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On a q^2-Srivastava-Attiya operator and related class of functions

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Abstract. An analogue of the well known Hurwitz-Lerch Zeta function is defined which in turn defines the q^2-Srivastava-Attiya operator. A new family of certain analytic functions involving the q^2-derivative and q^2-Srivastava-Attiya operator is taken into account and coefficient based results are obtained. Special cases of the results obtained are also discussed.

1. Preliminaries

The study of calculus without the notion of limits is known nowadays as quantum calculus. F.H. Jackson [5, 6] invented the quantum calculus, often known as the q-calculus, in the early 20th century. Because of its applicability in domains such as mathematics, mechanics, and physics, the field of q-calculus is seeing significant growth. The history of q-calculus may be seen in its many applications in quantum physics, analytic number theory, hypergeometric functions, operator theory, and, more recently, the theory of analytic univalent functions (see [16, 19]). While focusing on q-calculus and its applications in many domains of mathematics and physics, we will also look at q-analogues of several recent achievements in geometric function theory, namely theory of analytic univalent functions in the unit disc [15, 26]. For convenience, we have included some of the fundamental definitions and explanations of the q-calculus concepts utilized in this research [8]. We shall assume that q meets the requirement $0 < q < 1$ for the purpose of this paper. Let us recall some basic definitions.

Definition 1.1. *[8] Let $\lambda \in \mathbb{C}$ and $0 < q < 1$. The q-number denoted by $[\lambda]_q$, is defined as*

$$[\lambda]_q = \frac{1-q^\lambda}{1-q}.$$

For $\lambda = n \in \mathbb{N} \cup \{0\}$, the q-number $[n]_q$ is defined as

$$[n]_q = \frac{1-q^n}{1-q} = \begin{cases} 1+q+\cdots+q^{n-1}, & \text{if } n \neq 0 \\ 0, & \text{if } n = 0 \end{cases}.$$

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It is easy to observe that $\lim_{q\to 1^-}[n]_q = n$.

Definition 1.2. *[3] Let $\lambda \in \mathbb{C}$ and $0 < q < 1$. The symmetric q-number denoted by $\widetilde{[\lambda]}_q$ is defined as*

$$\widetilde{[\lambda]}_q = \frac{q^\lambda - q^{-\lambda}}{q - q^{-1}}.$$

For $\lambda = n \in \mathbb{N} \cup \{0\}$, the symmetric q-number $\widetilde{[n]}_q$ is defined as

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \begin{cases} \frac{1}{q^{n-1}}(1 + q^2 + \cdots + q^{2n-2}), & if\ \ n \neq 0 \\ 0, & if\ \ n = 0 \end{cases}.$$

Remark 1.3. *Observe that the so called "symmetric" q-number $\widetilde{[\lambda]}_q$ in Definition 1.2, is a trivial and inconsequential variation of the q-number $[\lambda]_{q^2}$ by multiplying the latter by $q^{1-\lambda}$. Therefore, from now onwards we use the notation $[\lambda]_{q^2}$ for $\widetilde{[\lambda]}_q$ and call it q^2-number.*

It is easy to observe that $\lim_{q\to 1^-} [n]_{q^2} = n$. Also note that $[n]_{q^2}$, which commonly arises in the investigation of q-deformed quantum mechanical simple harmonic oscillator [2], do not reduce to $[n]_q$. For $n \in \mathbb{N} \cup \{0\}$ and $0 < q < 1$, the q^2-number satisfies the following relations:

$$[n+1]_{q^2} = \frac{[n]_{q^2}}{q} + q^n \tag{1}$$

and

$$[n]_{q^2} = \frac{[n+b]_{q^2}\, q^n - [b]_{q^2}}{q^{n+b}}\ (b \in \mathbb{N}). \tag{2}$$

We now turn towards defining the q^2-derivative.

Definition 1.4. *For a function f, the q^2-derivative denoted as $D_{q^2}f(z)$ is defined as*

$$D_{q^2}f(z) = \begin{cases} \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} & for\ \ z \neq 0 \\ f'(0) & for\ \ z = 0 \end{cases}.$$

Note that $D_{q^2}z^n = [n]_{q^2}\, z^{n-1}$.

Remark 1.5. *The above q^2-derivative D_{q^2} was earlier defined by Brahim and Sidomou in [3].*

Let $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$ denotes the open unit disk and $\mathcal{A}$ be the class of analytic functions in $\mathbb{U}$ such that

$f(0) = 0$ and $f'(0) = 1$.

So $f \in \mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{3}$$

Hence, for $f(z)$ of the form (3),

$$D_{q^2}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{q^2}\, a_n z^{n-1} \tag{4}$$

Let $\mathcal{S}$ denotes the subclass of univalent functions in $\mathcal{A}$. A starlike function is one which maps unit disk conformally onto a starike domain with respect to origin. Let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ having starlike functions such that whenever $f \in \mathcal{S}^*$,

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ (z \in \mathbb{U})$$

holds.
Denote by $\mathcal{S}^*(\gamma)$, the class of starlike functions of order γ, consists of $f \in \mathcal{A}$ satisfying the condition,

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \ (0 \leq \gamma < 1; z \in \mathbb{U})$$

holds.
The convolution of two functions

$$g(z) = \sum_{n=1}^{\infty} b_n z^n \text{ and } h(z) = \sum_{n=1}^{\infty} c_n z^n$$

convergent in the unit disk $\mathbb{U}$ is defined as the function $k = g * h$ with convergent power series

$$k(z) = \sum_{n=1}^{\infty} b_n c_n z^n \ (z \in \mathbb{U}).$$

Definition 1.6. *[11] Let f and g be two analytic functions in $\mathbb{U}$. We say that f is subordinate to g (written as $f \prec g$) if*

$$f(z) = g(w(z)) \ (z \in \mathbb{U})$$

for some analytic function w in $\mathbb{U}$ such that $|w(z)| < 1$ and $w(0) = 0$. The superordinate function g need not be univalent.

If g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and image of $\mathbb{U}$ under g contains image of $\mathbb{U}$ under f. For comprehensive analysis of subordination one may refer to the monograph by Miller and Mocanu [11].

2. Introduction and Motivation

Investigation of different types of classes of analytic univalent functions in geometric function theory is indeed a trend, this suggests that researchers are focusing on exploring and characterizing various classes of functions within the context of geometric function theory. These classes encompasses a wide range of properties, behaviors, and geometric characteristics of analytic functions [1, 9, 12, 24, 27]. In a recent survey-cum-expository review article [15], an introductory overview of Bessel polynomials, generalized Bessel polynomials and q-extensions of Bessel polynomials have been given and each of these polynomials are investigated and applied in the existing literature on the subject. This survey also includes the investigation of various general families of hypergeometric functions together with the corresponding q-hypergeometric functions. Srivastava and Gaboury in 2015 [21] defined a new class of analytic functions by means of a generalization of Srivastava-Attiya operator and several geometric properties including coefficient inequalities, distortion theorems and the Fekete-Szegö problem for this function class were examined. In [22], coefficient inequalities, growth and distortion results, extreme points and solution to the Fekete-Szegö problem for a new class of analytic functions involving a generalization of the well known Srivastava-Attiya operator, were obtained. By utilizing the linear operator introduced and studied by Srivastava and Attiya [18], Srivastava et al. investigated suitable classes of admissible functions, presented the dual properties of third-order differential subordination and established various sandwich-type theorems for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform [25]. The systematic introduction and exploration of univalence criteria for a novel family of integral operators are aimed for in [23], utilizing a broadly examined form of the Srivastava-Attiya operator, several new conditions for univalence specific to this generalized Srivastava-Attiya operator are established, and its connection to prior related research are highlighted. For a normalized class of analytic functions which is defined by Srivastava-Attiya operator, Sim et al. [13] obtained the bounds for the real part, argument and made a comparison between these results and already known results for some specific cases.

The recent survey-cum-expository review article [14] comprehensively explore the systematic aspects of the Hurwitz-Lerch Zeta function $\phi(s,b;z)$, shedding light on its properties and characteristics. The general Hurwitz-Lerch Zeta function $\phi(s,b;z)$ given by

$$\phi(s,b;z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s}$$

($b \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$ and $\Re e(s) > 1$ when $|z| = 1$), can be continued meromorphically to whole complex s-plane except for a simple pole at $s = 1$ with residue 1. Consider the q-analogue of the Hurwitz-Lerch Zeta function [12] (see also [4, 20]) given by the following series:

$$\phi_q(s,b;z) = \sum_{n=0}^{\infty} \frac{z^n}{[n+b]_q{}^s},$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re e(s) > 1$ when $|z| = 1$. For $f \in \mathcal{A}$, the q-analogue of Srivastava-Attiya operator [12] is given by

$$J_{q,b}^s f(z) = \psi_q(s,b;z) * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_q}{[n+b]_q} \right)^s a_n z^n,$$

where

$$\psi_q(s,b;z) = [1+b]_q{}^s \left(\phi_q(s,b;z) - \frac{1}{[b]_q{}^s} \right) = z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_q}{[n+b]_q} \right)^s z^n,$$

$b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re e(s) > 1$ when $|z| = 1$. The q-Srivastava-Attiya operator $J_{q,b}^s$, widely and extensively utilized, generalizes several well known operators explored in prior investigations and holds numerous applications in the geometric function theory [26].

We now define the q^2-analogue of the well known Hurwitz-Lerch Zeta function by the following series:

$$\phi_{q^2}(s,b;z) = \sum_{n=0}^{\infty} \frac{z^n}{[n+b]_{q^2}^s}$$

which upon normalizing gives us

$$\begin{aligned} \psi_{q^2}(s,b;z) &= [1+b]_{q^2}^s \left(\phi_{q^2}(s,b;z) - \frac{1}{[b]_{q^2}^s} \right) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}} \right)^s z^n, \end{aligned} \tag{5}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re e(s) > 1$ when $|z| = 1$. Making use of the series (5) and (3), we now define the q^2-Srivastava-Attiya operator $J_{q^2,b}^s f(z) : \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned} J_{q^2,b}^s f(z) &= \psi_{q^2}(s,b;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}} \right)^s a_n z^n. \end{aligned} \tag{6}$$

Letting $q \to 1^-$, the q^2-Srivastava-Attiya operator $J^s_{q^2,b}f(z)$ transforms into the Srivastava-Attiya operator [18, 20]. Restricting b to the set of natural numbers and making use of (4) and (6), one may easily obtain the following identity:

$$zD_{q^2}J^{s+1}_{q^2,b}f(z) = \left(1 + \frac{[b]_{q^2}}{q^{b+1}}\right)J^s_{q^2,b}f(z) - q\left(\frac{[b]_{q^2}}{q^{b+1}}J^{s+1}_{q^2,b}f(q^{-1}z)\right). \tag{7}$$

Letting $q \to 1^-$ the identity (7) reduces to

$$zJ'_{s+1,b}f(z) = (1+b)J_{s,b}f(z) - bJ_{s+1,b}f(z)$$

which was obtained by Srivastava and Attiya in [18].
Taking motivation from the aforementioned work, we here define a new class of analytic univalent functions involving q^2-Srivastava-Attiya operator $J^s_{q^2,b}f(z)$ and the q^2-derivative D_{q^2} as follows:

Definition 2.1. *Let $f \in \mathcal{A}$. We say that $f \in SJ^s_{q^2,b}(\mu, A, B)$ if f satisfies the following condition:*

$$\frac{zD_{q^2}J^s_{q^2,b}f(z)}{(1-\mu)J^s_{q^2,b}f(z) + \mu z} \prec \frac{1+Az}{1+Bz} \; (-1 \le B < A \le 1; z \in \mathbb{U}), \tag{8}$$

where $0 \le \mu \le 1$, $b \in \mathbb{C} \setminus \mathbb{Z}^-_0$ and $s \in \mathbb{C}$.

We now mention some of the special cases of the class $SJ^s_{q^2,b}(\mu, A, B)$. For $\mu = 0$ in the Definition 2.1, we obtain a subclass of starlike functions associated with the q^2-Srivastava-Attiya operator.

Definition 2.2. *Let $f \in \mathcal{A}$. We say $f \in SJ^s_{q^2,b}(0, A, B)$ if f satisfies*

$$\frac{zD_{q^2}J^s_{q^2,b}f(z)}{J^s_{q^2,b}f(z)} \prec \frac{1+Az}{1+Bz} \; (-1 \le B < A \le 1; z \in \mathbb{U}),$$

where $0 < q < 1, b \in \mathbb{C} \setminus \mathbb{Z}^-_0$ and $s \in \mathbb{C}$.

Remark 2.3. *As $q \to 1^-$, $SJ^s_{q^2,b}(0, A, B) := SJ^s_b(A, B)$ the Janowski class of starlike functions associated with Srivastava-Attiya operator, and if we further put $s = 0$ in $SJ^s_b(A, B)$ we obtain the well known Janowski class of starlike functions which was introduced by Janowski in 1973 [7].*

For $\mu = 1$ in the Definition 2.1, we obtain a class $SJ^s_{q^2,b}(1, A, B)$, which shows a subordination relationship between $D_{q^2}J^s_{q^2,b}f(z)$ and the function $\frac{1+Az}{1+Bz}$.

Definition 2.4. *Let $f \in \mathcal{A}$. We say that $f \in SJ^s_{q^2,b}(1, A, B)$ if f satisfies*

$$D_{q^2}J^s_{q^2,b}f(z) \prec \frac{1+Az}{1+Bz} \; (-1 \le B < A \le 1; z \in \mathbb{U}),$$

where $0 < q < 1, b \in \mathbb{C} \setminus \mathbb{Z}^-_0$ and $s \in \mathbb{C}$.

3. Convolution Conditions

Theorem 3.1. *Let $f \in \mathcal{A}$. Then f is said to be in $SJ^s_{q^2,b}(\mu, A, B)$ if and only if f satisfies the condition*

$$f(z) * g(z) \neq (1 + A\xi)\mu z,$$

where

$$g(z) = z\left[(1+B\xi) - (1+A\xi)(1-\mu)\right] + \sum_{n=2}^{\infty}\left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s ([n]_{q^2}(1+B\xi) - (1+A\xi)(1-\mu))z^n \tag{9}$$

and $\xi \in \mathbb{C}, |\xi| = 1$.

Proof. Since $f \in SJ^s_{q^2,b}(\mu, A, B)$ if and only if for $\xi \in \mathbb{C}, |\xi| = 1$ and $z \in \mathbb{U}$,

$$\frac{zD_{q^2}J^s_{q^2,b}f(z)}{(1-\mu)J^s_{q^2,b}f(z)+\mu z} \neq \frac{1+A\xi}{1+B\xi}$$

or,

$$zD_{q^2}J^s_{q^2,b}f(z)\{(1+B\xi)\} - ((1-\mu)J^s_{q^2,b}f(z)+\mu z)\{1+A\xi\} \neq 0. \tag{10}$$

We know that $\frac{z}{1-z}$ is the identity function for the convolution. Thus, for $f \in \mathcal{A}$

$$f(z) = f(z) * \frac{z}{1-z}.$$

A simple computation yields that

$$zD_{q^2}f(z) = f(z) * \frac{z}{(1-qz)(1-q^{-1}z)}.$$

Hence, we may write

$$J^s_{q^2,b}f(z) = f(z) * \psi_{q^2} * \frac{z}{1-z}$$

and

$$zD_{q^2}J^s_{q^2,b}f(z) = f(z) * \psi_{q^2} * \frac{z}{(1-qz)(1-q^{-1}z)}.$$

Thus, condition (10) is equivalent to

$$\left\{f(z) * \psi_{q^2} * \frac{z}{(1-qz)(1-q^{-1}z)}\right\}(1+B\xi) - \left\{(1-\mu)f(z) * \psi_{q^2} * \frac{z}{1-z}\right\}(1+A\xi) \neq (1+A\xi)\mu z \tag{11}$$

or

$$f(z) * g(z) \neq (1+A\xi)\mu z,$$

where $g(z)$ is given by (9), which completes the proof. □

For $\mu = 0$ and $\mu = 1$ in Theorem 3.1, we have the following special cases of our result.

Corollary 3.2. *Let $f \in \mathcal{A}$. Then f is said to be in $SJ^s_{q^2,b}(0, A, B)$ if and only if*

$$f(z) * \left\{(A-B)\xi z - \sum_{n=2}^{\infty}\left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s \Big([n]_{q^2}(1+B\xi) - (1+A\xi)\Big)z^n\right\} \neq 0$$

for $\xi \in \mathbb{C}, |\xi| = 1$ and $z \in \mathbb{U}$.

Corollary 3.3. *Let $f \in \mathcal{A}$. Then f is said to be in $SJ^s_{q^2,b}(1, A, B)$ if and only if*

$$f(z) * \left\{(1+B\xi) + \sum_{n=2}^{\infty}\left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s [n]_{q^2}(1+B\xi)z^{n-1}\right\} \neq 1 + A\xi$$

for $\xi \in \mathbb{C}, |\xi| = 1$ and $z \in \mathbb{U}$.

4. Coefficient Bounds

We now make an attempt to obtain the bounds of the initial coefficients a_2 and a_3 for $f \in SJ^s_{q^2,b}(\mu, A, B)$.

Theorem 4.1. *If f be of the form (3) belongs to the class $SJ^s_{q^2,b}(\mu, A, B)$, then for $b \in \mathbb{C} \setminus \{-1\}$,*

$$|a_2| \leq \frac{A-B}{[2]_{q^2} - (1-\mu)} \left| \left(\frac{[2+b]_{q^2}}{[1+b]_{q^2}} \right)^s \right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2} - (1-\mu)} \left| \left(\frac{[3+b]_{q^2}}{[1+b]_{q^2}} \right)^s \right| \max\left\{1, \frac{\left|B\,[2]_{q^2} - A(1-\mu)\right|}{[2]_{q^2} - (1-\mu)}\right\}.$$

In case $b \in \mathbb{N}$,

$$|a_2| \leq \frac{A-B}{[2]_{q^2} - (1-\mu)} \left| \left(\frac{1}{q} + \frac{q^{b+1}}{[1+b]_{q^2}} \right)^s \right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2} - (1-\mu)} \left| \left\{ \left(\frac{1}{q} + \frac{q^{b+1}}{[1+b]_{q^2}} \right) \left(\frac{1}{q} + \frac{q^{b+2}}{[2+b]_{q^2}} \right) \right\}^s \right| \tag{12}$$

$$\times \max\left\{1, \frac{\left|B\,[2]_{q^2} - A(1-\mu)\right|}{[2]_{q^2} - (1-\mu)}\right\}. \tag{13}$$

Proof. Since $f \in SJ^s_{q^2,b}(\mu, A, B)$, hence

$$\frac{zD_{q^2} J^s_{q^2,b} f(z)}{(1-\mu) J^s_{q^2,b} f(z) + \mu z} \prec \frac{1+Az}{1+Bz}.$$

So, there exist a function $w(z)$ such that $|w(z)| < 1, w(0) = 0$ ($|z| < 1$) of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n = c_1 z + c_2 z^2 + \cdots \tag{14}$$

such that

$$\frac{zD_{q^2} J^s_{q^2,b} f(z)}{(1-\mu) J^s_{q^2,b} f(z) + \mu z} = \frac{1+Aw(z)}{1+Bw(z)}$$

or,

$$zD_{q^2} J^s_{q^2,b} f(z)(1 + Bw(z)) = (1 + Aw(z))\{(1-\mu) J^s_{q^2,b} f(z) + \mu z\}.$$

Using (4), (6) and (14), we may write

$$\left(z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}} \right)^s [n]_{q^2} a_n z^n \right) \left(1 + B \sum_{n=1}^{\infty} c_n z^n \right) = \left(1 + A \sum_{n=1}^{\infty} c_n z^n \right) \times \left(z + (1-\mu) \sum_{n=2}^{\infty} \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}} \right)^s a_n z^n \right).$$

On comparing coefficients of z^2 and z^3, we obtain

$$a_2 = \frac{c_1(A-B)\,[2+b]_{q^2}^s}{([2]_{q^2}-(1-\mu))\,[1+b]_{q^2}^s} \tag{15}$$

and

$$a_3 = \frac{(A-B)\,[3+b]_{q^2}^s}{([3]_{q^2}-(1-\mu))\,[1+b]_{q^2}^s}\left\{c_2-\left(\frac{B\,[2]_{q^2}-A(1-\mu)}{[2]_{q^2}-(1-\mu)}\right)c_1^2\right\}. \tag{16}$$

We know that coefficients of the bounded function w satisfies $|c_k| \leq 1$ $(k = 1, 2, 3, \cdots)$, from (15) we write

$$|a_2| \leq \frac{A-B}{[2]_{q^2}-(1-\mu)}\left|\left(\frac{[2+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right|.$$

In case $b \in \mathbb{N}$ and using (1), we may write

$$|a_2| \leq \frac{A-B}{[2]_{q^2}-(1-\mu)}\left|\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)^s\right|.$$

Since $w(z)$ is of the form (14), hence from the estimate $|c_2 - \alpha c_1^2| \leq \max\{1, |\alpha|\}$ $(\alpha \in \mathbb{C})$ given in [10], we obtain from (16) ,

$$|a_3| \leq \frac{A-B}{[3]_{q^2}-(1-\mu)}\left|\left(\frac{[3+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right|\max\left\{1\frac{\left|B\,[2]_{q^2}-A(1-\mu)\right|}{[2]_{q^2}-(1-\mu)}\right\}.$$

For $b \in \mathbb{N}$ and using (1), it is easy to observe

$$\frac{[3+b]_{q^2}}{[1+b]_{q^2}} = \left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)\left(\frac{1}{q}+\frac{q^{b+2}}{[2+b]_{q^2}}\right).$$

Hence, from (16) we may write

$$|a_3| = \frac{A-B}{[3]_{q^2}-(1-\mu)}\left|\left\{\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)\left(\frac{1}{q}+\frac{q^{b+2}}{[2+b]_{q^2}}\right)\right\}^s\right|\left|c_2-\left(\frac{B\,[2]_{q^2}-A(1-\mu)}{[2]_{q^2}-(1-\mu)}\right)c_1^2\right|.$$

Again making use of the estimate $|c_2 - \alpha c_1^2| \leq \max\{1, |\alpha|\}$ $(\alpha \in \mathbb{C})$ given in [10], we have

$$|a_3| \leq \frac{A-B}{[3]_{q^2}-(1-\mu)}\left|\left\{\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)\left(\frac{1}{q}+\frac{q^{b+2}}{[2+b]_{q^2}}\right)\right\}^s\right|\max\left\{1,\frac{\left|B\,[2]_{q^2}-A(1-\mu)\right|}{[2]_{q^2}-(1-\mu)}\right\}$$

which completes the proof. □

For $\mu = 0$ and $\mu = 1$ in Theorem 4.1, we obtain the following special cases of our result.

Corollary 4.2. *If f be of the form (3) belongs to the class $SJ^s_{q^2,b}(0, A, B)$, then for $b \in \mathbb{C} \setminus \{-1\}$,*

$$|a_2| \leq \frac{A-B}{[2]_{q^2}-1}\left|\left(\frac{[2+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2}-1}\left|\left(\frac{[3+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right| \max\left\{1, \frac{\left|B\,[2]_{q^2}-A\right|}{[2]_{q^2}-1}\right\}.$$

In case $b \in \mathbb{N}$,

$$|a_2| \leq \frac{A-B}{[2]_{q^2}-1}\left|\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)^s\right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2}-1}\left|\left\{\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)\left(\frac{1}{q}+\frac{q^{b+2}}{[2+b]_{q^2}}\right)\right\}^s\right| \tag{17}$$

$$\times \max\left\{1, \frac{\left|B\,[2]_{q^2}-A\right|}{[2]_{q^2}-1}\right\}. \tag{18}$$

Corollary 4.3. *If $f \in \mathcal{A}$ be of the form (3) belongs to the class $SJ^s_{q^2,b}(1, A, B)$, then for $b \in \mathbb{C} \setminus \{-1\}$,*

$$|a_2| \leq \frac{A-B}{[2]_{q^2}}\left|\left(\frac{[2+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2}}\left|\left(\frac{[3+b]_{q^2}}{[1+b]_{q^2}}\right)^s\right|.$$

In case $b \in \mathbb{N}$,

$$|a_2| \leq \frac{A-B}{[2]_{q^2}}\left|\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)^s\right|$$

and

$$|a_3| \leq \frac{A-B}{[3]_{q^2}}\left|\left\{\left(\frac{1}{q}+\frac{q^{b+1}}{[1+b]_{q^2}}\right)\left(\frac{1}{q}+\frac{q^{b+2}}{[2+b]_{q^2}}\right)\right\}^s\right|.$$

Lemma 4.4. *Let $p(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$. Then $\Re e(p(z)) > 0$.*

Proof. Since $p(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, which implies that $\frac{1+A}{1+B} > \Re e(p(z)) > \frac{1-A}{1-B} > 0$. □

We next obtain bounds of fixed n-th coefficient for the function $f_{n,\beta}(z) = z + \beta z^n$ $(n = 2, 3, \cdots; z \in \mathbb{U})$.

Theorem 4.5. *The function $f_{n,\beta}(z) = z + \beta z^n$ $(n = 2, 3, \cdots; z \in \mathbb{U})$ is in the class $SJ^s_{q^2,b}(\mu, A, B)$ if and only if for $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$,*

$$|\beta| \leq \min\left\{\frac{(A-B)\left|[n+b]_{q^2}\right|^s}{\left|[1+b]_{q^2}\right|^s\left((1-B)\,[n]_{q^2}-(1-A)(1-\mu)\right)}, \frac{(A-B)\left|[n+b]_{q^2}\right|^s}{\left|[1+b]_{q^2}\right|^s\left((1+B)\,[n]_{q^2}-(1+A)(1-\mu)\right)}\right\}. \tag{19}$$

Proof. Setting

$$F_{n,\beta}(z) = \frac{zD_{q^2} J^s_{q^2,b} f_{n,\beta}(z)}{(1-\mu) J^s_{q^2,b} f_{n,\beta}(z) + \mu z},$$

we obtain

$$F_{n,\beta}(z) = \frac{1 + \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s [n]_{q^2}\, \beta z^{n-1}}{1 + (1-\mu)\left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s \beta z^{n-1}}.$$

Since

$$F_{n,\beta}(z) \prec \frac{1+Az}{1+Bz},$$

we must have $F_{n,\beta}(z) \neq 0$. So we may assume that

$$\left| \left(\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right)^s [n]_{q^2}\, \beta \right| < 1.$$

Thus, we may write

$$\frac{1 - \left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s [n]_{q^2} \left|\beta\right|}{1 - (1-\mu)\left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s \left|\beta\right|} \leq \Re\mathfrak{e}(F_{n,\beta}(z)) \leq \frac{1 + \left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s [n]_{q^2} \left|\beta\right|}{1 + (1-\mu)\left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s \left|\beta\right|}.$$

From Lemma 4.4 we have,

$$\frac{1-A}{1-B} \leq \frac{1 - \left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s [n]_{q^2} \left|\beta\right|}{1 - (1-\mu)\left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s \left|\beta\right|} \text{ and } \frac{1 + \left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s [n]_{q^2} \left|\beta\right|}{1 + (1-\mu)\left|\frac{[1+b]_{q^2}}{[n+b]_{q^2}}\right|^s \left|\beta\right|} \leq \frac{1+A}{1+B}.$$

Hence, we obtain (19) which completes the proof. □

For $\mu = 0$ and $\mu = 1$ in Theorem 4.5, we obtain the following special cases of our result.

Corollary 4.6. *The function $f_{n,\beta}(z) = z + \beta z^n$ $(n = 2, 3, \cdots; z \in \mathbb{U})$ is in the class $SJ^s_{q^2,b}(0, A, B)$ if and only if for $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$,*

$$|\beta| \leq \min\left\{ \frac{(A-B)\left|[n+b]_{q^2}\right|^s}{\left|[1+b]_{q^2}\right|^s \left((1-B)\,[n]_{q^2} - (1-A)\right)}, \frac{(A-B)\left|[n+b]_{q^2}\right|^s}{\left|[1+b]_{q^2}\right|^s \left((1+B)\,[n]_{q^2} - (1+A)\right)} \right\}.$$

Corollary 4.7. *The function $f_{n,\beta}(z) = z + \beta z^n$ $(n = 2, 3, \cdots; z \in \mathbb{U})$ is in the class $SJ^s_{q^2,b}(1, A, B)$ if and only if for $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$,*

$$|\beta| \leq \min\left\{ \frac{(A-B)\left|[n+b]_{q^2}\right|^s}{(1-B)\,[n]_{q^2} \left|[1+b]_{q^2}\right|^s}, \frac{(A-B)\left|[n+b]_{q^2}\right|^s}{\left|[1+b]_{q^2}\right|^s (1+B)\,[n]_{q^2}} \right\}.$$

5. Concluding remarks

In our present investigation, a class of analytic functions with the parameters μ, A, B $(0 \leq \mu \leq 1; -1 \leq B < A \leq 1)$ involving the q^2-derivative and the q^2-Srivastava-Attiya operator is defined which further reduces into two more simpler classes on setting $\mu = 0$ and $\mu = 1$. In terms of convolution, necessary and sufficient conditions for a function to be in these classes are derived in Theorem 3.1, Corollary 3.2 and Corollary 3.3. Further, for the functions in these classes, bounds of the initial coefficients and bounds of the fixed n-th coefficient are obtained.

The concept of q^2-number $[\lambda]_{q^2}$ $(\lambda \in \mathbb{C})$ may further be extended by defining a (p^2, q^2)-number $[\lambda]_{(p^2,q^2)}$ $(0 < q \leq p \leqq 1)$ as follows:

$$[\lambda]_{(p^2,q^2)} = (pq)^{1-\lambda}\left(\frac{p^{2\lambda} - q^{2\lambda}}{p^2 - q^2}\right)$$

which is actually a trivial and inconsequential variation of $\left(\frac{q}{p}\right)^2$-number $[\lambda]_{\left(\frac{q}{p}\right)^2}$, so the extra parameter "p" is unnecessary. As it is emphasized in [16, p. 340] and [17, p. 1511-1512] that the q-analogues studied in a fairly large number of other earlier q-investigations may be readily transformed into the corresponding (p, q)-analogues $(0 < q \leq p \leqq 1)$ by using some simple parametric modifications, the additional forced-in parameter "p" is obviously redundant.

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