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Logarithmic coefficients for starlike functions associated with generalized telephone numbers

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Abstract. The objective of the present paper is to study the logarithmic coefficients of the class $\mathcal{S}_T^*(\mu)$ of starlike functions which is related with generalized telephone numbers, by using bounds on some coefficient functional for the family of functions with positive real part. We give a special result of main theorem.

1. Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let us denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing functions which are univalent in $\mathcal{U}$. An analytic function f is subordinate to an analytic function g (written as $f \prec g$) if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathcal{U}$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. For arbitrary fixed numbers A and B satisfying $-1 \leq B < A \leq 1$, denote by $\mathcal{P}[A, B]$ the class of analytic functions p such that $p(0) = 1$ and satisfy the subordination $p(z) \prec (1 + Az)/(1 + Bz)$ $(z \in \mathcal{U})$. Note that for $0 \leq \beta < 1$, $\mathcal{P}[1 - 2\beta, -1]$ is the class of analytic functions p with $p(0) = 1$ satisfying $\Re p(z) > \beta$ in $\mathcal{U}$. We call the functions in $\mathcal{P} = \mathcal{P}[1, -1]$ as Carathéodory functions. The class $\mathcal{S}^*[A, B]$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \in \mathcal{P}[A, B]$ for $z \in \mathcal{U}$. The functions in the class $\mathcal{S}^*[A, B]$ are called the Janowski starlike functions, introduced by Janowski [18]. For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β. Note that $\mathcal{S}^* = \mathcal{S}^*(0)$ is the classical class of starlike functions. Moreover the classes $\mathcal{S}^*[1-\beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\}$ and $\mathcal{S}^*[\beta, -\beta] := \widetilde{\mathcal{S}}^*(\beta) = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1|\}$ has been studied in [1, 3]. In terms of subordination, the class of starlike functions is given by $zf'(z)/f(z) \prec (1 + z)/(1 - z)$. Ma and Minda [24] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function $(1 + z)/(1 - z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps $\mathcal{U}$ onto univalently a region starlike with respect to 1 and symmetric with respect to the real axis. They introduced the following general class that envelopes several well-known classes as special cases: $\mathcal{S}^*[\varphi] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z)\}$. In literature, the functions belonging to this class is called Ma-Minda starlike function.

The logarithmic coefficients λ_n of $f \in \mathcal{S}$ are defined with the aid of the following series expansion:

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$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \lambda_n z^n, \quad z \in \mathcal{U}. \tag{1}$$

This coefficients play a important role for various estimates in the theory of univalent functions. In particular, the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\lambda_n = \frac{1}{n}$. It is clear that $|\lambda_1| \leq 1$ for each $f(z) \in \mathcal{S}$. The problem of the best upper bounds for $|\lambda_n|$ is still open. In fact even the proper order of magnitude is still not known, however, for the starlike functions that the best bounds is $|\lambda_n| \leq \frac{1}{n}$ and that this is not true in general [13]; [14]; [2] and [15]. In the paper [11] it is pointed out that the inequality $|\lambda_n| \leq An^{-1}\log n$ (A is an absolute constant) which holds for circularly symmetric functions. In a recent paper [16], it is presented that the inequality $|\lambda_n| \leq \frac{1}{n}$ holds also for close-to-convex functions. However, it is pointed out in [29] that there are some errors in the proof and, hence, the result is not substantiated. It is proved in [17] that there exist a close-to-convex function such that $|\lambda_n| > \frac{1}{n}$. Furthermore, it is proved in [35] that the inequality $|\lambda_n| \leq An^{-1}\log n$ holds for close-to-convex functions, where A is an absolute constant.

During the 1960's, Kayumov [19] solved Brennan's conjecture for conformal mappings with the help of studying the logarithmic coefficients. The significance of the logarithmic coefficients follows from Lebedev-Milin inequalities [25–27], where estimates of the logarithmic coefficients were applied to obtain bounds on the coefficients of f. Milin [25] conjectured in the inequality

$$\sum_{i=1}^{j}\sum_{n=1}^{i}\left(n\,|\lambda_n|^2 - \frac{1}{n}\right) \leq 0 \qquad (j = 1,\ 2,\ 3, ...)$$

that implies Robertson's conjecture [33] and hence Bieberbach's conjecture [6], which is the well-known coefficient problem in univalent function theory. De Branges [7] proved the Bieberbach's conjecture by establishing Milin's conjecture.

Recall that we can rewrite 1 in the series form as follows:

$$2\sum_{n=1}^{\infty} \lambda_n z^n = a_2 z + a_3 z^3 + a_4 z^4 + ... - \frac{1}{2}[a_2 z + a_3 z^2 + a_4 z^3 + ...]^2 + \frac{1}{3}[a_2 z + a_3 z^2 + a_4 z^3 + ...]^3 + ..., \ \ z \in \mathcal{U},$$

and considering the coefficients of z^n for $n = 1, 2, 3...$ it follows that

$$\begin{aligned} 2\lambda_1 &= a_2, \\ 2\lambda_2 &= a_3 - \frac{1}{2}a_2^2, \\ 2\lambda_3 &= a_4 - a_2 a_3 + \frac{1}{3}a_2^3, \\ 2\lambda_4 &= a_5 - a_2 a_4 + a_2^2 a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4. \end{aligned} \tag{2}$$

The geometry of analytic functions related with some familiar sequences of numbers has been explored by some researchers working in the theory. Especially, H. M. Srivastava and his co-authors have investigated the coefficient problem for some special generating functions. For example, Kumar et al. [22], Shafiq et al. [34] and Deniz [12] have studied the coefficient problem for certain classes related with Bell numbers, Fibonacci numbers and generalized Telephone numbers, respectively. The generating function for Van Der Pol numbers was recently used to introduce a subclass of starlike functions (see [31]), while the generating function for Bernoulli numbers is considered in [8] to investigate a subclass of $\mathcal{S}^*[\varphi]$. Recently, Kazımoğlu et al. [20] have studied starlike functions related with Gregory numbers.

In the present paper, our main focus on finding the upper bounds of logarithmic coefficients for starlike functions associted with generalized telephone numbers defined by 3.

Motivated by the above-cited works, we consider the function φ for which $\varphi(\mathcal{U})$ is starlike with respect to 1 and whose coefficients is the general telephone numbers. The classical telephone numbers, also known as involution numbers, are given by the recurrence relation $T(n) = T(n-1) + (n-1)T(n-2)$ for $n \geq 2$ with initial conditions $T(0) = T(1) = 1$. Relationships of these numbers with symmetric groups were observed for the first time in 1800 by Heinrich August Rothe, who pointed out that $T(n)$ is the number of involutions (self-inverse permutations) in the symmetric group (see, for example, [10, 21]). Because involutions correspond to standard Young tableaux it is clear that the n^{th} involution number is also the number of Young tableaux on the set $\{1, 2, \cdots, n\}$ (for details see [4]). According to John Riordan, above recurrence relation, in fact, produces the number of connection patterns in a telephone system with n subscribers (see [32]). In 2017, Włoch and Wołowiec-Musiał [36] introduced generalized telephone numbers $T(\mu, n)$ defined for integers $n \geq 0$ and $\mu \geq 1$ by the following recursion; $T(\mu, n) = \mu T(\mu, n-1) + (n-1)T(\mu, n-2)$ with initial conditions $T(\mu, 0) = 1$, $T(\mu, 1) = \mu$, and studied some properties. In 2019, Bednarz and Wołowiec-Musiał [5] introduced a new generalization of telephone numbers by $T_\mu(n) = T_\mu(n-1) + \mu(n-1)T_\mu(n-2)$ with initial conditions $T_\mu(0) = T_\mu(1) = 1$ for integers $n \geq 2$ and $\mu \geq 1$. They gave the generating function, direct formula, and matrix generators for these numbers. Moreover, they obtained interpretations and proved some properties of these numbers connected with congruences. In their paper, authors derived the exponential generating function and the summation formula for generalized telephone numbers $T_\mu(n)$ as follows:

$$e^{\left(x+\mu \frac{x^2}{2}\right)} = \sum_{n=0}^{\infty} T_\mu(n) \frac{x^n}{n!} \quad (\mu \geq 1). \tag{3}$$

As we can observe, if $\mu = 1$, then we obtain classical telephone numbers $T(n)$. Clearly, $T_\mu(n)$ for some values of n as $T_\mu(0) = T_\mu(1) = 1$, $T_\mu(2) = 1 + \mu$, $T_\mu(3) = 1 + 3\mu$, $T_\mu(4) = 1 + 6\mu + 3\mu^2$, $T\mu(5) = 1 + 10\mu + 15\mu^2$ and $T_\mu(6) = 1 + 15\mu + 45\mu^2 + 15\mu^3$. We now consider the function $\Psi(z) := e^{\left(z+\mu \frac{z^2}{2}\right)}$ with its domain of definition as the open unit disk $\mathcal{U}$. Very recently, Deniz [12] has defined the class $\mathcal{S}_T^*(\mu) := \{f : f \in \mathcal{S} \text{ and } zf'(z)/f(z) \prec \Psi(z)\}$ and obtained some coefficient estimates for this class. In this study, authors obtained upper bounds for logarithmic coefficients λ_n $(n = 1, 2, 3, 4)$ of functions belonging to the class $\mathcal{S}_T^*(\mu)$. In special case of $\mu = 1$, we write $\mathcal{S}_T^* = \mathcal{S}_T^*(1)$.

We give following the lemmas that use in the next sections.

Lemma 1.1. *[23] If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{4}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)\left(1 - |x|^2\right) y, \tag{5}$$

for some $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1.2. *[28] Let $\overline{\mathcal{U}} = \{z : |z| \leq 1\}$. Also, for any real numbers a, b and c, let the quantity $Y(a, b, c) = \max_{z \in \overline{\mathcal{U}}} \left\{ \left| a + bz + cz^2 \right| + 1 - |z|^2 \right\}$. If $ac \geq 0$, then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c| & |b| \geq 2(1 - |c|) \\ 1 + |a| + \frac{b^2}{4(1-|c|)} & |b| < 2(1 - |c|) \end{cases}.$$

Furthermore, if $ac < 0$, then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)} & \left(-4ac(c^{-2} - 1) \leq b^2;\ |b| < 2(1 - |c|)\right) \\ 1 + |a| + \frac{b^2}{4(1+|c|)} & b^2 < \min\left\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\right\} \\ R(a, b, c) & (otherwise) \end{cases},$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c| & (|c|\,(|b| + 4\,|a|) \leq |ab|) \\ -|a| + |b| + |c| & (|ab| \leq |c|\,(|b| - 4\,|a|)) \\ (|a| + |c|)\sqrt{1 - \frac{b^2}{4ac}} & (otherwise) \end{cases}.$$

Lemma 1.3. *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ $(p_1 \geq 0)$, *then*

$$p_n \leq 2 \tag{6}$$

and if $Q \in [0,1]$ *and* $Q(2Q-1) \leq R \leq Q$, *then*

$$\left| p_3 - 2Qp_1p_2 + Rp_1^3 \right| \leq 2. \tag{7}$$

also

$$\begin{aligned} \left| p_{n+k} - \delta p_n p_k \right| &\leq 2\max\{1, |2\delta - 1|\} \\ &= 2\begin{cases} 1, & \text{for } 0 \leq \delta \leq 1, \\ |2\delta-1|, & \text{otherwise} \end{cases}. \end{aligned} \tag{8}$$

The inequalities (6), (7) and (8) are taken from [9, 23] and [30], respectively.

2. Logarithmic Coefficient Estimates

Our main result as follows.

Theorem 2.1. *Let* $f(z) \in \mathcal{S}_T^*(\mu)$ *and* $\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \lambda_n z^n$. *Then*

$$\begin{aligned} |\lambda_1| &\leq \frac{1}{2}, \\ |\lambda_2| &\leq \frac{\mu+1}{8}, \end{aligned}$$

$$|\lambda_3| \leq \begin{cases} \frac{1}{6}; & 1 \leq \mu \leq \frac{5}{3} \\ \frac{3\mu+1}{36}; & \frac{5}{3} \leq \mu \end{cases},$$

$$|\lambda_4| \leq \begin{cases} \frac{3\mu^2+30\mu+25}{192}; & 1 \leq \mu < \mu_4 \\ \frac{3\mu-1}{16}; & \mu_4 \leq \mu \leq \mu_5 \\ \frac{3\mu^2+42\mu-11}{192}; & \mu > \mu_5 \end{cases},$$

where, $\mu_4 \approx 1.76208$ *and* $\mu_5 \approx 9.54606$ *are positive roots of* $3\mu^3 + 12\mu^2 - 35\mu + 8$ *and* $3\mu^2 - 30\mu + 13$, *respectively.*

Proof. Since $f \in \mathcal{S}_T^*(\mu)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{zf'(z)}{f(z)} = \Psi(w(z)) = e^{\left(w(z) + \mu\frac{w^2(z)}{2}\right)} \quad (z \in \mathcal{U}). \tag{9}$$

Define the functions p by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathcal{U})$$

or equivalently,

$$\begin{aligned} w(z) &= \frac{p(z)-1}{p(z)+1} = \frac{p_1}{2}z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 \\ &+ \frac{1}{2}\left(p_4 - p_1p_3 + \frac{3p_1^2p_2}{4} - \frac{p_2^2}{2} - \frac{p_1^4}{8}\right)z^4 \\ &+ \frac{1}{2}\left(p_5 - \frac{1}{2}p_1^3p_2 + \frac{3}{4}p_1p_2^2 + \frac{3}{4}p_1^2p_3 - p_2p_3 - p_1p_4 + \frac{1}{16}p_1^5\right)z^5 + \cdots \end{aligned} \tag{10}$$

in $\mathcal{U}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and has positive real part in $\mathcal{U}$. By using (10) together with $e^{\left(w(z)+\mu\frac{w^2(z)}{2}\right)}$, it is evident that

$$\begin{aligned}\Psi(w(z)) &= 1+\frac{p_1 z}{2}+\frac{1}{8}\left((-1+\mu)p_1^2+4p_2\right)z^2 \\ &+\frac{1}{48}\left((1-3\mu)p_1^3+12(-1+\mu)p_1p_2+24p_3\right)z^3+ \\ &+\frac{1}{384}\begin{pmatrix}\left(1+6\mu+3\mu^2\right)p_1^4+24(1-3\mu)p_1^2p_2 \\ +96(-1+\mu)p_1p_3+48\left((-1+\mu)p_2^2+4p_4\right)\end{pmatrix}z^4 \\ &+\frac{\begin{pmatrix}\left(-19+10\mu-45\lambda^2\right)p_1^5+40\left(1+6\mu+3\lambda^2\right)p_1^3p_2-240(-1+3\mu)p_1^2p_3 \\ -240p_1\left((-1+3\mu)p_2^2-4(-1+\mu)p_4\right)+960\left((-1+\mu)p_2p_3+2p_5\right)\end{pmatrix}}{3840}z^5+\cdots .\end{aligned} \tag{11}$$

Since

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1+a_2z+\left(-a_2^2+2a_3\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3 \\ &+\left(-a_2^4+4a_2^2a_3-2a_3^2-4a_2a_4+4a_5\right)z^4 \\ &+\left(a_2^5-5a_2^3a_3+5a_2a_3^2+5a_2^2a_4-5a_3a_4-5a_2a_5+5a_6\right)z^5+\cdots ,\end{aligned} \tag{12}$$

it follows by (9), (11) and (12) that

$$\begin{aligned}a_2 &= \frac{p_1}{2}, \\ a_3 &= \frac{1}{16}\left((1+\mu)p_1^2+4p_2\right), \\ a_4 &= \frac{1}{288}\left((3\mu-1)p_1^3+12(1+2\mu)p_1p_2+48p_3\right), \\ a_5 &= \frac{\left(2-6\mu+9\mu^2\right)p_1^4+24(\mu-1)p_1^2p_2+48(1+3\mu)p_1p_3+72\mu p_2^2+288p_4}{2304}.\end{aligned} \tag{13}$$

By using 13 in 2, we can obtain

$$\begin{aligned}\lambda_1 &= \frac{p_1}{4}, \\ \lambda_2 &= \frac{1}{32}\left((\mu-1)p_1^2+4p_2\right) \\ \lambda_3 &= \frac{1}{288}\left((1-3\mu)p_1^3+12(\mu-1)p_1p_2+24p_3\right) \\ \lambda_4 &= \frac{\left(1+6\mu+3\mu^2\right)p_1^4+24(1-3\mu)p_1^2p_2+96(\mu-1)p_1p_3-48\left(-4p_4+(1-\mu)p_2^2\right)}{3072}.\end{aligned}$$

Using $\left|p_n\right| \leq 2$, we can obtain

$$|\lambda_1| \leq \frac{1}{2} \quad \text{and} \quad |\lambda_2| \leq \frac{\mu+1}{8}.$$

We now investigate upper bounds of $|\lambda_3|$ and $|\lambda_4|$.

By using Lemma 1.1 and supposing that $s = p_1 \in [0, 2]$, λ_3 can be rewriten as follows:

$$\lambda_3 = \frac{4-s^2}{24}\left[\frac{(3\mu+1)s^3}{12\left(4-s^2\right)}+\frac{(\mu+1)s}{2}x-\frac{s}{2}x^2+(1-|x|^2)y\right] := H(s).$$

We now investigate upper bound of the $|H(s)|$ according to s.

Taking $s = 0$ and $s = 2$, respectively, we have

$$|H(s)| \leq \frac{1}{6} \tag{14}$$

and

$$|H(s)| = \frac{3\mu + 1}{36}. \tag{15}$$

We now assume that $s \in (0, 2)$. Then, we can write

$$\begin{aligned} |H(s)| &\leq \frac{4-s^2}{24}\left[\left|\frac{(3\mu+1)s^3}{12\,(4-s^2)} + \frac{(\mu+1)s}{2}x - \frac{s}{2}x^2\right| + 1 - |x|^2\right] \\ &= \frac{4-s^2}{24}\left[\left|a + bx + cx^2\right| + 1 - |x|^2\right], \end{aligned}$$

where

$$a = \frac{(3\mu+1)s^3}{12\,(4-s^2)},\; b = \frac{(\mu+1)s}{2},\; c = -\frac{s}{2}.$$

In the next process of the proof, we use Lemma 1.2. Considering $s \in (0, 2)$ and $\mu \geq 1$, it is obvious that $ac = -\frac{(3\mu+1)s^4}{24(4-s^2)} < 0$. Therefore, we use the second part of the Lemma 1.2.

Case I After some calculations, we see that the conditions $-4ac(c^{-2} - 1) \leq b^2$ and $|b| - 2(1 - |c|) < 0$ hold true for all $s \in \left(0, \frac{4}{\mu+3}\right)$. Not that, the inequality $\frac{4}{\mu+3} < 2$ true for $\mu \geq 1$. Therefore, we conclude that

$$\begin{aligned} |H(s)| &\leq \frac{4-s^2}{24}\left[1 - |a| + \frac{b^2}{4(1-|c|)}\right] \\ &= \frac{1}{24}\left[\frac{(3\mu^2+1)s^3}{24} + \frac{(\mu+3)(\mu-1)s^2}{4} + 4\right] \\ &:= H_0(s). \end{aligned} \tag{16}$$

Since all the coefficients of powers of s are positive, H_0 takes maximum value at $\frac{4}{\mu+3}$. Thus, we have

$$\max H_0\,(s) = H_0\left(\frac{4}{\mu+3}\right) = \frac{3\mu^3 + 24\mu^2 + 45\mu + 28}{9(\mu+3)^3} := I_0(\mu). \tag{17}$$

For all $s \in (0, 2)$ the inequality $b^2 < \min\left\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\right\}$ doesn't satisfied.

Let $\left[\frac{4}{\mu+3}, 2\right)$. On the other hand, from some calculations we see that

$$|ab| - |c|\,(|b| + 4\,|a|) = \frac{s^2}{24(4-s^2)}\left[\left(-3\mu^2 + 2\mu - 3\right)s^2 + 24\,(\mu+1)\right] \leq 0$$

holds true for $s \in (s_0, 2)$, where $s_0 = \sqrt{\frac{24(\mu+1)}{3\mu^2-2\mu+3}}$ and $\frac{4}{\mu+3} < s_0 < 2$ for $\mu > 3$. Therefore, we have

$$|H(s)| \leq \frac{4-s^2}{24}\,[|a| + |b| - |c|] = \frac{1}{288}\left[(1-3\mu)\,s^3 + 24\mu s\right] := H_1(s).$$

The positive root of the equation $H_1'(s) = \frac{1}{288}\left[3\,(1-3\mu)\,s^2 + 24\mu\right] = 0$ is $s_1 = \sqrt{\frac{8\mu}{3\mu-1}}$. We now consider the situations $s_0 - s_1 > 0$ and $s_0 - s_1 < 0$.

Let $3 < \mu \le \mu_0 \approx 3.85866$, where μ_0 is the root of the $\sqrt{\frac{24(\mu+1)}{3\mu^2-2\mu+3}} - \sqrt{\frac{8\mu}{3\mu-1}}$. Then we have $s_0 - s_1 > 0$ and H_1 takes maximum value at $\sqrt{\frac{24(\mu+1)}{3\mu^2-2\mu+3}}$. Hence we conclude that

$$\begin{aligned} \max H_1(s) &= H_1(s_0) \\ &= \frac{\mu+1}{12(3\mu^2-2\mu+3)}\left[(1-3\mu)\sqrt{\frac{24(\mu+1)}{3\mu^2-2\mu+3}}+24\mu\right] := I_1(\mu). \end{aligned} \tag{18}$$

Similarly, let $\mu_0 \le \mu$. Thus $s_0 - s_1 < 0$ and H_1 takes maximum value at s_1. Hence, we get

$$\begin{aligned} \max H_1(s) &= H_1(s_1) \\ &= \frac{\mu}{36(3\mu-1)}\left[(1-3\mu)\sqrt{\frac{8\mu}{3\mu-1}}+24\mu\right] := I_2(\mu). \end{aligned} \tag{19}$$

We can easily see that

$$|ab| - |c|(|b| - 4|a|) = \frac{s^2}{24(4-s^2)}\left[\left(3\mu^2+22\mu+11\right)s^2 - 24(\mu+1)\right] \le 0$$

holds true for $s \in \left[\frac{4}{\mu+3}, s_2\right)$, where $s_2 = \sqrt{\frac{24(\mu+1)}{3\mu^2+22\mu+11}}$ and $s_2 < s_0$ for $\mu \ge 1$. Therefore, we have

$$\begin{aligned} |H(s)| &\le \frac{(4-s^2)}{24}\left[-|a|+|b|+|c|\right] \\ &= \frac{s\left[s^2(-9\mu-13)+24(\mu+2)\right]}{24} := H_2(s). \end{aligned}$$

The positive root of the function H_2' is $s_3 = \sqrt{\frac{8\mu+16}{9\mu+13}}$. Simple calculations show that $s_3 - s_2 > 0$ for $\mu \ge \mu_1 \approx 2.31227$, where μ_1 is positive root of $3\mu^3 + \mu^2 - 11\mu - 17$. Therefore, we have

$$\begin{aligned} \max H_2(s) &= \max\left\{H_2\left(\frac{4}{\mu+3}\right), H_2(s_2)\right\} = H_2(s_2) \\ &= 2\sqrt{\frac{6(\mu+1)}{3\mu^2+22\mu+11}}\left(\mu+2-\frac{(8\mu+13)(\mu+1)}{3\mu^2+22\mu+11}\right) := I_3(\mu). \end{aligned} \tag{20}$$

On the other hand, the inequality $s_3 - s_2 < 0$ is true for $\mu < \mu_1$. Therefore, we obtain

$$\begin{aligned} \max H_2(s) &= H_2(s_3) \\ &= \frac{4\sqrt{2}(\mu+2)}{3}\sqrt{\frac{\mu+2}{9\mu+13}} := I_4(\mu) \end{aligned}$$

for $\mu < \mu_1$.

Finally, let $s \in (s_2, s_0)$. From Lemma 1.2 we have

$$\begin{aligned} |H(s)| &\le \frac{4-s^2}{24}\left((|a|+|c|)\sqrt{1-\frac{b^2}{4ac}}\right) \\ &= \frac{1}{288\sqrt{2(3\mu+1)}}\left((3\mu-5)s^2+24\right)\sqrt{(-3\mu^2-1)s^2+12(\mu+1)^2} := H_3(s). \end{aligned}$$

The smallest positive root of the function H_3' is $s_4 = 2\sqrt{2}\sqrt{\frac{3\mu^3-2\mu^2-7\mu-6}{9\mu^3-15\mu^2+3\mu-5}}$ for $1 \leq \mu < \mu_2 \approx 1.66667$ and $\mu \geq \mu_3 \approx 2.16823$.

Since $\mu_1 < \mu < \mu_0$, we obtain $s_4 \in (s_2, s_0)$. Then, H_3 takes maximum value at s_4 and we get

$$\begin{aligned} \max H_3(s) &= H_3(s_4) \\ &= \frac{1}{18}\sqrt{\frac{3\mu^3+7\mu^2-7\mu-3}{2(3\mu-5)(1+3\mu)}}\frac{\left(3\mu^3+7\mu^2-7\mu-3\right)}{1+3\mu^2} := I_5(\mu). \end{aligned} \tag{21}$$

If $1 \leq \mu < \mu_2$ and $\mu \geq \mu_0$, where μ_2 is positive root of $9\mu^3 - 15\mu^2 + 3\mu - 5$, then we have

$$\begin{aligned} \max H_3(s) &= H_3(s_0) \\ &= \frac{(\mu-1)\left(3\mu^2-2\mu-1\right)}{(3\mu^2-2\mu+3)\sqrt{18\mu+6}}\sqrt{\frac{3\mu^2+4\mu+1}{3\mu^2-2\mu+3}} := I_6(\mu). \end{aligned}$$

If $\mu_3 \leq \mu < \mu_1$,where μ_3 is positive root of $3\mu^3 - 2\mu^2 - 7\mu - 6$, then we have

$$\begin{aligned} \max H_3(s) &= H_3(s_2) \\ &= \frac{(3+\mu)\left(3\mu^2+10\mu+3\right)}{(3\mu^2+22\mu+11)\sqrt{18\mu+6}}\sqrt{\frac{3\mu^2+4\mu+1}{3\mu^2+22\mu+11}} := I_7(\mu). \end{aligned}$$

Consequently, from (14), (15) and $I_0(\mu) - I_7(\mu)$ we obtain bound of $|\lambda_3|$.

Finally, we investigate the upper bound of $|\lambda_4|$.

Let $1.76208 \approx \mu_4 \leq \mu \leq \mu_5 \approx 9.54606$, where μ_4 and μ_5 are positive roots of the $3\mu^3 + 12\mu^2 - 35\mu + 8$ and $3\mu^2 - 30\mu + 13$, respectively.

After some calculations for λ_4 we have

$$\lambda_4 = \frac{(\mu-1)p_1}{32}\left[p_3 - \frac{(3\mu-1)}{4(\mu-1)}p_1p_2 + \frac{\left(1+6\mu+3\mu^2\right)}{96(\mu-1)}p_1^3\right] + \frac{4p_4+(\mu-1)p_2^2}{64}$$

and thus

$$|\lambda_4| \leq \frac{(\mu-1)p_1}{32}\left|p_3 - \frac{(3\mu-1)}{4(\mu-1)}p_1p_2 + \frac{\left(1+6\mu+3\mu^2\right)}{96(\mu-1)}p_1^3\right| + \left|\frac{4p_4+(\mu-1)p_2^2}{64}\right|.$$

If we take

$$Q = \frac{3\mu-1}{8(\mu-1)}$$

and

$$R = \frac{3\mu^2+6\mu+1}{96(\mu-1)}$$

in (7), from (6) we obtain

$$|\lambda_4| \leq \frac{3\mu-1}{16}. \tag{22}$$

On the other hand if we rearrenge the λ_4, we can write

$$\lambda_4 = \frac{(\mu-1)p_1}{32}\left[p_3 - \frac{(3\mu-1)}{4(\mu-1)}p_1p_2\right] + \frac{\left(1+6\mu+3\mu^2\right)}{3072}p_1^4 + \frac{4p_4+(\mu-1)p_2^2}{64}.$$

For $1 \leq \mu < 3$ and $\mu \geq 3$ in (8) and from (6) we have

$$|\lambda_4| \leq \frac{3\mu^2 + 30\mu + 25}{192} \tag{23}$$

and

$$|\lambda_4| \leq \frac{3\mu^2 + 42\mu - 11}{192}, \tag{24}$$

respectively. Consequently, from (22), (23) and (24) we obtain the result for $|\lambda_4|$. □

For $\mu = 1$ in Theorem 2.1 we obtained the following result.

Corollary 2.2. *Let $f(z) \in \mathcal{S}_T^*(1)$. Then*

$$|\lambda_k| \leq \frac{1}{2k}, \ (k = 1, 2, 3)$$

$$|\lambda_4| \leq \frac{29}{96}.$$

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