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n-fractional polynomial p-convex functions and related inequalities

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Abstract. In this paper, we introduce a new class of convex functions called n-fractional polynomial p-convex functions. We discuss some properties and present Hermite-Hadamard type inequalities for this generalization. Also when $p = -1$, our results establish a new definition and Hermite-Hadamard inequalities for n-fractional polynomial harmonically convex functions.

1. Introduction and preliminaries

The idea of convexity is not a new one even it occured in some other form in Archimedes' treatment of orbit length. Nowadays, the application of several works on convexity can be directly or indirectly seen in various subjects like real analysis, functional analysis, linear algebra and geometry. Convexity theory has appears as a powerful technique to study a wide class of unrelated problems in pure and applied science. Many articles have been written by many mathematicians on convex functions and inequalities for different classes. It is well known in mathematical analysis that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}, I \neq \emptyset$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. A number of inequalities have been written for convex functions, one of the most famous is the Hermite-Hadamard inequality which is stated as follows (*see, e.g.,* [7]):

If $f : [a, b] \to \mathbb{R}$ is a convex function, then the following inequality is known as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Definition 1.1. *[9] The interval $\mathbb{I}$ is said to be a p-convex set if $(tu^p + (1-t)v^p)^{\frac{1}{p}} \in \mathbb{I}$ for all $u, v \in \mathbb{I}$, $p > 0$ and $t \in [0, 1]$.*

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Definition 1.2. *[9] A function $\psi : I \to \mathbb{R}$ is said to be p-convex if the inequality*

$$\psi[(tu^p + (1-t)v^p)^{\frac{1}{p}}] \leq t\psi(u) + (1-t)\psi(v) \tag{2}$$

holds for all $u, v \in \mathbb{I} = [c, d]$ and $t \in [0, 1]$ where $p > 0$.

It can be easily seen that, for $p = 1$, p-convexity is reduced to the classical convexity of functions defined on $\mathbb{I} \subseteq (0, \infty)$.

Remark 1.3. *[9] An interval I is said to be p-convex set if $(tu^p + (1-t)v^p)^{\frac{1}{p}} \in I$ for all $t, u \in I$ and $t \in [0, 1]$, where $p = 2k + 1$ or $p = m/n$, $n = 2r + 1$, $m = 2t + 1$ and $k, r, t \in N$.*

Remark 1.4. *If $I \subset (0, \infty)$ be a real interval and $p \in R - \{0\}$, then $(tu^p + (1-t)v^p)^{\frac{1}{p}} \in I$ for all $t, u \in I$ and $t \in [0, 1]$.*

According to Remark 1.4, in [5] İşcan gives a different version of the definition of p-convex function as follow:

Definition 1.5. *Let $I \subset (0, \infty)$ be a real interval and $p \in R - \{0\}$. A function $\psi : I \to R$ is said to be a p-convex function, if*

$$\psi\left((tu^p + (1-t)v^p)^{\frac{1}{p}}\right) \leq t\psi(u) + (1-t)\psi(v) \tag{3}$$

for all $t, u \in I$ and $t \in [0, 1]$. If the inequality in (3) is reversed, then ψ is said to be p-concave.

Example 1.6. *Let $\psi : (0, \infty) \to R$, $\psi(u) = u^p$, $p \neq 0$ and $\phi : (0, \infty) \to R$, $\psi(v) = c$, $c \in R$ then ψ and ϕ are both p-convex and p-concave functions.*

Definition 1.7. *[3] Let $\mathbb{I} \subseteq \mathbb{R} \setminus \{0\}$ be an interval. Then a real-valued function $\psi : I \to \mathbb{R}$ is said to be harmonically convex if*

$$\psi\left(\frac{uv}{tu + (1-t)v}\right) \leq t\psi(v) + (1-t)\psi(u) \tag{4}$$

holds for all $u, v \in \mathbb{I} = [c, d]$ and $t \in [0, 1]$.

Definition 1.8. *[2] Let $n \in \mathbb{N}$. A non-negative function $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex function if the ineqaulity*

$$\psi(tu + (1-t)v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u) + \sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v) \tag{5}$$

holds for all $u, v \in I$ and $t \in [0, 1]$. That ψ belongs to the class FPC(I).

We note that every nonnegative convex function is also n-fractional polynomial convex function[2].

Now we give basic definition for n-fractional polynomial p-convex functions and it's properties. Also in special case our definition establish n-fractional polynomial harmonically convex functions. After that we establish Hermite-Hadamard type inequalities for this type convex functions.

2. Main Results

Definition 2.1. *Let $p \in \mathbb{R} \setminus \{0\}$. A non-negative function $\psi : I \subseteq (0, \infty) \to \mathbb{R}$ is called n-fractional polynomial p-convex function if the ineqaulity*

$$\psi\left([tx^p + (1-t)x^p]^{\frac{1}{p}}\right) \leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v) \tag{6}$$

holds for all $u, v \in I$ and $t \in [0, 1]$.

Remark 2.2. *If $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ is a n-fractional polynomial p-convex function then ψ is a non-negative function. Indeed, since ψ is a n-fractional polynomial p-convex function we can write,*

$$\psi(x) = \psi\left([tx^p + (1-t)x^p]^{\frac{1}{p}}\right) \leq \frac{1}{n}\left[\sum_{i=1}^{n} t^{\frac{1}{i}} + \sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\right]\psi(x)$$

for all $x \in I$ and $t \in [0,1]$. Therefore, we have

$$\left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}} + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}} - 1\right]\psi(x) \geq 0 \tag{7}$$

for all $x \in I$ and $t \in [0,1]$. But, since $\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}} \geq t$ and $\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}} \geq 1-t$ for all $t \in [0,1]$. In inequality (7) , $\left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}} + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}} - 1\right] \geq 0$. Thus, we get $\psi(x) \geq 0$ for all $x \in I$.

Remark 2.3. *For $p = 1$ in Definition 2.1, the Definition 2.1 deduces to Definition 1.8.*

Remark 2.4. *For $p = 1$ and $n = 1$ in Definition 2.1, the Definition 2.1 deduces to the definition of classical convex functions.*

Remark 2.5. *If we take $p = -1$ in Definition 2.1, then we get*

$$\psi\left(\frac{uv}{tu + (1-t)v}\right) \leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u) + \sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v). \tag{8}$$

We will call this type functions as n-fractional harmonically convex functions.

Remark 2.6. *For $n = 1$ in Definition 2.1, the definition deduces to Definition 3. So we can say every p-convex function is also a 1-fractional polynomial p-convex function.*

The following proposition shows the sum of two n-fractional polynomial p-convex functions is also n-fractional polynomial p-convex function.

Proposition 2.7. *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus \{0\}$ and $\psi, \phi : I \subseteq (0,\infty) \to \mathbb{R}$ be two n-fractional polynomial p-convex function and where for $n \in N$, $u, v \in I$, $p \geq 0$ and $t \in [0,1]$, then $\psi + \phi$ is also an n-fractional polynomial p-convex function.*

Proof. *Let ψ and ϕ be two n-fractional polynomial p-convex functions, then for all $u, v \in I$ and $t \in [0,1]$ we have;*

$$\begin{aligned}
(\psi+\phi)\left([tu^p + (1-t)v^p]^{\frac{1}{p}}\right) &= \left[\psi\left([tu^p + (1-t)v^p]^{\frac{1}{p}}\right)\right] + \left[\left([tu^p + (1-t)v^p]^{\frac{1}{p}}\right)\right] \\
&\leq \left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v)\right] \\
&+ \left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\phi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\phi(v)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}(\psi+\phi)(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}(\psi+\phi)(v)
\end{aligned}$$

which shows that $\psi + \phi$ is a n-fractional polynomial p-convex function. □

Now we show the n-fractional polynomial p-convexity of the multiplication on an n-fractional function p-convex function with a scalar.

Proposition 2.8. *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus \{0\}$ and $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ be a nonnegative n-fractional polynomial p-convex function and $\lambda \geq 0$. Then $\lambda\psi$ is also a n-fractional polynomial p-convex function.*

Proof. *Let ψ be an n-fractional polynomial p-convex functions, then for all $u, v \in I$ and $t \in [0,1]$ we have;*

$$\begin{aligned}(\lambda\psi)\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right) &= \lambda\left[\psi\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right)\right]\\ &\leq \lambda\left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\phi(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\phi(v)\right]\\ &= \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}(\lambda\psi)(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}(\lambda\psi)(v)\end{aligned}$$

which shows that $\lambda\psi$ is a n-fractional polynomial p-convex function. □

Proposition 2.9. *Let Let $n \in \mathbb{N}$, $p \in \mathbb{R}\setminus\{0\}$ and $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ be a nonnegative n-fractional polynomial p-convex function and then*

$$\psi = Sup\psi_i, i = 1,2,3,....,n$$

is also an n-fractional polynomial p-convex function.

Proof. *Take any $u, v \in I$ and $t \in [0,1]$. Denote $\psi = Sup\psi_i, i = 1,2,3,....,n$ then we get,*

$$\begin{aligned}\psi\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right) &= Sup\psi_i\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right), i = 1,2,3,....,n\\ &\leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}} Sup\psi_i(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}} Sup\psi_i(v)\\ &= \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v).\end{aligned}$$

So,$\psi = Sup\psi_i, i = 1,2,3,....,n$ is also a n-fractional polynomial p-convex function. □

Proposition 2.10. *Let $I \subseteq (0,\infty)$ be a real interval, $p \in \mathbb{R}\setminus\{0\}$ and $\psi : I \to \mathbb{R}$ is a function. Then;*

1. *If $p \leq 1$ and ψ is n-fractional polynomial convex and nondecreasing function, then ψ is n-fractional polynomial p-convex.*
2. *If $p \geq 1$ and ψ is n-fractional polynomial p-convex and nondecreasing function, then ψ is n-fractional polynomial convex.*
3. *If $p \geq 1$ and ψ is n-fractional polynomial convex and nonincreasing function, then ψ is n-fractional polynomial p-convex.*
4. *If $p \leq 1$ and ψ is n-fractional polynomial p-convex and nonincreasing function, then ψ is n-fractional polynomial convex.*

Proof.

1. *If $p \leq 1$ then,*

$$[tu^p+(1-t)v^p]^{\frac{1}{p}} \leq tu+(1-t)v.$$

Also if ψ is nondecreasing function and n-fractional polynomial convex function we have,

$$\begin{aligned}\psi\left([tu^p+(1-t)v^p]^{\frac{1}{p}}\right) &\leq \psi(tu+(1-t)v)\\ &\leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v).\end{aligned}$$

So ψ is a n-fractional polynomial p-convex function.

2. *If $p \geq 1$ then,*

$$[tu^p + (1-t)v^p]^{\frac{1}{p}} \geq tu + (1-t)v.$$

Also if ψ is nondecreasing function and n-fractional polynomial p-convex function we have,

$$\begin{aligned} \psi\left(tu + (1-t)v\right) &\leq \psi\left([tu^p + (1-t)v^p]^{\frac{1}{p}}\right) \\ &\leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(v). \end{aligned}$$

So ψ is a n-fractional polynomial convex function.

The proof of other enumerated items can be done in a similar way. □

Theorem 2.11. *Let $0 < a < b$, $\psi : [a,b] \to \mathbb{R}$ be an n-fractional polynomial p-convex functions. If $\psi \in L[a,b]$, where $p \in \mathbb{R} \setminus \{0\}$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{n\psi\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{2\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}} \leq \frac{p}{b^p - a^p}\int_a^b \frac{\psi(x)dx}{x^{1-p}} \leq \frac{\psi(a)+\psi(b)}{n}\sum_{i=1}^{n}\frac{i}{i+1}. \tag{9}$$

Proof. *Fix $a, b \in R$, $p > 0$ and $t \in [0,1]$. Then, for every $i \in I$, by the definition of n-fractional polynomial p-convex function of ψ, we have;*

$$\begin{aligned} \psi\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &= \psi\left(\left[\frac{ta^p + (1-t)b^p + (1-t)a^p + tb^p}{2}\right]^{\frac{1}{p}}\right) \\ &= \psi\left(\left[\frac{\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)^p}{2} + \frac{\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)^p}{2}\right]^{\frac{1}{p}}\right) \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}\psi\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{2}\right)^{\frac{1}{i}}\psi\left([tb^p + (1-t)a^p]^{\frac{1}{p}}\right). \end{aligned} \tag{10}$$

Integration in the last inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} \psi\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}\int_0^1 \psi\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)dt \\ &+ \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}\int_0^1 \psi\left([tb^p + (1-t)a^p]^{\frac{1}{p}}\right)dt \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}\left[\int_0^1 \psi\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)dt + \int_0^1 \psi\left([tb^p + (1-t)a^p]^{\frac{1}{p}}\right)dt\right]. \end{aligned}$$

So we get;

$$\frac{n\psi\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{2\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}} \leq \frac{p}{b^p - a^p}\int_a^b \frac{\psi(x)dx}{x^{1-p}}. \tag{11}$$

On the other hand we have;

$$\begin{aligned}\frac{p}{b^p-a^p}\int_a^b \frac{\psi(x)}{x^{1-p}}dx &= \int_0^1 \psi\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)dt \\ &\le \int_0^1 \left[\frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}}\psi(a)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}}\psi(b)\right]dt \\ &= \frac{\psi(a)}{n}\sum_{i=1}^{n}\int_0^1 t^{\frac{1}{i}}dt+\frac{\psi(b)}{n}\sum_{i=1}^{n}\int_0^1 (1-t)^{\frac{1}{i}}dt \\ &= \frac{\psi(a)+\psi(b)}{n}\sum_{i=1}^{n}\frac{i}{i+1}.\end{aligned} \tag{12}$$

So we get the desired result. □

Remark 2.12. *Imposing some conditions of Theorem2.11, we get different two versions of Hermite-Hadamard type inequality.*

1. *For $n = 1$ and $p = 1$, we obtain Hermite-Hadamard type inequality (1) for classical convex functions [7].*
2. *For $p = 1$, we obtain Hermite-Hadamard type inequality for n-fractional polynomial convex functions proved in [2].*

Corollary 2.13. *For $p = -1$ in Theorem 2.11 we get Hermite-Hadamard type inequality for n-fractional polynomial harmonically convex functions;*

$$\frac{n\psi\left(\frac{2ab}{a+b}\right)}{2\sum_{i=1}^{n}\frac{1}{2}^{\frac{1}{i}}} \le \frac{ab}{b-a}\int_a^b \frac{\psi(x)dx}{x^2} \le \frac{\psi(a)+\psi(b)}{n}\sum_{i=1}^{n}\frac{i}{i+1}. \tag{13}$$

Remark 2.14. *For $n = 1$ in Corollary 2.13, the inequality (13) deduces to Hermite-Hadamard type inequality for harmonically convex functions proved in [3] .*

In [8] the following lemma is given, which will be helpful for generating refinements of Hermite-Hadamard type inequality.

Lemma 2.15. *Let $0 < a < b$ and $\psi : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o. If $\psi' \in L[a, b]$, then*

$$\begin{aligned}&\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{\psi(x)}{x^{1-p}}dx \\ = \ &\frac{b^p-a^p}{2p}\int_0^1 (1-2t)(ta^p+(1-t)b^p)^{\frac{1}{p}-1}\psi'(ta^p+(1-t)b^p)^{\frac{1}{p}}dt.\end{aligned} \tag{14}$$

Theorem 2.16. *Let $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$ and assume that $|\psi'| \in L[a, b]$. If ψ' is an n-fractional polynomial p-convex function on the interval $[a, b]$, then the following inequality holds for $t \in [0, 1]$:*

$$\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_0^1 \frac{\psi(x)}{x^{1-p}}dx\right| \le \frac{b^p-a^p}{2np}\sum_{i=1}^{n}\left[|\psi'(a)|A_p(i)+|\psi'(b)|B_p(i)\right] \tag{15}$$

where

$$A_p(i) = \int_0^1 |1-2t|t^{\frac{1}{i}}|ta^p+(1-t)b^p|^{\frac{1}{p}-1}dt$$

and

$$B_p(i) = \int_0^1 |1-2t|(1-t)^{\frac{1}{i}}|ta^p+(1-t)b^p|^{\frac{1}{p}-1}dt.$$

Proof. *Using the definition of n-fractional polynomial p-convexity and Lemma2.15 we get;*

$$\begin{aligned}
&\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{\psi(x)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{2p}\int_0^1|1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}|\psi'(ta^p+(1-t)b^p)|^{\frac{1}{p}}dt \\
\leq\ & \frac{b^p-a^p}{2p}\left(\int_0^1|1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}\left[\frac{1}{n}\sum_{i=1}^n t^{\frac{1}{i}}|\psi'(a)|+\frac{1}{n}\sum_{i=1}^n(1-t)^{\frac{1}{i}}\right]dt\right) \\
=\ & \frac{b^p-a^p}{2np}\left(|\psi'(a)|\int_0^1|1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}\sum_{i=1}^n t^{\frac{1}{i}}dt\right) \\
+\ & \frac{b^p-a^p}{2np}\left(|\psi'(b)|\int_0^1|1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}\sum_{i=1}^n(1-t)^{\frac{1}{i}}dt\right) \\
=\ & \frac{b^p-a^p}{2np}\sum_{i=1}^n\left[|\psi'(a)|A_p(i)+|\psi'(b)|B_p(i)\right]. \qquad (16)
\end{aligned}$$

□

Remark 2.17. *Imposing some conditions of Theorem 2.16, we get two different versions of our inequality.*

1. *For $n = 1$ and $p = 1$, we obtain the inequality for classical convex functions [2].*
2. *For $p = 1$, we obtain the inequality for n-fractional polynomial convex functions proved in [2].*

Corollary 2.18. *For $p = -1$ in Theorem2.16 we get following inequality for n-fractional polynomial harmonically convex functions:*

$$\left|\frac{\psi(a)+\psi(b)}{2}-\frac{ab}{b-a}\int_0^1\frac{\psi(x)}{x^2}dx\right|\leq\frac{(b-a)ab}{2n}\sum_{i=1}^n\left[|\psi'(a)|A(i)+|\psi'(b)|B(i)\right] \qquad (17)$$

where

$$A(i)=\int_0^1|1-2t|t^{\frac{1}{i}}|tb+(1-t)a|^{-2}dt$$

and

$$B(i)=\int_0^1|1-2t|(1-t)^{\frac{1}{i}}|tb+(1-t)a|^{-2}dt.$$

Theorem 2.19. *Let $\psi : I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I^o, $a, b\in I^o$ with $a<b$, $q>1$, $\frac{1}{k}+\frac{1}{q}=1$ and assume that $\psi'\in L[a,b]$. If $|\psi'|^q$ is an n-fractional polynomial p-convex function on the interval $[a,b]$, then the following inequality holds for $t\in[0,1]$:*

$$\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{\psi(x)}{x^{1-p}}dx\right|\leq\frac{b^p-a^p}{2p}\left(C_{p,k}\right)^{\frac{1}{k}}\left(\frac{2}{n}\sum_{i=1}^n\frac{i}{i+1}\right)^{\frac{1}{q}}\left(\frac{|\psi'(a)|^q+|\psi'(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (18)$$

where

$$C_{p,k}=\int_0^1\frac{|1-2t|^k}{|(ta^p+(1-t)b^p)^{1-\frac{1}{p}}|^k}dt.$$

Proof. Using Lemma 2.15, the definition of n-fractional polynomial p-convexity and Hölder inequality we get;

$$\begin{aligned}
&\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{\psi(x)}{x^{1-p}}dx\right| \\
\leq\ & \left|\frac{b^p-a^p}{2p}\int_0^1 |1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}\psi'(ta^p+(1-t)b^p)^{\frac{1}{p}}dt\right| \\
\leq\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|1-2t|^k}{|(ta^p+(1-t)b^p)^{1-\frac{1}{p}}|^k}dt\right)^{\frac{1}{k}} \times \left(\int_0^1 |\psi'(ta^p+(1-t)b^p)|^q\right)^{\frac{1}{q}} dt \\
\leq\ & \frac{b^p-a^p}{2p}\left(C_{p,k}\right)^{\frac{1}{k}} \times \left(\frac{|\psi'(a)|^q}{n}\sum_{i=1}^n \int_0^1 t^{\frac{1}{i}}dt + \frac{|\psi'(b)|^q}{n}\sum_{i=1}^n \int_0^1 (1-t)^{\frac{1}{i}}dt\right)^{\frac{1}{q}} \\
=\ & \frac{b^p-a^p}{2p}\left(C_{p,k}\right)^{\frac{1}{k}}\left(\frac{2}{n}\sum_{i=1}^n \frac{i}{i+1}\right)^{\frac{1}{q}}\left(\frac{|\psi'(a)|^q+|\psi'(b)|^q}{2}\right)^{\frac{1}{q}}. \qquad (19)
\end{aligned}$$

□

Remark 2.20. *Imposing some conditions of Theorem 2.19, we get two different versions of our inequality.*

1. *For $n = 1$ and $p = 1$, we obtain the inequality for classical convex functions[2].*
2. *For $p = 1$, we obtain the inequality for n-fractional polynomial convex functions proved in [2].*

Corollary 2.21. *For $p = -1$ in Theorem 2.19 we get following inequality for n-fractional polynomial harmonically convex functions:*

$$\left|\frac{\psi(a)+\psi(b)}{2}-\frac{b-a}{ab}\int_a^b \frac{\psi(x)}{x^2}dx\right| \leq \frac{(b-a)ab}{2}\left(C_k\right)^{\frac{1}{k}}\left(\frac{2}{n}\sum_{i=1}^n \frac{i}{i+1}\right)^{\frac{1}{q}}\left(\frac{|\psi'(a)|^q+|\psi'(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (20)$$

$$C_k = \int_0^1 \frac{|1-2t|^k}{|(tb+(1-t)a)^2|^k}dt.$$

Theorem 2.22. *Let $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$, $q > 1$, and assume that $\psi' \in L[a,b]$. If $|\psi'|^q$ is an n-fractional polynomial p-convex function on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$:*

$$\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{\psi(x)}{x^{1-p}}dx\right| \leq \frac{b^p-a^p}{2p}\left(C_p\right)^{1-\frac{1}{q}}\left(\frac{|\psi'(a)|^q}{n}D_{p,i}+\frac{|\psi'(b)|^q}{n}E_{p,i}\right)^{\frac{1}{q}} \qquad (21)$$

where

$$C_p = \int_0^1 \frac{|1-2t|}{|(ta^p+(1-t)b^p)^{1-\frac{1}{p}}|}dt,$$

$$D_{p,i} = \sum_{i=1}^n \int_0^1 \frac{|1-2t|t^{\frac{1}{i}}}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}dt$$

and

$$E_{p,i} = \sum_{i=1}^n \int_0^1 \frac{|1-2t|(1-t)^{\frac{1}{i}}}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}dt.$$

Proof. *Using the definition of n-fractional polynomial p-convexity and Lemma 2.15 we get;*

$$\begin{aligned}
&\left|\frac{\psi(a)+\psi(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{\psi(x)}{x^{1-p}}dx\right| \\
\leq\ & \left|\frac{b^p-a^p}{2p}\int_0^1 |1-2t||ta^p+(1-t)b^p|^{\frac{1}{p}-1}\psi'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt\right| \\
\leq\ & \frac{b^p-a^p}{2p}\left(\int_0^1\frac{|1-2t|}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1\frac{|1-2t|}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}\left|\psi'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}dt \\
\leq\ & \frac{b^p-a^p}{2p}\left(C_p\right)^{1-\frac{1}{q}}\left[\frac{|\psi'(a)|^q}{n}\left(\sum_{i=1}^n\int_0^1\frac{|1-2t|t^{\frac{1}{i}}}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}dt\right)\right. \\
+\ & \left.\frac{|\psi'(b)|^q}{n}\left(\sum_{i=1}^n\int_0^1\frac{|1-2t|(1-t)^{\frac{1}{i}}}{|(ta^p+(1-t)b^p)|^{1-\frac{1}{p}}}dt\right)\right]^{\frac{1}{q}} \\
=\ & \frac{b^p-a^p}{2p}\left(C_p\right)^{1-\frac{1}{q}}\left(\frac{|\psi'(a)|^q}{n}D_{p,i}+\frac{|\psi'(b)|^q}{n}E_{p,i}\right)^{\frac{1}{q}}. \qquad (22)
\end{aligned}$$

□

Remark 2.23. *Imposing some conditions of Theorem 2.22, we get two different versions of our inequality.*

1. *For $n=1$ and $p=1$, we obtain the inequality for classical convex functions [2].*
2. *For $p=1$, we obtain the inequality for n-fractional polynomial convex functions [2].*

Corollary 2.24. *For $p=-1$ in Theorem 2.22 we get following inequality for n-fractional polynomial harmonically convex functions:*

$$\begin{aligned}
\left|\frac{\psi(a)+\psi(b)}{2}-\frac{(b-a)ab}{2}\int_a^b\frac{\psi(x)}{x^2}dx\right| \leq\ & \frac{b-a}{2ab}(C)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|}{|(tb+(1-t)a)^2|}dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\frac{|\psi'(a)|^q}{n}D_{p,i}+\frac{|\psi'(b)|^q}{n}E_{p,i}\right)^{\frac{1}{q}} \qquad (23)
\end{aligned}$$

where

$$C=\int_0^1\frac{|1-2t|}{|(tb+(1-t)a)^2|}dt,$$

$$D_i=\int_0^1\frac{|1-2t|}{|(tb+(1-t)a)^2|}\sum_{i=1}^n t^{\frac{1}{i}}dt$$

and

$$E_i=\int_0^1\frac{|1-2t|}{|(tb+(1-t)a)^2|}\sum_{i=1}^n(1-t)^{\frac{1}{i}}dt.$$

Remark 2.25. *For $n=1$ in Corollary 2.24, the inequality deduces to inequality for harmonically convex functions proved in [3] .*

3. Conclusion

In this article we introduce a new class of convex functions called n-fractional polynomial p-convex functions. And we derive some basic results and propositions related to our new generalization. Also in Definition2.1 setting $p = -1$ we get a new class of convex functions called n-fractional polynomial harmonically convex functions. We obtain Hermite-Hadamard type inequalities by using these definitions. Researchers also can derive new inequalities or generalizations for these definitions.

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