Filomat 38:20 (2024), 7091–7099
https://doi.org/10.2298/FIL2420091P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On pseudocompactness of remainders of certain spaces

Liang-Xue Peng[a,*], Xing-Yu Hu[a]

[a] *Department of Mathematics, School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124, China*

Abstract. Let $\mathcal{B}$ be a base for a nowhere locally compact Tychonoff space X and let bX be a compactification of X. Then the following two statements hold:

(1) The remainder $bX \setminus X$ of X is pseudocompact if and only if for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ there exists an accumulation point of the family $\mathcal{V}$ in $bX \setminus X$.

(2) If for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ the set of all accumulation points of the family $\mathcal{V}$ in X is not a nonempty compact set of X, then $bX \setminus X$ is pseudocompact.

Let $X = \prod_{i\in I} X_i$ be a product space and S be a subset of X satisfying the following condition:

($*$) For each nonempty countable set $J \subset I$, the projection $p_J : X \to \prod_{i\in J} X_i$ satisfies that $p_J(S) = X_J := \prod_{i\in J} X_i$.

If $\mathcal{B}$ is the canonical base for X and $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ is a countable infinite subfamily of $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is nonempty, then for any $a \in F$ there exists a countable subset J of I such that $p_J^{-1}(p_J(a)) \cap S = p_J^{-1}(p_J(a)) \cap F$ and for any $\alpha \in I \setminus J$, $p_\alpha(F) = X_\alpha$.

By the above conclusions, we can get two known results in [8]. We finally show that if $X = \prod_{i\in I} X_i$ is a product of a family $\{X_i : i \in I\}$ of Tychonoff spaces such that uncountably many of them are non-compact and Y is a dense subspace of X, then for every compactification bY of Y the remainder $bY \setminus Y$ is pseudocompact.

1. Introduction

A topological space X is called *pseudocompact* if X is a Tychonoff space and every continuous real-valued function defined on X is bounded [6]. Recall that a point x of a space X is an *accumulation point of a family* $\mathcal{V}$ of subsets of X if every open neighborhood V_x of x meets infinite elements of $\mathcal{V}$. A subset A of a space X is said to be *bounded in* X if every infinite family ξ of open subsets of X such that $V \cap A \neq \emptyset$, for every $V \in \xi$, has an accumulation point in X [4]. So a Tychonoff space X is pseudocompact if X is bounded in itself.

A *compactification* of a space X is any compact space bX containing X as a subspace such that X is dense in bX. In this note, a compactification of a Tychonoff space is a Hausdorff compactification. A *remainder* of a space X is the subspace $bX \setminus X$ of a compactification bX of X.

Recall that a *paratopological group* is a group with a topology such that the multiplication on the group is jointly continuous. A *topological group* G is a paratopological group such that the inverse mapping of G into itself associating x^{-1} with $x \in G$ is continuous [5]. Recall that a space X is of *countable type* if every compact subspace B of X is contained in a compact subspace $F \subset X$ that has a countable base of open neighborhoods

2020 *Mathematics Subject Classification*. Primary 54D40
Keywords. Pseudocompact, remainder, regular open set, compactification.
Received: 27 October 2023; Revised: 28 January 2024; Accepted: 01 February 2024
Communicated by Ljubiša D. R. Kočinac
Research supported by the National Natural Science Foundation of China (Grant No. 12171015).
* Corresponding author: Liang-Xue Peng
Email addresses: pengliangxue@bjut.edu.cn (Liang-Xue Peng), 2021650586@qq.com (Xing-Yu Hu)

in X [1]. All metrizable spaces and locally compact spaces are of countable type. In [7], M. Henriksen and J.R. Isbell proved that a Tychonoff space X is of countable type if and only if the remainder in any (or in some) Hausdorff compactification of X is Lindelöf. In [2], it was proved that each remainder of a topological group G is Lindelöf, or each remainder of G is pseudocompact. In [3], Arhangel'skii and Bella investigated, when a topological group G is pseudocompact at infinity, that is, when $bG \setminus G$ is pseudocompact, for each compactification bG of G. Let $X = \prod_{i\in I} X_i$ be a product space such that uncountably many of the factors X_i are non-compact. Also, let S be a subspace of X such that $p_J(S) = X_J$ for each countable set $J \subset I$, where $p_J : X \to X_J := \prod_{i\in J} X_i$ is the projection. If bS is a compactification of S, then the remainder $bS \setminus S$ of S is pseudocompact ([8], Theorem 2.4).

In this note, we also study when a remainder of Tychonoff space is pseudocompact. Recall that a subset U of a space X is a *regular open* if $U = \overline{U}^{\circ}$. We first discuss some properties of regular open subsets of a space. We mainly get the following conclusions. Let $\mathcal{B}$ be a base for a nowhere locally compact Tychonoff space X and let bX be a compactification of X. Then the following two statements hold:

(1) The remainder $bX \setminus X$ of X is pseudocompact if and only if for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ there exists an accumulation point of the family $\mathcal{V}$ in $bX \setminus X$.
(2) If for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ the set of all accumulation points of the family $\mathcal{V}$ in X is not a nonempty compact set of X, then $bX \setminus X$ is pseudocompact.

Let $X = \prod_{i\in I} X_i$ be a product space and S be a subset of X satisfying the following condition: (∗) For each nonempty countable set $J \subset I$, the projection $p_J : X \to \prod_{i\in J} X_i$ satisfies that $p_J(S) = X_J := \prod_{i\in J} X_i$. If $\mathcal{B}$ is the canonical base for X and $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ is a countable infinite subfamily of $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is nonempty, then for any $a \in F$ there exists a countable subset J of I such that $p_J^{-1}(p_J(a)) \cap S = p_J^{-1}(p_J(a)) \cap F$ and for any $\alpha \in I \setminus J$, $p_\alpha(F) = X_\alpha$. By the above conclusions, we can get two known results in [8]. We finally show that if $X = \prod_{i\in I} X_i$ is a product space of a family $\{X_i : i \in I\}$ of Tychonoff spaces such that uncountably many of them are non-compact and Y is a dense subspace of X, then for every compactification bY of Y the remainder $bY \setminus Y$ is pseudocompact.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{R}$ be the set of all reals with the natural topology. In notation and terminology we will follow [6]. Let X be a topological space and let Y be a dense subspace of X and $A \subset Y$. Then the closure of A in the subspace Y of X is denoted by $\overline{A}^{(Y)}$ and the interior of the set A in the subspace Y of X is denoted by $Int_Y A$. The closure of a subset A of a space X is denoted by $\overline{A}$ and the interior of the set A in X is denoted by A°.

2. Main results

Lemma 2.1. *Let Y be a Tychonoff topological space. Then Y is not pseudocompact if and only if there exists an infinite locally finite family $\mathcal{V}$ of nonempty regular open subsets of Y.*

Proof. Suppose that Y is not pseudocompact. Then there exists a continuous function $f : Y \to \mathbb{R}$ such that f is not bounded. For each $y \in Y$, there exists some $n \in \mathbb{Z}$ such that $f(y) \in (n, n+2)$. Since f is not bounded, the set $\Lambda = \{n \in \mathbb{Z} : f(Y) \cap (n, n+2) \neq \emptyset\}$ is infinite. For each $n \in \Lambda$, $f^{-1}((n, n+2))$ is a nonempty open subset of X and $f(\overline{f^{-1}((n, n+2))}^{\circ}) \subset \overline{(n, n+2)} = [n, n+2]$. Since f is not bounded, we have $|\{\overline{f^{-1}((n, n+2))}^{\circ} : n \in \Lambda\}| = \omega$. So there exists an infinite subfamily $\mathcal{V} \subset \{\overline{f^{-1}((n, n+2))}^{\circ} : n \in \Lambda\}$ such that $\mathcal{V}$ is a family of pairwise distinct sets. For each $y \in Y$, the set $\{n \in \Lambda : (f(y) - 1, f(y) + 1) \cap [n, n+2] \neq \emptyset\}$ is finite. Since the mapping f is continuous, the set $O_y = f^{-1}(f(y) - 1, f(y) + 1)$ is an open neighborhood of the point y in Y and $|\{V \in \mathcal{V} : O_y \cap V \neq \emptyset\}| < \omega$. So $\mathcal{V}$ is a locally finite family of nonempty regular open subsets of X such that $|\mathcal{V}| = \omega$.

For the converse, it follows from ([6], Theorem 3.10.22). □

Lemma 2.2. *Let X be a topological space and let Y be a dense subspace of X. If U and V are regular open subsets of X, then $U = V$ if and only if $U \cap Y = V \cap Y$.*

Proof. Assume that U and V are regular open subsets of X and $U \cap Y = V \cap Y$. Thus $\overline{U \cap Y} = \overline{V \cap Y}$. Since U and V are open in X and Y is dense in X, $\overline{U \cap Y} = \overline{U}$ and $\overline{V \cap Y} = \overline{V}$. Thus $\overline{U} = \overline{V}$. Since U and V are regular open subsets of X, $U = \overline{U}^{\circ}$ and $V = \overline{V}^{\circ}$. Thus $U = V$.

For the converse, it is obvious that $U \cap Y = V \cap Y$ if $U = V$. □

Lemma 2.3. *Let X be a topological space and let Y be a dense subspace of X. If $U \subset Y$ is a regular open subset of the subspace Y of X, then $\overline{U}^{\circ}$ is a regular open subset of X such that $U = \overline{U}^{\circ} \cap Y$ and $\overline{U}^{\circ} = \overline{V}^{\circ}$ whenever V is an open subset of X such that $V \cap Y = U$.*

Proof. Since U is a regular open subset of Y, $U = Int_Y\overline{U}^{(Y)}$. Since U is open in Y, there exists an open subset V of X such that $V \cap Y = U$. Since Y is dense in X, $\overline{U} = \overline{V}$. Thus $\overline{U}^{\circ} = \overline{V}^{\circ}$. Since $\overline{U}^{(Y)} = \overline{U} \cap Y = \overline{V} \cap Y$, the set $U = V \cap Y \subset \overline{V}^{\circ} \cap Y = \overline{U}^{\circ} \cap Y \subset \overline{U} \cap Y = \overline{U}^{(Y)}$. Since $U = Int_Y\overline{U}^{(Y)}$, we have $U = \overline{U}^{\circ} \cap Y$. By the above proof, we also know that $\overline{U}^{\circ} = \overline{V}^{\circ}$ whenever V is an open subset of X such that $V \cap Y = U$. Thus $\overline{U}^{\circ}$ is a regular open subset of X and $U = \overline{U}^{\circ} \cap Y$. □

Lemma 2.4. *Let X be a topological space and let Y be a dense subset of X such that $X \setminus Y$ is a regular dense subspace of X. If $\mathcal{U} = \{U_n : n \in \omega\}$ is a family of regular open subsets of Y such that $\mathcal{U}$ is point-finite in Y and $U_n \neq U_m$ whenever $n \neq m$, then $\{\overline{U_n}^{\circ} : n \in \omega\}$ is a family of pairwise distinct regular open subsets of X such that the following properties hold:*

(1) $\{\overline{U_n}^{\circ} \cap (X \setminus Y) : n \in \omega\}$ is a family of pairwise distinct sets;

(2) Every family $\{O_n : n \in \omega\}$ of open subsets of $X \setminus Y$ satisfying $O_n \subset \overline{U_n}^{\circ}$ for each $n \in \omega$ is infinite, and for each $m \in \omega$ the set $\{n \in \omega : O_m \subset \overline{U_n}^{\circ}\}$ is finite.

Proof. Since $\overline{Y} = X$ and U_n is a regular open subset of Y for each $n \in \omega$, it follows from Lemma 2.3 that $\overline{U_n}^{\circ}$ is a regular open subset of X and $\overline{U_n}^{\circ} \cap Y = U_n$ for each $n \in \omega$. Since $\overline{U_n}^{\circ} \cap Y = U_n$ for each $n \in \omega$ and $U_n \neq U_m$ whenever $n \neq m$, we have $\overline{U_n}^{\circ} \neq \overline{U_m}^{\circ}$ whenever $n \neq m$. So $\{\overline{U_n}^{\circ} : n \in \omega\}$ is a family of pairwise distinct regular open subsets of X. Since $X \setminus Y$ is dense in X, by Lemma 2.2 $\overline{U_n}^{\circ} \cap (X \setminus Y) \neq \overline{U_m}^{\circ} \cap (X \setminus Y)$ whenever $n \neq m$. So $\{\overline{U_n}^{\circ} \cap (X \setminus Y) : n \in \omega\}$ is a family of pairwise distinct open subsets of $X \setminus Y$.

Now we assume that O_n is an open subset of $X \setminus Y$ such that $O_n \subset \overline{U_n}^{\circ}$ for each $n \in \omega$. Suppose $\{O_n : n \in \omega\}$ is finite. Then there exists some $m \in \omega$ such that $A = \{n \in \omega : O_m \subset \overline{U_n}^{\circ}\}$ is infinite.

Since $X \setminus Y$ is a regular dense subspace of X and O_m is a nonempty open subset of the subspace $X \setminus Y$ of X, there exists an nonempty open (in $X \setminus Y$) subset W such that $W \subset \overline{W}^{(X\setminus Y)} \subset O_m$. Thus $Int_{(X\setminus Y)}\overline{W}^{(X\setminus Y)}$ is a regular open subset of the subspace $X \setminus Y$ of X. If $V = Int_{(X\setminus Y)}\overline{W}^{(X\setminus Y)}$, then $V \subset \overline{U_n}^{\circ}$ for each $n \in A$. The set $X \setminus Y$ is dense in X and V is a regular open subset of the subspace $X \setminus Y$ of X. By Lemma 2.3, $\overline{V}^{\circ}$ is a regular open subset of X. So $\overline{V}^{\circ} \subset \overline{U_n}^{\circ}$ for each $n \in A$. Since Y is dense in X, the set $\overline{V}^{\circ} \cap Y \neq \emptyset$.

Take a point $z \in \overline{V}^{\circ} \cap Y$. Then $z \in \overline{U_n}^{\circ} \cap Y = U_n$ for each $n \in A$. This contradicts that $\{U_n : n \in \omega\}$ is point-finite. Thus $\{O_n : n \in \omega\}$ is infinite.

By the above proof, we know that $\{n \in \omega : O_m \subset \overline{U_n}^{\circ}\}$ is finite for each m. □

Theorem 2.5. *Let X be a nowhere locally compact Tychonoff space with a base $\mathcal{B}$ and let bX be a compactification of X and $Y = bX \setminus X$. If Y is not pseudocompact, then there exists a countable infinite family $\mathcal{V} \subset \mathcal{B}$ such that the set F of accumulation points of the family $\mathcal{V}$ in bX is a nonempty compact subset of X.*

Proof. Assume that Y is not pseudocompact. Then by Lemma 2.1 Y contains an infinite family $\mathcal{U} = \{U_n : n \in \omega\}$ of nonempty regular open subsets of Y such that $\mathcal{U}$ is locally finite in Y. We can assume that $U_n \neq U_m$ whenever $n \neq m$. Since X and Y are both dense in bX and $\mathcal{U}$ is point-finite in Y, the conditions of Lemma 2.4 are satisfied. So it follows from Lemma 2.4 that $\{Int_{bX}\overline{U_n}^{(bX)} : n \in \omega\}$ is an infinite family of pairwise distinct regular open subsets of bX. For each $n \in \omega$, the set $(Int_{bX}\overline{U_n}^{(bX)}) \cap X$ is a nonempty open subset of X. Since $\mathcal{B}$ is a base of X, there exists a family $\{B_n : n \in \omega\} \subset \mathcal{B}$ such that $B_n \neq \emptyset$ and $B_n \subset (Int_{bX}\overline{U_n}^{(bX)}) \cap X$ for each $n \in \omega$. By Lemma 2.4, the family $\{B_n : n \in \omega\}$ is infinite. Thus there exists an infinite subfamily $\mathcal{V} \subset \{B_n : n \in \omega\}$ such that $\mathcal{V}$ is a family of pairwise distinct sets.

For each $n \in \omega$, we let $V_n = Int_{bX}\overline{U_n}^{(bX)}$. Then by Lemma 2.3 V_n is a regular open subset of bX and $V_n \cap Y = U_n$ for each $n \in \omega$. Since $\overline{Y} = bX$, we have $\overline{V_n}^{(bX)} = \overline{U_n}^{(bX)}$ for each $n \in \omega$. Let $E = \{x \in bX : x$ is an accumulation point of the family $\{V_n : n \in \omega\}$ in $bX\}$. Then E is equal to $\{x \in bX : x$ is an accumulation point of the family $\{U_n : n \in \omega\}$ in $bX\}$. Since $\{U_n : n \in \omega\}$ is locally finite in Y, the set E is contained in X. By Lemma 2.4, we know that for each $O \in \mathcal{V}$, the set $\{n \in \omega : O \subset V_n \cap X\}$ is finite. Thus if a point $y \in bX$ is an accumulation point of the family $\mathcal{V}$ in bX, then y is an accumulation point of the family $\{V_n : n \in \omega\}$. Denote $M = \{x \in bX : x$ is an accumulation point of the family $\mathcal{V}$ in $bX\}$. Thus $M \subset E \subset X$. Since $|\mathcal{V}| = \omega$ and bX is compact, $M \neq \emptyset$. □

In fact, we have the following result.

Theorem 2.6. *Let X be a nowhere locally compact Tychonoff topological space and let bX be a compactification of X. Let $\{U_n : n \in \omega\}$ be any locally finite family of nonempty open subsets of $bX \setminus X$ such that $U_n \neq U_m$ whenever $n \neq m$. If W_n is an open subset of bX such that $W_n \cap (bX \setminus X) = U_n$ and V_n is a nonempty open subset of X such that $V_n \subset W_n \cap X$ for each $n \in \omega$, then $\mathcal{V} = \{V_n : n \in \omega\}$ is infinite and the set F of accumulation points of the family $\mathcal{V}$ in bX is nonempty and is contained in X.*

Proof. Suppose that $|\{V_n : n \in \omega\}| < \omega$. Then there exists some $m \in \omega$ such that $|\{n \in \omega : V_n = V_m\}| = \omega$. Let $V_m = O$ and $\{n \in \omega : V_n = O\} = \{k_i : i \in \omega\}$ such that $k_i \neq k_j$ whenever $i \neq j$. Then $V_{k_i} = O \subset W_{k_i}$ for each $i \in \omega$. Since O is an open subset of X, there exists an open subset O^* of bX such that $O^* \cap X = O$. Since $\overline{bX \setminus X} = bX$, we have $O^* \cap (bX \setminus X) \neq \emptyset$.

Let z be any point of $O^* \cap (bX \setminus X)$ and let M_z be any open subset of $bX \setminus X$ such that $z \in M_z$. Then there exists an open subset M_z^* of bX such that $M_z^* \cap (bX \setminus X) = M_z$. Thus $M_z^* \cap O^*$ is an open neighborhood of the point z in bX. Since $\overline{X} = bX$ and $O^* \cap X = O$, the set O is dense in O^*. Thus $(M_z^* \cap O^*) \cap O \neq \emptyset$. Let p be any point of $(M_z^* \cap O^*) \cap O$. Then $p \in O$ and $M_z^* \cap O^*$ is an open neighborhood of the point p in bX. For each $i \in \omega$, $O \subset W_{k_i}$ and $W_{k_i} \cap (bX \setminus X) = U_{k_i}$. Thus U_{k_i} is dense in W_{k_i} for each $i \in \omega$. So $M_z^* \cap O^* \cap U_{k_i} \neq \emptyset$ for each $i \in \omega$. Since $U_{k_i} \subset bX \setminus X$ for each $i \in \omega$, the set $M_z^* \cap O^* \cap U_{k_i} = M_z \cap O^* \cap U_{k_i} \neq \emptyset$. Thus $M_z \cap U_{k_i} \neq \emptyset$ for each $i \in \omega$. This contradicts with that $\{U_n : n \in \omega\}$ is locally finite in $bX \setminus X$. Thus the family $\mathcal{V} = \{V_n : n \in \omega\}$ is infinite. Since $\mathcal{V}$ is infinite, the set F of accumulation points of the family $\mathcal{V}$ in bX is nonempty.

By the proof above, we know that for each $n \in \omega$ the set $\{m \in \omega : V_n \subset W_m\}$ is finite. Then the set $F \subset \{x \in bX : x$ is an accumulation point of the family $\{W_n : n \in \omega\}\}$ is a nonempty subset of X. □

Lemma 2.7. *If Y is a dense subset of a space X and U is a regular open subset of X, then $\overline{U}^{\circ} \cap Y = U \cap Y$ is a regular open subset of Y.*

Proof. Since U is a regular open subset of X, we have $\overline{U}^{\circ} = U$. Thus $\overline{U}^{\circ} \cap Y = U \cap Y$. For any $x \in Int_Y(\overline{U \cap Y}^{(Y)})$, there exists an open subset O_x of the subspace Y of X such that $x \in O_x \subset \overline{U \cap Y}^{(Y)}$. Since Y is a dense subspace of X, we have $\overline{U \cap Y}^{(Y)} \subset \overline{U \cap Y} = \overline{U}$. Then $x \in O_x \subset \overline{U}$. Thus, there exists an open subset W_x of X such that $W_x \cap Y = O_x$. Since Y is dense in X, we have $\overline{W_x} = \overline{O_x}$. Thus $x \in W_x \subset \overline{W_x} \subset \overline{U}$. So $x \in \overline{U}^{\circ}$. Thus $Int_Y(\overline{U \cap Y}^{(Y)}) \subset \overline{U}^{\circ} \cap Y = U \cap Y$. It is obvious that $U \cap Y \subset Int_Y(\overline{U \cap Y}^{(Y)})$. Thus $Int_Y(\overline{U \cap Y}^{(Y)}) = U \cap Y$. Then $U \cap Y$ is a regular open subset of Y. □

Lemma 2.8. *Let Y_1 and Y_2 be dense subsets of a space X. If U and V are regular open subsets of Y_1 and $U \neq V$, then $\overline{U}^{\circ} \cap Y_2$ and $\overline{V}^{\circ} \cap Y_2$ are regular open subsets of Y_2 and the two sets $\overline{U}^{\circ} \cap Y_2$ and $\overline{V}^{\circ} \cap Y_2$ are distinct.*

Proof. Since U and V are regular open subsets of Y_1 and $\overline{Y_1} = X$, it follows from Lemma 2.3 that $\overline{U}^{\circ}$ and $\overline{V}^{\circ}$ are regular open subsets of X. Thus, by Lemma 2.7 the sets $\overline{U}^{\circ} \cap Y_2$ and $\overline{V}^{\circ} \cap Y_2$ are regular open subsets of Y_2.

By Lemma 2.3, we have $U = \overline{U}^{\circ} \cap Y_1$ and $V = \overline{V}^{\circ} \cap Y_1$. Since $U \neq V$, we have $\overline{U}^{\circ} \neq \overline{V}^{\circ}$. Since $\overline{U}^{\circ}$ and $\overline{V}^{\circ}$ are two distinct regular open subsets of X, it follows from Lemma 2.2 that the two sets $\overline{U}^{\circ} \cap Y_2$ and $\overline{V}^{\circ} \cap Y_2$ are distinct. □

Theorem 2.9. *Let X be a nowhere locally compact Tychonoff space and let $\mathcal{B}$ be a base for X. If bX is a compactification of X, then $bX \setminus X$ is pseudocompact if and only if for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ there exists an accumulation point of the family $\mathcal{V}$ in $bX \setminus X$.*

Proof. ($\Rightarrow$) Assume that $bX \setminus X$ is pseudocompact. Let $\mathcal{V} \subset \mathcal{B}$ be any countable infinite subfamily of $\mathcal{B}$. Without loss of generality, we assume that $\mathcal{V} = \{V_n : n \in \omega\}$ and $V_n \neq V_m$ whenever $n, m \in \omega$ and $n \neq m$. Since X is regular, for every $n \in \omega$ there exists a nonempty regular open subset U_n of X such that $U_n \subset \overline{U_n} \subset V_n$. If $|\{U_n : n \in \omega\}| < \omega$, then there exists $k \in \omega$ such that $|\{m \in \omega : U_m = U_k\}| = \omega$. If $z \in Int_{(bX)}\overline{U_k}^{(bX)} \cap (bX \setminus X)$, then z is an accumulation point of the family $\mathcal{V}$ in bX. Now we assume that for every $k \in \omega$, the set $\{m \in \omega : U_m = U_k\}$ is finite. Without loss of generality, we assume that $U_n \neq U_m$ whenever $n \neq m$.

It follows from Lemma 2.8 that $\{Int_{(bX)}\overline{U_n}^{(bX)} \cap (bX \setminus X) : n \in \omega\}$ is a family of regular open subsets of $bX \setminus X$ and $Int_{(bX)}\overline{U_n}^{(bX)} \cap (bX \setminus X) \neq Int_{(bX)}\overline{U_m}^{(bX)} \cap (bX \setminus X)$ whenever $n \neq m$. Since $bX \setminus X$ is pseudocompact, the family $\{Int_{(bX)}\overline{U_n}^{(bX)} \cap (bX \setminus X) : n \in \omega\}$ has an accumulation point z in $bX \setminus X$. Then the point z is an accumulation point of the family $\{Int_{(bX)}\overline{U_n}^{(bX)} : n \in \omega\}$. By Lemma 2.3, we have $U_n = Int_{(bX)}\overline{U_n}^{(bX)} \cap X$ for every $n \in \omega$. Thus the point z is an accumulation point of the family $\mathcal{V}$.

($\Leftarrow$) It follows from Theorem 2.5 that the remainder $bX \setminus X$ of X is pseudocompact. □

Recall that a *π-base* of a space X at a subset F of X is a family $\mathcal{V}$ of nonempty open subsets of X such that every open neighborhood of F contains at least one element of $\mathcal{V}$. A *strong π-base* of a space X at a subset F of X is an infinite family $\mathcal{V}$ of nonempty open subsets of X such that every open neighborhood of F contains all but finitely many elements of $\mathcal{V}$ ([2], p. 120).

Lemma 2.10. ([2], Lemma 2.1) *Suppose that X is a nowhere locally compact Tychonoff space, and bX is a compactification of X. Then the following two conditions are equivalent:*

(1) The remainder $Y = bX \setminus X$ is not pseudocompact;
(2) There exists a nonempty compact subspace F of X which has a strong countable π-base in X.

Lemma 2.11. *Let X be a regular space and $\mathcal{B}$ be a base for X. Then the following two conditions are equivalent:*

(1) There exists a countable infinite subfamily $\mathcal{V} = \{V_n : n \in \omega\} \subset \mathcal{B}$ such that the set $F = \{x \in X : x$ is an accumulation point of the family $\mathcal{V}$ in $X\}$ is a nonempty compact subset of X and any infinite family $\{W_n : n \in \omega\}$ of open subsets of X, with $W_n \subset V_n$ for every $n \in \omega$, has an accumulation point in X.
(2) There exists a nonempty compact subspace F of X which has a strong countable π-base $\mathcal{V}' \subset \mathcal{B}$.

Proof. (1) $\Rightarrow$ (2) Assume that there exists a countable infinite subfamily $\mathcal{V} = \{V_n : n \in \omega\} \subset \mathcal{B}$ such that the set $F = \{x \in X : x$ is an accumulation point of the family $\mathcal{V}$ in $X\}$ is a nonempty compact subset of X and any infinite family $\{W_n : n \in \omega\}$ of open subsets of X, with $W_n \subset V_n$ for every $n \in \omega$, has an accumulation point in X

Claim. The family $\mathcal{V}$ is a strong countable π-base at the compact subset F of X.

Proof of Claim. Take any open neighborhood O of the set F in X. Since X is regular and F is compact, there exists an open set W of X such that $F \subset \overline{W} \subset O$. Suppose $|\{V \in \mathcal{V} : V \setminus \overline{W} \neq \emptyset\}| = \omega$.

Case 1 $|\{V \setminus \overline{W} : V \in \mathcal{V}, V \setminus \overline{W} \neq \emptyset\}| = \omega$.

Then the family $\{V \setminus \overline{W} : V \in \mathcal{V}, V \setminus \overline{W} \neq \emptyset\}$ has an accumulation point y in X. Then $y \in F$. On the other hand, $V \setminus \overline{W} \subset X \setminus W$ for every $V \in \mathcal{V}$. Then the point $y \notin W$. This contradicts with $F \subset W$.

Case 2 $|\{V \setminus \overline{W} : V \in \mathcal{V}, V \setminus \overline{W} \neq \emptyset\}| < \omega$.

Since $|\{V \in \mathcal{V} : V \setminus \overline{W} \neq \emptyset\}| = \omega$, there exists a countable infinite subfamily $\mathcal{V}_1 \subset \mathcal{V}$ such that $|\{V \in \mathcal{V}_1 : V \setminus \overline{W} \neq \emptyset\}| = \omega$ and $|\{V \setminus \overline{W} : V \in \mathcal{V}_1\}| = 1$. For any $V \in \mathcal{V}_1$, take a point $x \in V \setminus \overline{W}$. Then $x \in \bigcap \mathcal{V}_1$. Thus the point x is an accumulation point of the family $\mathcal{V}_1$ in bX. Then $x \in F$. A contradiction.

Thus there exists a nonempty compact subspace F of X which has a strong countable π-base $\mathcal{V}' \subset \mathcal{B}$.

(2) $\Rightarrow$ (1) Let $\mathcal{V}' \subset \mathcal{B}$ be a strong countable π-base at a nonempty compact subspace F of X.

Case 1 If $\{V \in \mathcal{V}' : V \subset F\}$ is infinite, then there exists a countable infinite subfamily $\mathcal{V} = \{V_n : n \in \omega\} \subset \{V \in \mathcal{V}' : V \subset F\}$ such that $V_n \neq V_m$ whenever $n \neq m$. Since F is compact, the set $A = \{x \in X : x$ is an accumulation point of the family $\mathcal{V}\}$ is a nonempty closed subset of F. Then A is compact. It is obvious that for any infinite family $\{W_n : n \in \omega\}$ of open subsets of X with $W_n \subset V_n$ for every $n \in \omega$ has an accumulation point in X.

Case 2 Now we assume that $\{V \in \mathcal{V}' : V \setminus F \neq \emptyset\}$ is infinite. Without loss of generality, we assume that $V \setminus F \neq \emptyset$ for every $V \in \mathcal{V}'$. For every $V \in \mathcal{V}'$, there exists a nonempty set $O_V \in \mathcal{B}$ such that $O_V \subset V$ and $\overline{O_V} \cap F = \emptyset$. Since $\mathcal{V}'$ is a strong countable π-base at F and $\overline{O_V} \cap F = \emptyset$ for every $V \in \mathcal{V}'$, the set $\{O_V : V \in \mathcal{V}'\}$ is infinite. Thus there exists a subfamily $\{V_n : n \in \omega\} \subset \mathcal{V}'$ such that $O_{V_n} \neq O_{V_m}$ whenever $n \neq m$. Then $\mathcal{V} = \{O_{V_n} : n \in \omega\}$ is also a strong countable π-base at F. Since F is compact and $\mathcal{V} = \{O_{V_n} : n \in \omega\}$ is a strong countable π-base at F, the set A of accumulation points of the family $\mathcal{V}$ in X is a nonempty compact subset of X.

Let $\mathcal{W} = \{W_n : n \in \omega\}$ be any infinite family of open subsets of X with $W_n \subset O_{V_n}$ for every $n \in \omega$. Since $\mathcal{V} = \{O_{V_n} : n \in \omega\}$ is a strong countable π-base at the compact set F and $W_n \subset O_{V_n}$ for every $n \in \omega$, there exists an accumulation point $y \in F$ of the family $\mathcal{W}$ in X. Thus (1) holds. □

Theorem 2.12. *Let X be a nowhere locally compact Tychonoff space and let $\mathcal{B}$ be a base for X. Then for any compactification bX of X, the following two conditions are equivalent:*

(1) The remainder $Y = bX \setminus X$ is not pseudocompact;

(2) There exists a nonempty compact subspace F of X which has a strong countable π-base $\gamma \subset \mathcal{B}$.

Proof. (2) $\Rightarrow$ (1) Suppose that there exists a nonempty compact subspace F of X which has a strong countable π-base $\gamma \subset \mathcal{B}$. Then by Lemma 2.10, $bX \setminus X$ is not pseudocompact.

(1) $\Rightarrow$ (2) Now we prove the converse. Suppose that $bX \setminus X$ is not pseudocompact. By Theorem 2.9, there exists a countable infinite subfamily $\mathcal{V} \subset \mathcal{B}$ such that $\mathcal{V}$ has no accumulation points in $bX \setminus X$. We can assume that $\mathcal{V} = \{V_n : n \in \omega\}$ is such that $V_n \neq V_m$ whenever $n \neq m$.

Let $F_1 = \{x \in bX : x$ is an accumulation point of the family $\mathcal{V}$ in $bX\}$. Then F_1 is a nonempty closed compact subset of bX. Since the family $\mathcal{V}$ has no accumulation points in $bX \setminus X$, the set $F_1 \subset X$. Let $\mathcal{W} = \{W_n : n \in \omega\}$ by any infinite family of open subsets of X with $W_n \subset V_n$ for every $n \in \omega$. Then the family $\mathcal{W}$ has an accumulation point in bX. Then the set A of accumulation points of the family $\mathcal{W}$ in bX is a nonempty subset of F_1. Thus A is contained in X. Then the family $\mathcal{W}$ has an accumulation point in X. By Lemma 2.11, there exists a nonempty compact subspace F of X which has a strong countable π-base $\gamma \subset \mathcal{B}$. □

Theorem 2.13. *Let $\mathcal{B}$ be a base for a nowhere locally compact Tychonoff space X and bX be a compactification of X. If for any countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ the set of all accumulation points of the family $\mathcal{V}$ in X is not a nonempty compact set, then $bX \setminus X$ is pseudocompact.*

Proof. Suppose that $bX \setminus X$ is not pseudocompact. By Theorem 2.5, there exists a countable infinite subfamily $\mathcal{V}$ of $\mathcal{B}$ such that the set A of all accumulation points of the family $\mathcal{V}$ in bX is nonempty and contained in X. Since the set A is closed in bX, the set A is compact. Since $A \subset X$, the set A is equal to $\{x \in X : x$ is an accumulation point of the family $\mathcal{V}$ in $X\}$ and A is compact. A contradiction. □

Theorem 2.14. *Let $X = \prod_{i\in I} X_i$ be a product space and S be a subset of X satisfying the following condition:*

($*$) *For each nonempty countable set $J \subset I$, the projection $p_J : X \to \prod_{i\in J} X_i$ satisfies that $p_J(S) = X_J := \prod_{i\in J} X_i$. If $\mathcal{B}$ is the canonical base for X and $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ is a countable infinite subfamily of $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is nonempty, then for any $a \in F$ there exists a countable subset J of I such that $p_J^{-1}(p_J(a)) \cap S = p_J^{-1}(p_J(a)) \cap F$ and for any $\alpha \in I \setminus J$, $p_\alpha(F) = X_\alpha$.*

Proof. Let $\mathcal{B}$ be the canonical base for X and let $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$. Then $\mathcal{B}_S$ is a base for S. Let $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ be a countable infinite subfamily of $\mathcal{B}_S$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is nonempty.

For every $i \in \omega$, let $B_i = \bigcap_{\alpha\in A_i} p_\alpha^{-1}(U_\alpha)$ for some finite subset A_i of I and U_α is open in X_α for each $\alpha \in A_i$. If $J = \bigcup\{A_i : i \in \omega\}$, then $|J| \le \omega$ and $J \subset I$.

Since $F \neq \emptyset$, we take $a \in F$. Let $a_J = p_J(a)$ and let b be any element of $p_J^{-1}(a_J) \cap S$. In what follows, we show that $b \in F$. Let O_b be any open neighborhood of the point b in X and $O_b \in \mathcal{B}$. Assume that $O_b = \bigcap_{k\le n} p_{\alpha_k}^{-1}(O_{\alpha_k})$, where $n \in \mathbb{N}$, $\alpha_k \in I$ and O_{α_k} is open in X_{α_k} for each $k \le n$. We can assume that $n > 1$, $\{\alpha_1, ..., \alpha_i\} \subset J$ for some $1 \le i < n$ and $\{\alpha_{i+1}, ..., \alpha_n\} \subset I \setminus J$. If $C = \bigcap_{k\le i} p_{\alpha_k}^{-1}(O_{\alpha_k})$, then the set C is an open neighborhood of the point a in X.

Since $a \in F$, we have $|\{m \in \omega : C \cap B_m \cap S \neq \emptyset\}| = \omega$. If $m \in \omega$ and $C \cap B_m \cap S \neq \emptyset$, then let $y_m \in C \cap B_m \cap S$. For any $i + 1 \le k \le n$, we let $y_{\alpha_k} \in O_{\alpha_k}$. Since $J \cup \{\alpha_{i+1}, ..., \alpha_n\} \subset I$ is countable, there exists $x_m \in S$ such that $p_J(x_m) = p_J(y_m)$ and $p_{\alpha_{i+t}}(x_m) = y_{\alpha_{i+t}}$ for each $t \in \{1, ..., n-i\}$. Then $x_m \in O_b \cap B_m \cap S$. Then $|\{m \in \omega : O_b \cap B_m \cap S \neq \emptyset\}| = \omega$. Thus $b \in F \cap S = F$. Then we have proved $p_J^{-1}(a_J) \cap S \subset p_J^{-1}(a_J) \cap F$. Since $F \subset S$, we have $p_J^{-1}(a_J) \cap F \subset p_J^{-1}(a_J) \cap S$. Thus $p_J^{-1}(a_J) \cap S = p_J^{-1}(a_J) \cap F$.

Now we prove the last part of this result. Let $\alpha \in I \setminus J$, then $\{\alpha\} \cup J = J_1 \subset I$ is countable. If $x_\alpha \in X_\alpha$, then there exists $y \in S$ such that $p_J(y) = p_J(a)$ and $p_\alpha(y) = x_\alpha$. Then $y \in S \cap p_J^{-1}(p_J(a))$.

Since $p_J^{-1}(p_J(a)) \cap S = p_J^{-1}(p_J(a)) \cap F$, the point $y \in F$. Thus $x_\alpha \in p_\alpha(F)$. Hence $p_\alpha(F) = X_\alpha$ for each $\alpha \in I \setminus J$. □

Proposition 2.15. *Let X_i be a Tychonoff space for each $i \in I$ and $X = \prod_{i\in I} X_i$ be a product space. Let S be a subset of X satisfying the following conditions:*

(1) $p_J(S) = X_J := \prod_{i\in J} X_i$ for each nonempty countable subset $J \subset I$;

(2) for each nonempty countable subset $J \subset I$ and each $y \in X_J$, the intersection $p_J^{-1}(y) \cap S$ is not compact.

Then for the canonical base $\mathcal{B}$ for X and for any infinite family $\mathcal{V}_S$ of $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$, the set F of all accumulation points of the family $\mathcal{V}_S$ in S is not a nonempty compact set.

Proof. Suppose that there exists a countable infinite subfamily $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ of $\mathcal{B}_S$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is a nonempty compact subset of S. Then it follows from Theorem 2.14 that for any $a \in F$ there exists a countable subset J of I such that $p_J^{-1}(p_J(a)) \cap S = p_J^{-1}(p_J(a)) \cap F$. Since the set $p_J^{-1}(p_J(a)) \cap F$ is a closed subset of F and F is compact, the set $p_J^{-1}(p_J(a)) \cap F$ is compact. Then $p_J^{-1}(p_J(a)) \cap S$ is compact. A contradiction. □

Proposition 2.16. ([8], Corollary 2.7) *Let X_i be a Tychonoff space for each $i \in I$. Let $X = \prod_{i\in I} X_i$ be a product space and S be a subset of X satisfying the following conditions:*

(1) $p_J(S) = X_J := \prod_{i\in J} X_i$, for each nonempty countable subset $J \subset I$;

(2) for each nonempty countable subset $J \subset I$ and each $y \in X_J$, the intersection $p_J^{-1}(y) \cap S$ is not compact.

If bS is a compactification of S, then the remainder $Y = bS \setminus S$ is pseudocompact.

Proof. It can be gotten by Theorem 2.13 and Proposition 2.15. □

Theorem 2.17. ([8], Theorem 2.4) *Let $X = \prod_{i\in I} X_i$ be a product of Tychonoff spaces such that uncountably many of the factors X_i are non-compact. Also, let S be a subspace of X such that $p_J(S) = X_J$ for each countable set $J \subset I$, where $p_J : X \to X_J = \prod_{i\in J} X_i$ is the projection. If bS is a compactification of S, then the remainder $Y = bS \setminus S$ is pseudocompact.*

Proof. It is obvious that the subspace S of X is dense in X and it is nowhere locally compact. Let $\mathcal{B}$ be the canonical base for X and let $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$. Then $\mathcal{B}_S$ is a base for S. Suppose there exists a countable infinite subfamily $\mathcal{V}_S = \{B_i \cap S : i \in \omega\}$ of $\mathcal{B}_S$ such that the set F of all accumulation points of the family $\mathcal{V}_S$ in S is a nonempty compact subset of S. Then by Theorem 2.14 there exists a countable subset J of I such that for any $\alpha \in I \setminus J$, $p_\alpha(F) = X_\alpha$. Since the set F is compact and the mapping $p_J|S$ is continuous, the space X_α is compact for every $\alpha \in I \setminus J$. A contradiction.

Thus for any infinite subfamily $\mathcal{V}_S$ of $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}\}$, the set F of all accumulation points of the family $\mathcal{V}_S$ in S is not a nonempty compact set. Then it follows from Theorem 2.13 that $bS \setminus S$ is pseudocompact. □

Corollary 2.18. ([8], Corollary 2.5) *Let $\{X_i : i \in I\}$ be a family of Tychonoff spaces such that uncountably many of them are non-compact. If bX is a compactification of the product $X = \prod_{i\in I} X_i$, then the remainder $Y = bX \setminus X$ is pseudocompact.*

Lemma 2.19. *Let X be a regular space. If Y is a dense subspace of X and there exists a nonempty compact subspace F of Y which has a strong countable π-base in Y, then the set F has a strong countable π-base in X.*

Proof. Let $\mathcal{V} = \{V_n : n \in \omega\}$ be a family of nonempty open subsets of Y such that $\mathcal{V}$ is a strong countable π-base at a nonempty compact set F in Y.

For every $n \in \omega$, there exists an open subset U_n of X such that $U_n \cap Y = V_n$. Let O be any open neighborhood of F in X. By regularity of X and compactness of F, there exists an open set W of X such that $F \subset W \subset \overline{W} \subset O$. Then there exists $m \in \omega$ such that $V_n \subset W \cap Y$ for every $n \geq m$. Thus $\overline{V_n} \subset \overline{W} \subset O$ for every $n \geq m$. Since $\overline{Y} = X$ and U_n is open in X such that $U_n \cap Y = V_n$ for every $n \geq m$, we have $\overline{V_n} = \overline{U_n}$. Thus for every $n \geq m$, $U_n \subset \overline{U_n} \subset O$. Then $\{U_n : n \in \omega\}$ is a strong countable π-base at F in X. □

Theorem 2.20. *Let X be a Tychonoff space and let Y be a dense subspace of X. If X is a nowhere locally compact space such that for every compactification bX of X the remainder $bX \setminus X$ of X is pseudocompact, then for every compactification bY of Y the remainder $bY \setminus Y$ of Y is pseudocompact.*

Proof. Let bY be any compactification of Y. Since X is nowhere locally compact and Y is dense in X, the subspace Y of X is nowhere locally compact. Then $bY \setminus Y$ is dense in bY.

Suppose that the remainder $bY \setminus Y$ is not pseudocompact. By Lemma 2.10, there exists a nonempty compact subspace F of Y which has a strong countable π-base in Y. By Lemma 2.19, the set F has a strong countable π-base in X. If bX is a compactification of X, then it follows from Lemma 2.10 that the remainder $bX \setminus X$ of X is not pseudocompact. A contradiction. Thus the remainder $bY \setminus Y$ of Y is pseudocompact. □

By Corollary 2.18 and Theorem 2.20, we have the following result.

Theorem 2.21. *Let $\{X_i : i \in I\}$ be a family of Tychonoff spaces such that uncountably many of them are non-compact. If $X = \prod_{i\in I} X_i$ is a product space and Y is a dense subspace of X, then for every compactification bY of Y the remainder $bY \setminus Y$ is pseudocompact.*

In ([8], Theorem 3.7), it was proved that if X is an uncountable space and G is a non-compact topological group, then the remainder of $C_p(X, G)$ in any Hausdorff compactification is pseudocompact.

We denote the family of continuous functions from X to Y by $C(X, Y)$. The set with the topology inherited from the product space Y^X (that is, the pointwise convergence topology) is denoted by $C_p(X, Y)$. Every space of the form $C_p(X, Y)$ is assumed to be dense in Y^X ([8], p. 360). By Theorem 2.21, we have the following result.

Theorem 2.22. *Let Y be a non-compact Tychonoff space. If X is uncountable and $C_p(X, Y)$ is dense in Y^X, then for any compactification $bC_p(X, Y)$ of $C_p(X, Y)$, the remainder $bC_p(X, Y) \setminus C_p(X, Y)$ is pseudocompact.*

Proof. Since X is uncountable and Y is non-compact such that $C_p(X, Y)$ is dense Y^X, by Theorem 2.21, for any compactification $bC_p(X, Y)$ of $C_p(X, Y)$, the remainder $bC_p(X, Y) \setminus C_p(X, Y)$ is pseudocompact. □

Acknowledgement

The authors would like to thank the referee for his (or her) valuable remarks and suggestions which greatly improved the paper.

References

[1] A. V. Arhangel'skii, *A class of spaces containing all metric and all locally compact spaces*, Mat. Sb. **67(109)** (1965), 55–88 (in Russian); English translation: Amer. Math. Soc. Transl. **92** (1970), 1–39.
[2] A. V. Arhangel'skii, *Two types of remainders of topological groups*, Comment. Math. Univ. Carolin. **49** (2008), 119–126.
[3] A. V. Arhangel'skii, A. Bella, *On pseudocompactness of remainders of topological groups and some classes of mappings*, Topology Appl. **111** (2001), 21–33.
[4] A. V. Arhangel'skii, D. K. Burke, *Spaces with a regular G_δ-diagonal*, Topology Appl. **153** (2006), 1917–1929.
[5] A. V. Arhangel'skii, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Stud. Math., Vol. 1, Atlantis Press/World Scientific, Paris, Amsterdam, 2008.
[6] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed. 1989.
[7] M. Henriksen, J. R. Isbell, *Some properties of compactifications*, Duke Math. J. **25** (1958), 83–105.
[8] Á. Tamariz-Mascarúa, M. G. Tkachenko, *Remainders of products, topological groups and C_p-spaces*, Topology Appl. **258** (2019), 358–377.