Filomat 38:20 (2024), 7235–7244
https://doi.org/10.2298/FIL2420235Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new result on orthogonal factorizations in networks

Sizhong Zhou[a], Quanru Pan[a,*], Yang Xu[b]

[a]*School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*
[b]*Department of Mathematics, Qingdao Agricultural University, Qingdao, Shandong 266109, China*

Abstract. Let m, n, r, λ and k_i $(1 \le i \le m)$ be positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge (3\lambda - 1)r - 1$. Let G be a graph, and let H be an $m\lambda$-subgraph of G and $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ be a (g, f)-factorization of G. If for any partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ with $|A_i| = \lambda$, G admits a (g, f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ satisfying $A_i \subseteq E(F_i)$ for $1 \le i \le m$, then we say that $\mathcal{F}$ is randomly λ-orthogonal to H. Let $H_1, H_2, \cdots, H_r$ be r vertex-disjoint $n\lambda$-subgraphs of a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph G. In this paper, it is proved that a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph G contains a subgraph R such that R possesses a $[0, k_i]_1^n$-factorization randomly λ-orthogonal to every H_i, $1 \le i \le r$.

1. Introduction

Many real-world networks can be modelled by graphs or networks. An important example of such a network is a communication network with nodes representing cities and links corresponding to communication channels. Other examples include an aviation network with nodes modelling aviation stations and links representing air lines between two stations, or the World Wide Web with nodes corresponding to web pages and links modelling hyperlinks between web pages. Many real-life problems on network design and optimization, e. g. coding design, scheduling problems, the file transfer problems on computer networks, building blocks and so on, are related to the factors, factorizations and orthogonal factorizations in graphs [2]. A Room square of order $2n$ can be modelled as the orthogonal 1-factorization of K_{2n} which was first posed by Horton [9]. Euler [5] first discovered that a pair of orthogonal Latin squares of order n is related to two orthogonal 1-factorizations of $K_{n,n}$. A network can be represented by a graph, vertices of the graph corresponds to nodes and edges of the graph corresponds to links between the nodes. Henceforth we use the term *graph* instead of *network*.

The graphs discussed in this paper will be finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $g, f : V(G) \to \mathbb{Z}$ be two nonnegative functions satisfying $g(x) \le f(x)$ for each $x \in V(G)$. Let $d_G(x)$ denote the degree of a vertex x in G. A spanning subgraph F of G with $g(x) \le d_F(x) \le f(x)$ for every $x \in V(G)$ is called a (g, f)-factor of G. If G itself is a (g, f)-factor, then we call G a (g, f)-graph. Especially, if $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor and a (g, f)-graph is called an $[a, b]$-graph. A (g, f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ of G is a decomposition of the edge set $E(G)$ of G into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. A subgraph H

2020 *Mathematics Subject Classification*. Primary 05C70; Secondary 68M10, 68R10.
Keywords. network; factor; $[0, k_i]_{i=1}^n$-factorization; orthogonal.
Received: 09 November 2023; Accepted: 11 January 2024
Communicated by Paola Bonacini
* Corresponding author: Quanru Pan
Email addresses: zhousizhong@just.edu.cn (Sizhong Zhou), qrpana@163.com (Quanru Pan), xuyang_825@126.com (Yang Xu)

of G is said to be an m-subgraph if H possesses m edges in total. Assume that H is an $m\lambda$-subgraph of G and $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ is a (g, f)-factorization of G. A (g, f)-factorization $\mathcal{F}$ of G is λ-orthogonal to H if $|E(H) \cap E(F_i)| = \lambda$ for $1 \le i \le m$. If for any partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ with $|A_i| = \lambda$, G admits a (g, f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ satisfying $A_i \subseteq E(F_i)$ for $1 \le i \le m$, then we say that $\mathcal{F}$ is randomly λ-orthogonal to H. Let $k_1, k_2, \cdots, k_m$ be positive integers. A $[0, k_i]_1^m$-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ of G is a decomposition of the edge set $E(G)$ of G into edge-disjoint factors $F_1, F_2, \cdots, F_m$, where F_i is a $[0, k_i]$-factor, $1 \le i \le m$. If for any partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ with $|A_i| = \lambda$, G admits a $[0, k_i]_1^m$-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ satisfying $A_i \subseteq E(F_i)$ for $1 \le i \le m$, then we say that $\mathcal{F}$ is randomly λ-orthogonal to H. Note that randomly 1-orthogonal is equivalent to 1-orthogonal, and 1-orthogonal is simply called orthogonal.

Egawa and Kano [4] presented some sufficient conditions for graphs admitting (g, f)-factors. Zhou et al. [35, 38, 43–46], Wang and Zhang [26, 27], Wu [31] derived some results on $[1, 2]$-factors in graphs. Kouider and Lonc [13], Wang and Zhang [29] studied the existence of $[a, b]$-factors in graphs. Kano [10] derived some results on $[a, b]$-factorizations of graphs. Cai [3] showed some sufficient conditions for graphs having $[a, b]$-factorizations. Yan, Pan, Wong and Tokuda [33] put forward some sufficient conditions for a graph admitting a (g, f)-factorization. Ma and Gao [20] obtained some results for the existence of (g, f)-factorizations in graphs. The interested reader can discover many relevant results on factors and factorizations in graphs [1, 8, 11, 12, 21, 23, 28, 30, 34, 36, 37, 39–42]. Alspach, Heinrich and Liu [2] presented the following problem: Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G of certain type orthogonal to H? Li and Liu [16] claimed that every $(mg + m - 1, mf - m + 1)$-graph G admits a (g, f)-factorization orthogonal to any given m-subgraph of G. Lam et al. [14] verified that every $(mg + m - 1, mf - m + 1)$-graph G admits a (g, f)-factorization orthogonal to k vertex-disjoint m-subgraphs of G. Feng [6] proved that every $(0, mf - m + 1)$-graph G possesses a $(0, f)$-factorization orthogonal to any given m-subgraph of G. Feng and Liu [7] showed that every $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph G admits a $[0, k_i]_1^m$-factorization orthogonal to any given m-subgraph of G. Wang [25] demonstrated that there exists a subgraph R in an $(mg + k, mf - k)$-graph such that R has a (g, f)-factorization orthogonal to n vertex-disjoint k-subgraphs of R. Wang [24] studied the existence of subgraphs with orthogonal $[0, k_i]_1^n$-factorizations in $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graphs. Zhou, Zhang and Xu [47] claimed that there exists a subgraph R in a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph such that R possesses a $[0, k_i]_1^n$-factorization orthogonal to r vertex-disjoint n-subgraphs of R. Some other results on orthogonal factorizations can be discovered in [15, 17–19, 22, 32]. The following results on orthogonal factorizations of graphs are known.

Theorem 1.1 (Wang [24]). Let G be a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph, where m, n and k_i $(1 \le i \le m)$ are positive integers with $n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m$. Let H be an arbitrary n-subgraph of G. Then there exists a subgraph R of G such that R has a $[0, k_i]_1^n$-factorization orthogonal to H.

Zhou, Zhang and Xu [47] extended Theorem 1.1, and verified the following theorem.

Theorem 1.2 (Zhou, Zhang and Xu [47]). Let G be a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph, and let $H_1, H_2, \cdots, H_r$ be vertex-disjoint n-subgraphs of G, where m, n, r and k_i $(1 \le i \le m)$ are positive integers with $n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2r - 1$. Then there exists a subgraph R of G such that R possesses a $[0, k_i]_1^n$-factorization orthogonal to every H_i, $1 \le i \le r$.

We shall consider the following problem: Given r vertex-disjoint $n\lambda$-subgraphs $H_1, H_2, \cdots, H_r$ of G, does there exist a factorization $\mathcal{F}$ randomly λ-orthogonal to every H_i for $1 \le i \le r$? The purpose of this paper is to verify that for any r vertex-disjoint $n\lambda$-subgraphs $H_1, H_2, \cdots, H_r$ of a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph G, there exists a subgraph R such that R admits a $[0, k_i]_1^n$-factorization randomly λ-orthogonal to every H_i for $1 \le i \le r$, where m, n, r, λ and k_i $(1 \le i \le m)$ are positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2(2\lambda - 1)r - 1$.

2. Preliminary Lemmas

Let G be a graph. For a vertex subset S of G, we denote by $G[S]$ the subgraph of G induced by S, and write $G - S = G[V(G) \setminus S]$. For two disjoint vertex subsets S and T of G, we use $E_G(S, T)$ to denote the set of edges in G joining S and T, and use $e_G(S,T)$ to denote the cardinality of $E_G(S,T)$. For convenience, we write $\varphi(S) = \sum_{x\in S}\varphi(x)$ and $\varphi(\emptyset) = 0$ for any function φ. In particular, $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$.

Let S and T be two disjoint subsets of $V(G)$, and E_1 and E_2 be two disjoint subsets of $E(G)$. Put

$$U = V(G) \setminus (S \cup T),\ \ E(S) = \{xy \in E(G) : x, y \in S\}$$

and

$$E(T) = \{xy \in E(G) : x, y \in T\}.$$

Set

$$E_1' = E_1 \cap E(S),\ \ E_1'' = E_1 \cap E_G(S,U),$$
$$E_2' = E_2 \cap E(T),\ \ E_2'' = E_2 \cap E_G(T,U),$$
$$\alpha_G(S,T;E_1,E_2) = 2|E_1'| + |E_1''|,$$
$$\beta_G(S,T;E_1,E_2) = 2|E_2'| + |E_2''|.$$

With no danger of confusion, we use α and β to denote $\alpha_G(S,T;E_1,E_2)$ and $\beta_G(S,T;E_1,E_2)$, respectively. We easily see that $\alpha \le d_{G-T}(S)$ and $\beta \le d_{G-S}(T)$.

The proof of our main result in this paper depends heavily on the following result, which was first derived by Lam, Liu, Li and Shiu [14].

Lemma 2.1 (Lam, Liu, Li and Shiu [14]). Let G be a graph, and let $g, f : V(G) \to Z$ be two functions with $0 \le g(x) < f(x) \le d_G(x)$ for every $x \in V(G)$, and E_1 and E_2 be two disjoint subsets of $E(G)$. Then G possesses a (g,f)-factor F satisfying $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if

$$\gamma_G(S,T;g,f) = f(S) + d_{G-S}(T) - g(T) \ge \alpha_G(S,T;E_1,E_2) + \beta_G(S,T;E_1,E_2)$$

for any two disjoint subsets S and T of $V(G)$.

Next, we assume that m, n, r and k_i $(1 \le i \le m)$ are positive integers satisfying $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge (3\lambda - 1)r - 1$, and G is a $[0, k_1 + k_2 + \cdots + k_m - n + 1]$-graph. For every isolated vertex x of G and every $[0,k_i]$-factor F_i, we admit $d_{F_i}(x) = 0$. Let I be the set of all isolated vertices of G. If $G - I$ admits a $[0,k_i]$-factor, then G possesses also a $[0,k_i]$-factor. Consequently, we may assume that G does not admit isolated vertices. In what follows, we define

$$g(x) = \max\{0, d_G(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2)\}$$

and

$$f(x) = \min\{k_m, d_G(x)\}$$

for all $x \in V(G)$. According to the definitions of $g(x)$ and $f(x)$, we possess the following result.

Lemma 2.2. Let m be an integer with $m \ge 2$. Then

$$0 \le g(x) < f(x) = \min\{k_m, d_G(x)\} \le d_G(x)$$

for every vertex x of G.

We verify the following lemma, which will be used in the proof of our main theorem.

Lemma 2.3. Let G be a $[0, k_1 + k_2 + \cdots + k_m]$-graph, and let $H_1, H_2, \cdots, H_r$ be r vertex-disjoint λ-subgraphs of G, where m, r, λ and k_i $(1 \le i \le m)$ are positive integers with $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2(2\lambda - 1)r - 1$. Then G possesses a $[0,k_1]$-factor F_1 with $E(H_i) \subseteq E(F_1)$ for $1 \le i \le r$.

Proof. Set $E_1 = \bigcup_{i=1}^{r} E(H_i)$ and $E_2 = \emptyset$. We define α and β as before for two disjoint vertex subsets S and T of G. In light of the definitions of α and β, we derive

$$\alpha \le \min\{2\lambda r, \lambda|S|\}\ \ \textit{and}\ \ \beta = 0.$$

Consequently, we admit

$$\gamma_G(S,T;0,k_1) = k_1|S| + d_{G-S}(T) - 0\cdot|T| \geq (2(2\lambda-1)r-1)|S| \geq \lambda|S| \geq \alpha = \alpha+\beta$$

by $\lambda \geq 1$, $r \geq 1$ and $k_1 \geq 2(2\lambda-1)r-1$. Then it follows from Lemma 2.1 that G possesses a $[0,k_1]$-factor F_1 with $E(H_i) \subseteq E(F_1)$ for $1 \leq i \leq r$. Lemma 2.3 is verified. □

3. Main Result and its Proof

In what follows, we pose our main theorem in this paper.

Theorem 3.1. Let G be a $[0,k_1+k_2+\cdots+k_m-n+1]$-graph, and let $H_1,H_2,\cdots,H_r$ be r vertex-disjoint $n\lambda$-subgraphs of G, where m, n, r, λ and k_i ($1 \leq i \leq m$) are positive integers satisfying $1 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 2(2\lambda-1)r-1$. Then there exists a subgraph R of G such that R admits a $[0,k_i]_1^n$-factorization randomly λ-orthogonal to every H_i, $1 \leq i \leq r$.

Proof. In terms of Theorem 1.2, Theorem 3.1 holds for $\lambda = 1$. Next, we may assume that $\lambda \geq 2$.

We apply induction on m and n. According to Lemma 2.3, Theorem 3.1 holds for $n = 1$. Hence, we may consider that $n \geq 2$ in the following. For the inductive step, we assume that Theorem 3.1 holds for any $[0,k_1+k_2+\cdots+k_{m'}-n'+1]$-graph G' with $m' < m$, $n' < n$ and $1 \leq n' \leq m'$, and any r vertex-disjoint $n'\lambda$-subgraphs $H'_1,H'_2,\cdots,H'_r$ of G'. Next, we discuss a $[0,k_1+k_2+\cdots+k_m-n+1]$-graph G and any r vertex-disjoint $n\lambda$-subgraphs $H_1,H_2,\cdots,H_r$ of G.

We select any $A_i \subseteq E(H_i)$ with $|A_i| = \lambda$, $1 \leq i \leq r$. Write $E_1 = \bigcup_{i=1}^{r} A_i$ and $E_2 = \left(\bigcup_{i=1}^{r} E(H_i)\right) \setminus E_1$. Obviously, $|E_1| = \lambda r$ and $|E_2| = (n-1)\lambda r$. For two disjoint subsets S and T of $V(G)$, we define E'_1, E''_1, E'_2, E''_2, α and β as in Section 2. Thus, we derive

$$\alpha \leq \min\{2\lambda r, \lambda|S|\}$$

and

$$\beta \leq \min\{2(n-1)\lambda r, (n-1)\lambda|T|\}.$$

The definitions of $g(x)$ and $f(x)$ are identical to that in Section 2. Now, we select disjoint subsets S and T of $V(G)$ such that

(a) $\gamma_G(S,T;g,f) - \alpha_G(S,T;E_1,E_2) - \beta_G(S,T;E_1,E_2)$ is minimum.

(b) $|S|$ is minimum subject to (a).

We now demonstrate the following claim.

Claim 1. If $S \neq \emptyset$, then $f(x) \leq d_G(x) - 1$ for every $x \in S$, and so $f(x) = k_m$ for every $x \in S$.

Proof. Set $S_1 = \{x \in S : f(x) \geq d_G(x)\}$. Next, we verify $S_1 = \emptyset$.

Assume that $S_1 \neq \emptyset$. Then setting $S_0 = S \setminus S_1$. Hence, we admit

$$\begin{aligned}\gamma_G(S,T;g,f) &= f(S) + d_{G-S}(T) - g(T)\\ &= f(S_0) + f(S_1) + d_G(T) - e_G(S_0,T) - e_G(S_1,T) - g(T)\\ &= f(S_0) + d_{G-S_0}(T) - g(T) + f(S_1) - e_G(S_1,T)\\ &= \gamma_G(S_0,T;g,f) + f(S_1) - e_G(S_1,T)\\ &\geq \gamma_G(S_0,T;g,f) + d_G(S_1) - e_G(S_1,T)\\ &= \gamma_G(S_0,T;g,f) + d_{G-T}(S_1). \qquad (1)\end{aligned}$$

Note that

$$\alpha_G(S,T;E_1,E_2) + \beta_G(S,T;E_1,E_2) \leq \alpha_G(S_0,T;E_1,E_2) + \beta_G(S_0,T;E_1,E_2) + \alpha_G(S_1,T;E_1,E_2)$$

and

$$d_{G-T}(S_1) \geq \alpha_G(S_1,T;E_1,E_2).$$

Combining these with (1), we derive

$$\begin{aligned}
& \gamma_G(S,T;g,f) - \alpha_G(S,T;E_1,E_2) - \beta_G(S,T;E_1,E_2) \\
\geq\ & \gamma_G(S_0,T;g,f) + d_{G-T}(S_1) - \alpha_G(S_0,T;E_1,E_2) - \beta_G(S_0,T;E_1,E_2) - \alpha_G(S_1,T;E_1,E_2) \\
\geq\ & \gamma_G(S_0,T;g,f) - \alpha_G(S_0,T;E_1,E_2) - \beta_G(S_0,T;E_1,E_2),
\end{aligned}$$

which conflicts the choice of S. Hence, $S_1 = \emptyset$. And so if $S \neq \emptyset$, then $f(x) \leq d_G(x) - 1$ for every $x \in S$. Furthermore, we derive $f(x) = k_m$ for every $x \in S$. Claim 1 is verified. □

The remaining of the proof is dedicated to proving that G possesses a (g, f)-factor F_n with $E_1 \subseteq E(F_n)$ and $E_2 \cap E(F_n) = \emptyset$. According to Lemma 2.1 and the choice of S and T, it suffices to claim that $\gamma_G(S,T;g,f) \geq \alpha+\beta$.

Next, we let $\rho = k_1 + k_2 + \cdots + k_{m-1} - n + 2$, $T_1 = \{x : d_G(x) - \rho > 0, x \in T\}$ and $T_0 = T \setminus T_1$. We easily see that

$$g(x) = 0 \tag{2}$$

for every $x \in T_0$, and

$$g(x) = d_G(x) - \rho \tag{3}$$

for every $x \in T_1$. In terms of the definition of $\beta_G(S,T;E_1,E_2)$, we get

$$\beta_G(S,T_0;E_1,E_2) + \beta_G(S,T_1;E_1,E_2) = \beta_G(S,T;E_1,E_2). \tag{4}$$

Note that $\alpha \leq \min\{2\lambda r, \lambda|S|\} \leq \lambda|S|$ and $\beta \leq d_{G-S}(T)$. If $T_1 = \emptyset$, then we have

$$\begin{aligned}
\gamma_G(S,T;g,f) &= f(S) + d_{G-S}(T) - g(T) \\
&= k_m|S| + d_{G-S}(T) - g(T_0) - g(T_1) \\
&= k_m|S| + d_{G-S}(T) \\
&\geq (2(2\lambda-1)r-1)|S| + d_{G-S}(T) \\
&\geq \lambda|S| + d_{G-S}(T) \\
&\geq \alpha+\beta
\end{aligned}$$

by (2), Claim 1 and $\lambda \geq 2$.

If $S = \emptyset$, then $\alpha = 0$. It follows from (2), (3), (4), $r \geq 1$, $\lambda \geq 2$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 2(2\lambda-1)r-1$ that

$$\begin{aligned}
\gamma_G(S,T;g,f) &= f(S) + d_{G-S}(T) - g(T) \\
&= d_G(T) - g(T_1) \\
&= d_G(T_0) + d_G(T_1) - (d_G(T_1) - \rho|T_1|) \\
&= d_G(T_0) + \rho|T_1| \\
&\geq d_G(T_0) + ((m-1)(2(2\lambda-1)r-1) - n + 2)|T_1| \\
&\geq d_G(T_0) + ((n-1)(2(2\lambda-1)r-1) - n + 2)|T_1| \\
&\geq d_G(T_0) + (n-1)\lambda|T_1| \\
&\geq \beta_G(\emptyset,T_0;E_1,E_2) + \beta_G(\emptyset,T_1;E_1,E_2) \\
&= \beta_G(\emptyset,T;E_1,E_2) = \beta = \alpha+\beta.
\end{aligned}$$

In what follows, we always assume that $S \neq \emptyset$ and $T_1 \neq \emptyset$. To demonstrate Theorem 3.1, we shall consider two cases.

Case 1. $|S| \leq |T_1| - 1$.

Using (2), (3) and Claim 1, we get

$$\begin{aligned}\gamma_G(S,T;g,f) &= f(S)+d_{G-S}(T)-g(T)\\ &= f(S)+d_{G-S}(T_0)+d_{G-S}(T_1)-g(T_1)\\ &= k_m|S|+d_{G-S}(T_0)+d_G(T_1)-e_G(S,T_1)-g(T_1)\\ &= k_m|S|+d_{G-S}(T_0)+\rho|T_1|-e_G(S,T_1),\end{aligned}$$

namely,

$$\gamma_G(S,T;g,f)=k_m|S|+d_{G-S}(T_0)+\rho|T_1|-e_G(S,T_1). \tag{5}$$

Subcase 1.1. $|T_1|\le k_m-\lambda$.

According to (4), (5), $r\ge 1$, $\lambda\ge 2$, $2\le n\le m$ and $k_1\ge k_2\ge\cdots\ge k_m\ge 2(2\lambda-1)r-1$, we have

$$\begin{aligned}\gamma_G(S,T;g,f) &= k_m|S|+d_{G-S}(T_0)+\rho|T_1|-e_G(S,T_1)\\ &\ge k_m|S|+d_{G-S}(T_0)+\rho|T_1|-|S||T_1|\\ &= (k_m-|T_1|)|S|+d_{G-S}(T_0)+\rho|T_1|\\ &\ge \lambda|S|+d_{G-S}(T_0)+\rho|T_1|\\ &\ge \lambda|S|+d_{G-S}(T_0)+((m-1)k_m-n+2)|T_1|\\ &\ge \lambda|S|+d_{G-S}(T_0)+((m-1)(2(2\lambda-1)r-1)-n+2)|T_1|\\ &\ge \lambda|S|+d_{G-S}(T_0)+((n-1)(2(2\lambda-1)r-1)-n+2)|T_1|\\ &\ge \lambda|S|+d_{G-S}(T_0)+(n-1)\lambda|T_1|\\ &\ge \alpha+\beta_G(S,T_0;E_1,E_2)+\beta_G(S,T_1;E_1,E_2)\\ &= \alpha+\beta_G(S,T;E_1,E_2)=\alpha+\beta.\end{aligned}$$

Subcase 1.2. $|T_1|\ge k_m-\lambda+1$.

Subcase 1.2.1. $|S|\le 2n-4$.

We easily prove that $\rho-|S|>0$. Then it follows from (5), $r\ge 1$, $\lambda\ge 2$, $2\le n\le m$ and $k_1\ge k_2\ge\cdots\ge k_m\ge 2(2\lambda-1)r-1$ that

$$\begin{aligned}\gamma_G(S,T;g,f) &= k_m|S|+d_{G-S}(T_0)+\rho|T_1|-e_G(S,T_1)\\ &\ge k_m|S|+\rho|T_1|-|S||T_1|\\ &= k_m|S|+(\rho-|S|)|T_1|\\ &\ge k_m|S|+(\rho-|S|)(k_m-\lambda+1)\\ &> k_m|S|+(\rho-|S|)(k_m-\lambda)\\ &= \lambda|S|+(k_m-\lambda)\rho\\ &\ge \lambda|S|+(k_m-\lambda)((m-1)k_m-n+2)\\ &\ge \lambda|S|+(2(2\lambda-1)r-1-\lambda)((n-1)(2(2\lambda-1)r-1)-n+2)\\ &= \lambda|S|+((4r-1)\lambda-2r-1)((n-1)(2(2\lambda-1)r-1)-n+2)\\ &\ge \lambda|S|+(2(4r-1)-2r-1)((n-1)(4\lambda-3)-n+2)\\ &= \lambda|S|+(6r-3)((n-1)\lambda+3(n-1)(\lambda-1)-n+2)\\ &\ge \lambda|S|+3r((n-1)\lambda+3(n-1)-n+2)\\ &= \lambda|S|+3r((n-1)\lambda+2n-1)\\ &> \lambda|S|+2(n-1)\lambda r\\ &\ge \alpha+\beta.\end{aligned}$$

Subcase 1.2.2. $|S|\ge 2n-3$.

Note that G is a $[0, k_1+k_2+\cdots+k_m-n+1]$-graph. Thus, we get $d_G(S) \le (k_1+k_2+\cdots+k_m-n+1)|S| = (\rho+k_m-1)|S|$. In terms of (2), (3), Claim 1, $|S| \le |T_1|-1$, $r \ge 1$, $\lambda \ge 2$, $2 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2(2\lambda-1)r-1$, we derive

$$\begin{aligned}
\gamma_G(S,T;g,f) &= f(S) + d_{G-S}(T) - g(T)\\
&= f(S) + d_G(T) - e_G(S,T) - g(T_1)\\
&= k_m|S| + d_G(T) - e_G(S,T) - (d_G(T_1) - \rho|T_1|)\\
&\ge k_m|S| + \rho|T_1| - e_G(S,T)\\
&= \rho(|T_1| - |S|) + (k_m + \rho)|S| - e_G(S,T)\\
&\ge \rho + |S| + d_G(S) - e_G(S,T)\\
&= \rho + |S| + d_{G-T}(S)\\
&\ge (m-1)k_m - n + 2 + 2n - 3 + \alpha\\
&\ge (n-1)(2(2\lambda-1)r-1) + n - 1 + \alpha\\
&> \alpha + 2(n-1)\lambda r\\
&\ge \alpha + \beta.
\end{aligned}$$

Case 2. $|S| \ge |T_1|$.
Note that G is a $[0, k_1+k_2+\cdots+k_m-n+1]$-graph. Thus, we get $d_G(T_1) \le (k_1+k_2+\cdots+k_m-n+1)|T_1| = (\rho+k_m-1)|T_1|$. According to (2), (3), Claim 1, $|S| \ge |T_1|$, $r \ge 1$, $\lambda \ge 2$, $2 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge 2(2\lambda-1)r-1$, we derive

$$\begin{aligned}
\gamma_G(S,T;g,f) &= f(S) + d_{G-S}(T) - g(T)\\
&= f(S) + d_{G-S}(T) - g(T_1)\\
&= k_m|S| + d_{G-S}(T) - d_G(T_1) + \rho|T_1|\\
&= k_m(|S| - |T_1|) + (k_m + \rho)|T_1| + d_{G-S}(T) - d_G(T_1)\\
&\ge k_m(|S| - |T_1|) + d_G(T_1) + |T_1| + d_{G-S}(T) - d_G(T_1)\\
&= k_m(|S| - |T_1|) + |T_1| + d_{G-S}(T)\\
&\ge (2(2\lambda-1)r-1)(|S| - |T_1|) + |T_1| + d_{G-S}(T)\\
&\ge (4\lambda-3)(|S| - |T_1|) + |T_1| + d_{G-S}(T)\\
&\ge (\lambda+1)(|S| - |T_1|) + |T_1| + d_{G-S}(T),
\end{aligned}$$

that is,

$$\gamma_G(S,T;g,f) \ge (\lambda+1)(|S| - |T_1|) + |T_1| + d_{G-S}(T). \tag{6}$$

Subcase 2.1. $|S| \ge 2\lambda r$.
It follows from (6) and $|S| \ge |T_1|$ that

$$\begin{aligned}
\gamma_G(S,T;g,f) &\ge (\lambda+1)(|S| - |T_1|) + |T_1| + d_{G-S}(T)\\
&\ge |S| + d_{G-S}(T)\\
&\ge 2\lambda r + d_{G-S}(T)\\
&\ge \alpha + \beta.
\end{aligned}$$

Subcase 2.2. $|S| \le 2\lambda r - 1$.
Note that $T_1 \ne \emptyset$. Hence, we consider the following two subcases.
Subcase 2.2.1. $|T_1| = 1$.

We write $T_1 = \{x\}$. Using (3), (4), (6), Claim 1, $r \geq 1$, $\lambda \geq 2$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 2(2\lambda-1)r-1$, we have

$$\begin{aligned}
\gamma_G(S,T;g,f) &\geq (\lambda+1)(|S|-|T_1|)+|T_1|+d_{G-S}(T)\\
&= \lambda|S|-\lambda|T_1|+|S|+d_{G-S}(T_1)+d_{G-S}(T_0)\\
&= \lambda|S|+|S|+d_{G-S}(x)+d_{G-S}(T_0)-\lambda\\
&\geq \lambda|S|+d_G(x)+d_{G-S}(T_0)-\lambda\\
&\geq \lambda|S|+\rho+1+d_{G-S}(T_0)-\lambda\\
&\geq \lambda|S|+(m-1)k_m-n+3+d_{G-S}(T_0)-\lambda\\
&\geq \lambda|S|+d_{G-S}(T_0)+(n-1)(2(2\lambda-1)r-1)-n+3-\lambda\\
&> \lambda|S|+d_{G-S}(T_0)+(n-1)\lambda\\
&= \lambda|S|+d_{G-S}(T_0)+(n-1)\lambda|T_1|\\
&\geq \alpha+\beta_G(S,T_0;E_1,E_2)+\beta_G(S,T_1;E_1,E_2)\\
&= \alpha+\beta_G(S,T;E_1,E_2)\\
&= \alpha+\beta.
\end{aligned}$$

Subcase 2.2.2. $|T_1| \geq 2$.

Claim 2. $4(n-1)(2\lambda-1)r-4n+9-2\lambda r > 2n\lambda r$.

Proof. By $r \geq 1$, $\lambda \geq 2$ and $n \geq 2$, we admit

$$\begin{aligned}
&\quad 4(n-1)(2\lambda-1)r-4n+9-2\lambda r-2n\lambda r\\
&= 8(n-1)\lambda r-4(n-1)r-4n+9-2\lambda r-2n\lambda r\\
&= (8n-8-2-2n)\lambda r-4(n-1)r-4n+9\\
&= (6n-10)\lambda r-4(n-1)r-4n+9\\
&\geq 2(6n-10)r-4(n-1)r-4n+9\\
&= (8n-16)r-4n+9\\
&\geq 8n-16-4n+9\\
&= 4n-7 \geq 1 > 0,
\end{aligned}$$

namely,

$$4(n-1)(2\lambda-1)r-4n+9-2\lambda r > 2n\lambda r.$$

Claim 2 is proved. □

Since $|T_1| \geq 2$, there exist $x, y \in T_1$. It follows from (3), (6), Claim 2, $|S| \geq |T_1|$, $|S| \leq 2\lambda r-1$, $r \geq 1$, $\lambda \geq 2$, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 2(2\lambda-1)r-1$ that

$$\begin{aligned}
\gamma_G(S,T;g,f) &\geq (\lambda+1)(|S|-|T_1|)+|T_1|+d_{G-S}(T)\\
&\geq |S|+d_{G-S}(T_1)\\
&\geq 2|S|+d_{G-S}(T_1)-2\lambda r+1\\
&\geq d_G(x)+d_G(y)-2\lambda r+1\\
&\geq 2\rho+2-2\lambda r+1\\
&\geq 2((m-1)k_m-n+2)-2\lambda r+3\\
&\geq 2((n-1)(2(2\lambda-1)r-1)-n+2)-2\lambda r+3\\
&= 4(n-1)(2\lambda-1)r-4n+9-2\lambda r\\
&> 2n\lambda r\\
&= 2\lambda r+2(n-1)\lambda r\\
&\geq \alpha+\beta.
\end{aligned}$$

In conclusion, $\gamma_G(S,T;g,f) \geq \alpha_G(S,T;E_1,E_2) + \beta_G(S,T;E_1,E_2)$. According to the choice of S and T, we derive $\gamma_G(S',T';g,f) \geq \alpha_G(S',T';E_1,E_2) + \beta_G(S',T';E_1,E_2)$ for any disjoint vertex subsets S' and T' of G. Using Lemma 2.1, G has a (g,f)-factor F_n with $E_1 \subseteq E(F_n)$ and $E_2 \cap E(F_n) = \emptyset$, and F_n is also a $[0,k_n]$-factor of G. By the definitions of $g(x)$ and $f(x)$, we admit

$$d_{G-F_n}(x) = d_G(x) - d_{F_n}(x) \geq d_G(x) - f(x) \geq 0$$

and

$$\begin{aligned} d_{G-F_n}(x) &= d_G(x) - d_{F_n}(x) \leq d_G(x) - g(x) \\ &\leq d_G(x) - (d_G(x) - (k_1 + k_2 + \cdots + k_{m-1} - n + 2)) \\ &= k_1 + k_2 + \cdots + k_{m-1} - (n-1) + 1 \end{aligned}$$

for any $x \in V(G)$. Therefore, $G-F_n$ is a $[0,k_1+k_2+\cdots+k_{m-1}-(n-1)+1]$-graph. Write $H_i' = H_i - A_i$ for $1 \leq i \leq r$. Obviously, $H_1', H_2', \cdots, H_r'$ are r vertex-disjoint $(n-1)\lambda$-subgraphs of $G - F_n$. By the induction hypothesis, there exists a subgraph R' of $G - F_n$ such that R' admits a $[0,k_i]_1^{n-1}$-factorization randomly λ-orthogonal to every H_i', $1 \leq i \leq r$. Let R be the subgraph of G induced by $E(R') \cup E(F_n)$. Consequently, R is a subgraph of G such that R possesses a $[0,k_i]_1^n$-factorization randomly λ-orthogonal to every H_i, $1 \leq i \leq r$. We finish the proof of Theorem 3.1. □

If we set $r = 1$ and $\lambda = 1$ in Theorem 3.1, then we promptly derive Theorem 1.1. If we let $\lambda = 1$ in Theorem 3.1, then instantly gain Theorem 1.2. Consequently, Theorem 3.1 is a generalization of Theorems 1.1 and 1.2. If we set $r = 1$ in Theorem 3.1, then we get the following corollary.

Corollary 3.1. Let G be a $[0,k_1 + k_2 + \cdots + k_m - n + 1]$-graph, and let H be an $n\lambda$-subgraph of G, where m, n, λ and k_i $(1 \leq i \leq m)$ are positive integers satisfying $1 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq 4\lambda - 3$. Then there exists a subgraph R of G such that R admits a $[0,k_i]_1^n$-factorization randomly λ-orthogonal to H.

Acknowledgements

The authors would like to express their sincere gratitude to the anonymous referees for their careful reading and insightful suggestions on this paper.

References

[1] J. Akiyama, M. Kano, *Almost regular factorization of graphs*, J. Graph Theory **9** (1985), 123–128.
[2] B. Alspach, K. Heinrich, G. Liu, *Contemporary Design Theory–A Collection of Surveys*, John Wiley and Sons, New York, 1992, 13–37.
[3] M. Cai, *[a, b]-factorizations of graphs*, J. Graph Theory **15** (1991), 283–301.
[4] Y. Egawa, M. Kano, *Sufficient conditions for graphs to have (g, f)-factors*, Discrete Math. **151** (1996), 87–90.
[5] L. Euler, *Recherches sur une nouveau espece de quarres magiques*, in Leonhardi Euleri Opera Omnia. Ser. Prima. **7** (1923), 291–392.
[6] H. Feng, *On orthogonal (0, f)-factorizations*, Acta Math. Sci. Ser. B Engl. Ed. **19** (1999), 332–336.
[7] H. Feng, G. Liu, *Orthogonal factorizations of graphs*, J. Graph Theory **40** (2002), 267–276.
[8] W. Gao, W. Wang, D. Dimitrov, *Toughness condition for a graph to be all fractional (g, f, n)-critical deleted*, Filomat **33** (2019), 2735–2746.
[9] J. Horton, *Room designs and one-factorizations*, Aequationes Math. **22** (1981), 56–63.
[10] M. Kano, *[a, b]-factorizations of a graph*, J. Graph Theory **9** (1985), 129–146.
[11] M. Kano, *A sufficient condition for a graph to have [a, b]-factors*, Graphs Combin. **6** (1990), 245–251.
[12] P. Katerinis, *Toughness of graphs and the existence of factors*, Discrete Math. **80** (1990), 81–92.
[13] M. Kouider, Z. Lonc, *Stability number and [a, b]-factors in graphs*, J. Graph Theory **46** (2004), 254–264.
[14] P. C. B. Lam, G. Liu, G. Li, W. Shiu, *Orthogonal (g, f)-factorizations in networks*, Networks **35** (2000), 274–278.
[15] G. Li, C. Chen, G. Yu, *Orthogonal factorizations of graphs*, Discrete Math. **245** (2002), 173–194.
[16] G. Li, G. Liu, *(g, f)-factorizations Orthogonal to a Subgraph in Graphs*, Science China, Series A **49** (1998), 267–272.
[17] G. Liu, *Orthogonal (g, f)-factorizations in graphs*, Discrete Math. **143** (1995), 153–158.
[18] G. Liu, H. Long, *Randomly orthogonal (g, f)-factorizations in graphs*, Acta Math. Appl. Sin. Engl. Ser. **18** (2002), 489–494.
[19] G. Liu, B. Zhu, *Some problems on factorizations with constraints in bipartite graphs*, Discrete Appl. Math. **128** (2003), 421–434.
[20] R. Ma, H. Gao, *On (g, f)-factorizations of graphs*, Appl. Math. Mech. (English Ed.) **18** (1997), 407–410.
[21] H. Matsuda, *Ore-type conditions for the existence of even [2, b]-factors in graphs*, Discrete Math. **304** (2005), 51–61.
[22] Z. Sun, S. Zhou, *A generalization of orthogonal factorizations in digraphs*, Inform. Process. Lett. **132** (2018), 49–54.
[23] W. Wallis, *One factorizations*, Kluwer Academic Publishers, Dordreht, Boston, 1997.

[24] C. Wang, *Orthogonal factorizations in networks*, Int. J. Comput. Math. **88** (2011), 476–483.
[25] C. Wang, *Subgraphs with orthogonal factorizations and algorithms*, European J. Combin. **31** (2010), 1706–1713.
[26] S. Wang, W. Zhang, *Degree conditions for the existence of a* $\{P_2, P_5\}$*-factor in a graph*, RAIRO Oper. Res. **57** (2023), 2231–2237.
[27] S. Wang, W. Zhang, *Isolated toughness for path factors in networks*, RAIRO Oper. Res. **56** (2022), 2613–2619.
[28] S. Wang, W. Zhang, *Research on fractional critical covered graphs*, Probl. Inf. Transm. **56** (2020), 270–277.
[29] S. Wang, W. Zhang, *Some results on star-factor deleted graphs*, Filomat **38** (2024), 1101–1107.
[30] J. Wu, *A sufficient condition for the existence of fractional* (g, f, n)*-critical covered graphs*, Filomat **38** (2024), 2177–2183.
[31] J. Wu, *Path-factor critical covered graphs and path-factor uniform graphs*, RAIRO Oper. Res. **56** (2022), 4317–4325.
[32] G. Yan, *A new result on Alspach's problem*, Graphs Combin. **15** (1999), 365–371.
[33] G. Yan, J. Pan, C. Wong, T. Tokuda, *Decomposition of graphs into* (g, f)*-factors*, Graphs Combin. **16** (2000), 117–126.
[34] S. Zhou, *A neighborhood union condition for fractional* (a, b, k)*-critical covered graphs*, Discrete Appl. Math. **323** (2022), 343–348.
[35] S. Zhou, *Degree conditions and path factors with inclusion or exclusion properties*, Bull. Math. Soc. Sci. Math. Roumanie **66** (2023), 3–14.
[36] S. Zhou, *Path factors and neighborhoods of independent sets in graphs*, Acta Math. Appl. Sin. Engl. Ser. **39** (2023), 232–238.
[37] S. Zhou, *Remarks on restricted fractional* (g, f)*-factors in graphs*, Discrete Appl. Math. **54** (2024), 271–278.
[38] S. Zhou, *Some results on path-factor critical avoidable graphs*, Discuss. Math. Graph Theory **43** (2023), 233–244.
[39] S. Zhou, Q. Bian, Z. Sun, *Two sufficient conditions for component factors in graphs*, Discuss. Math. Graph Theory **43** (2023), 761–766.
[40] S. Zhou, H. Liu, *Two sufficient conditions for odd* $[1, b]$*-factors in graphs*, Linear Algebra Appl. **661** (2023), 149–162.
[41] S. Zhou, H. Liu, Y. Xu, *A note on fractional ID-*$[a, b]$*-factor-critical covered graphs*, Discrete Appl. Math. **319** (2022), 511–516.
[42] S. Zhou, Q. Pan, L. Xu, *Isolated toughness for fractional* $(2, b, k)$*-critical covered graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **24** (2023), 11–18.
[43] S. Zhou, Z. Sun, Q. Bian, *Isolated toughness and path-factor uniform graphs (II)*, Indian J. Pure Appl. Math. **54** (2023), 689–696.
[44] S. Zhou, Z. Sun, H. Liu, *Some sufficient conditions for path-factor uniform graphs*, Aequationes Math. **97** (2023), 489–500.
[45] S. Zhou, Z. Sun, F. Yang, *A result on* $P_{\geq 3}$*-factor uniform graphs*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **23** (2022), 3–8.
[46] S. Zhou, J. Wu, Q. Bian, *On path-factor critical deleted (or covered) graphs*, Aequationes Math. **96** (2022), 795–802.
[47] S. Zhou, T. Zhang, Z. Xu, *Subgraphs with orthogonal factorizations in graphs*, Discrete Appl. Math. **286** (2020), 29–34.