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A degree approach to fuzzy convexity in vector subspaces

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Abstract.
Based on a completely distributive lattice L, the degree to which an L-subset in a vector space is an L-fuzzy subspace is introduced via the implication operation on L. By using four kinds of cut sets, several characterizations of L-fuzzy subspace degree are given. Further, it is shown that the L-fuzzy subspace degree of a vector space can induce an L-fuzzy convexity in a natural way and the linear mappings between vector spaces are L-fuzzy convexity-preserving and L-fuzzy convex-to-convex mappings between the induced L-fuzzy convex spaces.

1. Introduction

Convexity as an important mathematical property originated from Euclidean spaces. The concept of convexity was substantially defined and studied in $\mathbb{R}^n$ in the works of the pioneers like Newton and Minkowski as described in [1, 3, 28]. Generally, a set in an n-dimensional Euclidean space is convex if and only if it contains all the segments which join each two of its points. Due to the extensional existence of convex sets, convexities could also be found in many other mathematical structures such as vector spaces, lattices, graphs, matroids, median algebras and so on [2, 25–27, 31].

Motivated by the axiomatic approach, the concept of convex structures (also called convexities) was proposed. Concretely, a convexity on a set X is defined as a family $\mathbb{C}$ of subsets of X, when it contains both X itself and the empty set $\emptyset$ and it is closed under arbitrary intersections and nested unions. The members of $\mathbb{C}$ are called convex sets [28]. So this kind of convex set exists with respect to a convex structure. In this sense, more different types of convex sets emerge in mathematical environments. For example, a subspace of a vector space can be treated as a convex set which is different from the standard convex set since the family of all subspaces of a vector space is exactly a convexity on the vector space.

Since Zadeh introduced fuzzy sets, many mathematical structures have been endowed with fuzzy set theory [4, 7, 8, 10, 11, 37], which leads to different types of fuzzy convexities. Rosa [20] first introduced the concept of fuzzy convex spaces with the real unit interval [0,1] as the lattice background. Later, Maruyama [13] generalized the interval [0,1] to a completely distributive lattice L and proposed the concept of L-fuzzy convexity spaces. These two kinds of fuzzy convex structures are called L-convex structures nowadays [6, 16, 35, 38]. In recent years, a new approach to the fuzzification of convex spaces was introduced in

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[22], known as M-fuzzifying convex spaces, in which each subset of X can be seen as a convex set to some degree. Up to now, M-fuzzifying convex spaces have deserved more and more attention [30, 32–34]. In 2017, Shi and Xiu [23] extended convexity to a more general case, known as (L, M)-fuzzy convex spaces [14, 15, 36]. Note that (L, M)-fuzzy convex spaces can include L-convex spaces and M-fuzzifying convex spaces as special cases.

Katsaras and Liu [10] first generalized the concept of vector spaces to the fuzzy case. Afterwards, many scholars developed the theory of fuzzy vector spaces [8, 11, 12]. In [39], Zhong and Shi proposed the concept of an L-fuzzy subspace of a vector space. In this sense, an L-fuzzy set in a vector space is either an L-fuzzy subspace or not. It lacks "fuzziness" to some extent. By this motivation, we will consider equipping each L-subset in a vector space with some degree to become an L-fuzzy subspace. Since a subset of a vector space can be treated as a convex subset from the axiomatic aspect, we will treat the degree to which an L-subset of a vector space becomes an L-fuzzy subspace as a "convexity degree". Following this trend, we will construct an L-fuzzy convexity in a natural way from a vector space. In other words, there exists naturally an L-fuzzy convexity in a vector space. The aim of this paper is to define the degree to which an L-subset in a vector space is an L-fuzzy subspace via the implication operation on the lattice background L. Then an L-fuzzy convexity on a vector space is naturally constructed. Also, the relationships between linear mapping, L-fuzzy convexity-preserving and L-fuzzy convex-to-convex mappings are discussed.

2. Preliminaries

Throughout this paper, unless otherwise stated, L denotes a completely distributive lattice. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. Let X be a vector space over the real field $\mathbb{K}$. The family of all L-subsets on X is denoted by L^X. L^X is also a complete lattice when it inherits the structure of the lattice L in a pointwise way. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X, respectively. Also, we adopt the convention that $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \perp$.

An element a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant c$ or $a \leqslant b$ [5]. The set of non-zero co-prime elements in L is denoted by $J(L)$. An element a in L is called prime if $b \wedge c \leqslant a$ implies $c \leqslant a$ or $b \leqslant a$. The set of non-unit prime elements in L is denoted by $P(L)$.

For each $a, b \in L$, we say that a is wedge below b in L, in symbols $a \prec b$, if for every subset $D \subseteq L$, $b \leqslant \bigvee D$ implies $a \leqslant d$ for some $d \in D$. The set $\{a \in L \mid a \prec b\}$, denoted by $\beta(b)$, is called the greatest minimal family of b in the sense of [29]. Dually, we can define a binary relation $\prec^{op}$ as follows: for all $a, b \in L$, $a \prec^{op} b$ if and only if for every subset $D \subseteq L$, $\bigwedge D \leqslant a$ implies $d \leqslant b$ for some $d \in D$. The set $\{b \in M \mid a \prec^{op} b\}$, denoted by $\alpha(a)$, is called the greatest maximal family of a in the sense of [29]. For convenience, let $\beta^*(b) = \beta(b) \cap J(L)$. and $\alpha^*(b) = \alpha(b) \cap P(L)$. A complete lattice L is a completely distributive if and only if $b = \bigvee \beta(b)$ (resp. $b = \bigvee \beta^\star(b)$) for each $b \in L$ if and only if $a = \bigwedge \alpha(a)$ (resp. $a = \bigwedge \alpha^\star(a)$) for each $a \in L$.

Theorem 2.1. ([29]) *Let* $\{a_i \mid i \in \Omega\} \subseteq L$. *Then*

(1) $\alpha(\bigwedge\limits_{i\in\Omega} a_i) = \bigcup\limits_{i\in\Omega} \alpha(a_i)$.

(2) $\beta(\bigvee\limits_{i\in\Omega} a_i) = \bigcup\limits_{i\in\Omega} \beta(a_i)$.

In [21], Shi introduced four kinds of cut sets of an L-subset as a basic theoretical tool to study fuzzy set theory.

Definition 2.2. ([21]) Let $\mu \in L^X$ and $a \in L$. We define

(1) $\mu_{[a]} = \{x \in X \mid a \leqslant \mu(x)\}$.

(2) $\mu_{(a)} = \{x \in X \mid a \in \beta(\mu(x))\}$.

(3) $\mu^{(a)} = \{x \in X \mid \mu(x) \nleqslant a\}$.

(4) $\mu^{[a]} = \{x \in X \mid a \notin \alpha(\mu(x))\}$.

Lemma 2.3. Let $x, y \in L$. Then the following statements are equivalent:

(1) $x \leqslant y$;

(2) for each $a \in J(L)$, $a \leqslant x \Rightarrow a \leqslant y$;

(3) for each $a \in \alpha(\bot)$, $a \notin \alpha(x) \Rightarrow a \notin \alpha(y)$;

(4) for each $a \in P(L)$, $x \nleqslant a \Rightarrow y \nleqslant a$;

(5) for each $a \in \beta(\top)$, $a \in \beta(x) \Rightarrow a \in \beta(y)$.

Proof. It is trivial and omitted here. □

In a completely distributive lattice L, there exists an implication operator

$$\rightarrow: L \times L \longrightarrow L.$$

The operator is the right adjoint for the meet operation $\wedge$ by

$$a \rightarrow b = \bigvee\{c \in L \mid a \wedge c \leqslant b\}.$$

We list some properties of the implication operation in the following lemma.

Lemma 2.4. ([7]) *For all $a, b, c \in L$, $\{a_i\}_{i \in I} \subseteq L$, the following statements hold:*

(1) $\top \rightarrow a = a$;

(2) $c \leqslant a \rightarrow b \Leftrightarrow a \wedge c \leqslant b$;

(3) $a \rightarrow b = \top \Leftrightarrow a \leqslant b$;

(4) $a \rightarrow (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(a \rightarrow a_i)$, *hence* $a \rightarrow b \leqslant a \rightarrow c$ *whenever* $b \leqslant c$;

(5) $(\bigvee_{i \in I} a_i) \rightarrow b = \bigwedge_{i \in I}(a_i \rightarrow b)$, *hence* $b \rightarrow c \leqslant a \rightarrow c$ *whenever* $a \leqslant b$;

(6) $(a \rightarrow c) \wedge (c \rightarrow b) \leqslant a \rightarrow b$;

(7) $(a \rightarrow b) \wedge (c \rightarrow d) \leqslant a \wedge c \rightarrow b \wedge d$.

In what follows, we recall the concept of (L, M)-fuzzy convexities.

Definition 2.5. ([23]) A mapping $\mathcal{C} : L^X \longrightarrow M$ is called an (L, M)-*fuzzy convexity* on X if it satisfies the following conditions:

(LMC1) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$;

(LMC2) if $\{A_i \mid i \in \Omega\} \subseteq L^X$ is nonempty, then $\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \leqslant \mathcal{C}(\bigwedge_{i \in \Omega} A_i)$;

(LMC3) if $\{A_i \mid i \in \Omega\} \subseteq L^X$ is nonempty and totally ordered, then $\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \leqslant \mathcal{C}(\bigvee_{i \in \Omega} A_i)$.

For an (L, M)-*fuzzy convexity* on X, the pair $(X, \mathcal{C})$ is called an (L, M)-*fuzzy convex space.* An (L, L)-fuzzy convex space is called an *L-fuzzy convex space* for short.

Let 2 denote the two element chain, that is, $2 = \{0, 1\}$. Then an $(L, 2)$-fuzzy convexity is an *L-convexity* in [13, 17, 18], an $(I, 2)$-fuzzy convexity is a *fuzzy convexity* in [20], where I=[0,1], a $(2, M)$-fuzzy convexity is an *M-fuzzifying convexity* in [22], a $(2, 2)$-fuzzy convexity is a *convexity* in [9, 24, 28].

Given a mapping $f : X \longrightarrow Y$, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ [19] by

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x), \;\; f_L^{\leftarrow}(B) = B \circ f$$

for all $A \in L^X$, $y \in Y$, $B \in L^Y$ and $x \in X$.

Definition 2.6. ([39]) Let X be a vector space over $\mathbb{K}$. $\mu \in L^X$ is called an L-fuzzy subspace of X if for all $x, y \in X$ and for all $k, l \in \mathbb{K}$, $\mu(kx + ly) \geqslant \mu(x) \wedge \mu(y)$.

3. Degrees of *L*-fuzzy subspaces

According to Definition 2.6, we know that an L-subset μ of a vector space X is either an L-fuzzy subspace or not. In this sense, there is no fuzziness to describe this notion. Adhering to the essential idea of fuzzy set theory, it is necessary to endow a "fuzziness degree" with an L-fuzzy subspace.

By means of the implication operation on L, we will equip an L-subset in a vector space some "fuzziness degree" to characterize the degree to which an L-subset in a vector space is an L-fuzzy subspace.

Definition 3.1. Let X be a vector space over $\mathbb{K}$ and $\mu \in L^X$. The degree to which μ is an L-fuzzy subspace on X (or called the L-fuzzy subspace degree of μ) is defined as follows:

$$\mathcal{D}(\mu) = \bigwedge_{x,y \in X, k,l \in \mathbb{K}} \left(\mu(x) \wedge \mu(y) \rightarrow \mu(kx + ly)\right).$$

Remark 3.2. It is obvious that $\mathcal{D}(\mu) = \top$ if and only if μ is an L-fuzzy subspace on X.

Example 3.3. Let $X = \mathbb{K}$ and define $\mu \in [0, 1]^{\mathbb{K}}$ as follows:

$$\mu(x) = \begin{cases} 0.7, & \text{if } x \in \mathbb{K}, x \neq 0, \\ 0.4, & \text{if } x = 0 \in \mathbb{K}. \end{cases}$$

Then it is easy to check that μ is not a fuzzy subspace of $\mathbb{K}$, but we can easily compute that $\mathcal{D}(\mu) = 0.4$.

Lemma 3.4. *Let X be a vector space over $\mathbb{K}$. Then for each $\mu \in L^X$ and $a \in L$, $a \leqslant \mathcal{D}(\mu)$ if and only if for all $x, y \in X$ and for each $k, l \in \mathbb{K}$, $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$.*

Proof. By Lemma 2.4(2), it is straightforward. □

Theorem 3.5. *Let X be a vector space over $\mathbb{K}$. Then for each $\mu \in L^X$, we have*

$$\mathcal{D}(\mu) = \bigvee \left\{a \in L \,\middle|\, \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly), \forall x, y \in X, \forall k, l \in \mathbb{K}\right\}.$$

Proof. By Lemma 3.4, it is obvious. □

In what follows, we will provide some characterizations of the L-fuzzy subspace degree via four kinds of cut sets.

Theorem 3.6. *Let X be a vector space over $\mathbb{K}$ and $\mu \in L^X$. Then*

$$\mathcal{D}(\mu) = \bigvee \left\{a \in L \mid \forall b \leqslant a, \mu_{[b]} \text{ is a subspace of } X\right\}.$$

Proof. Take any $a \in L$ such that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$ for any $x, y \in X$ and for any $k, l \in \mathbb{K}$. For any $b \leqslant a$ and $x, y \in \mu_{[b]}$, we have $\mu(kx + ly) \geqslant \mu(x) \wedge \mu(y) \wedge a \geqslant b \wedge a = b$, i.e., $kx + ly \in \mu_{[b]}$. Hence $\mu_{[b]}$ is a subspace of X. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \leqslant \bigvee \{a \in L \mid \forall b \leqslant a, \mu_{[b]} \text{ is a subspace of } X\}.$$

Conversely, suppose that $\mu_{[b]}$ is a subspace of X for any $b \leqslant a$. It suffices to verify that for all $x, y \in X$ and for all $k, l \in \mathbb{K}$, $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$.

Suppose that $b \in L$ with $b \leqslant \mu(x) \wedge \mu(y) \wedge a$. Then $b \leqslant \mu(x)$, $b \leqslant \mu(y)$ and $b \leqslant a$. So $x, y \in \mu_{[b]}$. Since $\mu_{[b]}$ is a subspace of X for any $b \leqslant a$, it follows that $kx + ly \in \mu_{[b]}$, i.e., $\mu(kx + ly) \geqslant b$. By the arbitrariness of b, we can obtain $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \geqslant \bigvee \{a \in L \mid \forall b \leqslant a, \mu_{[b]} \text{ is a subspace of } X\}.$$

□

Theorem 3.7. *Let X be a vector space over $\mathbb{K}$ and $\mu \in L^X$. Then*

$$\mathcal{D}(\mu) = \bigvee \left\{a \in L \,\middle|\, \forall b \in P(L), a \nleqslant b, \mu^{(b)} \textit{ is a subspace of } X\right\}.$$

Proof. Take any $a \in L$ such that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$ for any $x, y \in X$ and for any $k, l \in \mathbb{K}$. For each $b \in P(L)$ with $a \nleqslant b$ and for all $x, y \in \mu^{(b)}$, we have $\mu(x) \wedge \mu(y) \wedge a \nleqslant b$ by $\mu(x) \nleqslant b$ and $\mu(y) \nleqslant b$. Since $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$, it follows that $\mu(kx + ly) \nleqslant b$, i.e., $kx + ly \in \mu^{(b)}$. Hence $\mu^{(b)}$ is a subspace of X. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \leqslant \bigvee \left\{a \in L \,\middle|\, \forall b \in P(L), a \nleqslant b, \mu^{(b)} \text{ is a subspace of } X\right\}.$$

Conversely, suppose that $\mu^{(b)}$ is a subspace of X for any $b \in P(L)$ with $a \nleqslant b$. It remains to verify that for all $x, y \in X$ and for all $k, l \in \mathbb{K}$, $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$.

Suppose that $b \in P(L)$ with $\mu(x) \wedge \mu(y) \wedge a \nleqslant b$. Then $\mu(x) \nleqslant b$, $\mu(y) \nleqslant b$ and $a \nleqslant b$. So $x, y \in \mu^{(b)}$. Since $\mu^{(b)}$ is a subspace of X for any $b \in P(L)$ with $a \nleqslant b$, it follows that $kx + ly \in \mu^{(b)}$, i.e., $\mu(kx + ly) \nleqslant b$. By the arbitrariness of b, we can obtain $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \geqslant \bigvee \left\{a \in L \,\middle|\, \forall b \in P(L), a \nleqslant b, \mu^{(b)} \text{ is a subspace of } X\right\}.$$

□

Theorem 3.8. *Let X be a vector space over $\mathbb{K}$ and $\mu \in L^X$. Then*

$$\mathcal{D}(\mu) = \bigvee \left\{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \textit{ is a subspace}\right\}.$$

Proof. Take any $a \in L$ such that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$ for any $x, y \in X$ and for any $k, l \in \mathbb{K}$. For any $b \notin \alpha(a)$ and for all $x, y \in \mu^{[b]}$, i.e., $b \notin \alpha(\mu(x))$, $b \notin \alpha(\mu(y))$, we have

$$b \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a).$$

By $\alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a) = \alpha(\mu(x) \wedge \mu(y) \wedge a)$ we obtain $b \notin \alpha(\mu(x) \wedge \mu(y) \wedge a)$. Further, because $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$, we can know that $b \notin \alpha(\mu(kx + ly))$, i.e., $kx + ly \in \mu^{[b]}$. Hence $\mu^{[b]}$ is a subspace of X. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \leqslant \bigvee \left\{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subspace}\right\}.$$

Conversely, suppose that $\mu^{[b]}$ is a subspace of X for all $b \notin \alpha(a)$. It suffices to prove that for all $x, y \in X$ and for all $k, l \in \mathbb{K}$, $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$.

Suppose that $b \in L$ with $b \notin \alpha(\mu(x) \wedge \mu(y) \wedge a)$. We obtain $b \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a)$ by $\alpha(\mu(x) \wedge \mu(y) \wedge a) = \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a)$. Then $b \notin \alpha(\mu(x))$, $b \notin \alpha(\mu(y))$ and $b \notin \alpha(a)$. So $x, y \in \mu^{[b]}$. Since $\mu^{[b]}$ is a subspace of X, it follows that $kx + ly \in \mu^{[b]}$, i.e. $, b \notin \alpha(\mu(kx + ly))$. By the arbitrariness of b, we have $\alpha(\mu(kx + ly)) \subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a)$. Then it follows that

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &= \bigwedge \alpha(\mu(x) \wedge \mu(y) \wedge a) \\ &\leqslant \bigwedge \alpha(\mu(kx + ly)) \\ &= \mu(kx + ly). \end{aligned}$$

By Theorem 3.5, we know $\mathcal{D}(\mu) \geqslant \bigvee\{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subspace}\}$. □

Theorem 3.9. *Let X be a vector space over $\mathbb{K}$. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for all $a, b \in L$, then for each $\mu \in L^X$, we have*

$$\mathcal{D}(\mu) = \bigvee\{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{ is a subspace}\}.$$

Proof. Take any $a \in L$ such that $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx+ly)$ for any $x, y \in X$ and $k, l \in \mathbb{K}$. For any $b \in \beta(a)$ and for all $x, y \in \mu_{(b)}$, we have $b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a)$ by $b \in \beta(\mu(x))$ and $b \in \beta(\mu(y))$. We obtain $b \in \beta(\mu(x) \wedge \mu(y) \wedge a)$ by $\beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a) = \beta(\mu(x) \wedge \mu(y) \wedge a)$. Since $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$, it follows that

$$b \in \beta(\mu(kx + by)), \text{ i.e., } kx + ly \in \mu_{(b)}.$$

Hence $\mu_{(b)}$ is a subspace of X. By Theorem 3.5, we know

$$\mathcal{D}(\mu) \leqslant \bigvee\{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{ is a subspace}\}.$$

Conversely, suppose that $\mu_{(b)}$ is a subspace of X for any $b \in \beta(a)$. It remains to prove that for all $x, y \in X$ and for all $k, l \in \mathbb{K}$, $\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(kx + ly)$.

Suppose that $b \in L$ with $b \in \beta(\mu(x) \wedge \mu(y) \wedge a)$. We have $b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a)$ by $\beta(\mu(x) \wedge \mu(y) \wedge a) = \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a)$. Then $b \in \beta(\mu(x))$, $b \in \beta(\mu(y))$ and $b \in \beta(a)$. So we have $x, y \in \mu_{(b)}$. Since $\mu_{(b)}$ is a subspace of X for any $b \in \beta(a)$, it follows that $kx + ly \in \mu_{(b)}$, i.e., $b \in \beta(\mu(kx + ly))$. By the arbitrariness of b, we have $\beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(kx + ly))$ and

$$\begin{aligned} \mu(x) \wedge \mu(y) \wedge a &= \bigvee \beta(\mu(x) \wedge \mu(y) \wedge a) \\ &\leqslant \bigvee \beta(\mu(kx + ly)) \\ &= \mu(kx + ly). \end{aligned}$$

By Theorem 3.5, we know $\mathcal{D}(\mu) \geqslant \bigvee\{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{ is a subspace}\}$. □

4. *L*-fuzzy convexity induced by *L*-fuzzy subspace degree

In the classical case, all the subspaces of a vector space consist a convexity on the vector space. In the above section, we considered the degree to which an L-subset of a vector space becomes a subspace. This motivates us to consider the relationships between the L-fuzzy subspace degrees and fuzzy convexities. In this section, by means of an L-fuzzy subspace degree, we construct an L-fuzzy convexity on a vector space. Furthermore, we shall explore its corresponding L-fuzzy convexity preserving mapping and L-fuzzy convex-to-convex mapping.

By the definition of $\mathcal{D}(\mu)$, we know that $\mathcal{D}$ can be naturally considered as a mapping $\mathcal{D} : L^X \longrightarrow L$ defined by $\mu \longmapsto \mathcal{D}(\mu)$. The following theorem shows that $\mathcal{D}$ is an L-fuzzy convexity on X.

Theorem 4.1. *Let X be a vector space over $\mathbb{K}$. Then the mapping $\mathcal{D}$: $L^X \longrightarrow L$ defined by $\mu \longmapsto \mathcal{D}(\mu)$ is an L-fuzzy convexity on X, which is called the L-fuzzy convexity induced by L-fuzzy subspace degree on X.*

Proof. It is enough to show that $\mathcal{D}$ satisfies (LMC1)–(LMC3).

(LMC1) It is straightforward.

(LMC2) Take any nonempty subfamily $\{A_i \mid i \in \Omega\} \subseteq L^X$ and $a \in L$ such that $a \leqslant \bigwedge_{i\in\Omega} \mathcal{D}(A_i)$. By Lemma 3.4, for all $i \in \Omega$, we have $A_i(x) \wedge A_i(y) \wedge a \leqslant A_i(kx + ly)$ for all $x, y \in X$ and for each $k, l \in \mathbb{K}$. This implies

$$\begin{aligned}\left(\bigwedge_{i\in\Omega} A_i(x)\right) \wedge \left(\bigwedge_{j\in\Omega} A_j(y)\right) \wedge a &\leqslant \bigwedge_{i\in\Omega}(A_i(x) \wedge A_i(y) \wedge a)\\ &\leqslant \bigwedge_{i\in\Omega} A_i(kx + ly).\end{aligned}$$

By Lemma 3.4, we have $a \leqslant \mathcal{D}(\bigwedge_{i\in\Omega} A_i)$. By the arbitrariness of a, we obtain $\bigwedge_{i\in\Omega} \mathcal{D}(A_i) \leqslant \mathcal{D}(\bigwedge_{i\in\Omega} A_i)$.

(LMC3) Take any nonempty and totally ordered subfamily $\{A_i \mid i \in \Omega\} \subseteq L^X$ and $a \in L$ such that $a \leqslant \bigwedge_{i\in\Omega} \mathcal{D}(A_i)$. By Lemma 3.4, for all $i \in \Omega$, we have $A_i(x) \wedge A_i(y) \wedge a \leqslant A_i(kx + ly)$ for all $x, y \in X$ and $k, l \in \mathbb{K}$. Take any $b \in J(L)$ such that

$$b \prec \left(\bigvee_{i\in\Omega} A_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} A_i(y)\right) \wedge a.$$

Then we have $b \prec \bigvee_{i\in\Omega} A_i(x)$, $b \prec \bigvee_{i\in\Omega} A_i(y)$ and $b \leqslant a$. Hence there exists some $i, j \in \Omega$ such that $b \leqslant A_i(x)$, $b \leqslant A_j(y)$ and $b \leqslant a$. Since $\{A_i \mid i \in \Omega\}$ is totally ordered, we assume $A_j \leqslant A_i$. Then it follows that $b \leqslant A_i(x) \wedge A_i(y) \wedge a$. By $A_i(x) \wedge A_i(y) \wedge a \leqslant A_i(kx + ly)$, we obtain $b \leqslant A_i(kx + ly)$. Hence $b \leqslant \bigvee_{i\in\Omega} A_i(kx + ly)$. By the arbitrariness of b, we have

$$\left(\bigvee_{i\in\Omega} A_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} A_i(y)\right) \wedge a \leqslant \bigvee_{i\in\Omega} A_i(kx + ly).$$

Combining Lemma 3.4, we have $a \leqslant \mathcal{D}\left(\bigvee_{i\in\Omega} A_i\right)$. By the arbitrariness of a, we obtain

$$\bigwedge_{i\in\Omega} \mathcal{D}(A_i) \leqslant \mathcal{D}\left(\bigvee_{i\in\Omega} A_i\right).$$

This shows that $\mathcal{D}$ is an L-fuzzy convexity on X. □

Example 4.2. Let $X = \{a, b, c\}$ be a vector space with a field $\mathbb{K}$ and $L = \{0, \frac{1}{2}, 1\}$. For each $\mu \in L^X$, denote its range by $\mathrm{ran}(\mu)$. By the formula

$$\mathcal{D}(\mu) = \bigwedge_{x,y\in X,k,l\in\mathbb{K}} (\mu(x) \wedge \mu(y) \to \mu(kx + ly)),$$

we can compute the values of $\mathcal{D}(\mu)$ according to $\mathrm{ran}(\mu)$ as follows:

$$\mathcal{D}(\mu) = \begin{cases} 0, & \text{if } \mathrm{ran}(\mu) = \{0, \frac{1}{2}\} \text{ or } \{0, 1\} \text{ or } \{0, \frac{1}{2}, 1\},\\ \frac{1}{2}, & \text{if } \mathrm{ran}(\mu) = \{\frac{1}{2}, 1\},\\ 1, & \text{if } \mathrm{ran}(\mu) = \{0\} \text{ or } \{\frac{1}{2}\} \text{ or } \{1\}.\end{cases}$$

Then $\mathcal{D}$ is an L-fuzzy convexity on X.

Convexity-preserving mappings play an important role in the theory of convexity theory. In the framework of fuzzy convexities, Shi and Xiu [23] introduced the notion of (L, M)-fuzzy convexity-preserving mappings.

Definition 4.3. ([23]) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. A mapping $f : X \longrightarrow Y$ is called an (L, M)-*fuzzy convexity-preserving mapping* provided $\mathcal{D}(\mu) \leqslant \mathcal{C}(f_L^{\leftarrow}(\mu))$ for all $\mu \in L^Y$.

An (L, L)-fuzzy convexity-preserving mapping is called an L-*fuzzy convexity-preserving mapping* for short.

An $(L, 2)$-fuzzy convexity preserving mapping is exactly an L-*convexity preserving mapping* in [13].

Linear mappings are important to establish the relationships between vector spaces, while L-fuzzy convexity-preserving mappings are important to establish the relationships between L-fuzzy convex spaces. This motivates us to consider the relationships between linear mappings and L-fuzzy convexity-preserving mappings.

Theorem 4.4. *Let X and Y be two vector spaces over $\mathbb{K}$ and $\mathcal{D}_X$ and $\mathcal{D}_Y$ be the L-fuzzy convexities induced by L-fuzzy subspace degrees on X and Y, respectively. If $f : X \longrightarrow Y$ is a linear mapping between vector spaces, then $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is an L-fuzzy convexity-preserving mapping.*

Proof. Take any $a \in L$ such that $a \leqslant \mathcal{D}_Y(\mu)$. That is,

$$a \leqslant \bigwedge_{y_1, y_2 \in Y, k, l \in \mathbb{K}} \left(\mu(y_1) \wedge \mu(y_2) \to \mu(ky_1 + ly_2)\right).$$

By Lemma 3.4, we have that for any $y_1, y_2 \in Y$ and $k, l \in \mathbb{K}$, $\mu(y_1) \wedge \mu(y_2) \wedge a \leqslant \mu(ky_1 + ly_2)$. Note that f is a linear mapping, we have that for all $x_1, x_2 \in X$,

$$\begin{aligned} f_L^{\leftarrow}(\mu)(x_1) \wedge f_L^{\leftarrow}(\mu)(x_2) \wedge a &= \mu(f(x_1)) \wedge \mu(f(x_2)) \wedge a \\ &\leqslant \mu(kf(x_1) + lf(x_2)) \\ &= \mu(f(kx_1 + lx_2)) \\ &= f_L^{\leftarrow}(\mu)(kx_1 + lx_2). \end{aligned}$$

By Lemma 3.4, we have $a \leqslant \mathcal{D}_X(f_L^{\leftarrow}(\mu))$. By the arbitrariness of a, we obtain $\mathcal{D}_Y(\mu) \leqslant \mathcal{D}_X(f_L^{\leftarrow}(\mu))$, which shows that $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is an L-fuzzy convexity-preserving mapping. □

In the classical case, if $f : X \longrightarrow Y$ is a surjective and linear mapping between vector spaces, then Z is a subspace of Y if and only if $f^{\leftarrow}(Z) = \{x \mid f(x) \in Z\}$ is a subspace of X. Now we give its fuzzy counterpart.

Theorem 4.5. *Let X and Y be two vector spaces over $\mathbb{K}$ and $\mathcal{D}_X$ and $\mathcal{D}_Y$ be the L-fuzzy convexities induced by L-fuzzy subspace degrees on X and Y, respectively. If $f : X \longrightarrow Y$ is a surjective and linear mapping between vector spaces, then $\mathcal{D}_X(f_L^{\leftarrow}(\mu)) = \mathcal{D}_Y(\mu)$.*

Proof. Since f is surjective, we have

$$\begin{aligned} \mathcal{D}_X(f_L^{\leftarrow}(\mu)) &= \bigwedge_{x_1, x_2 \in X, k, l \in \mathbb{K}} \left(f_L^{\leftarrow}(\mu)(x_1) \wedge f_L^{\leftarrow}(\mu)(x_2) \to f_L^{\leftarrow}(\mu)(kx_1 + lx_2)\right) \\ &= \bigwedge_{x_1, x_2 \in X, k, l \in \mathbb{K}} \left(\mu(f(x_1)) \wedge \mu(f(x_2)) \to \mu(f(kx_1 + lx_2))\right) \\ &= \bigwedge_{x_1, x_2 \in X, k, l \in \mathbb{K}} \left(\mu(f(x_1)) \wedge \mu(f(x_2)) \to \mu(kf(x_1) + lf(x_2))\right) \\ &= \bigwedge_{y_1, y_2 \in Y, k, l \in \mathbb{K}} \left(\mu(y_1) \wedge \mu(y_2) \to \mu(ky_1 + ly_2)\right) \\ &= \mathcal{D}_Y(\mu). \end{aligned}$$

□

Convex-to-convex mappings also play an important role in the theory of convexity theory. In the framework of fuzzy convexities, Shi and Xiu [23] introduced the notion of (L, M)-fuzzy convex-to-convex mappings.

Definition 4.6. ([23]) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. A mapping $f : X \longrightarrow Y$ is called an *(L, M)-fuzzy convex-to-convex mapping* provided $\mathcal{C}(\mu) \leqslant \mathcal{D}(f_L^{\rightarrow}(\mu))$ for all $\mu \in L^X$.

An (L, L)-fuzzy convex-to-convex mapping is called an *L-fuzzy convex-to-convex mapping* for short.

An $(L, 2)$-fuzzy convex-to-convex mapping is exactly an *L-convex-to-convex mapping* in [13].

L-fuzzy convexity-preserving mappings are also important to establish the relationships between L-fuzzy convex spaces. This motivates us to consider the relationships between linear mappings and L-fuzzy convex-to-convex mappings.

Theorem 4.7. *Let X and Y be two vector spaces over $\mathbb{K}$ and $\mathcal{D}_X$ and $\mathcal{D}_Y$ be the L-fuzzy convexities induced by L-fuzzy subspace degrees on X and Y, respectively. If $f : X \longrightarrow Y$ is a linear mapping between vector spaces, then $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is an L-fuzzy convex-to-convex mapping.*

Proof. Take any $a \in L$ such that $a \leqslant \mathcal{D}_X(\lambda)$. By Lemma 3.4, we have

$$\lambda(x_1) \wedge \lambda(x_2) \wedge a \leqslant \lambda(kx_1 + lx_2)$$

for any $x_1, x_2 \in X$ and $k, l \in \mathbb{K}$. Then for all $y_1, y_2 \in Y$, we have

$$\begin{aligned}
f_L^{\rightarrow}(\lambda)(y_1) \wedge f_L^{\rightarrow}(\lambda)(y_2) \wedge a &= \bigvee_{f(x_1)=y_1} \lambda(x_1) \wedge \bigvee_{f(x_2)=y_2} \lambda(x_2) \wedge a \\
&= \bigvee \{\lambda(x_1) \wedge \lambda(x_2) \wedge a \mid f(x_1) = y_1, f(x_2) = y_2\} \\
&\leqslant \bigvee \{\lambda(kx_1 + lx_2) \mid f(x_1) = y_1, f(x_2) = y_2\} \\
&\leqslant \bigvee \{\lambda(kx_1 + lx_2) \mid f(kx_1 + lx_2) = ky_1 + ly_2\} \\
&= f_L^{\rightarrow}(\lambda)(ky_1 + ly_2).
\end{aligned}$$

By Lemma 3.4, we have $a \leqslant \mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right)$. By the arbitrariness of a, we obtain $\mathcal{D}_X(\lambda) \leqslant \mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right)$, which shows that $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is an L-fuzzy convex-to-convex mapping. □

In the classical case, if $f : X \longrightarrow Y$ is an injective and linear mapping between vector spaces, then Z is a subspace of X if and only if $f^{\rightarrow}(X) = \{f(x) \mid x \in Z\}$ is a subspace of Y. Now we give the fuzzy case of this conclusion.

Theorem 4.8. *Let X and Y be two vector spaces over $\mathbb{K}$ and $\mathcal{D}_X$ and $\mathcal{D}_Y$ be the L-fuzzy convexities induced by L-fuzzy subspace degrees on X and Y, respectively. If $f : X \longrightarrow Y$ is a injective and linear mapping between vector spaces, then $\mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right) = \mathcal{D}_X(\lambda)$.*

Proof. By Theorem 4.7, it suffices to prove that $\mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right) \leqslant \mathcal{D}_X(\lambda)$. Take any $a \in L$ such that $a \leqslant \mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right)$. By Lemma 3.4, we have $f_L^{\rightarrow}(\lambda)(y_1) \wedge f_L^{\rightarrow}(\lambda)(y_2) \wedge a \leqslant f_L^{\rightarrow}(\lambda)(ky_1 + ly_2)$ for all $y_1, y_2 \in Y$ and $k, l \in \mathbb{K}$. For any $x_1, x_2 \in X$, let $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since f is injective and linear, it follows that

$$f_L^{\rightarrow}(\lambda)(f(x_1)) = \bigvee_{f(x)=f(x_1)} \lambda(x) = \lambda(x_1),$$

$$f_L^{\rightarrow}(\lambda)(f(x_2)) = \bigvee_{f(x)=f(x_2)} \lambda(x) = \lambda(x_2) \text{ and}$$

$$f_L^{\rightarrow}(\lambda)(ky_1 + ly_2)) = \bigvee_{f(x)=f(kx_1+lx_2)} \lambda(x) = \lambda(kx_1 + lx_2).$$

Hence

$$\begin{aligned}\lambda(x_1) \wedge \lambda(x_2) \wedge a &= f_L^{\rightarrow}(\lambda)(f(x_1)) \wedge f_L^{\rightarrow}(\lambda)(f(x_2)) \wedge a \\ &\leqslant f_L^{\rightarrow}(\lambda)(kf(x_1) + lf(x_2)) \\ &= \lambda(kx_1 + lx_2).\end{aligned}$$

Then it follows from Lemma 3.4 that $a \leqslant \mathcal{D}_X(\lambda)$. We obtain $\mathcal{D}_Y\left(f_L^{\rightarrow}(\lambda)\right) \leqslant \mathcal{D}_X(\lambda)$ by the arbitrariness of a. □

Definition 4.9. ([23]) Let (X, C) and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. A mapping $f : X \longrightarrow Y$ is called an (L, M)-*fuzzy isomorphism* provided f is bijective, (L, M)-fuzzy convexity preserving and (L, M)-fuzzy convex-to-convex.

An (L, L)-fuzzy isomorphism is called an *L-fuzzy isomorphism* for short.

An $(L, 2)$-fuzzy isomorphism is exactly an *L-isomorphism* [13].

According to Theorems 4.4 and 4.7, we have the following theorem.

Theorem 4.10. *Let X and Y be two vector spaces over $\mathbb{K}$ and $\mathcal{D}_X$ and $\mathcal{D}_Y$ be the L-fuzzy convexities induced by L-fuzzy subspace degrees on X and Y, respectively. If $f : X \longrightarrow Y$ is a bijective and linear mapping between vector spaces, then $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is an L-fuzzy isomorphism.*

5. Conclusions

In this paper, the concept of degree to which an L-subset of a vector space is an L-fuzzy subspace was proposed. Then an L-fuzzy convexity on a vector space is naturally constructed and some properties of this kind of L-fuzzy convexity are studied.

It is worth noting that the same thought can be applied to different algebraic systems such as groups, rings, fields and so on. Thus L-fuzzy convexities can be induced by different algebraic systems. The above facts will be useful to help further investigations.

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