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On circulant and r-circulant matrices with Ducci sequences and Lucas numbers

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Abstract. A Ducci sequence is a sequence $\{S, DS, D^2S, ...\}$, where the map $D : \mathbb{Z}^n \to \mathbb{Z}^n$ takes each $S = (s_1, s_2, s_3, ..., s_{n-1}, s_n)$ to $(|s_1 - s_2|, |s_2 - s_3|, ..., |s_{n-1} - s_n|, |s_n - s_1|)$. In this paper, we study norms of r-circulant matrices $Circ_r(DL)$ and $Circ_r(D^2L)$, where L is an n-tuple of Lucas numbers. Then we examine some properties of circulant matrices $Circ(DL)$ and $Circ(D^2L)$.

1. Introduction

Matrices have extensive applications in various branches of mathematics and other scientific fields like engineering, statistics, physics, and economics. Therefore, researchers have defined different types of matrices and extensively studied their diverse properties. One of these matrices is called the Toeplitz matrix.

A Toeplitz matrix [20, 33], also known as a diagonal-constant matrix, is named after Otto Toeplitz. It is characterized by having constant values along each descending diagonal from left to right. A circulant matrix [14], a particular kind of Toeplitz matrix, is a square matrix where all row vectors consist of the same elements. What makes it remarkable is that each row vector is shifted one element to the right compared to the previous row vector. An $n \times n$ circulant matrix C is of the form

$$C = Circ(c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Let $r \in \mathbb{C} \setminus \{0\}$. An $n \times n$ r-circulant matrix [12] C_r is of the form

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$$C_r = Circ_r(c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \dots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \dots & rc_{n-1} & c_0 \end{pmatrix}.$$

Note that an r-circulant matrix, for $r = 1$, is a circulant matrix. Circulant and r-circulant matrices find wide applications in various scientific areas, including coding theory, signal processing, image processing, time-series analysis, etc. (see, for instance, [3, 9, 16, 25, 36]).

The Fibonacci sequence is a well-known sequence of integers that is defined recursively by the relation

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, n \geq 2. \tag{1}$$

Another well-known sequence, called the Lucas sequence, is defined recursively by the relation

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, n \geq 2. \tag{2}$$

For further information about Fibonacci and Lucas numbers, we refer to [21]. Let F_n be the nth Fibonacci number and L_n be the nth Lucas number. The following properties hold [18, 21]:

$$L_{-n} = (-1)^n L_n, \tag{3}$$

$$F_{n-1} + F_{n+1} = L_n, \tag{4}$$

$$L_n + L_{n+2} = 5F_{n+1}, \tag{5}$$

$$L_n^2 + L_{n-1}^2 = 5F_{2n-1}, \tag{6}$$

$$L_n L_{n+1} = L_{2n+1} + (-1)^n, \tag{7}$$

$$\sum_{k=0}^{n} L_k^2 = L_n L_{n+1} + 2 = L_{2n+1} + (-1)^n + 2. \tag{8}$$

In recent years, there have been several papers focusing on circulant and r-circulant matrices that contain special entries like Fibonacci-type numbers. These matrices' norms have been extensively studied. Solak [28, 29] introduced the $n \times n$ circulant matrices $A = [a_{ij}]$ such that $a_{ij} \equiv F_{(mod(j-i,n))}$ and $B = [b_{ij}]$ such that $b_{ij} \equiv L_{(mod(j-i,n))}$. In other words, the matrix A has the form $Circ(F_0, F_1, F_2, ..., F_{n-2}, F_{n-1})$, where F_{n-1} is the $(n-1)$th Fibonacci number. Also, the matrix B has the form $Circ(L_0, L_1, L_2, ..., L_{n-2}, L_{n-1})$, where L_{n-1} is the $(n-1)$th Lucas number. The author also gave some bounds for the spectral norms of the matrices A and B. Then, in [26], Shen and Cen gave upper and lower bounds for the spectral norms of r-circulant matrices $Circ_r(F_0, F_1, F_2, ..., F_{n-2}, F_{n-1})$ and $Circ_r(L_0, L_1, L_2, ..., L_{n-2}, L_{n-1})$. Thus, the authors generalized Solak's findings [28, 29] by applying them to r-circulant matrices. For some other related studies, see, for example, [1, 2, 13, 22, 23, 27, 34, 35].

Ducci sequences were first introduced in 1937 [10]. Ducci sequences' discovery is attributed to Italian mathematician Enrico Ducci. From then on, Ducci sequences have been examined in several papers. We refer to [4–8, 11, 15, 17, 30, 31] and the references given therein. A Ducci sequence is defined as follows:

Let n be a positive integer, and $S = (s_1, s_2, s_3, ..., s_{n-1}, s_n)$ be an n-tuple of integers. A Ducci sequence generated by $S = (s_1, s_2, s_3, ..., s_{n-1}, s_n)$ is a sequence $\{S, DS, D^2S, ...\}$ obtained by iterating the map (called the Ducci map) $D : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by

$$DS = D(s_1, s_2, s_3, ..., s_{n-1}, s_n) = (|s_1 - s_2|, |s_2 - s_3|, ..., |s_{n-1} - s_n|, |s_n - s_1|).$$

A cycle is formed by every Ducci sequence $\{S, DS, D^2S, ...\}$, meaning that there exist integers i and j with $0 \le i < j$ with $D^iS = D^jS$. When i and j are as small as possible, a Ducci sequence is said to have a period $(j - i)$ [4].

In [30], Solak and Bahşi applied the Ducci map to each row of the circulant matrix $Circ(a_1, a_2, a_3, ..., a_{n-1}, a_n)$. Moreover, the authors established relationships between the Frobenius (or Euclidean) norm, spectral norm, l_p norm, determinant, and eigenvalues of the matrix $Circ(a_1, a_2, a_3, ..., a_{n-1}, a_n)$ and its image under the Ducci map. Also, the authors gave a numerical example in terms of Fibonacci numbers (see [30] for details). After, Solak et al. [31] examined some properties of circulant matrices $Circ(F)$, $Circ(DF)$, and $Circ(D^2F)$, where $F = (F_1, F_2, F_3, ..., F_{n-1}, F_n)$ such that F_n denotes the nth Fibonacci number.

Let $L = (L_1, L_2, L_3, ..., L_{n-1}, L_n) \in \mathbb{Z}^n$, where L_n denotes the nth Lucas number. Then, considering Eqs. (2) and (3), we have

$$\begin{aligned} DL &= (|L_1 - L_2|, |L_2 - L_3|, |L_3 - L_4|, ..., |L_{n-1} - L_n|, |L_n - L_1|) \\ &= (L_0, L_1, L_2, ..., L_{n-2}, L_n - 1) \end{aligned}$$

and

$$\begin{aligned} D^2L &= D(DL) \\ &= (|L_0 - L_1|, |L_1 - L_2|, |L_2 - L_3|, ..., |L_{n-2} - (L_n - 1)|, |L_n - 1 - L_0|) \\ &= (1, L_0, L_1, ..., L_{n-4}, L_{n-1} - 1, L_n - 3). \end{aligned}$$

In the present study, let

$$Circ_r(L) = \begin{pmatrix} L_1 & L_2 & L_3 & \dots & L_{n-1} & L_n \\ rL_n & L_1 & L_2 & \dots & L_{n-2} & L_{n-1} \\ rL_{n-1} & rL_n & L_1 & \dots & L_{n-3} & L_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_3 & rL_4 & rL_5 & \dots & L_1 & L_2 \\ rL_2 & rL_3 & rL_4 & \dots & rL_n & L_1 \end{pmatrix}$$

(see [26]),

$$Circ_r(DL) = \begin{pmatrix} L_0 & L_1 & L_2 & \dots & L_{n-2} & L_n - 1 \\ r(L_n - 1) & L_0 & L_1 & \dots & L_{n-3} & L_{n-2} \\ rL_{n-2} & r(L_n - 1) & L_0 & \dots & L_{n-4} & L_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_2 & rL_3 & rL_4 & \dots & L_0 & L_1 \\ rL_1 & rL_2 & rL_3 & \dots & r(L_n - 1) & L_0 \end{pmatrix}$$

and

$$Circ_r(D^2L) = \begin{pmatrix} 1 & L_0 & L_1 & \dots & L_{n-1} - 1 & L_n - 3 \\ r(L_n - 3) & 1 & L_0 & \dots & L_{n-4} & L_{n-1} - 1 \\ r(L_{n-1} - 1) & r(L_n - 3) & 1 & \dots & L_{n-5} & L_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rL_1 & rL_2 & rL_3 & \dots & 1 & L_0 \\ rL_0 & rL_1 & rL_2 & \dots & r(L_n - 3) & 1 \end{pmatrix}$$

be r-circulant matrices. Note that if we take $r = 1$, we obtain circulant matrices $Circ(L)$, $Circ(DL)$ and $Circ(D^2L)$, respectively.

The main idea of this paper is to present upper and lower bounds for the spectral norms of the r-circulant matrices $Circ_r(DL)$ and $Circ_r(D^2L)$. Furthermore, we establish relationships between the Frobenius norms, l_p norms, and determinants of the circulant matrices $Circ(DL)$ and $Circ(D^2L)$ in a similar way to Solak et al. [31].

Below are some preliminaries that are relevant to our study. Let A be any $m \times n$ matrix.

The l_p $(1 < p < \infty)$ norm of matrix A is

$$||A||_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}.$$

When we take $p = 2$ in the l_p norm, we have the Frobenius (or Euclidean) norm of matrix A as

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

The spectral norm of matrix A is

$$||A||_2 = \sqrt{\max_i \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ are eigenvalues of $A^H A$ and A^H is conjugate transpose of matrix A. The following relation between spectral norm and Frobenius norm holds [32]:

$$\frac{1}{\sqrt{n}} ||A||_F \leq ||A||_2 \leq ||A||_F. \tag{9}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the Hadamard product [19] of matrices A and B, denoted by $A \circ B$, is defined by

$$A \circ B = [a_{ij} b_{ij}].$$

Lemma 1.1. *[19, 24] Let A and B be two $m \times n$ matrices. Then we have*

$$||A \circ B||_2 \leq r_1(A) c_1(B),$$

where

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$

Lemma 1.2. *[30] The determinant of the circulant matrix $Circ(DA)$ satisfies*

$$|\det Circ(DA)| \leq \frac{1}{n^{\frac{n}{2}}} ||Circ(DA)||_F^n,$$

where $A = (a_1, a_2, a_3, ..., a_{n-1}, a_n)$ is an n-tuple of integers, and D denotes the Ducci map.

2. Main Results

Let us first give the following lemma that will be used in the proofs of the next theorems.

Lemma 2.1. *For r-circulant matrices $Circ_r(DL)$ and $Circ_r(D^2L)$, we have*

$$||Circ_r(DL)||_F^2 = \sum_{t=0}^{n-2}(n-t)L_t^2 + \sum_{t=1}^{n-2} t|r|^2L_t^2 + (n-1)|r|^2(L_n-1)^2 + (L_n-1)^2$$

and

$$||Circ_r(D^2L)||_F^2 = n + \sum_{t=0}^{n-4}(n-1-t)L_t^2 + \sum_{t=0}^{n-4}(t+1)|r|^2L_t^2 + (n-2)|r|^2(L_{n-1}-1)^2 + 2(L_{n-1}-1)^2$$
$$+ (n-1)|r|^2(L_n-3)^2 + (L_n-3)^2.$$

Proof. The proof is readily obtained by the definition of the Frobenius norm. □

The following theorem gives us the upper and lower bounds for the spectral norm of the r-circulant matrix $Circ_r(DL)$.

Theorem 2.2. *Let $\Delta_1 = L_{n-2}L_{n-1} + (L_n-1)^2 + 2$. For the r-circulant matrix $Circ_r(DL)$, we have*

(i) If $|r| \geq 1$, then

$$\sqrt{\Delta_1} \leq ||Circ_r(DL)||_2 \leq \sqrt{(|r|^2(n-1)+1)\Delta_1}.$$

(ii) If $|r| < 1$, then

$$|r|\sqrt{\Delta_1} \leq ||Circ_r(DL)||_2 \leq \sqrt{n\Delta_1}.$$

Proof. (i) Let $|r| \geq 1$. From Lemma 2.1 and Eq. (8), we have

$$||Circ_r(DL)||_F^2 \geq \sum_{t=0}^{n-2} nL_t^2 + n(L_n-1)^2 = n(L_{n-2}L_{n-1} + (L_n-1)^2 + 2).$$

It follows that

$$\frac{1}{\sqrt{n}}||Circ_r(DL)||_F \geq \sqrt{L_{n-2}L_{n-1} + (L_n-1)^2 + 2}.$$

From Eq. (9), we get

$$||Circ_r(DL)||_2 \geq \sqrt{L_{n-2}L_{n-1} + (L_n-1)^2 + 2}$$

that is

$$\sqrt{\Delta_1} \leq ||Circ_r(DL)||_2. \tag{10}$$

In order to find an upper bound, let matrices A and B be as

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ r & r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & 1 & 1 \\ r & r & r & \dots & r & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} L_0 & L_1 & L_2 & \dots & L_{n-2} & L_n-1 \\ L_n-1 & L_0 & L_1 & \dots & L_{n-3} & L_{n-2} \\ L_{n-2} & L_n-1 & L_0 & \dots & L_{n-4} & L_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_2 & L_3 & L_4 & \dots & L_0 & L_1 \\ L_1 & L_2 & L_3 & \dots & L_n-1 & L_0 \end{pmatrix}$$

such that $Circ_r(DL) = A \circ B$, where $A \circ B$ is the Hadamard product of A and B. Then we obtain

$$r_1(A) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{nj}|^2} = \sqrt{|r|^2(n-1)+1}$$

and

$$c_1(B) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{i1}|^2} = \sqrt{\sum_{k=0}^{n-2} L_k^2 + (L_n-1)^2} = \sqrt{\Delta_1}.$$

Considering Lemma 1.1, we can write

$$||Circ_r(DL)||_2 \le \sqrt{(|r|^2(n-1)+1)\Delta_1}. \tag{11}$$

Hence, from Eqs. (10) and (11), we have

$$\sqrt{\Delta_1} \le ||Circ_r(DL)||_2 \le \sqrt{(|r|^2(n-1)+1)\Delta_1}.$$

(ii) Let $|r| < 1$. From Lemma 2.1 and Eq. (8), we have

$$||Circ_r(DL)||_F^2 \ge \sum_{t=0}^{n-2} n|r|^2 L_t^2 + n|r|^2(L_n-1)^2 = n|r|^2(L_{n-2}L_{n-1} + (L_n-1)^2 + 2).$$

It follows that

$$\frac{1}{\sqrt{n}}||Circ_r(DL)||_F \ge |r|\sqrt{L_{n-2}L_{n-1} + (L_n-1)^2 + 2}.$$

From Eq. (9), we can write

$$||Circ_r(DL)||_2 \ge |r|\sqrt{L_{n-2}L_{n-1} + (L_n-1)^2 + 2} = |r|\sqrt{\Delta_1}. \tag{12}$$

Since $Circ_r(DL) = A \circ B$ for the matrices A and B defined as in part (i), we obtain

$$r_1(A) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{n}$$

and

$$c_1(B) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{k=0}^{n-2} L_k^2 + (L_n-1)^2} = \sqrt{\Delta_1}.$$

From Lemma 1.1, we can write

$$||Circ_r(DL)||_2 \le \sqrt{n\Delta_1}. \tag{13}$$

From Eqs. (12) and (13), we have

$$|r|\sqrt{\Delta_1} \le ||Circ_r(DL)||_2 \le \sqrt{n\Delta_1}.$$

□

The following theorem gives us the upper and lower bounds for the spectral norm of the r-circulant matrix $Circ_r(D^2L)$.

Theorem 2.3. *Let $\Delta_2 = L_{n-4}L_{n-3} + (L_{n-1} - 1)^2 + (L_n - 3)^2 + 3$. For the r-circulant matrix $Circ_r(D^2L)$, we have*

(i) If $|r| \geq 1$, then

$$\sqrt{\Delta_2} \leq \|Circ_r(D^2L)\|_2 \leq \sqrt{(|r|^2(n-1)+1)\Delta_2}.$$

(ii) If $|r| < 1$, then

$$|r|\sqrt{\Delta_2} \leq \|Circ_r(D^2L)\|_2 \leq \sqrt{n\Delta_2}.$$

Proof. (i) Let $|r| \geq 1$. From Lemma 2.1 and Eq. (8), we have

$$\|Circ_r(D^2L)\|_F^2 \geq n + \sum_{t=0}^{n-4} nL_t^2 + n(L_{n-1}-1)^2 + n(L_n-3)^2 = n(L_{n-4}L_{n-3} + (L_{n-1}-1)^2 + (L_n-3)^2 + 3).$$

It follows that

$$\frac{1}{\sqrt{n}}\|Circ_r(D^2L)\|_F \geq \sqrt{L_{n-4}L_{n-3} + (L_{n-1}-1)^2 + (L_n-3)^2 + 3}.$$

From Eq. (9), we can write

$$\sqrt{\Delta_2} \leq \|Circ_r(D^2L)\|_2. \tag{14}$$

In order to find an upper bound, let matrices C and D be as

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ r & r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & 1 & 1 \\ r & r & r & \dots & r & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & L_0 & L_1 & \dots & L_{n-1}-1 & L_n-3 \\ L_n-3 & 1 & L_0 & \dots & L_{n-4} & L_{n-1}-1 \\ L_{n-1}-1 & L_n-3 & 1 & \dots & L_{n-5} & L_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_1 & L_2 & L_3 & \dots & 1 & L_0 \\ L_0 & L_1 & L_2 & \dots & L_n-3 & 1 \end{pmatrix}$$

such that $Circ_r(D^2L) = C \circ D$. Then we obtain

$$r_1(C) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{|r|^2(n-1)+1}$$

and

$$c_1(D) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |d_{ij}|^2} = \sqrt{1 + (L_{n-1}-1)^2 + (L_n-3)^2 + \sum_{k=0}^{n-4} L_k^2} = \sqrt{\Delta_2}.$$

Considering Lemma 1.1, we can write

$$\|Circ_r(D^2L)\|_2 \leq \sqrt{(|r|^2(n-1)+1)\Delta_2}. \tag{15}$$

Thus, from Eqs. (14) and (15), we have

$$\sqrt{\Delta_2} \leq \|Circ_r(D^2L)\|_2 \leq \sqrt{(|r|^2(n-1)+1)\Delta_2}.$$

(ii) Let $|r| < 1$. From Lemma 2.1 and Eq. (8), we have

$$\begin{aligned}\|Circ_r(D^2L)\|_F^2 &\geq n|r|^2 + \sum_{t=0}^{n-4} n|r|^2 L_t^2 + n|r|^2(L_{n-1}-1)^2 + n|r|^2(L_n-3)^2 \\ &= n|r|^2(L_{n-4}L_{n-3} + (L_{n-1}-1)^2 + (L_n-3)^2 + 3).\end{aligned}$$

It follows that

$$\frac{1}{\sqrt{n}}\|Circ_r(D^2L)\|_F \geq |r|\sqrt{L_{n-4}L_{n-3} + (L_{n-1}-1)^2 + (L_n-3)^2 + 3}.$$

From Eq. (9), we can write

$$\|Circ_r(D^2L)\|_2 \geq |r|\sqrt{\Delta_2}. \tag{16}$$

On the other hand, since $Circ_r(D^2L) = C \circ D$ for the matrices C and D defined as in part (i), we get

$$r_1(C) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2} = \sqrt{n}$$

and

$$c_1(D) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|d_{ij}|^2} = \sqrt{1 + \sum_{k=0}^{n-4} L_k^2 + (L_{n-1}-1)^2 + (L_n-3)^2} = \sqrt{\Delta_2}.$$

From Lemma 1.1, we obtain

$$\|Circ_r(D^2L)\|_2 \leq \sqrt{n\Delta_2}. \tag{17}$$

Considering Eqs. (16) and (17), we have

$$|r|\sqrt{\Delta_2} \leq \|Circ_r(D^2L)\|_2 \leq \sqrt{n\Delta_2}.$$

□

Now, we consider the circulant matrices

$$Circ(DL) = \begin{pmatrix} L_0 & L_1 & L_2 & \dots & L_{n-2} & L_n - 1 \\ L_n - 1 & L_0 & L_1 & \dots & L_{n-3} & L_{n-2} \\ L_{n-2} & L_n - 1 & L_0 & \dots & L_{n-4} & L_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_2 & L_3 & L_4 & \dots & L_0 & L_1 \\ L_1 & L_2 & L_3 & \dots & L_n - 1 & L_0 \end{pmatrix}$$

and

$$Circ(D^2L) = \begin{pmatrix} 1 & L_0 & L_1 & \dots & L_{n-1} - 1 & L_n - 3 \\ L_n - 3 & 1 & L_0 & \dots & L_{n-4} & L_{n-1} - 1 \\ L_{n-1} - 1 & L_n - 3 & 1 & \dots & L_{n-5} & L_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_1 & L_2 & L_3 & \dots & 1 & L_0 \\ L_0 & L_1 & L_2 & \dots & L_n - 3 & 1 \end{pmatrix}.$$

Theorem 2.4. *For the Frobenius norms of the matrices* $Circ(DL)$ *and* $Circ(D^2L)$*, we have*

$$||Circ(DL)||_F^2 - ||Circ(D^2L)||_F^2 = n(L_n^2 - 5L_{2n-3} - 2L_n + 10F_{n+1} - 10),$$

where F_{n+1} *is the* $(n+1)$*th Fibonacci number, and* L_n *is the nth Lucas number.*

Proof. By virtue of the definition of the Frobenius norm, we have

$$||Circ(DL)||_F^2 = n\left(\sum_{k=0}^{n-2} L_k^2 + (L_n - 1)^2\right) \tag{18}$$

and

$$||Circ(D^2L)||_F^2 = n\left(1 + \sum_{k=0}^{n-4} L_k^2 + (L_{n-1} - 1)^2 + (L_n - 3)^2\right). \tag{19}$$

Then, from Eqs. (1), (2) and Eqs. (4)-(8), we get

$$\begin{aligned}||Circ(DL)||_F^2 - ||Circ(D^2L)||_F^2 &= n\left(\sum_{k=0}^{n-2} L_k^2 + (L_n - 1)^2 - 1 - \sum_{k=0}^{n-4} L_k^2 - (L_{n-1} - 1)^2 - (L_n - 3)^2\right)\\ &= n(L_{n-2}L_{n-1} - L_{n-4}L_{n-3} - L_{n-1}^2 + 4L_n + 2L_{n-1} - 10)\\ &= n(L_n^2 + L_{2n-3} - L_{2n-7} - 2L_n - 5F_{2n-1} + 10F_{n+1} - 10)\\ &= n(L_n^2 - 2L_n + 5F_{2n-5} - 5F_{2n-1} + 10F_{n+1} - 10)\\ &= n(L_n^2 - 5L_{2n-3} - 2L_n + 10F_{n+1} - 10).\end{aligned}$$

So the proof is completed. □

Theorem 2.5. *For the* l_p *norms of the matrices* $Circ(DL)$ *and* $Circ(D^2L)$*, we have*

$$||Circ(DL)||_p^p - ||Circ(D^2L)||_p^p = n(L_{n-3}^p + L_{n-2}^p + (L_n - 1)^p - (L_{n-1} - 1)^p - (L_n - 3)^p - 1).$$

Proof. By virtue of the definition of the l_p norm, we have

$$||Circ(DL)||_p^p = n\left(\sum_{k=0}^{n-2} L_k^p + (L_n - 1)^p\right) \tag{20}$$

and

$$||Circ(D^2L)||_p^p = n\left(1 + \sum_{k=0}^{n-4} L_k^p + (L_{n-1} - 1)^p + (L_n - 3)^p\right). \tag{21}$$

Using Eqs. (20) and (21), we get

$$\begin{aligned}||Circ(DL)||_p^p - ||Circ(D^2L)||_p^p &= n\left(\sum_{k=0}^{n-2} L_k^p + (L_n - 1)^p - \sum_{k=0}^{n-4} L_k^p - (L_{n-1} - 1)^p - (L_n - 3)^p - 1\right)\\ &= n\left(\sum_{k=0}^{n-4} L_k^p + L_{n-3}^p + L_{n-2}^p + (L_n - 1)^p - \sum_{k=0}^{n-4} L_k^p - (L_{n-1} - 1)^p - (L_n - 3)^p - 1\right)\\ &= n(L_{n-3}^p + L_{n-2}^p + (L_n - 1)^p - (L_{n-1} - 1)^p - (L_n - 3)^p - 1).\end{aligned}$$

□

Theorem 2.6. *For circulant matrices $Circ(DL)$ and $Circ(D^2L)$, we have*

$$|\det Circ(DL)| \leq \left(L_n^2 + L_{2n-3} - 2L_n + (-1)^n + 3\right)^{\frac{n}{2}}$$

and

$$|\det Circ(D^2L)| \leq (L_{2n-7} + 5F_{2n-1} - 10F_{n+1} + (-1)^n + 13)^{\frac{n}{2}}.$$

Proof. Considering Lemma 1.2 and Eqs. (7), (8), and (18), we get

$$\begin{aligned}|\det Circ(DL)| &\leq \frac{1}{n^{\frac{n}{2}}}\|Circ(DL)\|_F^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left(\sqrt{n\left(\sum_{k=0}^{n-2} L_k^2 + (L_n - 1)^2\right)}\right)^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left(\sqrt{n\left(L_n^2 + L_{2n-3} - 2L_n + (-1)^n + 3\right)}\right)^n \\ &= \left(L_n^2 + L_{2n-3} - 2L_n + (-1)^n + 3\right)^{\frac{n}{2}}.\end{aligned}$$

On the other hand, considering Lemma 1.2, Eqs. (4)-(8), and Eq. (19), we get

$$\begin{aligned}|\det Circ(D^2L)| &\leq \frac{1}{n^{\frac{n}{2}}}\|Circ(D^2L)\|_F^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left(\sqrt{n\left(1 + \sum_{k=0}^{n-4} L_k^2 + (L_{n-1} - 1)^2 + (L_n - 3)^2\right)}\right)^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left(\sqrt{n\,(L_{2n-7} + 5F_{2n-1} - 10F_{n+1} + (-1)^n + 13)}\right)^n \\ &= (L_{2n-7} + 5F_{2n-1} - 10F_{n+1} + (-1)^n + 13)^{\frac{n}{2}}.\end{aligned}$$

□

Example 2.7. *Let the Lucas sequence be $L = (L_1, L_2, L_3, L_4) = (1, 3, 4, 7)$ 4-tuple. Then*

$$DL = D(1, 3, 4, 7) = (|1 - 3|, |3 - 4|, |4 - 7|, |7 - 1|) = (2, 1, 3, 6)$$

and

$$D^2L = D(DL) = D(2, 1, 3, 6) = (|2 - 1|, |1 - 3|, |3 - 6|, |6 - 2|) = (1, 2, 3, 4).$$

For considering circulant matrices

$$Circ(DL) = \begin{pmatrix} 2 & 1 & 3 & 6 \\ 6 & 2 & 1 & 3 \\ 3 & 6 & 2 & 1 \\ 1 & 3 & 6 & 2 \end{pmatrix} \quad \text{and} \quad Circ(D^2L) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

it is easy to obtain that $\det Circ(DL) = -624$ *and* $\det Circ(D^2L) = -160$.

On the other hand,

$$\frac{1}{16}\|Circ(DL)\|_F^4 = \left(L_4^2 + L_5 - 2L_4 + (-1)^4 + 3\right)^2 = (49 + 11 - 14 + 1 + 3)^2 = 2500$$

and

$$\frac{1}{16}||Circ(D^2L)||_F^4 = \left(L_1 + 5F_7 - 10F_5 + (-1)^4 + 13\right)^2 = (1 + 65 - 50 + 1 + 13)^2 = 900.$$

Clearly, $|-624| < 2500$ *and* $|-160| < 900$. *Thus,* $|\det Circ(DL)| < \frac{1}{16}||Circ(DL)||_F^4$ *and* $|\det Circ(D^2L)| < \frac{1}{16}||Circ(D^2L)||_F^4$.

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