Considering Lemma 6.3 and Lemma 6.4, we can deduce that

$$x \in W_0(P_\varphi, I) \cap l_\infty(I).$$

The theorem is proved. □

Theorem 6.6. *Let M be an Orlicz function satisfying the Δ_2-condition and P_φ be a regular power-summable family method. Then,*

1) $W(P_\varphi, M, I) \cap l_\infty(I) = W(P_\varphi, I) \cap l_\infty(I) = st(P_\varphi, I) \cap l_\infty(I)$.

2) $W(P_\varphi, I) \subset W(P_\varphi, M, I) \subset st(P_\varphi, I)$.

Proof. 1) Note that every bounded family is P_φ uniformly bounded. Combining this fact with Theorem 3.6 and Theorem 6.5, we can deduce that

$$W(P_\varphi, M, I) \cap l_\infty(I) = W(P_\varphi, I) \cap l_\infty(I) = \mathrm{st}(P_\varphi, I) \cap l_\infty(I).$$

2) It follows from Proposition 6.1 that

$$W(P_\varphi, I) \subset W(P_\varphi, M, I).$$

We continuously prove the inclusion $W(P_\varphi, M, I) \subset \mathrm{st}(P_\varphi, I)$. Let $\varepsilon > 0$ and $x \in W_0(P_\varphi, M, I)$. We consider a family $y = (y_i)$ to be defined by

$$y_i = \begin{cases} \dfrac{1}{x_i}, & \text{if } |x_i| \geq \varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, y is a bounded family and

$$xy = \chi_{K(x;\varepsilon)} \in W_0(P_\varphi, M, I) \cap l_\infty(I).$$

From Theorem 6.5, we infer that

$$\chi_{K(x;\varepsilon)} \in W_0(P_\varphi, I) \cap l_\infty(I).$$

Applying Theorem 3.6, we can conclude that $x \in \mathrm{st}(P_\varphi) \cap l_\infty(I)$. □

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