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Fractional Simpson's majorization inequality pertaining twice differentiable function with applications

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Abstract. Over the past three decades, fractional calculus has gained increasing importance and practical relevance in various fields of science and engineering. This article aims to develop enhanced estimations based on the fractional Simpson's rule for functions that are twice differentiable. Leveraging majorization theory, we introduce a novel auxiliary identity by making use of fractional integral operators. To derive the novel bounds presented in this manuscript, we employ the notion of convex functions in conjunction with the Niezgoda Jensen Mercer (JM) inequality for majorized tuples, as well as some core inequalities, including Young's, Power mean, and Hölder's inequalities. Furthermore, this study encompasses the application of quadrature rules and provides illustrative examples related to special functions. Notably, the primary contributions of this research involve the extension and generalization of numerous well-established findings found in the current body of literature.

1. Introduction and Preliminaries

Let's start by reflecting on the relevant ideas related to convex mappings and other important concepts discussed in this paper.

Definition 1.1. *Let* $g : [\zeta_1, \zeta_2] \subseteq \Re \to \Re$ *is convex hold for all* $c, d \in [\zeta_1, \zeta_2]$ *and* $\kappa \in [0, 1]$

$$g(\kappa c + (1 - \kappa)d) \le \kappa g(c) + (1 - \kappa)g(d) \tag{1}$$

Convex functions are crucial to a wide range of mathematical disciplines. This theory offers a fantastic starting point and foundation for developing numerical tools for tackling and exploring difficult mathematical problems. Considering all of their useful qualities, they are particularly magical in the realm of optimization theory. Convex function theory and mathematical inequalities have a lovely relationship.

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Here is one of the most fundamental inequality that is considered as the expansion of a convex function. Let $0 < d_1 \leq d_2 \leq ... \leq d_n$ and let $\omega = (\omega_1, \omega_2, ..., \omega_\lambda)$ be the weight which can not be negative such that $\sum_{\jmath=1}^{\lambda} \omega_\jmath = 1$ and if the function $g : [\zeta_1, \zeta_2] \to \Re$ on the given interval is convex then Jensen's inequality given as [1]:

$$g\left(\sum_{\jmath=1}^{\lambda} \omega_\jmath \, d_\jmath\right) \leq \left(\sum_{\jmath=1}^{\lambda} \omega_\jmath \, g\left(d_\jmath\right)\right), \tag{2}$$

For all $d_\jmath \in [\zeta_1, \zeta_2], \omega_\jmath \in [0, 1]$ for $(\jmath = 1, 2, ..., \lambda)$. The JM inequality has been applied in several disciplines, particularly statistics, machine learning, and economics. It is commonly utilised in a variety of fields to establish essential limits and verify crucial findings. Jensen's inequality has numerous applications in finance, optimisation, economics and statistics , but it is especially useful in information theory for forecasting the estimations of the bounds of distance functions [2–4].

McD Mercer [5] presented a fascinating perspective on Jensen's inequality designated as Jensen Mercer (in short JM) inequality in the year 2003, given as:
Let g is convex function on $[\zeta_1, \zeta_2]$, then

$$g\left(\zeta_1 + \zeta_2 - \sum_{\jmath=1}^{\lambda} \omega_\jmath \, d_\jmath\right) \leq g(\zeta_1) + g(\zeta_2) - \sum_{\jmath=1}^{\lambda} \omega_\jmath \, g\left(d_\jmath\right), \tag{3}$$

is valid for all finite positive increasing sequence $d_\jmath \in [\zeta_1, \zeta_2]$, for $(\jmath = 1, 2, ..., \lambda)$ together with weights $\omega_\jmath \in [0, 1]$ defined in (2).

Definition 1.2. *[6] Let* $c = (c_1, ..., c_\ell)$ *and* $d = (d_1, ..., d_\ell)$ *be two tuple with its arrangements* $c_{[\ell]} \leq c_{[\ell-1]} \leq ... \leq c_{[1]}, d_{[\ell]} \leq d_{[\ell-1]} ... \leq d_{[1]}$ *where each of them is a real number then* c *is considered to be majorize* d *(or* d *is a said to be majorize by* c*, in symbolic terms* $c \succ d$), *if* :

$$\sum_{\mathfrak{s}=1}^{\lambda} d_{[\mathfrak{s}]} \leq \sum_{\mathfrak{s}=1}^{\lambda} c_{[\mathfrak{s}]} \quad for \ \lambda = 1, 2, ..., \ell - 1 \tag{4}$$

and

$$\sum_{\mathfrak{s}=1}^{\ell} d_{[\mathfrak{s}]} = \sum_{\mathfrak{s}=1}^{\ell} c_{[\mathfrak{s}]}.$$

In [7], an idea of extended JM inequality in context of majorization presented by Niezgoda is stated as follows:

Theorem 1.3. *Let* $(d_{\jmath\mathfrak{s}})$ *be a* $\lambda \times \ell$ *real matrix and* $\zeta = (\zeta_1, \zeta_2, ..., \zeta_\ell)$ *be* ℓ *tuple such that* $\zeta_\mathfrak{s}, d_{\jmath\mathfrak{s}}, \forall \ \jmath = 1, 2, ..., \lambda$, $\mathfrak{s} \in \{1, ..., \ell\}$ *and function* g *be a convex defined in I. Moreover,* $\omega_\jmath \geq 0$ *for* $\jmath = 1, 2, ..., \lambda$ *with* $\sum_{\jmath=1}^{\lambda} \omega_\jmath = 1$*.If* ζ *majorizes each row of* $d_{\jmath\mathfrak{s}}$ *then*

$$g\Big(\sum_{\mathfrak{s}=1}^{\ell} \zeta_\mathfrak{s} - \sum_{\mathfrak{s}} = 1^{\ell-1} \sum_{\jmath=1}^{\lambda} \omega_\jmath d_{\jmath\mathfrak{s}}\Big) \leq \sum_{\mathfrak{s}=1}^{\ell} g(\zeta_\mathfrak{s}) - \sum_{\mathfrak{s}=1}^{\ell-1} \sum_{\jmath=1}^{\lambda} \omega_\jmath g(d_{\jmath\mathfrak{s}}). \tag{5}$$

Many researchers have studied and investigated the JM inequality over the years in a variety of ways, including raising its dimension, obtaining it for convex operators together with its numerous refinements,

operator variants for superquadratic functions, improved features and various generalisations with applications in information theory, see the papers [26]-[33].
Simpson's inequality is widely recognized for its significant geometrical implications and extensive range of practical applications, as mentioned in the subsequent discussion [8].

$$\left|\frac{1}{3}\left\{\frac{g(\zeta_1)+g(\zeta_2)}{2}+2g\left(\frac{\zeta_1+\zeta_2}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} g(x)\,dx\right\}\right| \leq \frac{1}{2880}\left\|g^4\right\|_\infty (\zeta_2-\zeta_1)^4 .$$

where $g : [\zeta_1, \zeta_2] \to \Re$ is a mapping which is continuously differentiable four times on (ζ_1, ζ_2) and $\left\|g^{(4)}\right\|_\infty = \sup_{x\in(\zeta_1,\zeta_2)} \left|g^{(4)}(x)\right| < \infty$.

In recent years, there has been a notable expansion in research on inequalities of Simpson-type applicable to the functions that can be differentiated twice. Sarikaya et al. pioneered the utilization of Riemann-Liouville (in short RL)-fractional integrals to derive various inequalities of trapezoidal and Hermite-Hadamard-types. For differentiable s-convex functions and convex functions which are twice differentiable, Sarikaya et al. [9, 10] proven the general form of Simpson's type inequality. For instance, Hezenci et al. [11] introduced an identity regarding functions which are differentiable two times employing RL-fractional integral operators, leading to a series of inequalities of Simpson-type. In [12], Hezenci et al. established a generalized fractional integral identity and employed it to derive inequalities Simpson's-formula-type for convex functions which are differentiable two times. These outcomes can be expressed in Riemann integral, RL- and k-RL fractional integral forms. Subsequently, Zhou et al. [13] utilized fractional integrals incorporating exponential kernels to present several parameterized integral inequalities associated with convex functions. These inequalities encompass the averaged midpoint-trapezoid inequality, the trapezoid inequality and Simpson's inequality. Furthermore, Zhou et al. [14] proposed weighted parameterized integral inequalities relevant to twice differentiable functions, unifying midpoint-, Simpson-, Bullen-, and trapezoid-type inequalities and Faisal et al. established variants of Hermite Hadamard type inequalities using the majorization technique in [15–18].

In recent years, numerous studies have placed significant emphasis on the concept of convexity and its various manifestations. Specifically, the investigation of convexity within the framework of integral inequalities has emerged as a compelling research topic. Hermite's inequality, Hadamard's inequality, and Jensen's inequality, as well as Hilbert's inequality and Hardy's inequality, are among the most notable inequalities pertaining to the convex function's integral mean.
In the past thirty years, fractional calculus, which encompasses the study of integrals and derivatives with arbitrary real or complex orders, has gained significant attention. Its increasing popularity can be attributed to its proven utility across a diverse spectrum of scientific and engineering disciplines. This field offers promising techniques for addressing a multitude of challenges, including solving differential and integral equations, as well as tackling a variety of mathematical physics problems, special functions, and its expansions and generalisations in either one or more variables. One of the ways to build fractional calculus is as an application of the concept of a derivative operator, which may be expanded from integer order to any other (non-integral) order. The field of fractional calculus encompasses theories related to differential, integral, and integro-differential equations, alongside specialized mathematical physics functions, extending into one or more variables. Its applications are diverse and encompass various domains. Some of the present areas where fractional calculus finds utility consist of rheology, fluid dynamics, diffusive transport resembling diffusion, dynamic processes within identical and porous structures, electrical network analysis, statistics and probability, viscoelastic behavior, control theory for dynamic systems, electrochemical corrosion studies, optics, chemical physics and signal processing, among others. Numerous investigations have demonstrated that fractional operators are capable of explaining complex long-memory and multi scale processes in materials which are challenging to model utilizing conventional mathematical techniques, such as classical differential calculus [19]. The RL-fractional integral operators are the first and foremost flexible in terms of local kernels are defined as:

Let g be the integrable function on $[\zeta_1, \zeta_2]$, then Riemann Liouvile fractional integral of order α such that $\alpha > 0$ are stated subsequently:

$$J^{\alpha}_{\zeta_1^+} g(x) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^{x} (x-t)^{\alpha-1}\, g(t)\, dt, \quad x > \zeta_1$$

and

$$J^{\alpha}_{\zeta_2^-} g(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\zeta_2} (t-x)^{\alpha-1}\, g(t)\, dt, \quad x < \zeta_2.$$

where Γ is termed as Gamma function.

The core aim of this manuscript is to provide a study of fractional Simpson inequality estimates by using the majorization approach. We come up with a fractional auxiliary outcome, then by using Neizgoda JM inequality related to majorization and convexity, we provide a range of novel estimates for Simpson's fractonal inequalities. In order to further enhance the elegance of connections, we present Simpson's estimations utilizing special q-digamma and Bessel functions.

2. Main Results

This section contains Simpson's type lemma via majorization for the RL-integral operator for differentiable functions on the given interval I. We start with the following lemma:
Here $\zeta = (\zeta_1, \zeta_2, ..., \zeta_\ell)$, $(c_1, c_2, ..., c_\ell)$ and $(d_1, d_2, ..., d_\ell)$ be the three ℓ-tuples that will be used in this paper.

Lemma 2.1. *Supposing that $\zeta_s, c_s, d_s \in I$ for all $s \in \{1, 2, 3, ..., \ell\}$ be three tuples such that $c_\ell > d_\ell$, $\alpha > 0$ and function g be the continuous as well as differentiable on I such that $\Re \supseteq I$. If $g' \in L(I)$ and ζ majorizes both c and d, then the following identity:*

$$\begin{aligned}
S_\alpha(c_s, d_s, \zeta_s, \ell; g) &= \frac{1}{6}\left\{ g\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1} c_s\right) + 4g\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1}\frac{c_s + d_s}{2}\right) + g\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1} d_s\right)\right\} \\
&- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{s=1}^{\ell-1}(d_s - c_s)\right)^{\alpha}} \times \left\{ J^{\alpha}_{\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1} c_s\right)^-} g\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1}\frac{c_s + d_s}{2}\right)\right. \\
&\left. + J^{\alpha}_{\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1} d_s\right)^+} g\left(\sum_{s=1}^{\ell}\zeta_s - \sum_{s=1}^{\ell-1}\frac{c_s + d_s}{2}\right)\right\} \\
&= \frac{\left(\sum_{s=1}^{\ell-1}(d_s - c_s)\right)^2}{8(\alpha+1)} \int_0^1 \left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{(\alpha+1)}\right) \times \\
&\quad \left[g''\left(\sum_{s=1}^{\ell}\zeta_s - \left[\frac{1-\kappa}{2}\sum_{s=1}^{\ell-1} c_s + \frac{1+\kappa}{2}\sum_{s=1}^{\ell-1} d_s\right]\right) + \right. \\
&\quad \left. g''\left(\sum_{s=1}^{\ell}\zeta_s - \left[\frac{1-\kappa}{2}\sum_{s=1}^{\ell-1} d_s + \frac{1+\kappa}{2}\sum_{s=1}^{\ell-1} c_s\right]\right)\right] d\kappa
\end{aligned} \tag{6}$$

satisfies for $\kappa \in [0, 1]$.

Proof. Let us first consider L.H.S

$$
\begin{aligned}
&= \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^{2}}{8(\alpha+1)}\int_{0}^{1}\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\\
&\times\left[\mathrm{g}''\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left[\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}+\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right]\right)+\mathrm{g}''\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left[\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}+\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}\right]\right)\right]d\kappa\\
&= \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^{2}}{8(\alpha+1)}\{I_1+I_2\}.
\end{aligned}
$$

Where I_1 is given as,

$$I_1=\Big[\int_0^1\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\mathrm{g}''\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left[\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}+\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right]\right)d\kappa.$$

Now by applying integrating by parts on I_1, we have

$$
\begin{aligned}
I_1 &= \left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\frac{\mathrm{g}'\left(\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left[\frac{1-\kappa}{2}\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}+\frac{1+\kappa}{2}\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right]\right)}{-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}}}{2}}\Bigg|_0^1\\
&\int_0^1\frac{\mathrm{g}'\left(\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left[\frac{1-\kappa}{2}\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}+\frac{1+\kappa}{2}\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right]\right)}{-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}}}{2}}\left(\frac{2(\alpha+1)}{3}-(\alpha+1)\kappa^{\alpha}\right)d\kappa.\\
I_1 &= \frac{2(1-\alpha)}{3\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})}\left[\mathrm{g}'\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_{\mathfrak{s}}+\mathrm{d}_{\mathfrak{s}}}{2}\right)\right]+\\
&\frac{4(1+\alpha)}{3\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^{2}}\mathrm{g}\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right)+\frac{8(\alpha+1)}{3}\mathrm{g}\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_{\mathfrak{s}}+\mathrm{d}_{\mathfrak{s}}}{2}\right)\\
&-\frac{\alpha(\alpha+1)}{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^{2}}\int_0^1\mathrm{g}\left[\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\left(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_{\mathfrak{s}}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right)\right]\kappa^{\alpha-1}d\kappa. \qquad (7)
\end{aligned}
$$

By substituting the variables, we get

$$
\begin{aligned}
I_1 &= \frac{2(1-2\alpha)}{3\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})}\left[\mathrm{g}'\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_{\mathfrak{s}}+\mathrm{d}_{\mathfrak{s}}}{2}\right)\right]+\\
&\frac{4(1-\alpha)}{3\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^{2}}\mathrm{g}\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_{\mathfrak{s}}\right)+\frac{8(\alpha+1)}{3}\mathrm{g}\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_{\mathfrak{s}}+\mathrm{d}_{\mathfrak{s}}}{2}\right)-
\end{aligned}
$$

$$\frac{2^{(\alpha+2)}\alpha(\alpha+1)}{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})\right)^{\alpha}}\int_{\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}}^{\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}}\left[\left[\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}\right]-P\right]^{\alpha-1}g(P)dP.$$

Similarly for I_2, by applying integration by parts we have

$$I_2=-\frac{2(1-2\alpha)}{3\sum\limits_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})}\left[g'\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}\right)\right]+$$

$$\frac{4(1-\alpha)}{3\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})\right)^2}g\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}\right)+\frac{8(\alpha+1)}{3}g\left(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}\right)$$

$$-\frac{2^{(\alpha+2)}\alpha(\alpha+1)}{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})\right)^{\alpha}}\int_{\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}}^{\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}}\left[P-\left[\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}\right]\right]^{\alpha-1}g(P)dP. \tag{8}$$

Before applying the fractional integral's definition, we demonstrate that

$$\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}>\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}$$

and

$$\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{c_\mathfrak{s}+d_\mathfrak{s}}{2}<\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}$$

From the given condition,

$$c_\ell>d_\ell \implies c_\ell-d_\ell>0$$

As $c\prec\zeta$ and $d\prec\zeta$ then by applying the definition of majorization

$$\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}+d_\ell=\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}+c_\ell \implies \sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}=c_\ell-d_\ell.$$

$$\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}>\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s} \implies -\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}<-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}.$$

$$-\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}<\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}-2\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s} \implies -\sum_{\mathfrak{s}=1}^{\ell-1}\frac{(c_\mathfrak{s}+d_\mathfrak{s})}{2}<-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}.$$

Adding $\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}$ on both sides

$$\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{(c_\mathfrak{s}+d_\mathfrak{s})}{2}<\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}.$$

Similarly,

$$\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}<\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{(c_\mathfrak{s}+d_\mathfrak{s})}{2}.$$

Adding I_1 and I_2, we have

$$I_1 + I_2 = \frac{4(\alpha+1)}{3\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s}-\mathrm{c}_\mathfrak{s})\right)^2}\Big[\mathrm{g}\Big[\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_\mathfrak{s}\Big]+4\mathrm{g}\Big[\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{(\mathrm{c}_\mathfrak{s}+\mathrm{d}_\mathfrak{s})}{2}\Big]+$$
$$\mathrm{g}\Big[\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_\mathfrak{s}\Big]\Big]-\frac{2^{(\alpha+2)}(\alpha+1)\Gamma(\alpha+1)}{\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s}-\mathrm{c}_\mathfrak{s})^{\alpha+2}}\Big[J^{\alpha}_{(\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_\mathfrak{s})^-}\mathrm{g}\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_\mathfrak{s}+\mathrm{d}_\mathfrak{s}}{2}\Big)$$
$$+J^{\alpha}_{(\sum\limits_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum\limits_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_\mathfrak{s})^+}\mathrm{g}\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\sum_{\mathfrak{s}=1}^{\ell-1}\frac{\mathrm{c}_\mathfrak{s}+\mathrm{d}_\mathfrak{s}}{2}\Big)\Big]. \tag{9}$$

Multiply by $\dfrac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s}-\mathrm{c}_\mathfrak{s})\right)^2}{8(\alpha+1)}$ on both sides of (9), we obtain (6) .

Remark 2.2. *By substituting $\ell = 2$ in Lemma 6, the above mentioned identity becomes:*

$$S_\alpha(\mathrm{c}_\mathfrak{s},\mathrm{d}_\mathfrak{s},\zeta_\mathfrak{s},2;\mathrm{g}) = \frac{1}{6}\Big[\mathrm{g}(\zeta_1+\zeta_2-\mathrm{c})+4\mathrm{g}\Big(\zeta_1+\zeta_2-\frac{\mathrm{c}+\mathrm{d}}{2}\Big)+\mathrm{g}(\zeta_1+\zeta_2-\mathrm{d})\Big]-$$
$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathrm{d}-\mathrm{c})^\alpha}\times\Big[J^{\alpha}_{\zeta_1+\zeta_2-\mathrm{c}^-}\mathrm{g}\Big(\zeta_1+\zeta_2-\frac{\mathrm{c}+\mathrm{d}}{2}\Big)+J^{\alpha}_{\zeta_1+\zeta_2-\mathrm{d}^+}\mathrm{g}\Big(\zeta_1+\zeta_2-\frac{\mathrm{c}+\mathrm{d}}{2}\Big)\Big]$$
$$=\frac{(\mathrm{d}-\mathrm{c})^2}{8(\alpha+1)}\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\Big[\mathrm{g}''\left(\zeta_1+\zeta_2-\Big(\frac{1-\kappa}{2}\mathrm{c}+\frac{1+\kappa}{2}\mathrm{d}\Big)\right)$$
$$+\mathrm{g}''\left(\zeta_1+\zeta_2-\Big(\frac{1-\kappa}{2}\mathrm{d}+\frac{1+\kappa}{2}\mathrm{c}\Big)\right)d\kappa\Big],$$

The above mentioned equality is known as Mercer equality pertaining to RL-fractional integral and is a novel concept in the field of inequalities.

Remark 2.3. *In above Remark 2.2, for $\alpha = 1$ we attain the traditional Simpson Mercer form given below:*

$$S_1(\mathrm{c}_\mathfrak{s},\mathrm{d}_\mathfrak{s},\zeta_\mathfrak{s},2;\mathrm{g}) = \frac{1}{6}\Big[\mathrm{g}(\zeta_1+\zeta_2-\mathrm{c})+4\mathrm{g}\Big(\zeta_1+\zeta_2-(\frac{\mathrm{c}+\mathrm{d}}{2})+\mathrm{g}(\zeta_1+\zeta_2-\mathrm{d})\Big]-$$
$$\frac{1}{\mathrm{d}-\mathrm{c}}\int_{\zeta_1+\zeta_2-\mathrm{d}}^{\zeta_1+\zeta_2-\mathrm{c}}\mathrm{g}(P)dP = \frac{(\mathrm{d}-\mathrm{c})^2}{48}\int_0^1\Big(4\kappa-3\kappa^2-1\Big)$$
$$\Big[\mathrm{g}''\Big(\zeta_1+\zeta_2-\Big(\frac{1-\kappa}{2}\mathrm{c}+\frac{1+\kappa}{2}\mathrm{d}\Big)+\mathrm{g}''\Big(\zeta_1+\zeta_2-\Big(\frac{1-\kappa}{2}\mathrm{d}+\frac{1+\kappa}{2}\mathrm{c}\Big)\Big]d\kappa.$$

Remark 2.4. *Here for $\zeta_1 = \mathrm{c}$ and $\zeta_2 = \mathrm{d}$ in Remark 2.2, we obtain an equality via Riemann Liouville which is proved in [20].*

Remark 2.5. *By using $\alpha = 1$ and $\zeta_1 = \mathrm{c}$ and $\zeta_2 = \mathrm{d}$ in Remarks 2.2, we obtain the traditional Simpson Lemma that has been proven by Butt et al. in [21].*

□

Based on Lemma 2.1, some new majorization-based Simpson's type inequalities results for convex functions are given below.

Theorem 2.6. *According to Lemma's 2.1 assumptions, if the function $|g''|$ on I is continuous as well as convex , then $\forall\ \alpha > 0$, the subsequent inequality for fractional integral inequality:*

$$\left|S_\alpha(c_s, d_s, \zeta_s, \ell; g)\right| \leq \frac{\left(\sum\limits_{s=1}^{\ell-1}(d_s - c_s)\right)^2}{8(\alpha+1)} Y_1(\alpha)\left[2\left|g''\Big(\sum_{s=1}^{\ell}\zeta_s\Big)\right| - \left\{\left|g''\Big(\sum_{s=1}^{\ell-1}c_s\Big)\right| + \left|g''\Big(\sum_{s=1}^{\ell-1}d_s\Big)\right|\right\}\right]$$

where

$$Y_1(\alpha) = \begin{cases} \frac{1-\alpha^2}{3(\alpha+2)} & 0 < \alpha \leq \frac{1}{2} \\ 2\left(\frac{(\rho_\alpha)^{\alpha+2}}{\alpha+2} - \frac{(1-2\alpha)\rho_\alpha + (\alpha+1)(\rho_\alpha)^2}{3}\right) + \frac{1-\alpha^2}{3(\alpha+2)} & \alpha > \frac{1}{2} \end{cases}$$

satisfies for $\kappa \in [0, 1]$.

Proof. By considering the modulus on both sides of Lemma 2.1, we have

$$\left|S_\alpha(c_s, d_s, \zeta_s, \ell; g)\right| \leq \frac{\left(\sum\limits_{s=1}^{\ell-1}(d_s - c_s)\right)^2}{8(\alpha+1)} \int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{\alpha+1}\right)\right| \times$$
$$\left[\left|g''\Big(\sum_{s=1}^{\ell}\zeta_s - \Big(\frac{1+\kappa}{2}\sum_{s=1}^{\ell-1}c_s + \frac{1-\kappa}{2}\sum_{s=1}^{\ell-1}d_s\Big)\Big)\right| + \left|g''\Big(\sum_{s=1}^{\ell}\zeta_s - \Big(\frac{1+\kappa}{2}\sum_{s=1}^{\ell-1}d_s + \frac{1-\kappa}{2}\sum_{s=1}^{\ell-1}c_s\Big)\Big)\right|\right] d\kappa.$$

By using (5) for $\lambda = 2$, $\omega_1 = \frac{1-\kappa}{2}$ and $\omega_2 = \frac{1+\kappa}{2}$, we have

$$\leq \frac{\left(\sum\limits_{s=1}^{\ell-1}(d_s - c_s)\right)^2}{8(\alpha+1)} \int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{\alpha+1}\right)\right| \times$$
$$\left\{\sum_{s=1}^{\ell}\left|g''(\zeta_s)\right| - \left[\frac{1+\kappa}{2}\sum_{s=1}^{\ell-1}\left|g''(c_s)\right| + \frac{1-\kappa}{2}\sum_{s=1}^{\ell-1}\left|g''(d_s)\right|\right] + \right.$$
$$\left.\sum_{s=1}^{\ell}\left|g''(\zeta_s)\right| - \left[\frac{1+\kappa}{2}\sum_{s=1}^{\ell-1}\left|g''(d_s)\right| + \frac{1-\kappa}{2}\sum_{s=1}^{\ell-1}\left|g''(c_s)\right|\right]\right\} d\kappa$$
$$= \frac{\left(\sum\limits_{s=1}^{\ell-1}(d_s - c_s)\right)^2}{8(\alpha+1)} Y_1(\alpha)\left\{2\sum_{s=1}^{\ell}\left|g''(\zeta_s)\right| - \left(\sum_{s=1}^{\ell-1}\left|g''(c_s)\right| + \sum_{s=1}^{\ell-1}\left|g''(d_s)\right|\right)\right\}$$

which finishes the proof. □

Remark 2.7. *By substituting $\ell = 2$ in Theorem 2.6, one can obtain Mercer estimates of inequality:*

$$\left|S_\alpha(c_s, d_s, \zeta_s, 2; g)\right| \leq \frac{(d-c)^2}{8(\alpha+1)} Y_1(\alpha)\Big\{2\,|\,g''(\zeta_1)\,| + 2\,|\,g''(\zeta_2)\,| - |\,g''(c)\,| - |\,g''(d)\,|\Big\}.$$

Remark 2.8. *For $\alpha = 1$, $\rho_\alpha = \frac{1}{3}$ and $\ell = 2$ in Theorem 2.6, we obtain classical Simpson estimates that is proved in [21].*

Remark 2.9. *By using $\zeta_1 = \mathrm{c}$ and $\zeta_2 = \mathrm{d}$ in above Remark 2.7, we originate Simpson type inequality fractional estimates proved in [20].*

Remark 2.10. *Here by using $\zeta_1 = \mathrm{c}$, $\zeta_2 = \mathrm{d}$, $\rho_\alpha = \frac{1}{3}$ and $\alpha = 1$ Remark 2.7, for a convex function which is twice differentiable, we develop traditional Simpson estimates, which is proved in [10].*

Theorem 2.11. *Based on the assumptions of Lemma 2.1, if the function $|g''|^q$ on I is continuous convex for $q > 1$, then the subsequent inequality for fractional integral inequality $\forall\ \alpha > 0$ holds:*

$$\Big|S_\alpha(\mathrm{c}_\mathfrak{s}, \mathrm{d}_\mathfrak{s}, \zeta_\mathfrak{s}, \ell; \mathrm{g})\Big| \leq \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s} - \mathrm{c}_\mathfrak{s})\right)^2}{8(\alpha+1)} \Upsilon(\alpha, p) \times$$
$$\Big\{\sum_{\mathfrak{s}=1}^{\ell}\left|\mathrm{g}''(\zeta_\mathfrak{s})\right|^q - \Big[\frac{3}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{c}_\mathfrak{s})\right|^q + \frac{1}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{d}_\mathfrak{s})\right|^q\Big]^{1/q}$$
$$+ \sum_{\mathfrak{s}=1}^{\ell}\left|\mathrm{g}''(\zeta_\mathfrak{s})\right|^q - \Big[\frac{1}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{c}_\mathfrak{s})\right|^q + \frac{3}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{d}_\mathfrak{s})\right|^q\Big]^{1/q}\Big\}$$

where $\kappa \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$, also $\Upsilon(\alpha, p)$ is defined as:

$$\Upsilon(\alpha, p) = \int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{(\alpha+1)}\right)\right|^p d\kappa\Big)^{\frac{1}{p}}.$$

Proof. From Lemma 2.1, taking modulus on both sides, we have

$$\Big|S_\alpha(\mathrm{c}_\mathfrak{s}, \mathrm{d}_\mathfrak{s}, \zeta_\mathfrak{s}, \ell; \mathrm{g})\Big| \leq \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s} - \mathrm{c}_\mathfrak{s})\right)^2}{8(\alpha+1)} \int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{(\alpha+1)}\right)\right|$$
$$\Big[\Big|\mathrm{g}''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s} - \big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_\mathfrak{s} + \frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_\mathfrak{s}\big)\Big)\Big| +$$
$$\Big|\mathrm{g}''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s} - \big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_\mathfrak{s} + \frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_\mathfrak{s}\big)\Big)\Big|\Big] d\kappa. \tag{10}$$

By applying Hölder's inequality on (10), we have

$$\leq \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s} - \mathrm{c}_\mathfrak{s})\right)^2}{8(\alpha+1)} \Big(\int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{(\alpha+1)}\right)\right|^p d\kappa\Big)^{\frac{1}{p}}$$
$$\Big(\Big|\mathrm{g}''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s} - \big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{c}_\mathfrak{s} + \frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\mathrm{d}_\mathfrak{s}\big)\Big)\Big|^q d\kappa\Big)^{\frac{1}{q}} +$$
$$\Big(\int_0^1 \left|\left(\frac{1-2\alpha}{2} + \frac{2(\alpha+1)}{3}\kappa - \kappa^{(\alpha+1)}\right)\right|^p d\kappa\Big)^{\frac{1}{p}} \times$$

$$\left(\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_{\mathfrak{s}}-\Big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_{\mathfrak{s}}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_{\mathfrak{s}}\Big)\right|^{q}d\kappa\right)^{\frac{1}{q}}.$$

By using (5) $\lambda = 2$, $\omega_1 = \frac{1+\kappa}{2}$ and $\omega_2 = \frac{1-\kappa}{2}$, we have

$$\leq \frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(d_{\mathfrak{s}}-c_{\mathfrak{s}})\right)^2}{8(\alpha+1)}\left(\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\right|^p d\kappa\right)^{\frac{1}{p}}\times$$
$$\left[\left(\int_0^1\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_{\mathfrak{s}})\right|^q-\left(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_{\mathfrak{s}})\right|^q+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_{\mathfrak{s}})\right|^q\right)d\kappa\right)^{\frac{1}{q}}+\right.$$
$$\left.\left(\int_0^1\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_{\mathfrak{s}})\right|^q-\left(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_{\mathfrak{s}})\right|^q+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_{\mathfrak{s}})\right|^q\right)d\kappa\right)^{\frac{1}{q}}\right].$$
$$=\frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(d_{\mathfrak{s}}-c_{\mathfrak{s}})\right)^2}{2}\Upsilon(\alpha,p)\times\Big[\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_{\mathfrak{s}})\right|^q-\Big(\frac{3}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_{\mathfrak{s}})\right|^q$$
$$+\frac{1}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_{\mathfrak{s}})\right|^q\Big)\Big]^{\frac{1}{q}}+\Big[\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_{\mathfrak{s}})\right|^q-\left(\frac{3}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_{\mathfrak{s}})\right|^q+\frac{1}{4}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_{\mathfrak{s}})\right|^q\right)\Big]^{\frac{1}{q}}.$$

Remark 2.12. *By substituting $\ell = 2$ in Theorem 2.11, it reduces to new fractional Mercer estimates given as:*

$$\left|S_\alpha(c_{\mathfrak{s}},d_{\mathfrak{s}},\zeta_{\mathfrak{s}},2;g)\right|\leq\frac{(d-c)^2}{8(\alpha+1)}\Upsilon(\alpha,p)\times\Big[\left|g''(\zeta_1)\right|^q+\left|g''(\zeta_2)\right|^q-\Big[\frac{3}{4}\left|g''(x)\right|^q+\frac{1}{4}\left|g''(d)\right|^q\Big]\Big]^{\frac{1}{q}}$$
$$+\Big[\left|g''(\zeta_1)\right|^q+\left|g''(\zeta_2)\right|^q-\Big[\frac{3}{4}\left|g''(d)\right|^q+\frac{1}{4}\left|g''(c)\right|^q\Big]\Big]^{\frac{1}{q}}.$$

Remark 2.13. *By using $\zeta_1 = c$ and $\zeta_2 = d$ in above mentioned Remark 2.12, we establish estimates of the fractional Simpson inequality for RL-integral operators, as illustrated in [20].*

Remark 2.14. *For $\alpha = 1$ and $\rho_\alpha = \frac{1}{3}$ in Remark 2.12, we find the bound for the traditional Mercer inequality.:*

$$\left|S_1(c_{\mathfrak{s}},d_{\mathfrak{s}},\zeta_{\mathfrak{s}},2;g)\right|\leq\frac{(d-c)^2}{162}\Upsilon(1,p)\times\Big[\left|g''(\zeta_1)\right|^q+\left|g''(\zeta_2)\right|^q-\Big[\frac{3}{4}\left|g''(c)\right|^q+\frac{1}{4}\left|g''(d)\right|^q\Big]\Big]^{\frac{1}{q}}$$
$$+\Big[\left|g''(\zeta_1)\right|^q+\left|g''(\zeta_2)\right|^q-\Big[\frac{3}{4}\left|g''(d)\right|^q+\frac{1}{4}\left|g''(c)\right|^q\Big]\Big]^{\frac{1}{q}}.$$

Remark 2.15. *For $\alpha = 1$, $\rho_\alpha = \frac{1}{3}$, $\zeta_1 = c$ and $\zeta_2 = d$ we obtain classical Simpson estimates proved in [20].*
□

Theorem 2.16. *Based on the assumptions of Lemma 2.1, if the function $\left|g''\right|^q$ on I is continuous convex, then for $q > 1$ the subsequent inequality holds:*

$$\left|S_\alpha(c_{\mathfrak{s}},d_{\mathfrak{s}},\zeta_{\mathfrak{s}},\ell;g)\right|\leq\frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(d_{\mathfrak{s}}-c_{\mathfrak{s}})\right)^2}{8(\alpha+1)}\Big(\Upsilon_1(\alpha)\Big)^{1-\frac{1}{q}}\times$$
$$\left[\left(\Upsilon_1(\alpha)\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_{\mathfrak{s}})\right|^q-\left\{\left(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_{\mathfrak{s}})\right|^q+\left(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_{\mathfrak{s}})\right|^q\right\}\right)^{\frac{1}{q}}\right.$$

$$+\Bigg(\Upsilon_1(\alpha)\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_\mathfrak{s})\right|^q-\Bigg\{\left(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_\mathfrak{s})\right|^q+\left(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_\mathfrak{s})\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\Bigg]$$

for $\kappa\in[0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *where*

$$\Upsilon_2(\alpha)=\begin{cases}\frac{3+\alpha-2\alpha^2}{18(\alpha+3)} & 0<\alpha\le\frac{1}{2}\\ 2\left(\frac{(\rho_\alpha)^{\alpha+3}}{\alpha+3}-\frac{3(1-2\alpha)(\rho_\alpha)^2+4(\alpha+1)(\rho_\alpha)^3}{18}\right)+\frac{3+\alpha-2\alpha^2}{18(\alpha+3)} & \alpha>\frac{1}{2}.\end{cases}$$

Proof. From Lemma 2.1, by taking modulus on both sides we have

$$\begin{aligned}\left|S_\alpha(c_\mathfrak{s},d_\mathfrak{s},\zeta_\mathfrak{s},\ell;g)\right|\le&\frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})\right)^2}{8(\alpha+1)}\Bigg[\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\right|\\&\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}\big)\Big)\right|d\kappa\\&+\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\right|\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}\big)\Big)\right|d\kappa\Bigg]\end{aligned}\tag{11}$$

By employing Power mean inequality on R.H.S of (11), we have

$$\begin{aligned}\le&\frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(d_\mathfrak{s}-c_\mathfrak{s})\right)^2}{2}\Bigg[\Bigg(\Big(\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)d\kappa\right|^{1-\frac{1}{q}}\times\Big(\int_0^1\Bigg|\frac{1-2\alpha}{2}\\&+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\Bigg|\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}\big)\Big)\right|^q d\kappa\Big)^{\frac{1}{q}}\\&+\Big(\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\right|\Big)^{1-\frac{1}{q}}\Big(\int_0^1\left|\left(\frac{1-2\alpha}{2}+\frac{2(\alpha+1)}{3}\kappa-\kappa^{(\alpha+1)}\right)\right|\\&\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}\big)\Big)\right|^q d\kappa\Big)^{\frac{1}{q}}\Bigg].\end{aligned}$$

By utilizing (5) for $\lambda=2$, $\omega_1=\frac{1+\kappa}{2}$ and $\omega_2=\frac{1-\kappa}{2}$, we have

$$\begin{aligned}&\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}\big)\Big)\right|^q\le\\&\qquad\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_\mathfrak{s})\right|^q-\big[\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_\mathfrak{s})\right|^q+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_\mathfrak{s})\right|^q\big].\\&\left|g''\Big(\sum_{\mathfrak{s}=1}^{\ell}\zeta_\mathfrak{s}-\big(\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}d_\mathfrak{s}+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}c_\mathfrak{s}\big)\Big)\right|^q\le\\&\qquad\sum_{\mathfrak{s}=1}^{\ell}\left|g''(\zeta_\mathfrak{s})\right|^q-\left[\frac{1+\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(c_\mathfrak{s})\right|^q+\frac{1-\kappa}{2}\sum_{\mathfrak{s}=1}^{\ell-1}\left|g''(d_\mathfrak{s})\right|^q\right].\end{aligned}$$

$$= \frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^2}{8(\alpha+1)}\Big(\Upsilon_1(\alpha)\Big)^{1-\frac{1}{q}} \times \Bigg(\Upsilon_1(\alpha)\sum_{\mathfrak{s}=1}^{\ell}\left|\mathrm{g}''(\zeta_{\mathfrak{s}})\right|^q - \Bigg\{\left(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{c}_{\mathfrak{s}})\right|^q +$$
$$\left(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{d}_{\mathfrak{s}})\right|^q\Bigg\}\Bigg)^{\frac{1}{q}} + \Bigg(\Upsilon_1(\alpha)\sum_{\mathfrak{s}=1}^{\ell}\left|\mathrm{g}''(\zeta_{\mathfrak{s}})\right|^q$$
$$-\Bigg\{\left(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{c}_{\mathfrak{s}})\right|^q + \left(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\right)\sum_{\mathfrak{s}=1}^{\ell-1}\left|\mathrm{g}''(\mathrm{d}_{\mathfrak{s}})\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}.$$

□

Remark 2.17. *In above Theorem 2.16, for $\ell = 2$ one can obtain the following inequality:*

$$\left|S_\alpha(\mathrm{c}_{\mathfrak{s}},\mathrm{d}_{\mathfrak{s}},\zeta_{\mathfrak{s}},2;\mathrm{g})\right| \le \frac{(\mathrm{d}-\mathrm{c})^2}{8(\alpha+1)}\Big(\Upsilon_1(\alpha)\Big)^{1-\frac{1}{q}} \times \Big[\Big(\left|\mathrm{g}''(\zeta_1)\right|^q + \left|\mathrm{g}''(\zeta_2)\right|^q\Big)\Upsilon_1(\alpha)$$
$$-\left(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\right)\left|\mathrm{g}''(\mathrm{c})\right|^q + \Big(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\Big)\left|\mathrm{g}''(\mathrm{d})\right|^q\Big]\Big]^{\frac{1}{q}} +$$
$$\Big[\Big(\left|\mathrm{g}''(\zeta_1)\right|^q + \left|\mathrm{g}''(\zeta_2)\right|^q\Big)\Upsilon_1(\alpha) - \left(\frac{\Upsilon_1(\alpha)-\Upsilon_2(\alpha)}{2}\right)\left|\mathrm{g}''(\mathrm{c})\right|^q + \Big(\frac{\Upsilon_1(\alpha)+\Upsilon_2(\alpha)}{2}\Big)\left|\mathrm{g}''(\mathrm{d})\right|^q\Big]\Big]^{\frac{1}{q}}.$$

Remark 2.18. *If we substitute $\zeta_1 = \mathrm{c}$ and $\zeta_2 = \mathrm{d}$ in Remark 2.17, then we obtain the inequality that is proved in [20].*

Remark 2.19. *If $\alpha = 1$ and $\rho_\alpha = \frac{1}{3}$ is picked in Remark 2.17, we attain traditional Simpson estimates proved in [21].*

Remark 2.20. *Here by using $\zeta_1 = \mathrm{c}$, $\zeta_2 = \mathrm{d}$, $\alpha = 1$ and $\rho_\alpha = \frac{1}{3}$ in Remark 2.17, We establish the bounds for the standard Simpson inequality for convex functions which are twice differentiable, that is presented in [10].*

3. Applications

3.1. Simpson-like quadrature formula

In the following section, we will explore the applications of the integral inequalities established in the previous part. These inequalities can be effectively harnessed to approximate composite quadrature rules, resulting substantial decrease in error compared to traditional procedures.

Proposition 3.1. *Let the function $\mathrm{g} : [\zeta_1, \zeta_2] \to \mathfrak{R}$ is a bounded. If $I_\imath \in \zeta_1 = \vartheta_0, \vartheta_1, ..., \vartheta_{\imath-1}, \vartheta_\imath = \zeta_2$ is the interval and $\vartheta_{\gamma,1}, \vartheta_{\gamma,2} \in [\vartheta_\gamma, \vartheta_{\gamma+1}]$ with $\chi_\gamma = \vartheta_{\gamma+1} - \vartheta_\gamma$ $\forall\ \gamma = 0, 1, ..., \imath - 1$ then we will have,*

$$\int_{\vartheta_0+\vartheta_\imath-\vartheta_2}^{\vartheta_0+\vartheta_\imath-\vartheta_1}\mathrm{g}(P)dP = B(I_\imath,\mathrm{g}) + R(I_\imath,\mathrm{g})$$

where

$$B(I_\imath,\mathrm{g}) = \frac{1}{6}\Bigg[\sum_{\gamma=0}^{\imath-1}\mathrm{g}[\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,1}]\chi_\gamma + 4\sum_{\gamma=0}^{\imath-1}\mathrm{g}[\vartheta_\gamma+\vartheta_{\gamma+1}-\frac{\vartheta_{\gamma,1}+\vartheta_{\gamma,2}}{2}]\chi_\gamma +$$
$$\sum_{\gamma=0}^{\imath-1}\mathrm{g}[\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,2}]\chi_\gamma\Bigg]$$

and remainder term satisfies

$$\left|R(I_\iota,\mathrm{g})\right| \leq \frac{1}{162}\Big[\chi_\gamma^2 2\sum_{\gamma=0}^{\iota-1}\Big[\left|\mathrm{g}''(\vartheta_\gamma)\right|+\left|\mathrm{g}''(\vartheta_{\gamma+1})\right|\Big]-\Big[\chi_\gamma^2\sum_{\gamma=0}^{\iota-1}\left|\mathrm{g}''(\vartheta_{\gamma,1})\right|+\chi_\gamma^2\sum_{\gamma=0}^{\iota-1}\left|\mathrm{g}''(\vartheta_{\gamma,2})\right|\Big]\Big].$$

Proof. Applying the Theorem 2.6, with $\ell=2$ and $\alpha=1$ on interval $[\vartheta_\gamma,\vartheta_{\gamma+1}]$, $\gamma=0,1,...,\iota-1$, we get

$$\Big|\frac{1}{6}\Big[\mathrm{g}[\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,1}]h_\gamma+4\mathrm{g}[\vartheta_\gamma+\vartheta_{\vartheta+1}-\frac{\vartheta_{\gamma,1}+\vartheta_{\gamma,2}}{2}]\chi_\gamma+\mathrm{g}[\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,2}]\chi_\gamma\Big]-$$
$$\int_{\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,2}}^{\vartheta_\gamma+\vartheta_{\gamma+1}-\vartheta_{\gamma,1}}\mathrm{g}(P)dP \leq$$
$$\frac{1}{162}\Big[\chi_\gamma^2 2\sum_{\gamma=0}^{\iota-1}\Big[\left|\mathrm{g}''(\vartheta_\gamma)\right|+\left|\mathrm{g}''(\vartheta_{\gamma+1})\right|\Big]-\Big[\chi_\gamma^2\sum_{\gamma=0}^{\iota-1}\left|\mathrm{g}''(\vartheta_{\gamma,1})\right|+\chi_\gamma^2\sum_{\gamma=0}^{\iota-1}\left|\mathrm{g}''(\vartheta_{\gamma,2})\right|\Big]\Big]$$

$\forall\ \gamma=0,1,...,\iota-1$. Taking summation across 0 to $\iota-1$ and considering the triangle inequality, we get the aforementioned outcome. □

3.2. $\mathbf{Q}$-*digamma function(in short* $\mathbf{Q}$-*DF)*

$\Psi_{\mathbf{Q}}$-digamma function which is described as logarithmic derivative of $\mathbf{Q}$-DF. Some studies were also used to investigate the monotonicity and full monotonicity features of functions associated with the $\mathbf{Q}$-gamma function (in short $\mathbf{Q}$-GF) and $\mathbf{Q}$-DF, which result in surprising inequalities [22, 23] given as:

Suppose $0<\mathbf{q}<1$, the

$$\Psi_{\mathbf{q}} = -\ln(1-\mathbf{q})+\ \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+\xi}}{1-\mathbf{q}^{k+\xi}}$$
$$= -\ln(1-\mathbf{q})+\ \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k\xi}}{1-\mathbf{q}^{k\xi}}.$$

$\mathbf{q}-$DF $\Psi_{\mathbf{q}}$ for $\xi>0$ and $\mathbf{q}\geq 1$ can be given as:

$$\Psi_{\mathbf{q}} = -\ln(\mathbf{q}-1)+\ \ln\mathbf{q}\Big[\xi-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\xi)}}{1-\mathbf{q}^{-(k+\xi)}}\Big]$$
$$= -\ln(\mathbf{q}-1)+\ \ln\mathbf{q}\Big[\xi-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\xi}}{1-\mathbf{q}^{-k\xi}}\Big].$$

is monotonic on given interval $(0,\infty)$ and is convex.

Proposition 3.2. *Let assume* $\mathrm{c}_\mathfrak{s},\mathrm{d}_\mathfrak{s},\zeta_\mathfrak{s}\in I\subset\mathfrak{R}$ *Let* $\mathrm{c}_\mathfrak{s}\geq\mathrm{d}_\mathfrak{s}$ *for all* $\mathfrak{s}=\{1,...,\ell\}$ *and* $1\geq \boldsymbol{q}\geq 0$ *then we have below mentioned inequality:*

$$\left|S_\alpha(\mathrm{c}_\mathfrak{s},\mathrm{d}_\mathfrak{s},\zeta_\mathfrak{s},2;\Psi_q)\right| \leq \frac{\left(\sum_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_\mathfrak{s}-\mathrm{c}_\mathfrak{s})\right)^2}{8(\alpha+1)}$$
$$\Upsilon_1(\alpha)\Big[2\sum_{\mathfrak{s}=1}^{\ell}\left|\Psi_q^{(3)}(\zeta_\mathfrak{s})\right|-\Big\{\sum_{\mathfrak{s}=1}^{\ell-1}\left|\Psi_q^{(3)}(\mathrm{c}_\mathfrak{s})\right|+\sum_{\mathfrak{s}=1}^{\ell-1}\left|\Psi_q^{(3)}(\mathrm{d}_\mathfrak{s})\right|\Big\}\Big]$$

where

$$\Upsilon_1(\alpha) = \begin{cases} \frac{1-\alpha^2}{3(\alpha+2)} & 0 < \alpha \le \frac{1}{2} \\ 2\left(\frac{(\rho_\alpha)^{\alpha+2}}{\alpha+2} - \frac{(1-2\alpha)\rho_\alpha + (\alpha+1)(\rho_\alpha)^2}{3}\right) + \frac{1-\alpha^2}{3(\alpha+2)} & \alpha > \frac{1}{2}. \end{cases}$$

Proof. According to the definition of $\mathbf{Q}$-DF, it is straightforward to know that for each $\mathbf{q} \in (0,1)$, the mapping $g \to \Psi'_{\mathbf{q}}$ on $(0,\infty)$ is completely monotonic. It is to say that on $(0,\infty)$, $g''(\xi) = \Psi'''_{\mathbf{q}}(\xi)$ is non-negative convex mapping. We get the required result by utilising this substitution in Theorem 2.6. □

Remark 3.3. *If $\ell = 2$, $\zeta_1 = c$ and $\zeta_2 = d$ is picked in Proposition 3.2, then we have subsequent inequality:*

$$\left|\frac{1}{6}\left[\Psi'_q(c) + 4\Psi'_q\left(\frac{c+d}{2}\right) + \Psi'_q(d)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(d-c)^\alpha}\right.$$
$$\left.\times J^{\alpha}_{c^-}\Psi'_q\left(\frac{c+d}{2}\right) + J^{\alpha}_{d^+}\Psi'_q\left(\frac{c+d}{2}\right)\right| \le \frac{(d-c)^2}{8(\alpha+1)}\Upsilon_1(\alpha)\left\{ \mid \Psi^{(3)}_q(c) \mid + \mid \Psi^{(3)}_q(d) \mid \right\}.$$

Remark 3.4. *If $\alpha = 1$ is picked in Remark 3.3, then we have subsequent inequality:*

$$\left|\frac{1}{6}\left\{\Psi'_q(c) + 4\Psi'_q\left(\frac{c+d}{2}\right) + \Psi'_q(d)\right\} - \frac{\Psi_q(d) - \Psi_q(c)}{d-c}\right|$$
$$\le \frac{(d-c)^2}{162}\left[\mid \Psi^{(3)}_q(c) \mid + \mid \Psi^{(3)}_q(d) \mid \right].$$

3.3. Modified Bessel Function

Bessel functions were called after Friedrich Wilhelm Bessel (1784 − 1846), however Daniel Bernoulli is often regarded as the first to introduce them.

In 1732, the Bessels notion is operational. Several findings about Bessel functions have been made utilizing its generating function.

We consider the modified Bessel function of the first kind Θ_γ, that has the series representation [24, 25].

$$\Theta_\gamma(\xi) = \Sigma_{n\ge 0}\frac{\left(\frac{\xi}{2}\right)^{\gamma+2n}}{n!\Gamma(\gamma+n+1)},$$

where $\gamma > -1$ and $\xi \in \mathfrak{R}$, whereas modified Bessel function of the second kind [25], is generally defined as

$$\Theta_\gamma(\xi) = \frac{\pi}{2}\,\frac{\mathfrak{I}_{-\gamma}(\xi) - \mathfrak{I}_\gamma(\xi)}{\sin\gamma\pi}. \tag{12}$$

taking into account the function $\Theta_\gamma(\xi) : \mathfrak{R} \to [1,\infty)$ given by

$$\Theta_\gamma(\xi) = 2^\gamma\Gamma(\gamma+1)\,\xi^{-\gamma}\hbar_\gamma(\xi),$$

where Γ represents gamma function.

Formula for the derivative of $\Theta_\gamma(\xi)$ of first order is stated by [25]:

$$\Theta'_\gamma(\xi) = \frac{\xi}{2(\gamma+1)}\Theta_{\gamma+1}(\xi) \tag{13}$$

and the 2nd derivative may be derived simply from (13) to be

$$\Theta''_\gamma(\xi) = \frac{\xi^2\Theta_{\gamma+2}(\xi)}{4(\gamma+1)(\gamma+2)} + \frac{\Theta_{\gamma+1}(\xi)}{2(\gamma+1)}. \tag{14}$$

Proposition 3.5. *Suppose for $\gamma > -1$ and $q > 1$ and $\mathrm{c}_{\mathfrak{s}}, \mathrm{d}_{\mathfrak{s}}, \zeta_{\mathfrak{s}} \in I \subset \mathfrak{R}$, then the subsequent inequality:*

$$\left|S_{\alpha}(\mathrm{c}_{\mathfrak{s}}, \mathrm{d}_{\mathfrak{s}}, \zeta_{\mathfrak{s}}, 2; \Theta_{\gamma})\right|$$
$$\leq \frac{\left(\sum\limits_{\mathfrak{s}=1}^{\ell-1}(\mathrm{d}_{\mathfrak{s}}-\mathrm{c}_{\mathfrak{s}})\right)^2}{8(\alpha+1)} \Upsilon_1(\alpha)\left[2\left\{\frac{\sum\limits_{\mathfrak{s}=1}^{\ell} \zeta_{\mathfrak{s}}{}^2\left|\Theta_{\gamma+2}(\zeta_{\mathfrak{s}})\right|}{4(\gamma+1)(\gamma+2)} + \frac{\sum\limits_{\mathfrak{s}=1}^{\ell}\left|\Theta_{\gamma+1}(\zeta_{\mathfrak{s}})\right|}{2(\gamma+1)}\right\} -\right.$$
$$\left.\left\{\frac{\sum\limits_{\mathfrak{s}=1}^{\ell-1} \mathrm{c}_{\mathfrak{s}}{}^2\left|\Theta_{\gamma+2}(\mathrm{c}_{\mathfrak{s}})\right|}{4(\gamma+1)(\gamma+2)} + \frac{\sum\limits_{\mathfrak{s}=1}^{\ell-1}\left|\Theta_{\gamma+1}(\mathrm{c}_{\mathfrak{s}})\right|}{2(\gamma+1)}\right\} - \left\{\frac{\sum\limits_{\mathfrak{s}=1}^{\ell-1} \mathrm{d}_{\mathfrak{s}}{}^2\left|\Theta_{\gamma+2}(\mathrm{d}_{\mathfrak{s}})\right|}{4(\gamma+1)(\gamma+2)} + \frac{\sum\limits_{\mathfrak{s}=1}^{\ell-1}\left|\Theta_{\gamma+1}(\mathrm{d}_{\mathfrak{s}})\right|}{2(\gamma+1)}\right\}\right]$$

holds.

Proof. The Bessel function $\mathrm{g} \to \Theta_{\gamma}$ can be used to attain the desired result. It is straightforward that utilising $\Theta_{\gamma}''(\xi), \xi > 0$ on $(0, \infty)$ is a convex function. We achieve the required result by examining this substitution and utilising (13) in Theorem 2.6. □

Remark 3.6. *If $\ell = 2$, $\alpha = 1$, $\zeta_1 = \mathrm{c}$ and $\zeta_2 = \mathrm{d}$ is picked in Proposition 3.5 then we arrive at the inequality shown below:*

$$\left|\frac{1}{6}\left\{\Theta_{\gamma}(\mathrm{c}) + 4\Theta_{\gamma}\left(\frac{\mathrm{c}+\mathrm{d}}{2}\right) + \Theta_{\gamma}(\mathrm{d})\right\} - \frac{\Theta_{\gamma}(\mathrm{c}) - \Theta_{\gamma}(\mathrm{d})}{\mathrm{d}-\mathrm{c}}\right|$$
$$\leq (\mathrm{d}-\mathrm{c})^2\left(\frac{1}{162}\right)\left[\left\{\frac{\mathrm{c}^2}{4(\gamma+1)(\gamma+2)}\left|\Theta_{\gamma+2}(\mathrm{c})\right| + \frac{1}{2(\gamma+1)}\left|\Theta_{\gamma+1}(\mathrm{c})\right|\right\}\right.$$
$$\left. + \left\{\frac{\mathrm{d}^2}{4(\gamma+1)(\gamma+2)}\left|\Theta_{\gamma+2}(\mathrm{d})\right| + \frac{1}{2(\gamma+1)}\left|\Theta_{\gamma+1}(\mathrm{d})\right|\right\}\right].$$

4. Conclusion

To the best of our understanding, this present study marks the exploration of the RL-fractional integral inequality of Simpson-type, specifically when considering differentiable functions in conjunction with a majorization approach. In our research, we have established a novel fractional integral identity for functions which are differentiable by leveraging the principles of majorization theory. We presented Simpson inequalities tailored for mappings that possess second derivatives and whose absolute value derivatives, raised to specific powers, exhibit convex properties. Our findings extend beyond conventional Simpson inequalities, incorporating connections to Bessel functions within the application section and relating to special functions theory, such as $\mathfrak{Q}$-DF. Furthermore, there is potential to expand this concept to encompass generalized integral operators with non-local and non-singular kernels by incorporating the principles of generalized convexities.

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