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Existence, decay and blow-up results for a plate viscoelastic equation with variable-exponent logarithmic terms

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Abstract. In the present study, we investigate the solution behavior for a class of fourth-order viscoelastic equation with variable-exponent logarithmic nonlinearities. First, we prove the global existence of solutions, and next, by constructing suitable auxiliary functionals, the general decay of solutions has been proved when the exponents satisfy appropriate conditions. Finally, under suitable conditions on data, we establish the finite time blow-up of solutions with positive initial energy. Our results extend and improve many results in the literature.

1. Introduction

In this paper, we are concerns with the following viscoelastic plate equation with logarithmic terms with variable exponents nonlinearities:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + div(\nabla u \ln|\nabla u|^{p(x)}) + u_t = |u|^{q(x)-2}u\ln|u|, \qquad x \in \Omega,\ t > 0 \tag{1}$$

$$u(x,t) = \Delta u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega \tag{3}$$

where $\Omega \subset R^n (n \geq 1)$ is an open bounded domain with a smooth boundary $\partial\Omega$. Here, the variable exponents and the kernel of the memory satisfying the following conditions:
$(A1)$ The exponents $p(\cdot), q(\cdot)$ are given measurable functions on $\bar{\Omega}$ such that

$$0 < p_1 \leq p(x) \leq p_2 < \begin{cases} \infty, & n < 5 \\ \frac{2(n-2)}{n-4}, & n \geq 5 \end{cases}$$

$$2 \leq q_1 \leq q(x) \leq q_2 < \begin{cases} \infty, & n < 5 \\ \frac{2(n-2)}{n-4}, & n \geq 5 \end{cases}$$

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with

$$p_1 := \operatorname{ess}\inf_{x\in\overline{\Omega}} p(x),\ \ p_2 := \operatorname{ess}\sup_{x\in\overline{\Omega}} p(x),$$
$$q_1 := \operatorname{ess}\inf_{x\in\overline{\Omega}} q(x),\ \ q_2 := \operatorname{ess}\sup_{x\in\overline{\Omega}} q(x).$$

$(A2)$ The kernel of the memory g is a non-increasing and non-negative function satisfying

$$g(t) \geq 0, \quad g'(t) \leq -g(t),\ \ 1 - \int_0^\infty g(t)dt = \ell > 0.$$

In recent years, many researchers investigated the behavior of solutions for equations with logarithmic nonlinearities. These models are widely applied in many branches, such as nuclear physics, and geophysics [8, 17]. For example, Liu [20] considered the following plate equation with nonlinear damping and a logarithmic source term

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u \log |u|^k,$$

in a bounded domain in $R^n(n \geq 1)$ and by the contraction mapping principle, he established the local existence. The global existence and decay estimate of the solution at subcritical initial energy are obtained. They also proved that the solution with negative initial energy blows up in finite time under suitable conditions. Moreover, they also give the blow-up in finite time of solution at the arbitrarily high initial energy for linear damping (i.e. $m = 2$).
It is well known, from the viscoelasticity theory, the viscoelastic materials undergoing deformation exhibit dual properties of viscosity and elasticity, which can keep memory of their history and show natural damping. So, the study of viscoelastic mechanical equations with logarithmic nonlinearity is of particular importance. In this regards, Ferreira et al. [12] concerned with the following viscoelastic Petrovsky type equation with logarithmic nonlinearity

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u \ln |u|,$$

and derived the blow-up results by the combination of the perturbation energy method, concavity method, and differential-integral inequality technique. Al-Gharabli et al. [3] considered the following plate equation:

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = ku \ln |u|,$$

and proved the existence and decay results of the solutions, imposing the following condition on the relaxation function:

$$g'(t) \leq -\xi(t)g^p(t),\ 1 \leq p < \frac{3}{2}.$$

In another study, Al-Gharabli et al. [2] investigated the long-time behavior of the following viscoelastic equation with a logarithmic source term and a nonlinear feedback localized on a part of the boundary:

$$\begin{cases} u_{tt} - \Delta u + u + \int_0^t g(t-s)\Delta u(s)ds = ku \ln |u|, & \text{in } \Omega \times R^+ \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t(t)) = 0, & \text{on } \Gamma_1 \times R^+ \\ u(t) = 0, & \text{on } \Gamma_0 \times R^+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

where u denotes the transverse displacement of waves, Ω is a bounded domain of $R^N (N \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that Γ_0 and Γ_1 are closed and disjoint, with meas.$(\Gamma_0) > 0$, ν is the unit outer normal to $\partial\Omega$, the constant k is a small positive real number, and g and h are specific functions. They proved the global existence and then, discussed the asymptotic behavior of the problem with a very general assumption on the behavior of the relaxation function g. Perreira et al. [1] proved the existence, uniqueness, exponential decay, and blow-up of solutions for the viscoelastic beam equation involving the p-Laplacian operator, strong damping, and a logarithmic source term, given by

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{r-2} u \ln |u|, \text{in } Q = \Omega \times R^+.$$

By using the Faedo–Galerkin approximation, they established the existence and uniqueness result for the global solutions, taking into account that the initial data must belong to an appropriate stability set created from the Nehari manifold. The study of the exponential decay was based on Nakao's method. Ha and Park in [18] considered the following viscoelastic wave equation with strong damping and logarithmic nonlinearity

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds - \Delta u_t(x,t) = |u(x,t)|^{p-2} u(x,t) \ln |u(x,t)|, \tag{4}$$

and proved the existence and uniqueness of local weak solutions by using Faedo–Galerkin's method and contraction mapping principle. Then, they established a finite time blow-up result for the solution with positive initial energy as well as non-positive initial energy. Next, the same authors in [19] considered the (4) without the strong damping term and proved the existence and general decay estimate of the solution using energy estimates and theory of convex functions. Recently, Tahamtani et al. [34] considered the following weak viscoelastic equation with acoustic boundary conditions and a logarithmic source term:

$$\begin{aligned}
&u_{tt} - \Delta u + \alpha(t) \int_0^t g(t-s)\Delta u(s)ds + u_t + u = u \ln |u|^k, && (x,t) \in \Omega \times (0,\infty),\\
&u = 0, && (x,t) \in \Gamma_0 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu} - \alpha(t) \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s)ds = y_t, && (x,t) \in \Gamma_1 \times (0,\infty),\\
&u_t + p(x)y_t + q(x)y = 0, && (x,t) \in \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x), && x \in \Omega,\\
&y(x,0) = y_0, && x \in \Gamma_1,
\end{aligned}$$

where $\Omega \subset R^n (n \geq 1)$ is a bounded domain with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$, Γ_0 and Γ_1 are disjoint and closed with positive measures, ν represents the unit out ward normal vector to Γ and the constant k is a positive number. They proved the global existence and general decay result with appropriate conditions on the initial data. For more information about the equations with logarithmic source term, see [4, 5, 9, 16, 21, 23, 35–37].

On the other hand, the problems with nonstandard growth conditions arise in various fields of sciences such as flows of electro-rheological fluids, nonlinear viscoelasticity, and image processing. Thus, the study of equations with variable-exponent nonlinearities has become a hot research field in the last decade. Shahrouzi [28] studied the behavior of solutions to the following initial-boundary value problem with variable-exponent nonlinearities

$$u_{tt} - \Delta u - div(|\nabla u|^{m(x)} \nabla u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(x,t,u,\nabla u) + \beta u_t = |u|^{p(x)} u, \ in\ \Omega \times (0,+\infty)$$

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0\\ \frac{\partial u}{\partial n}(x,t) = \int_0^t g(t-\tau)\frac{\partial u}{\partial n}(\tau)d\tau - |\nabla u|^{m(x)} \frac{\partial u}{\partial n} + \alpha u, & x \in \Gamma_1, t > 0 \end{cases}$$

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ in \ \ \Omega.$$

Under appropriate conditions, he proved a general decay result associated to solution energy. Moreover, regarding arbitrary positive initial energy, blow up of solutions has been proved. Shahrouzi et al. [29] studied the blow-up analysis for a class of plate viscoelastic $p(x)$-Kirchhoff type inverse source problem of the form:

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + \beta|u_t|^{m(x)-2}u_t = \alpha|u|^{q(x)-2}u + f(t)\omega(x).$$

Under suitable conditions on kernel of the memory, initial data and variable exponents, they proved the blow up of solutions in two cases: linear damping term ($m(x) \equiv 2$) and nonlinear damping term ($m(x) > 2$). Precisely, they showed that the solutions with positive initial energy blow up in a finite time when $m(x) \equiv 2$ and blow up at infinity if $m(x) > 2$. Recently, Ferreira et al. [15] considered a plate viscoelastic $p(x)-$Kirchhoff type equation with variable-exponent nonlinearities of the form

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-s)\Delta^2 u(s)ds + \beta\Delta^2 u_t + |u_t|^{m(x)-2}u_t = |u|^{q(x)-2}u,$$

associated with initial and boundary feedback. Under appropriate conditions on $p(\cdot), m(\cdot)$ and $q(\cdot)$, general decay result along the solution energy is proved. By introducing a suitable auxiliary function, it is also shown that regarding negative initial energy and a suitable range of variable exponents, solutions blow up in a finite time. Shahrouzi et al. [31] studied the following viscoelastic plate equation involving $(p(x), q(x))$-Laplacian operator

$$u_{tt} + \Delta^2 u - \mathrm{div}[(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2})\nabla u] - \int_0^t g(t-s)\Delta^2 u(s)ds - \xi\Delta u_t = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u,$$

where $\xi, \alpha, \beta \geq 0$ and Ω is a bounded domain in $R^n (n \geq 1)$. They proved the global existence of solutions, and next, they showed that the solutions are asymptotically stable if initial data, $p(x)$, and $q(x)$ are in the appropriate range. Moreover, under suitable conditions on initial data, they prove that there exists a finite time in which some solutions blow up with positive as well as negative initial energies. Recently, Shahrouzi et al. [33] investigated the following variable-exponent fourth order wave equation with logarithmic damping term

$$u_{tt} - \Delta u - a\Delta u_{tt} + \Delta^2 u + div\left(\nabla u \cdot \ln|\nabla u|^{p(x)}\right) - b\Delta u_t = |u|^{q(x)-2}u,$$

and proved the blow up of solutions for positive as well as negative initial energy. Regarding the equations with variable exponent nonlinearities, we refer the reader to [6, 13, 14, 24–27, 30, 32].

The outline of this paper is as follows: Firstly, in section 2, we give some assumptions and lemmas about the Sobolev spaces with variable exponent and some Lemmas needed in our proof of the results. Then, in section 3, we get the global existence result. Moreover, in section 4, the general decay result is established and finally, the blow-up result with positive initial energy has been proved in section 5.

2. Preliminaries

To prove our results for the problem (1)-(3), we need to present some theories about the function spaces with variable-exponents as Lebesuge and Sobolev. (See [7, 11]).

Suppose that Ω is a subset of R^n and the function $p : \Omega \to [1, \infty]$ is measurable. The variable exponent Lebesgue space is defined by:

$$L^{p(x)}(\Omega) = \left\{u \middle| \text{ u is measurable in } \Omega \text{ and } \int_\Omega |\lambda u(x)|^{p(x)}dx < \infty \text{ for some } \lambda > 0\right\}.$$

The Lebesgue space, $L^{p(\cdot)}(\Omega)$, is equipped with the below Luxembourg-type norm:

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0 \middle| \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\right\}.$$

Lemma 2.1. *[11] Let Ω be a bounded domain in R^n*
(i) the space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$.
(ii) For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds

$$\Big|\int_\Omega fg dx\Big| \leq \Big(\frac{1}{p_1} + \frac{1}{q_1}\Big)\|f\|_{p(\cdot)}\|g\|_{q(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

The following formula is used to determine the relationship between the modular $\int_\Omega |f|^{p(x)}dx$ and the norm

$$\min(\|f\|^{p_1}_{p(\cdot)}, \|f\|^{p_2}_{p(\cdot)}) \leq \int_\Omega |f|^{p(x)}dx \leq \max(\|f\|^{p_1}_{p(\cdot)}, \|f\|^{p_2}_{p(\cdot)}).$$

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla_x u \text{ exists and } |\nabla_x u| \in L^{p(\cdot)}(\Omega)\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla_x u\|_{p(\cdot)}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, an equivalent norm is defined as

$$\|u\|_{1,p(\cdot)} = \|\nabla_x u\|_{p(\cdot)}.$$

Let the log-Hölder continuity condition be satisfied by the variable component $p(.)$

$$|p(x) - p(y)| \leq \frac{-A}{log|x-y|}, \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta,$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.2. *(Sobolev-Poincaré inequality) Suppose that Ω is a bounded domain of R^n and the log-Hölder condition is satisfied by $p(\cdot)$. Then we have $H_0^2(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and*

$$\|u\|_{p(\cdot)} \leq c_*\|\nabla u\|_{p(\cdot)} \leq c_*\|\Delta u\|_{p(\cdot)}, \text{ for all } u \in H_0^2(\Omega), \tag{5}$$

where $c_* = c(p_1, p_2, |\Omega|) > 0$.

Lemma 2.3. *[10](Logarithmic Sobolev inequality) Let u be any function in $H_0^2(\Omega) \setminus 0$ and $a > 0$ be any number. Then*

$$2\int_\Omega |\nabla u|^2 \ln(\frac{|\nabla u|}{\|\nabla u\|^2})dx + n(1 + lna)\|\nabla u\|^2 \leq \frac{a^2}{\pi}\|\Delta u\|^2. \tag{6}$$

For completeness, the local existence result for the problem (1)-(3) is stated as follows. This theorem could be proved by the Faedo-Galerkin approximation and combination of the procedures that have been used in [13, 16, 22].

Theorem 2.4. *(Local existence) Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ be given. Assume that $(A1)$ and $(A2)$ are satisfied; then the problem (1)-(3) has at least one weak solution such that*

$$u \in C\Big((0,T), H_0^2(\Omega)\Big) \cap W_0^{1,p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega),$$
$$u_t \in C\Big((0,T), H_0^2(\Omega)\Big) \cap L^2(\Omega).$$

Define

$$E(t) =\frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{1}{2}(g * \Delta u)(t) + \frac{1}{4}\int_\Omega p(x)|\nabla u|^2 dx + \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}dx$$
$$- \frac{1}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ln|u|dx, \tag{7}$$

where

$$(g * \Delta u)(t) = \int_0^t g(t-\tau)\|\Delta u(t) - \Delta u(\tau)\|^2 d\tau.$$

Lemma 2.5. *(Monotonicity of energy) Assume that $u(x,t)$ be a local solution of* (1)-(3). *Then, along the solution, $E(t)$ is a nonincreasing functional and*

$$E'(t) = -\|u_t\|^2 + \frac{1}{2}(g' * \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2 \le 0 \tag{8}$$

Proof. Multiplying equation (1) by u_t and integrating it over Ω, then by using hypotheses on the variable exponents and kernel of the memory, desired result could be obtained for any weak solution. □

3. Global existence

In this section, we will state and prove Theorem 3.2, i.e. the existence of solutions for the problem (1)-(3). To prove the global existence of solutions for the problem (1)-(3), we define:

$$I(t) =(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + (g * \Delta u)(t) + \|\nabla u\|^2 - \int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}dx - \int_\Omega |u|^{q(x)}\ln|u|dx, \tag{9}$$

and

$$J(t) =\frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{1}{2}(g * \Delta u)(t) + \frac{1}{4}\int_\Omega p(x)|\nabla u|^2 dx + \int_\Omega \frac{1}{q^2(x)}|u|^{q(x)}dx$$
$$- \frac{1}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ln|u|dx. \tag{10}$$

Lemma 3.1. *Assume that the conditions of Theorem 2.4 are satisfied. If $I(0) > 0$ and for sufficiently small $a > 0$:*

$$0 \le \|\nabla u\|^2 < (ea)^{\frac{n}{2}} e^{\frac{1}{p_2}}. \tag{11}$$

Then, we have

$$I(t) > 0, \qquad \text{for all } t \in [0,T].$$

Proof. By using the continuity of $u(t)$, and since $I(0) > 0$, thus there exists $T^* < T$ such that $I(t) \ge 0$ for all $t \in [0,T^*]$. Also, by virtue of the conditions on the variable exponents and the fact that $1 - \int_0^t g(\tau)d\tau < 1 - \int_0^\infty g(\tau)d\tau = \ell$, we easily get

$$I(t) \ge \ell\|\Delta u\|^2 + (g * \Delta u)(t) + \|\nabla u\|^2 - p_2\int_\Omega |\nabla u|^2 \ln|\nabla u|dx - \int_\Omega |u|^{q(x)}\ln|u|dx. \tag{12}$$

Thanks to the Logarithmic Sobolev inequality, for any positive number a, we obtain

$$\int_\Omega |\nabla u|^2 \ln|\nabla u|dx \le \|\nabla u\|^2 \ln\|\nabla u\|^2 + \frac{a^2}{2\pi}\|\Delta u\|^2 - \frac{n}{2}(1+\ln a)\|\nabla u\|^2. \tag{13}$$

Utilizing inequality (13) into (12), we deduce

$$
\begin{aligned}
I(t) &\geq \left(\ell - \frac{a^2}{2\pi}\right)\|\Delta u\|^2 + (g * \Delta u)(t) + \left(1 + \frac{np_2}{2}(1+\ln a) - p_2 \ln \|\nabla u\|^2\right)\|\nabla u\|^2 - \int_\Omega |u|^{q(x)} \ln |u| dx \\
&\geq \left(\ell - \frac{a^2}{2\pi}\right)\|\Delta u\|^2 + (g * \Delta u)(t) - \int_\Omega |u|^{q(x)} \ln |u| dx, \qquad (14)
\end{aligned}
$$

where the assumption (11) has been used.
Suppose that

$$
\Omega_1 = \{u | u \in W_0^{1,q(x)}(\Omega), 0 < u < 1\}, \qquad \Omega_2 = \{u | u \in W_0^{1,q(x)}(\Omega), u \geq 1\},
$$

and thus, it is easy to see that for any $u \in \Omega_2$, we have $0 \leq u^{-2} \ln u \leq 1$.
Consequently, we deduce

$$
\begin{aligned}
\int_\Omega |u|^{q(x)} \ln |u| dx &= \int_{\Omega_1} |u|^{q(x)} \ln |u| dx + \int_{\Omega_2} |u|^{q(x)} \ln |u| dx \\
&\leq \int_{\Omega_2} |u|^{q(x)} \ln |u| dx \\
&\leq \frac{1}{2e} \int_{\Omega_2} |u|^{q(x)+2} dx \leq \frac{1}{2e} \int_\Omega |u|^{q(x)+2} dx \\
&\leq \frac{1}{2e} \max\{\left(\int_\Omega |u|^{q(x)+2} dx\right)^{q_1+2}, \left(\int_\Omega |u|^{q(x)+2} dx\right)^{q_2+2}\} \\
&\leq \frac{1}{2e} \max\{C_{*,2}^{q_1+2} \|\nabla u\|^{q_1+2}, C_{*,2}^{q_2+2} \|\nabla u\|^{q_2+2}\} \\
&= \frac{1}{2e} \max\{C_{*,2}^{q_1+2} \|\nabla u\|^{q_1}, C_{*,2}^{q_2+2} \|\nabla u\|^{q_2}\} \|\nabla u\|^2 \\
&\leq \frac{C_2^2}{2e} \max\{C_{*,2}^{q_1+2} \left(a^n e^{n+\frac{2}{p_2}}\right)^{\frac{q_1}{2}}, C_{*,2}^{q_2+2} \left(a^n e^{n+\frac{2}{p_2}}\right)^{\frac{q_2}{2}}\} \|\Delta u\|^2, \qquad (15)
\end{aligned}
$$

where $C_{*,2}$ is the best constant of embedding $H_0^1(\Omega) \hookrightarrow L^{q(x)+2}(\Omega)$, C_2 is the best constant of Poincaré inequality and (11) has been used.
Now, if we set

$$
\eta_0 := \frac{C_2^2}{2e} \max\{C_*^{q_1+2} \left(a^n e^{n+\frac{2}{p_2}}\right)^{\frac{q_1}{2}}, C_*^{q_2+2} \left(a^n e^{n+\frac{2}{p_2}}\right)^{\frac{q_2}{2}}\},
$$

then we obtain from (14)

$$
I(t) \geq \left(\ell - \frac{a^2}{2\pi} - \eta_0\right)\|\Delta u\|^2 + (g * \Delta u)(t). \qquad (16)
$$

Finally, we choose a sufficiently small such that $\ell - \frac{a^2}{2\pi} - \eta_0 > 0$ and therefore, we deduce

$$
I(t) > 0, \quad \forall t \in [0, T^*].
$$

By repeating this procedure, T^* extended to T and proof of Lemma 3.1 is completed. □

At this point, we state and prove the global existence result as follows:

Theorem 3.2. *Under the assumptions of Lemma 3.1, the solution of problem* (1)-(3) *is global in time if* $\ell \in (\frac{1}{2\pi}, 1)$ *and*

$$0 \leq \|\nabla u\|^2 < e^{\frac{n}{2}} \min\{a^{\frac{n}{2}} e^{\frac{1}{p_2}}, (\frac{1}{p_2})^{\frac{n}{4}}\}. \tag{17}$$

Proof. From definition of $J(t)$ and additional conditions on the variable exponents, we obtain

$$\begin{aligned} J(t) \geq &\frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{1}{2}(g * \Delta u)(t) + \frac{p_1}{4}\|\nabla u\|^2 + \frac{1}{q_2^2}\int_\Omega |u|^{q(x)}dx - \frac{1}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}dx \\ &- \frac{1}{q_1}\int_\Omega |u|^{q(x)} \ln|u|dx \\ \geq &\frac{q_1 - 2}{2q_1}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{q_1 - 2}{2q_1}(g * \Delta u)(t) + \frac{p_1 q_1 - 4}{4q_1}\|\nabla u\|^2 + \frac{1}{q_2^2}\int_\Omega |u|^{q(x)}dx \\ &- \frac{p_2(q_1 - 2)}{2q_1}\int_\Omega |\nabla u|^2 \ln|\nabla u|dx + \frac{1}{q_1}I(t). \end{aligned} \tag{18}$$

Recalling the Logarithmic Sobolev inequality, we get for $a = \frac{1}{\sqrt{p_2}}$

$$\int_\Omega |\nabla u|^2 \ln|\nabla u|dx \leq \|\nabla u\|^2 \ln\|\nabla u\|^2 + \frac{1}{2\pi p_2}\|\Delta u\|^2 - \frac{n}{2}(1 - \frac{1}{2}\ln p_2)\|\nabla u\|^2. \tag{19}$$

By applying inequality (19) into (18), and using the fact that $1 - \int_0^t g(\tau)d\tau > \ell$, it yields

$$\begin{aligned} J(t) \geq &\frac{(2\pi\ell - 1)(q_1 - 2)}{4\pi q_1}\|\Delta u\|^2 + \frac{q_1 - 2}{2q_1}(g * \Delta u)(t) + \frac{1}{q_2^2}\int_\Omega |u|^{q(x)}dx \\ &+ \left(\frac{p_1 q_1 - 4}{4q_1} + \frac{np_2(q_1 - 2)}{4q_1}(1 - \frac{1}{2}\ln p_2) - \frac{p_2(q_1 - 2)}{2q_1}\ln\|\nabla u\|^2\right)\|\nabla u\|^2 + \frac{1}{q_1}I(t) \\ \geq &\frac{(2\pi\ell - 1)(q_1 - 2)}{4\pi q_1}\|\Delta u\|^2 + \frac{q_1 - 2}{2q_1}(g * \Delta u)(t) + \frac{1}{q_2^2}\int_\Omega |u|^{q(x)}dx + \frac{p_1 q_1 - 4}{4q_1}\|\nabla u\|^2 + \frac{1}{q_1}I(t). \end{aligned} \tag{20}$$

Since by Lemma 3.1 we have $I(t) > 0$, if we set

$$\gamma := \min\{\frac{(2\pi\ell - 1)(q_1 - 2)}{4\pi q_1}, \frac{q_1 - 2}{2q_1}, \frac{1}{q_2^2}, \frac{p_1 q_1 - 4}{4q_1}\},$$

then we obtain

$$J(t) \geq \gamma\left(\|\Delta u\|^2 + \|\nabla u\|^2 + (g * \Delta u)(t) + \int_\Omega |u|^{q(x)}dx\right). \tag{21}$$

Thanks to the definition of the energy functional $E(t)$ and using (21) we deduce

$$\begin{aligned} E(0) \geq E(t) = &\frac{1}{2}\|u_t\|^2 + J(t) \\ \geq &\frac{1}{2}\|u_t\|^2 + \gamma\left(\|\Delta u\|^2 + \|\nabla u\|^2 + (g * \Delta u)(t) + \int_\Omega |u|^{q(x)}dx\right) \\ \geq &\gamma_0\left(\|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + (g * \Delta u)(t) + \int_\Omega |u|^{q(x)}dx\right), \end{aligned}$$

where $\gamma_0 := \min\{\frac{1}{2}, \gamma\}$. Therefore, it follows that

$$\|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + (g * \Delta u)(t) + \int_\Omega |u|^{q(x)}dx \leq \frac{E(0)}{\gamma_0}, \tag{22}$$

and the proof of the Theorem 3.2 is completed. □

4. General decay

This section aims at proving the general decay of the global solutions for the problem (1)-(3). To prove this result, we assume the following additional condition on the kernel of the memory:
($A3$) There exists a non-increasing differentiable function $\xi : R^+ \to R^+$ such that

$$\xi(0) \geq 0, \;\; g'(t) \leq -\xi(t)g(t), \;\; \int_0^\infty \xi(s)ds = +\infty.$$

Our main result in this section reads in the following theorem:

Theorem 4.1. *Let the conditions of Theorem 3.2 and ($A3$) are satisfied and moreover, for sufficiently large p_2*

$$0 \leq \|\nabla u\|^2 < e^{\frac{n-1}{2}} \left(\frac{\pi \ell}{2p_2}\right)^{\frac{n}{4}}. \tag{23}$$

Then the energy functional $E(t)$ of problem (1)-(3) is generally decay in the sense of the kernel of the memory and satisfies the following general estimate for the two constants k and K:

$$E(t) \leq KE(0)e^{-k\int_0^t \xi(s)ds}, \;\; t \geq 0. \tag{24}$$

Proof. To prove the result, we define the following functional

$$\phi(t) = E(t) + \varepsilon \int_\Omega u u_t dx, \tag{25}$$

where ε is an appropriate positive constant.
Differentiating $\phi(t)$, by using (8) we arrived at

$$\begin{aligned}\phi'(t) =& E'(t) + \varepsilon\|u_t\|^2 + \varepsilon \int_\Omega u u_{tt} dx \\ \leq& -(1-\varepsilon)\|u_t\|^2 - \varepsilon(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \varepsilon \int_0^t g(t-\tau) \int_\Omega \Delta u(\Delta u(\tau) - \Delta u)dxd\tau \\ &+ \varepsilon \int_\Omega |\nabla u|^2 \ln |\nabla u|^{p(x)} dx + \varepsilon \int_\Omega |u|^{q(x)} \ln |u| dx.\end{aligned} \tag{26}$$

By recalling the definition of $E(t)$ we find that

$$\begin{aligned}\phi'(t) \leq& -\varepsilon E(t) - (1 - \frac{3\varepsilon}{2})\|u_t\|^2 - \frac{\varepsilon}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{\varepsilon}{2}(g * \Delta u)(t) + \frac{\varepsilon p_2}{4}\|\nabla u\|^2 \\ &+ \frac{\varepsilon}{q_1^2}\int_\Omega |u|^{q(x)}dx + \varepsilon \int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau) - \Delta u)dxd\tau + \frac{\varepsilon p_2}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u| dx \\ &+ \frac{\varepsilon(q_2 - 1)}{q_2}\int_\Omega |u|^{q(x)} \ln |u| dx,\end{aligned} \tag{27}$$

where the conditions on the variable exponents have been used.
Now, by using the Young's inequality, it is easy to get

$$\varepsilon \int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau) - \Delta u)dxd\tau \leq \frac{\varepsilon \ell}{4}\|\Delta u\|^2 + \frac{\varepsilon(1-\ell)^2}{\ell}(g * \Delta u)(t). \tag{28}$$

Also, by virtue of the Logarithmic Sobolev inequality, for any $a_0 > 0$, we get

$$\frac{\varepsilon p_2}{2}\int_\Omega |\nabla u|^2 \ln |\nabla u| dx \leq \frac{\varepsilon p_2}{2}\|\nabla u\|^2 \ln \|\nabla u\|^2 + \frac{a_0^2 \varepsilon p_2}{4\pi}\|\Delta u\|^2 - \frac{n\varepsilon p_2}{4}(1 + \ln a_0)\|\nabla u\|^2. \tag{29}$$

Applying inequalities (28) and (29) into (27) to obtain

$$
\begin{aligned}
\phi'(t) \leq & -\varepsilon E(t) - (1 - \frac{3\varepsilon}{2})\|u_t\|^2 - \left(\frac{\varepsilon}{2}(1 - \int_0^t g(\tau)d\tau) - \frac{\varepsilon\ell}{4} - \frac{a_0^2\varepsilon p_2}{4\pi}\right)\|\Delta u\|^2 + \left(\frac{\varepsilon}{2} + \frac{\varepsilon(1-\ell)^2}{\ell}\right)(g * \Delta u)(t) \\
& - \left(\frac{n\varepsilon p_2}{4}(1 + \ln a_0) - \frac{\varepsilon p_2}{4} - \frac{\varepsilon p_2}{2}\ln\|\nabla u\|^2\right)\|\nabla u\|^2 + \frac{\varepsilon}{q_1^2}\int_\Omega |u|^{q(x)}dx + \frac{\varepsilon(q_2-1)}{q_2}\int_\Omega |u|^{q(x)}\ln|u|dx \\
\leq & -\varepsilon E(t) - (1 - \frac{3\varepsilon}{2})\|u_t\|^2 - \frac{\varepsilon}{4}\left(\ell - \frac{a_0^2 p_2}{\pi}\right)\|\Delta u\|^2 + \varepsilon\left(\frac{1}{2} + \frac{(1-\ell)^2}{\ell}\right)(g * \Delta u)(t) \\
& - \frac{\varepsilon p_2}{4}\left(n(1 + \ln a_0) - 1 - \ln\|\nabla u\|^2\right)\|\nabla u\|^2 + \frac{\varepsilon}{q_1^2}\int_\Omega |u|^{q(x)}dx + \frac{\varepsilon(q_2-1)}{q_2}\int_\Omega |u|^{q(x)}\ln|u|dx, \qquad (30)
\end{aligned}
$$

where $1 - \int_0^t g(\tau)d\tau > \ell$ has been used.
At this point, we choose $\varepsilon < \frac{2}{3}$, and $a_0 = \sqrt{\frac{\pi\ell}{2p_2}}$ to get

$$
\begin{aligned}
\phi'(t) \leq & -\varepsilon E(t) - \frac{\varepsilon\ell}{8}\|\Delta u\|^2 + \varepsilon\left(\frac{1}{2} + \frac{(1-\ell)^2}{\ell}\right)(g * \Delta u)(t) - \frac{\varepsilon p_2}{4}\left(n - 1 + \frac{n}{2}\ln\frac{\pi\ell}{2p_2} - 2\ln\|\nabla u\|^2\right)\|\nabla u\|^2 \\
& + \frac{\varepsilon}{q_1^2}\int_\Omega |u|^{q(x)}dx + \frac{\varepsilon(q_2-1)}{q_2}\int_\Omega |u|^{q(x)}\ln|u|dx. \qquad (31)
\end{aligned}
$$

On the other hand, thanks to the assumption (23), we have

$$
\begin{aligned}
\frac{\varepsilon}{q_1^2}\int_\Omega |u|^{q(x)}dx \leq & \frac{\varepsilon}{q_1^2}\max\{\left(\int_\Omega |u|^{q(x)}dx\right)^{q_1}, \left(\int_\Omega |u|^{q(x)}dx\right)^{q_2}\} \\
\leq & \frac{\varepsilon}{q_1^2}\max\{C_*^{q_1}\|\nabla u\|^{q_1}, C_*^{q_2}\|\nabla u\|^{q_2}\} \\
= & \frac{\varepsilon}{q_1^2}\max\{C_*^{q_1}\|\nabla u\|^{q_1-2}, C_*^{q_2}\|\nabla u\|^{q_2-2}\}\|\nabla u\|^2 \\
\leq & \frac{\varepsilon}{q_1^2}\max\{C_*^{q_1}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_1-2}{2}}, C_*^{q_2}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_2-2}{2}}\}\|\nabla u\|^2 \\
\leq & \frac{C_2^2\varepsilon}{q_1^2}\max\{C_*^{q_1}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_1-2}{2}}, C_*^{q_2}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_2-2}{2}}\}\|\Delta u\|^2 \\
= & \eta_1\|\Delta u\|^2, \qquad (32)
\end{aligned}
$$

where C_* is the best constant of embedding $H^1(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. Also,

$$\begin{aligned}
\frac{\varepsilon(q_2-1)}{q_2}\int_\Omega |u|^{q(x)}\ln|u|dx =&\frac{\varepsilon(q_2-1)}{q_2}\left(\int_{\Omega_1}|u|^{q(x)}\ln|u|dx+\int_{\Omega_2}|u|^{q(x)}\ln|u|dx\right)\\
\leq&\frac{\varepsilon(q_2-1)}{q_2}\int_{\Omega_2}|u|^{q(x)}\ln|u|dx\\
\leq&\frac{\varepsilon(q_2-1)}{2eq_2}\int_{\Omega_2}|u|^{q(x)+2}dx\leq\frac{\varepsilon(q_2-1)}{2eq_2}\int_{\Omega}|u|^{q(x)+2}dx\\
\leq&\frac{\varepsilon(q_2-1)}{2eq_2}\max\{\left(\int_\Omega|u|^{q(x)+2}dx\right)^{q_1+2},\left(\int_\Omega|u|^{q(x)+2}dx\right)^{q_2+2}\}\\
\leq&\frac{\varepsilon(q_2-1)}{2eq_2}\max\{C_{*,2}^{q_1+2}\|\nabla u\|^{q_1+2},C_{*,2}^{q_2+2}\|\nabla u\|^{q_2+2}\}\\
=&\frac{\varepsilon(q_2-1)}{2eq_2}\max\{C_{*,2}^{q_1+2}\|\nabla u\|^{q_1},C_{*,2}^{q_2+2}\|\nabla u\|^{q_2}\}\|\nabla u\|^2\\
\leq&\frac{\varepsilon C_2^2(q_2-1)}{2eq_2}\max\{C_{*,2}^{q_1+2}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_1}{2}},C_{*,2}^{q_2+2}\left(e^{\frac{n-1}{2}}\left(\frac{\pi\ell}{2p_2}\right)^{\frac{n}{4}}\right)^{\frac{q_2}{2}}\}\|\Delta u\|^2,\\
=&\eta_2\|\Delta u\|^2,
\end{aligned}\tag{33}$$

where $C_{*,2}$ is the best constant of embedding $H^1(\Omega) \hookrightarrow L^{q(x)+2}(\Omega)$.
Utilizing (23), (32) and (33) into (31) we deduce

$$\phi'(t)\leq-\varepsilon E(t)-\left(\frac{\varepsilon\ell}{8}-\eta_1-\eta_2\right)\|\Delta u\|^2+\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)(g*\Delta u)(t).$$

Now, since $\varepsilon<\frac{2}{3}$, we choose p_2 large enough to guaranty $\eta_1+\eta_2<\frac{\varepsilon\ell}{8}$. Therefore, we obtain

$$\phi'(t)\leq-\varepsilon E(t)+\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)(g*\Delta u)(t).\tag{34}$$

Thanks to the assumption $(A3)$, by multiplying (34) by $\xi(t)$ we have

$$\begin{aligned}
\xi(t)\phi'(t)\leq&-\varepsilon\xi(t)E(t)-\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)(g'*\Delta u)(t)\\
\leq&-\varepsilon\xi(t)E(t)-\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)E'(t),
\end{aligned}\tag{35}$$

where Lemma 2.5 has been used.
Let define $L(t)=\xi(t)\phi(t)+\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)E(t)$. It is easy to see that there exist two positive constants $\check{\gamma}$ and $\hat{\gamma}$, such that

$$\check{\gamma}E(t)\leq L(t)\leq\hat{\gamma}E(t).\tag{36}$$

Thus, by using (35) we get

$$\begin{aligned}
L'(t)=&\xi'(t)\phi(t)+\xi(t)\phi'(t)+\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)E'(t)\\
\leq&\xi(t)\phi'(t)+\varepsilon\left(\frac{1}{2}+\frac{(1-\ell)^2}{\ell}\right)E'(t)\\
\leq&-\varepsilon\xi(t)E(t)\\
\leq&-\frac{\varepsilon}{\hat{\gamma}}\xi(t)L(t),
\end{aligned}\tag{37}$$

where (36) and non-increasingness of $\xi(\cdot)$ have been used.
So by a simple integration over $(0, t)$ we then obtain

$$L(t) \leq L(0)e^{-\frac{\varepsilon}{\gamma}\int_0^t \xi(s)ds}, \qquad \forall t \geq 0. \tag{38}$$

A combination of (36) and (38) completes the proof. □

5. Blow-up

In this section, we assume that the problem (1)-(3) has positive initial energy and prove that under some suitable conditions on data, there exists a finite time such that the solutions blow up at this time. Our blow-up result with positive initial energy reads in the following theorem:

Theorem 5.1. *Suppose that the assumptions of Theorem 2.4 hold and* $q_1 > \max\{2,\ 2\sqrt{1+4C_*^2},\ -1+\sqrt{1+\frac{4(1-\ell)^2}{\ell}}\}$. *Moreover,* $E(0) > 0$ *is a given initial energy level. If we choose initial data* u_0, u_1 *satisfying*

$$\int_\Omega u_0(x)u_1(x)dx \geq q_1 E(0), \tag{39}$$

then the solution of (1)-(3) *blows up in finite time, i.e., there exists* $t^* < +\infty$ *such that*

$$\lim_{t\to t^*} E(t) = +\infty.$$

Proof. Assume that u is a global solution, i.e. satisfy Theorem 3.2. Let define

$$H(t) = \int_\Omega u u_t dx - q_1 E(t), \tag{40}$$

where $E(t)$ satisfies (8).
Differentiating $H(t)$, we obtain

$$\begin{aligned} H'(t) =& \|u_t\|^2 + \int_\Omega u u_{tt} dx - q_1 E'(t) \\ \geq& \|u_t\|^2 + \int_\Omega u u_{tt} dx, \end{aligned} \tag{41}$$

where Lemma 2.5 has been used.
By multiplying (1) in u and integrating over Ω, (41) is rewritten as:

$$\begin{aligned} H'(t) \geq& \|u_t\|^2 - (1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau) - \Delta u)dxd\tau + \int_\Omega |\nabla u|^2 \ln|\nabla u|^{p(x)}dx \\ &+ \int_\Omega |u|^{q(x)}\ln|u|dx - \int_\Omega u u_t dx. \end{aligned} \tag{42}$$

Taking into account the definition of $H(t)$ and $(A1)$ into (42), we infer

$$\begin{aligned} H'(t) \geq& H(t) + \frac{q_1+2}{2}\|u_t\|^2 + \frac{q_1-2}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u\|^2 + \frac{q_1 p_1}{4}\|\nabla u\|^2 + \frac{q_1}{2}(g * \Delta u)(t) + \frac{q_1}{q_2^2}\int_\Omega |u|^{q(x)}dx \\ &- 2\int_\Omega u u_t dx - \frac{p_2(q_1-2)}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u|dx + \int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau) - \Delta u)dxd\tau. \end{aligned} \tag{43}$$

By using the Young and Logarithmic Sobolev inequalities, we could estimate the last three terms on the right hand side of (43) as follows:

$$
\begin{aligned}
|\int_\Omega uu_t dx| \leq & \frac{(q_1-2)\ell}{16C_*^2}\|u\|^2 + \frac{4C_*^2}{(q_1-2)\ell}\|u_t\|^2 \\
\leq & \frac{(q_1-2)\ell}{16}\|\Delta u\|^2 + \frac{4C_*^2}{(q_1-2)\ell}\|u_t\|^2. \qquad (44)
\end{aligned}
$$

$$
\int_\Omega |\nabla u|^2 \ln|\nabla u| dx \leq \|\nabla u\|^2 \ln\|\nabla u\|^2 + \frac{\ell}{2p_2}\|\Delta u\|^2 - \frac{n}{2}(1+\frac{1}{2}\ln\frac{\pi\ell}{p_2})\|\nabla u\|^2. \qquad (45)
$$

$$
|\int_0^t g(t-\tau)\int_\Omega \Delta u(\Delta u(\tau)-\Delta u)dxd\tau| \leq \frac{(q_1-2)\ell}{8}\|\Delta u\|^2 + \frac{2(1-\ell)^2}{(q_1-2)\ell}(g*\Delta u)(t). \qquad (46)
$$

Applying (44)-(46) into (43) to obtain

$$
\begin{aligned}
H'(t) \geq & H(t) + \left(\frac{q_1+2}{2} - \frac{8C_*^2}{(q_1-2)\ell}\right)\|u_t\|^2 + \left(\frac{q_1-2}{2}(1-\int_0^t g(\tau)d\tau) - \frac{(q_1-2)\ell}{4}\right)\|\Delta u\|^2 \\
& + \left(\frac{q_1p_1}{4} + \frac{np_2(q_1-2)}{4}(1+\frac{1}{2}\ln\frac{\pi\ell}{p_2}) - \frac{p_2(q_1-2)}{2}\ln\|\nabla u\|^2\right)\|\nabla u\|^2 \\
& + \left(\frac{q_1}{2} - \frac{2(1-\ell)^2}{(q_1-2)\ell}\right)(g*\Delta u)(t) + \frac{q_1}{q_2^2}\int_\Omega |u|^{q(x)}dx \\
\geq & H(t) + \left(\frac{q_1+2}{2} - \frac{8C_*^2}{(q_1-2)\ell}\right)\|u_t\|^2 + \left(\frac{q_1p_1}{4} + \frac{np_2(q_1-2)}{4}(1+\frac{1}{2}\ln\frac{\pi\ell}{p_2}) - \frac{p_2(q_1-2)}{2}\ln\|\nabla u\|^2\right)\|\nabla u\|^2 \\
& + \left(\frac{q_1}{2} - \frac{2(1-\ell)^2}{(q_1-2)\ell}\right)(g*\Delta u)(t), \qquad (47)
\end{aligned}
$$

where $1-\int_0^t g(\tau)d\tau > 1-\int_0^\infty g(\tau)d\tau = \ell$ has been used.
On the other hand, by virtue of the assumption of the Theorem 5.1, if we choose

$$
q_1 > \max\{2,\ 2\sqrt{1+4C_*^2},\ -1+\sqrt{1+\frac{4(1-\ell)^2}{\ell}}\},
$$

then we get from (47)

$$
H'(t) \geq H(t) + \left(\frac{q_1p_1}{4} + \frac{np_2(q_1-2)}{4}(1+\frac{1}{2}\ln\frac{\pi\ell}{p_2}) - \frac{p_2(q_1-2)}{2}\ln\|\nabla u\|^2\right)\|\nabla u\|^2. \qquad (48)
$$

Now, since we assumed that u is global, so by applying the assumptions of Theorem 3.2, it is easy to see that if we choose a small enough and q_1 sufficiently large such that

$$
0 \leq \|\nabla u\|^2 < e^{\frac{n}{2}}\min\{a^{\frac{n}{2}}e^{\frac{1}{p_2}},\ (\frac{1}{p_2})^{\frac{n}{4}}\} \leq \left(\frac{e^2\pi\ell}{p_2}\right)^{\frac{n}{4}} e^{\frac{q_1p_1}{2p_2(q_1-2)}},
$$

then we arrive at

$$
H'(t) \geq H(t), \qquad (49)
$$

and since we have $H(0) > 0$, inequality (45) implies

$$
H(t) \geq e^t H(0) > 0, \quad \forall t \geq 0.
$$

This shows that the functional $H(t)$ exponentially growth when time goes to infinity.
From definition of $H(t)$ we have

$$e^t H(0) \leq H(t) = \int_\Omega u u_t dx - q_1 E(t) \leq \int_\Omega u u_t dx.$$

Thanks to the Hölder and Young inequalities, thus there exists a constant C such that

$$e^t H(0) \leq H(t) \leq C\left(\|\nabla u\|^2 + \|u_t\|^2\right). \tag{50}$$

Thus inequality (50) shows that $\|\nabla u\|^2$ or $\|u_t\|^2$ must grow exponentially.
On the other hand, since we assumed that u is a global solution so by recalling (22) we must have

$$\|u_t\|^2 + \|\nabla u\|^2 \leq \frac{E(0)}{\gamma_0},$$

which contradicts the previous result that $\|\nabla u\|^2$ or $\|u_t\|^2$ are exponentially growing. Therefore there exists a finite time t^* such that solutions of problem (1)-(3) blow up and proof of Theorem 5.1 is completed. □

6. Conclusions

In this paper, we studied global existence, general decay and blow-up of solutions for a class of plate viscoelastic equation with variable-exponent logarithmic nonlinearities. This type of nonlinearities were introduced in the non-relativistic wave equations describing spinning particles moving in an external electromagnetic field and also in the relativistic wave equation for spinless particles. Thus, the investigating the behavior of the solutions for such problems are important and to the best of our knowledge, this work is the first one regarding the viscoelastic equations with logarithmic terms and variable-exponent nonlinearities. In fact, we obtained global existence and general decay of solutions by constructing a suitable Lyapunov functional and next, blow up of solutions for the problem (1)-(3) in a finite time with arbitrary positive initial energy has been established by using the contradiction. We like to point out that the local existence, global existence and asymptotic behavior of solutions to problem (1)-(3) with $p(x) \equiv p(x, t)$ instead of $p(x) \equiv p(x)$ and $q(x) \equiv q(x, t)$ is still an open problem.
The problem (1)-(3) may be studied with variable exponents and logarithmic nonlinearity terms as follows:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + div(\nabla u \ln |\nabla u|^{p(x,t)}) + u_t = |u|^{q(x,t)-2} u \ln |u|,$$

Also, this equation could be studied under different initial and boundary conditions. Furthermore, different mathematical behavior such as local existence, attractor... etc. may be established.

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