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Chlodowsky variant of generalised Jain-Appell operators

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Abstract. In this paper we consider a Chlodowsky variant of Jain-Appell operators which includes many known and new defined operators such as Chlodowsky variant of generalised Jakimovski-Leviatan, Jain-Appell, Appell-Baskakov and Appell-Lupaş operators. These operators are constructed in terms of the function ϱ and their weighted approximation to the identity operator is given in the weighted space with weight $\varphi = 1 + \varrho^2$ by using the Korovkin set $\{1, \varrho, \varrho^2\}$. We investigate a quantitative error estimate of the operators by using Holhos' weighted modulus of continuity and obtain the local approximation properties in terms of the first and the second modulus of continuities and Lipschitz class maximal function. Furthermore, the Voronovskaja type asymptotic formula is also obtained.

1. Introduction

The task of finding approximation of functions by simpler functions such as polynomials is what approximation theory is all about. Bernstein [13] was the first to build a sequence of positive linear operators to prove the Weierstrass approximation theorem. Several linear positive operators have been constructed since then to examine approximation properties in various spaces such as Szasz, Baskakov, Lupaş, Meyer König and Zeller, Bleimann-Butzer-Hann operators. Especially in the last two decades, there has been an increasing interest in the investigation of certain linear positive operators ([1], [23], [24], [35], [42]).
In approximating a continuous function on the unbounded interval $[0, \infty)$, one of the famous operators is the Szasz-Mirakjan operators given by

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

where $n \in \mathbb{N}$, $x \in [0, \infty)$ and f is a sufficiently nice function that ensures convergence of the above sum and belongs to a subspace of $C[0, \infty)$, the space of continuous functions defined on $[0, \infty)$. Recently, several new results on various modifications or generalizations of Szasz-Mirakjan operators have been published. We list some of them as follows :
The Jain-Pethe [31] operators are defined by

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$$S_n^{(\alpha)}(f;x)=\frac{1}{(1+n\alpha)^{(x/\alpha)}}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\left(\frac{n}{1+n\alpha}\right)^k\frac{x^{(k,-\alpha)}}{k!},$$

where

$$\begin{aligned}x^{(k,-\alpha)} &= x(x+\alpha)(x+2\alpha)\cdots(x+(k-1)\alpha), \quad (k\in\mathbb{N}:=1,2,\cdots)\\ x^{(0,-\alpha)} &= 1.\end{aligned}$$

These operators are the gamma transform of the Szasz-Mirakjan operators.
Generalized Szasz-Mirakyan operators [9] are defined by

$$\begin{aligned}S_n^{p}(f;x) &= \exp(-n\varrho(x))\sum_{k=0}^{\infty}(f\circ\varrho^{-1})\left(\frac{k}{n}\right)\frac{(n\varrho(x))^k}{k!}\\ &= (S_n(f\circ\varrho^{-1})\circ\varrho)(x)=\sum_{k=0}^{\infty}f\left(\varrho^{-1}\left(\frac{k}{n}\right)\right)\mathcal{P}_{\varrho,n,k}(x)\end{aligned}$$

where

$$\mathcal{P}_{\varrho,n,k}(x)=\exp(-n\varrho(x))\frac{(n\varrho(x))^k}{k!}$$

and the function $\varrho(x)$ satisfies the following conditions:

(a) ϱ is continuously differentiable function on $\mathbb{R}^+$.
(b) $\varrho(0)=1$ and $\inf_{x\geq 0}\varrho'(x)\geq 1$.

It is clear that in the case $\varrho(x)=x$ in the above definition, we recover the usual Szazs-Mirakyan operators. There are other functions which satisfy properties (a) and (b) such as $\varrho_m(x)=\sum_{k=1}^{m}x^k$, $x\in\mathbb{R}^+$. The authors investigated the approximation properties of these operators in the space $C_\varphi^k(\mathbb{R}^+)$ (see section 2) for the definition where $\varphi=1+\varrho^2$, by using the Korovkin theorem proved in [25], which uses the test functions 1, ϱ and ϱ^2. Here $B_\varphi[0,\infty)=\left\{f:[0,\infty)\to[0,\infty)\mid \|f\|_\varphi=\sup_{x\geq 0}\frac{f(x)}{\varphi(x)}<\infty\right\}$ and $C_\varphi[0,\infty)$ denotes the subspace of all continuous function belonging to $B_\varphi[0,\infty)$ and $C_\varphi^k[0,\infty))$ denotes the subspace of all functions $f\in C_\varphi[0,\infty)$ with the property $\lim_{x\to\infty}\frac{f(x)}{\varphi(x)}=K_f$, where K_f is a constant depending on f.
Jakamovski and Leviatan [32] introduced the operators

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right),\quad x\geq 0, \tag{1}$$

by using Appell polynomials, where $\{p_k(x)\}_{k\geq 0}$ are the Appell polynomials which satisfy the property

$$p_k'(x)=kp_{k-1}(x).$$

An equivalent definition of the Appell polynomials can be given by means of the generating function

$$g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k,$$

where

$$g(u)=\sum_{n=0}^{\infty}a_nu^n,\quad g(1)\neq 1 \tag{2}$$

is an analytic function in the disc $|u|<r$, $(r>1)$.
They also have the explicit form (another equivalent definition) as

$$p_k(x)=\sum_{i=0}^{k}a_i\frac{x^{k-i}}{(k-i)!},\quad k\in\mathbb{N}.$$

Operators given in equation (1) are linear and positive operators when $a_i\geq 0$,($i=0,1,2,...$). Special Appell polynomials such as Apostol-Genocchi polynomials have been recently used as a basis in the construction of integral variant of Baskakov type operators in [43] and [21]. On the other hand different approaches about the Jakimovski-Leviatan type operators have been proposed in the very recent papers [3] and [4] which give rise to some interesting properties.
Given a function $\phi(x)$ on the interval $(0,\beta)$, $\beta>0$ and linear positive operator $L_n(\phi(y);x)$, the Chlodowsky's approach is to propose new operator $C_n\left(\phi(\beta y);\frac{x}{\beta}\right)$ with $\beta:=\beta_n$ such that $\beta_n\to\infty$ and $\frac{\beta_n}{n}\to 0$, then the operator $C_n(\phi)$ is investigated for some classes of function $\phi(x)$ on the unbounded interval. When β is finite, this approach has been used in the papers [10], [18], [39], [45], [46].
On the other hand, when ϕ is defined on the interval $(0,\infty)$, Chlodowsky's approach has been modified by the new operator $\mathtt{C}_n(\phi;x)=L_n\left(\phi(by);\frac{x}{b}\right)$ with the conditions $b:=b_n$ such that $b_n\to\infty$ and $\frac{b_n}{n}\to 0$. The modified approach has been taken in to account in the papers [11], [12], [14], [20], [28], [33], [38], [40], [41].
Chlodowsky type of Jakimovski-Leviatan operators [15] are defined by

$$C_n^{*}(f;x)=\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)f\left(\frac{k}{n}b_n\right). \tag{3}$$

Here $\{b_n\}$ is a positive increasing sequence which satisfies $lim_{n\to\infty}b_n=\infty$ and $\lim_{n\to\infty}\frac{b_n}{n}=0$.
Recently, Jain-Appell operators [42] have been defined by

$$C_n^{\alpha}(f;x)=\frac{1}{g(1)(1+n\alpha)^{(x/\alpha)}}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)p_k^{\sim(\alpha)}(x;n) \tag{4}$$

where

$$p_k^{\sim(\alpha)}(x;n)=\sum_{i=0}^{k}\frac{a_i}{(k-i)!}\left(\frac{n}{1+n\alpha}\right)^{k-i}x^{(k-i,-\alpha)}$$

and the cofficients a_i are given in (2).
These operators are the gamma transform of the Jakimovski-Leviatan operators. It is pointed out that these operators include the Jain-Pethe operators and many interesting new operators such as Appell-Baskakov and Appell-Lupaş operators. Therefore it has been shown that the Appell polynomials can be used to extend the usual Baskakov operators and Lupaş operators in the Jakimovski-Leviatan sense.
Inspired and motivated by the above mentioned operators, we introduce the Chlodowsky variant of the generalized Jain-Appell operators by

$$C_n^{\varrho,(\alpha)}(f;x)\quad=\quad\frac{1}{(1+\frac{n}{b_n}\alpha)^{(\varrho(x)/\alpha)}g(1)}\times\sum_{k=0}^{\infty}(f\circ\varrho^{-1})\left(\frac{k}{n}b_n\right)A_k^{\sim(\alpha)}\left(\varrho(x);\frac{n}{b_n}\right) \tag{5}$$

where

$$A_k^{\sim(\alpha)}\left(\varrho(x);\frac{n}{b_n}\right)=\sum_{i=0}^{k}\frac{a_i}{(k-i)!}\left(\frac{\frac{n}{b_n}}{1+\frac{n}{b_n}\alpha}\right)^{k-i}\varrho(x)^{(k-i,-\alpha)},$$

with $\varrho(x)$ satisfies (a) and (b) and the cofficients a_i comes from (2). We also assume that $\{b_n\}$ is a positive increasing sequence which satisfies $lim_{n\to\infty}b_n=\infty$ and $\lim_{n\to\infty}\frac{b_n}{n}=0$. It is obvious that, the operators are positive and linear when $a_i\geq 0$.
These operators include many known and new defined operators. We list some of them below.

- By taking $\varrho(x)=x$ in (5), we have the operators

$$C_n^{(\alpha)}(f;x)=\frac{1}{(1+\frac{n}{b_n}\alpha)^{(x/\alpha)}g(1)}\sum_{k=0}^{\infty}f\left(\frac{k}{n}b_n\right)A_k^{\sim(\alpha)}\left(x;\frac{n}{b_n}\right)\tag{6}$$

  where

$$A_k^{\sim(\alpha)}\left(x;\frac{n}{b_n}\right)=\sum_{i=0}^{k}\frac{a_i}{(k-i)!}\left(\frac{\frac{n}{b_n}}{1+\frac{n}{b_n}\alpha}\right)^{k-i}x^{(k-i,-\alpha)}.$$

  We call these operators as Chlodowsky variant of Jain-Appell operators.

- Taking $g(x)=1$ in (5), we get the Chlodowsky variant of generalised Jain-Pethe operators defined as

$$S_{\frac{n}{b_n}}^{\varrho,(\alpha)}(f;x)=\frac{1}{(1+\frac{n}{b_n}\alpha)^{(\varrho(x)/\alpha)}}\sum_{k=0}^{\infty}(f\circ\varrho^{-1})\left(\frac{k}{n}b_n\right)A_k^{\sim(\alpha)}(\varrho(x);\frac{n}{b_n}),\tag{7}$$

  and further, by taking $\varrho(x)=x$, we get the Chlodowsky variant of Jain-Pethe operators.

- Letting $\alpha\to 0^+$ in (5), we recover the generalised version of the Chlodowsky variant of Jakimovski-Leviatan operators given by

$$C_n^*(f;x)=\frac{e^{-\frac{n}{b_n}\varrho(x)}}{g(1)}\sum_{k=0}^{\infty}A_k\left(\frac{n}{b_n}\varrho(x)\right)(f\circ\varrho^{-1})\left(\frac{k}{n}b_n\right).\tag{8}$$

  Note that, by taking $\varrho(x)=x$, we get the Chlodowsky type Jakimovski-Leviatan operators given in (3).

- Choosing $\alpha=\alpha_n(x)=\frac{\varrho(x)}{n}$, where $x\geq 0$, $n\in\mathbb{N}$ in (5) , we get the following operators where we call them Chlodowsky variant of generalised Appell-Baskakov operators

$$B_n^{\varrho,C}(f;x):=C_n^{\varrho,\left(\frac{\varrho(x)}{n}\right)}(f;x)=\frac{1}{\left(\frac{b_n+\varrho(x)}{b_n}\right)^n g(1)}\sum_{k=0}^{\infty}(f\circ\varrho^{-1})\left(\frac{k}{n}b_n\right)\sum_{i=0}^{k}a_i\binom{n+k-i-1}{n-1}\left(\frac{\varrho(x)}{b_n+\varrho(x)}\right)^{k-i}.\tag{9}$$

Taking $\varrho(x) = x$ in the above operator, we get

$$B_n^C(f;x) := C_n^{\left(\frac{x}{n}\right)}(f;x) = \frac{1}{\left(\frac{b_n+x}{b_n}\right)^n g(1)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) \sum_{i=0}^{k} a_i \binom{n+k-i-1}{n-1} \left(\frac{x}{b_n+x}\right)^{k-i} \tag{10}$$

where we call them as Chlodowsky variant of Appell-Baskakov operators. Note that in the case $\varrho(x) = x$ and $g(1) = 1$, we recover the Chlodowsky variant of Baskakov operators.

- Choosing $\alpha = \alpha_n(x) = \frac{b_n}{n}$, where $x \geq 0$ and $n \in \mathbb{N}$ in (5), we have

$$L_n^{\varrho,C}(f;x) := C_n^{\varrho,\left(\frac{b_n}{n}\right)}(f;x) = \frac{1}{2^{\left(\frac{n\varrho(x)}{b_n}\right)} g(1)} \sum_{k=0}^{\infty} (f \circ \varrho^{-1}) \left(\frac{k}{n}b_n\right) A_k^{\sim\left(\frac{b_n}{n}\right)} \left(\varrho(x); \frac{n}{b_n}\right) \tag{11}$$

where

$$A_k^{\sim\left(\frac{b_n}{n}\right)} \left(\varrho(x); \frac{n}{b_n}\right) = \sum_{i=0}^{k} \frac{a_i}{(k-i)!} \left(\frac{n}{2b_n}\right)^{k-i} (\varrho(x))^{(k-i,-\alpha)}.$$

which can be called as Chlodowsky variant of generalised Appell-Lupaş operators. Taking $\varrho(x) = x$ in the above operator, we get

$$L_n^C(f;x) := C_n^{\left(\frac{b_n}{n}\right)}(f;x) = \frac{1}{2^{\left(\frac{nx}{b_n}\right)} g(1)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) A_k^{\sim\left(\frac{b_n}{n}\right)} \left(x; \frac{n}{b_n}\right) \tag{12}$$

where

$$A_k^{\sim\left(\frac{b_n}{n}\right)} \left(x; \frac{n}{b_n}\right) = \sum_{i=0}^{k} \frac{a_i}{(k-i)!} \left(\frac{n}{2}\right)^{k-i} x^{(k-i,-\alpha)}.$$

where we call them as Chlodowsky variant of Appell-Lupaş operator.

The paper is organized as follows:
In Section 2, we investigate weighted approximation properties of Chlodowsky variant of generalised Jain-Appell polynomials. In section 3, we study a weighted quantitative type theorem for the rate of this convergence by using Holhos's [29] modulus of continuity. In Section 4, we obtain some local approximation results related to Petree's K-functional, the first and the second modulus of continuity and Lipschitz class maximal function. In Section 5 , we obtain Voronovskaya type asymptotic formula for the operators $C_n^{(\alpha)}(f;x)$ our operators.

2. Weighted approximation properties

We start this section by showing that the operators given in (5) are the gamma transformation of the Chodowsky type Jakimovski-Leviatan operators. We then compute the first few moments and central moments of these operators.

Theorem 2.1. *For $\alpha > 0$ be fixed, the operators given in* (5) *can be represented as*

$$C_n^{\varrho,(\alpha)}(f;x) = \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)}\int_0^\infty e^{-t}t^{(\varrho(x)/\alpha)-1}C_n^*(f;\alpha t)dt$$

where the function $\varrho(x)$ satisfies the conditions (a) *and* (b) *and f is sufficiently nice function which guarantees the uniform convergence of the series in* (5)*.*

Proof. Under the hypothesis of the theorem, direct calculation yields

$$\begin{aligned}
C_n^{\varrho,(\alpha)}(f;x) &= \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)}\int_0^\infty e^{-t}t^{(\varrho(x)/\alpha)-1}C_n^*(f;\alpha t)dt \\
&= \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)}\int_0^\infty e^{-t}t^{(\varrho(x)/\alpha)-1}\frac{e^{-\frac{n}{b_n}\alpha t}}{g(1)}\sum_{k=0}^\infty p_k\left(\frac{n}{b_n}\alpha t\right)f\left(\frac{k}{n}b_n\right)dt \\
&= \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)}\int_0^\infty e^{-t}t^{(\varrho(x)/\alpha)-1}\frac{e^{-\frac{n}{b_n}\alpha t}}{g(1)}\times\sum_{k=0}^\infty\sum_{i=0}^k a_i\frac{(n\alpha t)^{k-i}}{b_n^{k-i}(k-i)!}f\left(\frac{k}{n}b_n\right)dt \\
&= \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)g(1)}\sum_{k=0}^\infty f\left(\frac{k}{n}b_n\right)\sum_{i=0}^k a_i\frac{(n\alpha)^{k-i}}{b_n^{k-i}(k-i)!}\times\int_0^\infty e^{-(1+\frac{n}{b_n}\alpha)t}t^{(\varrho(x)/\alpha)+k-i-1}dt \\
&= \frac{1}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)g(1)}\sum_{k=0}^\infty f\left(\frac{k}{n}b_n\right)\times\sum_{i=0}^k a_i\frac{(n\alpha)^{k-i}}{b_n^{k-i}(k-i)!}\frac{\Gamma(\frac{\varrho(x)}{\alpha}+k-i)}{(1+\frac{n}{b_n}\alpha)^{(\varrho(x)/\alpha)+k-i}}.
\end{aligned}$$

On the other hand since

$$\begin{aligned}
\frac{\Gamma(\frac{\varrho(x)}{\alpha}+k-i)}{\Gamma(\frac{\varrho(x)}{\alpha})} &= \frac{\left(\frac{\varrho(x)}{\alpha}+k-i-1\right)...\left(\frac{\varrho(x)}{\alpha}\right)\Gamma\left(\frac{\varrho(x)}{\alpha}\right)}{\Gamma\left(\frac{\varrho(x)}{\alpha}\right)} \\
&= \frac{\varrho(x)(\varrho(x)+\alpha)...(\varrho(x)+(k-i-1)\alpha)}{\alpha^{k-i}} = \frac{(\varrho(x))^{k-i,-\alpha}}{\alpha^{k-i}}
\end{aligned}$$

with $x^{(0,-\alpha)} = 1$, we get the following family of linear positive operators:

$$C_n^{\varrho,(\alpha)}(f;x) = \frac{1}{(1+\frac{n}{b_n}\alpha)^{(\varrho(x)/\alpha)}g(1)}\sum_{k=0}^\infty f\left(\frac{k}{n}b_n\right)A_k^{\sim(\alpha)}\left(\varrho(x);\frac{n}{b_n}\right)$$

where

$$A_k^{\sim(\alpha)}\left(\varrho(x);\frac{n}{b_n}\right) = \sum_{i=0}^k\frac{a_i}{(k-i)!}\left(\frac{\frac{n}{b_n}}{1+\frac{n}{b_n}\alpha}\right)^{k-i}\varrho(x)^{(k-i,-\alpha)},$$

which is exactly (5).

□

Lemma 2.2. *For the first few moments of operators defined in* (5), *we have*

$$C_n^{\varrho,(\alpha)}(1;x) = 1 \tag{13}$$

$$C_n^{\varrho,(\alpha)}(\varrho(t);x) = \varrho(x) + \frac{b_n}{n}\frac{g'(1)}{g(1)} \tag{14}$$

$$\begin{aligned} C_n^{\varrho,(\alpha)}(\varrho^2(t);x) &= \varrho^2(x) + \alpha\varrho(x) + \varrho(x)\frac{b_n}{n}\frac{g(1)+2g'(1)}{g(1)} \\ &+ \frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)} \end{aligned} \tag{15}$$

$$\begin{aligned} C_n^{\varrho,(\alpha)}(\varrho^3(t);x) &= \varrho^3(x) + 3\varrho^2(x)\alpha + 2\varrho(x)\alpha^2 \\ &+(\varrho^2(x)+\varrho(x)\alpha)\left(\frac{b_n}{n}\frac{4g(1)+3g'(1)}{g(1)}\right) \\ &+\varrho(x)\left(\frac{b_n^2}{n^2}\frac{g(1)+8g'(1)+3g''(1)}{g(1)}\right) \\ &+\left(\frac{b_n^3}{n^3}\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}\right) \end{aligned} \tag{16}$$

$$\begin{aligned} C_n^{\varrho,(\alpha)}(\varrho^4(t);x) &= \varrho^4(x) + 6\varrho^3(x)\alpha + 11\varrho^2(x)\alpha^2 + 6\varrho(x)\alpha^3 + \\ &+(\varrho^3(x)+3\varrho^2(x)\alpha+2\varrho(x)\alpha^2)\left(\frac{b_n}{n}\frac{10g(1)+4g'(1)}{g(1)}\right) \\ &+(\varrho^2(x)+\varrho(x)\alpha)\left(\frac{b_n^2}{n^2}\frac{14g(1)+30g'(1)+6g''(1)}{g(1)}\right) \\ &+\varrho(x)\left(\frac{b_n^3}{n^3}\frac{g(1)+26g'(1)+30g''(1)+4g'''(1)}{g(1)}\right) \\ &+\frac{b_n^4}{n^4}\frac{g'(1)+14g''(1)+10g'''(1)+g^{(4)}(1)}{g(1)}. \end{aligned} \tag{17}$$

Proof. Recalling the moments of the Chlodowsky variant of Jakimovski-Leviatan operators [15] :

$$\begin{aligned} C_n^*(e_0;x) &= 1 \\ C_n^*(e_1;x) &= x + \frac{b_n}{n}\frac{g'(1)}{g(1)} \\ C_n^*(e_2;x) &= x^2 + \frac{b_n}{n}\frac{g(1)+2g'(1)}{g(1)}x + \frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)} \\ C_n^*(e_3;x) &= x^3 + \frac{b_n}{n}\frac{4g(1)+3g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g(1)+8g'(1)+3g''(1)}{g(1)}x \\ &+\frac{b_n^3}{n^3}\frac{g'(1)+4g''(1)+g'''(1)}{g(1)} \\ C_n^*(e_4;x) &= x^4 + \frac{b_n}{n}\frac{10g(1)+4g'(1)}{g(1)x^3} + \frac{b_n^2}{n^2}\frac{14g(1)+30g'(1)+6g''(1)}{g(1)}x^2 \\ &+\frac{b_n^3}{n^3}\frac{g(1)+28g'(1)+30g''(1)+4g'''(1)}{g(1)}x \\ &+\frac{b_n^4}{n^4}\frac{g'(1)+14g''(1)+10g'''(1)+g^{(4)}(1)}{g(1)}, \end{aligned}$$

the moments of Chlodowsky variant of generalised Jain-Appell operators follow by using the representation of the operators given in Theorem (2.1). □

Lemma 2.3. *The first few r-th central moments of the operators*

$$M_{n,r}(x) = C_n^{\varrho,(\alpha)}((\varrho(t)-\varrho(x))^r;x), \quad r = 0,1,2... \tag{18}$$

are given for $n \in \mathbb{N}$ and $x \in [0,\infty)$, by

$$\begin{aligned}
M_{n,0}(x) = C_n^{\varrho,(\alpha)}((\varrho(t)-\varrho(x))^0;x) &= 1\\
M_{n,1}(x) = C_n^{\varrho,(\alpha)}((\varrho(t)-\varrho(x))^1;x) &= \frac{b_n}{n}\frac{g'(1)}{g(1)}\\
M_{n,2}(x) = C_n^{\varrho,(\alpha)}((\varrho(t)-\varrho(x))^2;x) &= \alpha\varrho(x)+\varrho(x)\frac{b_n}{n}+\frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)}\\
M_{n,4}(x) = C_n^{\varrho,(\alpha)}((\varrho(t)-\varrho(x))^4;x) &= \left(3\alpha^2+14\alpha\frac{b_n}{n}+\frac{10g(1)+4g'(1)}{g(1)}\frac{b_n^2}{n^2}\right)\varrho^2(x)\\
&+\left(6\alpha^3+\alpha^2\left(\frac{20g(1)+8g'(1)}{g(1)}\right)\frac{b_n}{n}\right.\\
&+\alpha\left(\frac{14g(1)+30g'(1)+6g''(1)}{g(1)}\right)\frac{b_n^2}{n^2}\\
&\left.+\left(\frac{g(1)+22g'(1)+14g''(1)}{g(1)}\right)\frac{b_n^3}{n^3}\right)\varrho(x)\\
&+\frac{g'(1)+14g''(1)+10g'''(1)+g^{(4)}}{g(1)}\frac{b_n^4}{n^4}.
\end{aligned}$$

Proof. The proof follows from the linearity of the operators and Lemma (2.3) and (2.2). □

We now proceed by obtaining the weighted approximation of the operator $C_n^{\varrho,(\alpha)}(f;x)$. To prove our weighted approximation theorem, we recall some notations and definitions which are needed. By choosing $\varrho(x)$ as a function that satisfies the condition (a) and the condition (b) given before, let $\varphi : I \subset \mathbb{R} \to (0,\infty)$, $\varphi(x) = 1+\varrho^2(x)$ be a weight function. Then the function spaces $B_\varphi[0,\infty)$, $C_\varphi[0,\infty)$, $C_\varphi^k[0,\infty)$ and $U_\varphi[0,\infty)$ are defined as follows [25]:

$$\begin{aligned}
B_\varphi[0,\infty) &= \left\{f:\mathbb{R}^+\to\mathbb{R} : \|f\|_\varphi = \sup_{x\geq 0}\frac{f(x)}{\varphi(x)} < \infty\right\},\\
C_\varphi[0,\infty) &= \left\{f \in B_\varphi(\mathbb{R}^+) : f \text{ is continuous}\right\},\\
C_\varphi^k[0,\infty) &= \left\{f \in C_\varphi(\mathbb{R}^+) : \lim_{x\to\infty}\frac{f(x)}{\varphi(x)} = K_f < +\infty\right\},\\
U_\varphi[0,\infty) &= \left\{f \in C_\varphi(\mathbb{R}^+) : \frac{f}{\varphi} \quad \text{is uniformly continuous}\right\}.
\end{aligned}$$

It can be easily seen that

$$C_\varphi^k[0,\infty) \subset U_\varphi[0,\infty) \subset C_\varphi[0,\infty) \subset B_\varphi[0,\infty).$$

Lemma 2.4. *[26] The linear positive operator sequence $(L_n)_{n\geq 1}$ is a sequence of transformations from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ if and only if*

$$|L_n(\varphi)(x)| \leq K\varphi(x)$$

where K is a positive constant.

Theorem A. *[27] Let $L_n : C_\varphi \to B_\varphi$ be a sequence of linear positive operators. If*

$$\lim_{n\to\infty} \| L_n(\varrho^i) - \varrho^i \|_\varphi = 0, \quad i = 0,1,2 \quad (*)$$

then for all $f \in C_\varphi^k$ we have

$$\lim_{n\to\infty} \| L_n f - f \|_\varphi = 0. \tag{19}$$

In the rest of the paper, we assume that

(iii) $\alpha := (\alpha_n(x))$ such that $0 \le \frac{\varrho(x)\alpha_n(x)}{1+\varrho^2(x)} \le c_n$ with $c_n \to 0$ as $n \to \infty$.

Theorem 2.5. *Let $\alpha := \alpha_n(x)$ satisfies (iii) and $(b_n)_{n\ge 0}$ be a non-negative sequence such that $lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{n} = 0$. Then for all $f \in C_\varrho^k[0,\infty)$, we have*

$$\lim_{n\to\infty} \| C_n^{\varrho,(\alpha)} f - f \|_\varphi = 0 \tag{20}$$

where $\varphi(x) = 1 + \varrho^2(x)$.

Proof. We start by showing that the operator $C_n^{(\alpha_n)}$ is a transformation from $C_\varphi[0,\infty)$ to $B_\varphi[0,\infty)$. Let $\varphi(x) = 1 + \varrho^2(x)$, then we have

$$\begin{aligned}
\| C_n^{\varrho,(\alpha_n)}(f) \|_{\varphi=1+\varrho^2} &= \sup_{x\in(0,\infty)} \frac{\|C_n^{\varrho,(\alpha_n(x))} f(x)\|}{\varphi(x)} \\
&\le \| f \|_\varphi \sup_{x\in(0,\infty)} \frac{\|C_n^{\varrho,(\alpha_n)}(\varphi)\|}{\varphi(x)} \\
&= \| f \|_\varphi \sup_{x\in(0,\infty)} \frac{1 + \varrho^2(x) + \alpha_n(x)\varrho(x) + \varrho(x)\frac{b_n}{n}\frac{g(1)+2g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)}}{1+\varrho^2(x)} \\
&\le (1+M) \| f \|_\varphi .
\end{aligned}$$

Therefore the operators $C_n^{\varrho,(\alpha_n)}$ are uniformly bounded. Moreover it is obvious that $\| C_n^{\varrho,(\alpha_n)}(1) - 1 \|_\varphi = 0$. On the other hand,

$$\begin{aligned}
\| C_n^{\varrho,(\alpha_n)}(\varrho) - \varrho \|_\varphi &= \sup_{x\in(0,\infty)} \frac{\varrho(x) + \frac{b_n}{n}\frac{g'(1)}{g(1)} - \varrho(x)}{1+\varrho^2(x)} \\
&= \sup_{x\in(0,\infty)} \frac{b_n g'(1)}{n g(1)(1+\varrho^2(x))} \to 0
\end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned}
\| C_n^{\varrho,(\alpha_n)}(\varrho)^2 - \varrho^2 \|_\varphi &= \sup_{x\in(0,\infty)} \frac{\varrho^2(x) + \alpha_n(x)\varrho(x) + \varrho(x)\frac{b_n}{n}\frac{g(1)+2g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)} - \varrho^2(x)}{1+\varrho^2(x)} \\
&= \sup_{x\in(0,\infty)} \left(\frac{\alpha_n(x)\varrho(x)}{1+\varrho^2(x)} + \frac{\varrho(x) b_n (g(1)+2g'(1))}{n g(1)(1+\varrho^2(x))} + \frac{b_n^2 g'(1) + g''(1)}{n^2 g(1)(1+\varrho^2(x))} \right) \to 0
\end{aligned}$$

as $n \to \infty$. Hence, from Theorem (A), we have

$$\lim_{n\to\infty} \| C_n^{\varrho,(\alpha_n)}(f) - f \|_{\varphi=1+\varrho^2} = 0$$

for all $f \in C_\varrho^k[0,\infty)$. □

3. Weighted quantitative estimates

Now, we calculate the rate of convergence of the operators given in (5) by using the modulus of continuity $\omega_\varrho(f;\delta)$, which was proposed by Adrian Holhos in [29] as

$$\omega_\varrho(f,\delta) = \sup_{\substack{x,y\geq 0\\ \|\varrho(x)-\varrho(y)\|\leq\delta}} \frac{|f(x)-f(y)|}{\varphi(x)+\varphi(y)},$$

for each $f \in C_\varphi(\mathbb{R}^+)$ and for every $\delta \geq 0$, where $\varrho(x)$ satisfies (a) and (b).
We see that α is chosen to be certain function sequence while configuring the special instances. We put the following constraint on α since we will be working in space $C_{\varphi=1+x^2}(\mathbb{R}^+)$. $\alpha := (\alpha_n(x))$ satisfies (iii).

Theorem B. *[29] Let $L_n : C_\varrho(\mathbb{R}_+) \to B_\varrho(\mathbb{R}_+)$ be a sequence of linear operators satifying*

$$\begin{aligned}
\| L_n\varrho^0 - \varrho^0 \|_{\varphi^0} &= j_n,\\
\| L_n\varrho - \varrho \|_{\varphi^{\frac{1}{2}}} &= k_n,\\
\| L_n\varrho^2 - \varrho^2 \|_{\varphi} &= l_n,\\
\| L_n\varrho^3 - \varrho^3 \|_{\varphi^{\frac{3}{2}}} &= m_n
\end{aligned}$$

where j_n, k_n, l_n and m_n tend to zero as $n \to \infty$. Then we have

$$\| L_nf - f \|_{\varphi^{\frac{3}{2}}} \leq (7 + 4j_n + 2l_n)\omega_\varrho(f,\delta_n) + \| f \|_\varphi \; j_n$$

for all $f \in C_\varrho(\mathbb{R}_+)$, where

$$\delta_n = 2\sqrt{(j_n + 2k_n + l_n)(1 + j_n)} + j_n + 3k_n + 3l_n + m_n.$$

Theorem 3.1. *For all $f \in C^k_\varrho(\mathbb{R}^+)$ we have*

$$\| C_n^{\varrho,(\alpha_n)}(f) - f \|_{\varphi^{\frac{3}{2}}} \leq \left(7 + 2c_n + \frac{b_n}{n}\frac{2g(1)+4g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{2g'(1)+2g''(1)}{g(1)}\right)\omega_\varrho(f,\delta_n)$$

where

$$\begin{aligned}
\delta_n = {} & 2\sqrt{c_n + \frac{g(1)+4g'(1)}{g(1)}\frac{b_n}{n} + \frac{g'(1)+g''(1)}{g(1)}\frac{b_n^2}{n^2}} + 6c_n + 2\sqrt{2}c_n^2\\
&+\frac{(7+4c_n)g(1)+(12+3c_n)g'(1)}{g(1)}\frac{b_n}{n} + \frac{g(1)+11g'(1)+6g''(1)}{g(1)}\frac{b_n^2}{n^2}\\
&+\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}\frac{b_n^3}{n^3}.
\end{aligned}$$

Proof. We first determine the sequences j_n, k_n, l_n, and m_n from (13), (14), (15) and (16) , and then use Theorem B to get the proposed results. Direct calculations give

$$\| C_n^{\varrho,(\alpha_n)}(1) - 1 \|_{\varphi 0} = 0 = j_n,$$

$$\| C_n^{\varrho,(\alpha_n)}(\varrho) - \varrho \|_{\varphi^{1/2}} = \sup_{x\in(0,\infty)} \frac{b_n g'(1)}{ng(1)(1+\varrho^2(x))^{1/2}} \leq \frac{b_n}{n}\frac{g'(1)}{g(1)} = k_n,$$

$$\begin{aligned}\| C_n^{\varrho,(\alpha_n)}(\varrho)^2 - \varrho^2 \| &= \sup_{x\in(0,\infty)}\left(\frac{\alpha_n(x)\varrho(x)}{(1+\varrho^2(x))^2} + \frac{\varrho(x)b_n(g(1)+2g'(1))}{ng(1)(1+\varrho^2(x))} + \frac{b_n^2 g'(1)+g''(1)}{n^2 g(1)(1+\varrho^2(x))}\right)\\ &\le c_n + \frac{b_n}{n}\frac{g(1)+2g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g'(1)+g''(1)}{g(1)} = l_n,\end{aligned}$$

and

$$\begin{aligned}\| C_n^{\varrho,(\alpha_n)}(\varrho)^3 - \varrho^3 \|_{\varphi^{3/2}} &= \sup_{x\in(0,\infty)} \frac{3\varrho^2(x)\alpha_n(x) + 2\varrho(x)\alpha_n^2(x) + (\varrho^2(x)+\varrho(x)\alpha_n(x))\left(\frac{b_n}{n}\frac{4g(1)+3g'(1)}{g(1)}\right)}{(1+\varrho^2(x))^{3/2}}\\ &\quad + \frac{\varrho(x)\left(\frac{b_n^2}{n^2}\frac{g(1)+8g'(1)+3g''(1)}{g(1)}\right) + \frac{b_n^3}{n^3}\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}}{(1+\varrho^2(x))^{3/2}}\\ &= 3c_n + 2\sqrt{2}c_n^2 + (1+c_n)\left(\frac{b_n}{n}\frac{4g(1)+3g'(1)}{g(1)}\right) + \left(\frac{b_n^2}{n^2}\frac{g(1)+8g'(1)+3g''(1)}{g(1)}\right)\\ &\quad + \left(\frac{b_n^3}{n^3}\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}\right) = m_n\end{aligned}$$

where j_n, k_n, l_n and m_n tend to zero as $n\to\infty$ and $\alpha\to\infty$. Then we have

$$\begin{aligned}\| C_n^{\varrho,(\alpha_n)}(f) - f \|_{\varphi^{\frac{3}{2}}} &\le (7+4j_n+2l_n)\omega_\varrho(f;\delta_n) + \| f \|_\varphi \, j_n\\ &\le \left(7 + 2c_n + \frac{b_n}{n}\frac{2g(1)+4g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{2g'(1)+2g''(1)}{g(1)}\right)\omega_\varrho(f,\delta_n)\end{aligned}$$

for all $f \in C_\varrho(\mathbb{R}_+)$, where

$$\begin{aligned}\delta_n &= 2\sqrt{(j_n+2k_n+l_n)(1+j_n)} + j_n + 3k_n + 3l_n + m_n\\ &= 2\sqrt{c_n + \frac{g(1)+4g'(1)}{g(1)}\frac{b_n}{n} + \frac{g'(1)+g''(1)}{g(1)}\frac{b_n^2}{n^2}}\\ &\quad + 6c_n + 2\sqrt{2}c_n^2 + \frac{(7+4c_n)g(1)+(12+3c_n)g'(1)}{g(1)}\frac{b_n}{n}\\ &\quad + \frac{g(1)+11g'(1)+6g''(1)}{g(1)}\frac{b_n^2}{n^2} + \frac{g'(1)+4g''(1)+g'''(1)}{g(1)}\frac{b_n^3}{n^3}.\end{aligned}$$

□

4. Local approximation properties

In this section we investigate the local approximation properties of the operators by means of Lipschitz type maximal functions and two different types of modulus of continuity. Lets denote the bounded continuous functions space with $C_B[0,\infty)$ on $[0,\infty)$ given with the norm

$$\| h \|_{C_B[0,\infty)} := \sup\{|h(x)| : x \in [0,\infty)\}$$

and the space $C_B^2(\mathbb{R}^+) := \{f \in C_B(\mathbb{R}^+) : f', f'' \in C_B(\mathbb{R}^+)\}$ endowed with the norm

$$\| f \|_{C_B^2(\mathbb{R}^+)} = \| f \|_{C_B[0,\infty)} + \| f' \|_{C_B[0,\infty)} + \| f'' \|_{C_B[0,\infty)} .$$

The first type modulus of continuity which is the classical type for $h \in C_B[0,\infty)$ is defined as follows

$$\omega(h;\delta) = \sup_{|k|\leq\delta}\{|h(x+k) - h(x)| : x \in [0,\infty)\}. \tag{21}$$

where $\delta > 0$ [6] and for function $h \in C_B[0,\infty)$ the second-order modulus of smoothness is defined by

$$\omega_2(h,\delta) := \sup_{0<k\leq\delta}\{|h(x+2k) - 2h(x+k) + h(x)| : x \in [0,\infty)\} \tag{22}$$

for $\delta > 0$ [6].
The Petree's K-functional is defined by

$$K(h,\delta) = inf_{g\in C_B^2(\mathbb{R}^+)}\{\| h - f \|_{C_B[0,\infty)} + \delta \| f \|_{C_B^2(\mathbb{R}^+)}\} \tag{23}$$

where $\delta > 0$. The Peetre's K-functional $K(h;\delta)$ and the second order modulus of smoothness ω_2 have a relation given by

$$K(h,\delta) \leq C\omega_2(h, \sqrt{\delta}). \tag{24}$$

for $h \in C_B[0,\infty)$.

Proposition 4.1. *For all $h \in C_B[0,\infty)$, we have*

$$\| C_n^{\varrho,(\alpha)}(h;\cdot) \|_{C_B[0,\infty)} \leq \| h \|_{C_B[0,\infty)} . \tag{25}$$

Proof. Since $C_n^{(\alpha)}(\varrho^0(t);x) = 1$, we obtain

$$\begin{aligned} |C_n^{\varrho,(\alpha)}(h;x)| &\leq \| h \circ \varrho^{-1} \|_{C_B[0,\infty)} C_n^{\varrho,(\alpha)}(\varrho^0;x) \\ &\leq \| h \circ \varrho^{-1} \|_{C_B[0,\infty)} \leq \| h \|_{C_B[0,\infty)} . \end{aligned}$$

□

Theorem 4.2. *For all $h \in C_B^1[0,\infty) := \{h \in C_B[0,\infty) : h' \in C_B[0,\infty)\}$, we have*

$$|C_n^{\varrho,(\alpha)}(h;x) - h(x)| \leq \sqrt{(M_{n,1}(x))^2(f\circ\varrho^{-1})'(\varrho(x))^2} + 2\sqrt{M_{n,2}(x)}\omega\left((h\circ\varrho^{-1})'; \sqrt{M_{n,2}(x)}\right).$$

where $M_{(n,1)}(x)$ and $M_{(n,2)}(x)$ are given in Lemma (2.3).

Proof. Let $h \in [0,\infty)$. From Taylor's theorem we can write

$$\begin{aligned} h(t) - h(x) &= (h\circ\varrho^{-1})(\varrho(t)) - (h\circ\varrho^{-1})(\varrho(x)) \\ &= (\varrho(t) - \varrho(x))(h\circ\varrho^{-1})'(\varrho(x)) + \int_{\varrho(x)}^{\varrho(t)} \left\{(h\circ\varrho^{-1})'(s) - (h\circ\varrho^{-1})'(\varrho(x))\right\} ds \end{aligned} \tag{26}$$

for $x, t \in [0,\infty)$. Using the inequality

$$|h(t) - h(x)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(h;\delta).$$

we can write that

$$\left|\int_{\varrho(x)}^{\varrho(t)} \left\{(h \circ \varrho^{-1})'(s) - (h \circ \varrho^{-1})'(\varrho(x))\right\} ds\right| \leq \omega\left((h \circ \varrho^{-1})';\delta\right)\left\{\frac{(\varrho(t)-\varrho(x))^2}{\delta} + |\varrho(t)-\varrho(x)|\right\}.$$

Therefore, using the above inequality (26) and then applying the operators $C_n^{\varrho,(\alpha)}$ on both sides of the resultant inequality we have

$$\begin{aligned}\left|C_n^{\varrho,(\alpha)}(h;x) - h(x)\right| \leq\ & |M_{n,1}(x)||(h \circ \varrho^{-1})'(\varrho(x))| \\ & + \omega\left((h \circ \varrho^{-1})';\delta\right)\left\{\frac{M_{n,2}(x)}{\delta} + C_n^{\varrho,(\alpha)}(|\varrho(t)-\varrho(x)|;x)\right\}.\end{aligned}$$

Finally, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|C_n^{\varrho,(\alpha)}(h;x) - h(x)\right| \leq\ & \sqrt{(M_{n,1}(x))^2(h \circ \varrho^{-1})'(\varrho(x))^2} \\ & + \omega\left((h \circ \varrho^{-1})';\delta\right)\left\{\frac{M_{n,2}(x)}{\delta} + \sqrt{M_{n,2}(x)}\right\}.\end{aligned}$$

Hence the result follows by choosing $\delta := \delta_n : \sqrt{M_{n,2}(x)}$ and Lemma (2.3). □

In the following theorem we obtain quantitative type estimate in terms of the Petree's K-functional.

Theorem 4.3. *For all $h \in C_B[0,\infty)$, Chlodowsky variant of generalised Jain-Appell operators $C_n^{\varrho,(\alpha)}$ satisfies the following inequality*

$$|C_n^{\varrho,(\alpha)}(h;x) - f(h)| \leq K\left(h;\left\{\frac{M_{n,1}(x)}{2} + \frac{M_{n,2}(x)\max\left\{1, \|\varrho''\|_{C_B[0,\infty)}\right\}}{4}\right\}\right) \tag{27}$$

where $M_{n,1}(x)$ and $M_{n,2}(x)$ are given in Lemma (2.3).

Proof. Using the Taylor's theorem, we have

$$(k \circ \varrho^{-1})(\varrho(t)) = (k \circ \varrho^{-1})(\varrho(x)) + (\varrho(t)-\varrho(x))(k \circ \varrho^{-1})'(\varrho(x)) + \frac{(\varrho(t)-\varrho(x))^2}{2!}(k \circ \varrho^{-1})''(\varrho(c)),$$

where $c \in (x,t)$. Considering the identities

$$(k \circ \varrho^{-1})'(\varrho(x)) = \frac{k'(x)}{\varrho'(x)}.$$

$$(k \circ \varrho^{-1})''(\varrho(x)) = \frac{k''(x)}{(\varrho'(x))^2} - \frac{k'(x)\varrho''(x)}{(\varrho'(x))^3},$$

we get,

$$k(t)-k(x)=\frac{k'(x)}{\varrho'(x)}(\varrho(t)-\varrho(x))+\frac{(\varrho(t)-\varrho(x))^2}{2}\left\{\frac{k''(c)}{(\varrho'(c))^2}+\frac{k'(c)\varrho''(c)}{(\varrho'(c))^3}\right\}.$$

Applying $C_n^{\varrho,(\alpha)}$ on both sides of the above equality and taking into account that $\inf_{x\geq 0}\varrho'(x)\geq 1$, we get

$$\begin{aligned}
|C_n^{\varrho,(\alpha)}(k;x)-k(x)| &\leq M_{n,1}(x)\frac{k'(x)}{\varrho'(x)}+\frac{M_{n,2}(x)}{2}\left\{\frac{k''(c)}{(\varrho'(c))^2}+\frac{k'(c)\varrho''(c)}{(\varrho'(c))^3}\right\}\\
&\leq M_{n,1}(x)(\| k'' \|_{C_B[0,\infty)}+\| k' \|_{C_B[0,\infty)}+\| k \|_{C_B[0,\infty)})\\
&\quad +\frac{M_{n,2}(x)}{2}(\| k'' \|_{C_B[0,\infty)}+\| k' \|_{C_B[0,\infty)}\| \varrho'' \|_{C_B[0,\infty)}+\| k \|_{C_B[0,\infty)})\\
&\leq M_{n,1}(x)\| k \|_{C_B^2[0,\infty)}+\frac{M_{n,2}(x)}{2}\max\left\{1,\| \varrho'' \|_{C_B[0,\infty)}\right\}\| k \|_{C_B^2[0,\infty)}\\
&= \left\{M_{n,1}(x)+\frac{M_{n,2}(x)}{2}\max\left\{1,\| \varrho'' \|_{C_B[0,\infty)}\right\}\right\}\| k \|_{C_B^2[0,\infty)}.
\end{aligned}$$

Now for $h\in C_B[0,\infty)$, there exists $k\in C_B^2[0,\infty)$ such that $\| h-k \|_{C_B[0,\infty)}\leq \varepsilon$. Therefore using triangle inequality, (25) and the above inequality, we have

$$\begin{aligned}
|C_n^{\varrho,(\alpha)}(h;x)-h(x)| &\leq C_n^{(\alpha)}(|h-k|;x)+|h(x)-k(x)|+|C_n^{(\alpha)}(k;x)-k(x)|\\
&\leq 2\| h-k \|_{C_B[0,\infty)}\\
&\quad +\left\{M_{n,1}(x)+\frac{M_{n,2}(x)}{2}\max\left\{1,\| \varrho'' \|_{C_B[0,\infty)}\right\}\right\}\| k \|_{C_B^2[0,\infty)}.
\end{aligned}$$

From the Peetre's K-functional (23) definition we get the desired result as

$$|C_n^{\varrho,(\alpha)}(h;x)-h(x)| \leq K\left(h;\left\{\frac{M_{n,1}(x)}{2}+\frac{M_{n,2}(x)\max\left\{1,\| \varrho'' \|_{C_B[0,\infty)}\right\}}{4}\right\}\right). \tag{28}$$

□

The following theorem states the quantitative estimate by means of the second order modulus of smoothness.

Theorem 4.4. *We have the following inequality*

$$|C_n^{\varrho,(\alpha)}(h;x)-h(x)| \leq C\{\omega_2(h,\sqrt{\delta_n}+\min(1,\delta_n)\| h \|_{C_B[0,\infty)}\} \tag{29}$$

where $h\in_{C_B[0,\infty)}$, the positive constant C is independent of n and

$$\delta_n=\frac{M_{n,1}(x)}{2}+\frac{M_{n,2}(x)\max\left\{1,\| \varrho'' \|_{C_B[0,\infty)}\right\}}{4}.$$

Proof. The proof follows by using relation (24) in Theorem (4.3). □

Recall the Lipschitz class functional that was described in [25]. With the choice of the function ϱ that satisfy conditions given before, the set of all functions h satisfying

$$|h(t)-h(x)| \leq M|\varrho(t)-\varrho(x)|^{\eta}, \quad x,t \geq 0, \tag{30}$$

is said to be of class $Lip_M(\varrho(x);\eta)$ for $\eta \in (0,1]$ and $M > 0$.
Let $E \subset [0,\infty)$. The function $h \in C[0,\infty)$ belongs to $Lip_M(\varrho(x);\eta)$, $\eta \in (0,1]$ if the following holds true

$$|h(t)-h(x)| \leq M_{\eta,h}|\varrho(t)-\varrho(x)|^{\eta}, \quad x \in E \text{ and } t \geq 0, \tag{31}$$

where $M_{\eta,h}$ is a constant depending on h and α.

Theorem 4.5. *Let E be any bounded subset of* $[0,\infty)$*.* ϱ *be a function satisfying conditions* (a) *and* (b)*. Then we have,*

$$\begin{aligned} &|C_n^{\varrho,(\alpha)}(h;x)-h(x)| \leq M_{\eta,h}\left(M_{n,2}(x)^{\frac{\eta}{2}}+2(\varrho'(\xi))^{\eta}d^{\eta}(x,E)\right), \\ &\quad x\in(0,\infty), \quad n\in\mathbb{N}, \end{aligned} \tag{32}$$

for any $h \in Lip_M(\varrho(x);\eta)$ *on E and* $\eta \in (0,1]$ *where the distance between the point x and the set E is* $d(x,E) = \inf\{|x-y| : y \in E\}$ *,* $M_{\eta,h}$ *is a constant depending on* η *and h,* ξ *is a part of the interval with terminal points x and* x_0 *.*

Proof. We use $\overline{E}$ to denote the closure of the subset E in $[0,\infty)$, than for at least one $x_0 \in \overline{E}$ we have $d(x,E) = \|x-x_0\|$. The hypothesis on h and using the monotonicity of $C_n^{(\alpha)}$, we get

$$\begin{aligned} |C_n^{\varrho,(\alpha)}(h;x)-h(x)| &\leq C_n^{\varrho,(\alpha)}(|h(t)-h(x_0)|;x)+C_n^{\varrho,(\alpha)}(|h(t)-h(x_0)|;x) \\ &\leq M_{\eta,h}\{C_n^{\varrho,(\alpha)}(|\varrho(t)-\varrho(x_0)|^{\eta};x)+|\varrho(x)-\varrho(x_0)|^{\eta}\} \\ &\leq M_{\eta,h}\{C_n^{\varrho,(\alpha)}(|\varrho(t)-\varrho(x)|^{\eta};x)+2|\varrho(x)-\varrho(x_0)|^{\eta}\}. \end{aligned}$$

From the Holder's inequality with $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$ and the fact $\|\varrho(x)-\varrho(x_0)\| = \varrho'(\xi)\|x-x_0\|$ where ξ belongs to the interval whose terminal points are x and x_0, we easily conclude

$$\begin{aligned} |C_n^{\varrho,(\alpha)}(h;x)-h(x)| &\leq M_{\eta,h}\{[C_n^{\varrho,(\alpha)}(|\varrho(t)-\varrho(x)|^2;x)]^{\frac{\eta}{2}}+2(\varrho'(\xi)|x-x_0|)^{\eta}\} \\ &\leq M_{\eta,h}\{(M_{n,2})^{\frac{\eta}{2}}+2(\varrho'(\xi))^{\eta}d^{\eta}(x,E)\}. \end{aligned}$$

□

Proposition 4.6. *Let* ϱ *satisfy the conditions* (a) *and* (b)*. Then for any* $h \in Lip_M(\varrho(x);\eta)$*, we have*

$$|C_n^{\varrho,(\alpha)}(h;x)-h(x)| \leq M(M_{n,2}(x))^{\frac{\eta}{2}}. \tag{33}$$

In [30] generalized Lipschitz-type maximal function of order η is defined as

$$\omega_{\eta}^{\sim\varrho}(h;x) = \sup_{x\neq t, t\in[0,\infty)} \frac{|h(t)-h(x)|}{|\varrho(t)-\varrho(x)|^{\eta}}. \tag{34}$$

Theorem 4.7. *Let* $h \in C_B(0,\infty)$ *and* $0 < \eta \leq 1$*. Then for all* $x \in [0,\infty)$*, we have*

$$|C_n^{(\eta)}(h;x)-h(x)| \leq \omega_{\eta}^{\sim\varrho}(h;x)(M_{n,2})^{\frac{\eta}{2}}. \tag{35}$$

Proof. From the definition of η^{th} order generalized Lipschitz-type maximal function given in (34), we have

$$|C_n^{\varrho,(\eta)}(h;x) - h(x)| \leq \omega_\beta^{\tilde{\varrho}}(h;x) C_n^{\varrho,(\eta)}(|\varrho(t) - \varrho(x)|^\eta; x).$$

Applying the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\eta}$, we have

$$\begin{aligned} |C_n^{\varrho,(\eta)}(h;x) - h(x)| &\leq \omega_\eta^{\tilde{\varrho}}(h;x)[C_n^{\varrho,(\eta)}((\varrho(t) - \varrho(x))^2; x)]^{\frac{\eta}{2}} \\ &\leq \omega_\eta^{\tilde{\varrho}}(h;x)(M_{n,2})^{\frac{\eta}{2}}. \end{aligned}$$

which is (35). □

5. A Voronovskaya asymptotic formula

For the $C_n^{(\alpha)}(f;x)$ we will give Voronovskaya type theorem in this section.

Theorem 5.1. *Let $h \in C_\varphi(\mathbb{R}^+)$, $x \in \mathbb{R}^+$ and with the assumption of $(h \circ \varrho^{-1})$'s first and second derivatives exist at $\varrho(x)$. We also assume that $b_n \to \infty$ and $\frac{b_n}{n} \to 0$, $\frac{n\alpha_n(x)}{b_n} \to 0$, for fixed $[0,\infty)$. If $(h \circ \varrho^{-1})$'s second derivative is bounded on $(0,\infty)$, then we have*

$$\lim_{n\to\infty} \frac{n}{b_n}\left[C_n^{\varrho,(\alpha_n(x))}(h;x) - h(x)\right] = \frac{g'(1)}{g(1)}(h \circ \varrho^{-1})'(\varrho(x)) + \varrho^2(x)\frac{1}{2}(f \circ \varrho^{-1})''$$

for fixed $x \in [0,\infty)$.

Proof. With the Taylor expansion at the point $\varrho(x) \in \mathbb{R}^+$ of $h \circ \varrho^{-1}$, there is a ξ between x and t such that

$$\begin{aligned} h(t) &= (h \circ \varrho^{-1})(\varrho(x)) + (h \circ \varrho^{-1})'(\varrho(x))(\varrho(t) - \varrho(x)) \\ &\quad + \frac{1}{2}(h \circ \varrho^{-1})''(\varrho(x))(\varrho(t) - \varrho(x))^2 + \lambda_x(t)(\varrho(t) - \varrho(x))^2 \end{aligned} \tag{36}$$

where

$$\lambda_x(t) = \frac{(h \circ \varrho^{-1})''(\varrho(\xi)) - (h \circ \varrho^{-1})''(\varrho(x))}{2}. \tag{37}$$

Here, the assumption on h together with (37) guarantees $|\lambda_x(t)| \leq M$ for all t and $|\lambda_x(t)| \to 0$ as $t \to x$. Applying $C_n^{\alpha_n}$ to (36), we get

$$\begin{aligned} C_n^{\varrho,(\alpha_n(x))}(h;x) - h(x) &= (h \circ \varrho^{-1})'(\varrho(x)) C_n^{\varrho,(\alpha_n(x))}(\varrho(t) - \varrho(x); x) \\ &\quad + \frac{1}{2}(h \circ \varrho^{-1})''(\varrho(x)) C_n^{\varrho,(\alpha_n(x))}((\varrho(t) - \varrho(x))^2; x) \\ &\quad + C_n^{\varrho,(\alpha_n(x))}(\lambda_x(t)(\varrho(t) - \varrho(x))^2; x). \end{aligned}$$

Clearly

$$\lim_{n\to\infty} \frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}(\varrho(t) - \varrho(x); x) = \lim_{n\to\infty} \frac{n}{b_n}\left(\frac{b_n}{n}\frac{g'(1)}{g(1)}\right) = \lim_{n\to\infty}\left(\frac{g'(1)}{g(1)}\right) = \frac{g'(1)}{g(1)}.$$

Since $\lim_{n\to\infty} \frac{n\alpha_n(x)}{b_n} = 0$ for fixed $x \in [0,\infty)$, we have

$$\lim_{n\to\infty} \frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}((\varrho(t) - \varrho(x))^2; x) = \lim_{n\to\infty} \frac{n}{b_n}\left[\alpha_n(x)\varrho(x) + \varrho(x)\frac{b_n}{n} + \frac{b_n^2}{n^2}\frac{g'(1) + g''(1)}{g(1)}\right] = \varrho(x).$$

Now let's estimate $\lim_{n\to\infty} \frac{n}{b_n} C_n^{\varrho,(\alpha_n)}(\lambda_x(t)(\varrho(t)-\varrho(x))^2;x)$ term. Choose $\delta > 0$ and $\varepsilon > 0$ such that $|\lambda_x(t)| < \varepsilon$ for $|t-x| < \delta$. Recall that, by the condition (b),

$$|t-x| \leq \varrho'(\mu)|t-x| = |\varrho(t)-\varrho(x)|.$$

Consequently, $|\varrho(t)-\varrho(x)| < \delta$, implies $|\lambda_x(t)(\varrho(t)-\varrho(x))^2| < \varepsilon(\varrho(t)-\varrho(x))^2$, while if $|\varrho(t)-\varrho(x)| \geq \delta$, using the fact that $|\lambda_x(t)| < M$, we find

$$|\lambda_x(t)|(\varrho(t)-\varrho(x))^2| < \frac{M}{\delta^2}(\varrho(t)-\varrho(x))^4.$$

Thus , we can write that

$$\begin{aligned}\frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}(\lambda_x(t)((\varrho(t)-\varrho(x))^2;x) \quad &< \quad \varepsilon \frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}((\varrho(t)-\varrho(x))^2;x) \\ &\quad + \frac{n}{b_n}\frac{M}{\delta^2} C_n^{\varrho,(\alpha_n(x))}((\varrho(t)-\varrho(x))^4;x).\end{aligned}$$

Since $\frac{n\alpha_n(x)}{b_n} \to 0$ for fixed $x \in [0,\infty)$, we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}((\varrho(t)-\varrho(x))^4;x) \\ = \quad & lim_{n\to\infty} \frac{n}{b_n}\Bigg[\left(3\alpha_n^2(x) + 14\alpha_n(x)\frac{b_n}{n} + \frac{10g(1)+4g'(1)}{g(1)}\frac{b_n^2}{n^2}\right)\varrho^2(x) \\ & + \Bigg(\left(6\alpha_n^3(x) + \alpha_n^2(x)\left(\frac{20g(1)+8g'(1)}{g(1)}\right)\right)\frac{b_n}{n} \\ & + \alpha_n(x)\left(\frac{14g(1)+30g'(1)+6g''(1)}{g(1)}\right)\frac{b_n^2}{n^2} \\ & + \left(\frac{g(1)+22g'(1)+14g''(1)}{g(1)}\right)\frac{b_n^3}{n^3}\Bigg)\varrho(x) \\ & + \left(\frac{g'(1)+14g''(1)+10g'''(1)+g^{(4)}}{g(1)}\frac{b_n^4}{n^4}\right)\Bigg] \\ = \quad & 0\end{aligned}$$

therefore, we conclude from (38) that

$$\lim_{n\to\infty} \frac{n}{b_n} C_n^{\varrho,(\alpha_n(x))}(\lambda_x(t)(\varrho(t)-\varrho(x))^2;x) = 0$$

Since ε is arbitrary it follows that,

$$\lim_{n\to\infty} \frac{n}{b_n}\left[C_n^{\varrho,(\alpha_n(x))}(h;x) - h(x)\right] \quad = \quad \frac{g'(1)}{g(1)}(h\circ\varrho^{-1})'(\varrho(x)) + \varrho^2(x)\frac{1}{2}(h\circ\varrho^{-1})''.$$

□

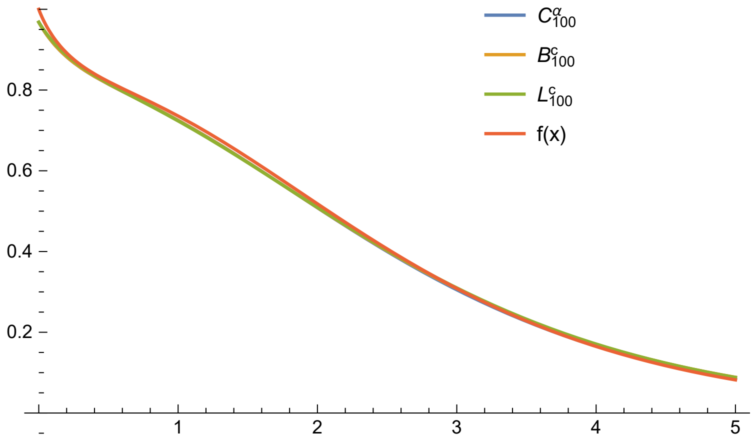

Figure 1: Approximation of $h(x) = (x^{3/2} + 1)e^{-x}$ on $[0, 5]$, when $n = 100$.

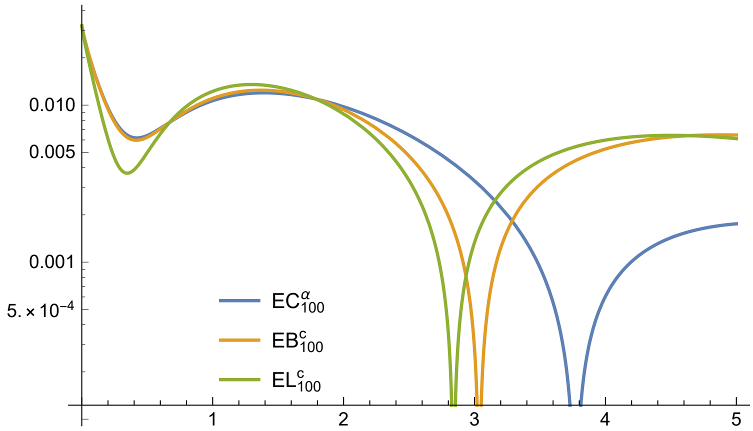

Figure 2: Graph of errors when $h(x) = (x^{3/2} + 1)e^{-x}$ on $[0, 5]$, when $n = 100$.

6. Illustrative examples

Figures 1 illustrates the approximations to the function $h(x) = (x^{3/2} + 1)e^{-x}$ by the Chlodowsky variant of Jain-Appell, Chlodowsky variant of Appell-Baskakov and Chlodowsky variant of Appell-Lupaş operators with $f(x) = e^x$, $b_n = n^{\frac{1}{3}}$, for $n = 100$. Figure 2 illustrates the approximation error of function $h(x) = (x^{3/2}+1)e^{-x}$ by the same three operators with $f(x) = e^x$, $b_n = n^{\frac{1}{3}}$, for $n = 100$. In both figures, the blue curve represents the $C_n^{\varrho,(\alpha)}(h; x)$ operators given in (5), the yellow curve represents the $B_n^{\varrho,C}(h; x)$ operators given in (9) and the green curve represents the $L_n^{\varrho,C}(h; x)$ operators given in (11). It should be mentioned that, increasing n will cause much efficient approximation to a given function.

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