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Property (R) and hypercyclicity for bounded linear operators

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Abstract. Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. In this paper, we mainly characterize these bounded linear operators T on $\mathcal{H}$ and their function calculus that satisfy property (R) by the new spectrum originated from the single-valued extension property. Meanwhile, the relationship between property (R) and hypercyclic property is also explored.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and the set of all non-negative integers, respectively. Let $\mathcal{H}$ be a complex infinite dimensional separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. The unit closed disk on the complex plane $\mathbb{C}$ is denoted by $\mathbb{D}$. For $T \in \mathcal{B}(\mathcal{H})$, T^*, $N(T)$ and $R(T)$ stand for the adjoint, the kernel and the range of T, respectively. If $R(T)$ is closed and $n(T) < \infty$, then we call T is an upper semi-Fredholm operator, while T is said to be lower semi-Fredholm if $d(T) < \infty$, where $n(T)$ and $d(T)$ denote the dimension of $N(T)$ and the codimension of $R(T)$, respectively. $T \in \mathcal{B}(\mathcal{H})$ is a semi-Fredholm operator if T is either an upper semi-Fredholm operator or a lower semi-Fredholm operator, while $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if T is both an upper semi-Fredholm operator and a lower semi-Fredholm operator. If T is semi-Fredholm, the index of T is defined as $\text{ind}(T) = n(T) - d(T)$. In particular, we call $T \in \mathcal{B}(\mathcal{H})$ is a bounded below operator if T is upper semi-Fredholm with $n(T) = 0$. If T is semi-Fredholm with $\text{ind}(T) = 0$, then T is said to be a Weyl operator. The ascent and descent of T are defined respectively by $\text{asc}(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and $\text{des}(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. If the infimum does not exist, then we write $\text{asc}(T) = \infty$ (resp. $\text{des}(T) = \infty$). T is called a Browder operator if it is Fredholm of finite ascent and descent, equivalently, T is semi-Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. T is called an upper semi-Weyl operator if it is upper semi-Fredholm with $\text{ind}(T) \leq 0$, while T is called an upper semi-Browder operator if it is upper semi-Fredholm of finite ascent. The spectrum $\sigma(T)$, the approximate point spectrum $\sigma_a(T)$, the upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$, the semi-Fredholm spectrum $\sigma_{SF}(T)$, the essential approximate

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point spectrum $\sigma_{ea}(T)$, the Browder essential approximate point spectrum $\sigma_{ab}(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined respectively by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an invertible operator}\},$$

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a bounded below operator}\},$$

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Fredholm operator}\},$$

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Weyl operator}\},$$

$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Browder operator}\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\},$$

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a semi-Fredholm operator}\}.$$

Let $\rho(T) = \mathbb{C} \setminus \sigma(T)$, $\rho_a(T) = \mathbb{C} \setminus \sigma_a(T)$, $\rho_w(T) = \mathbb{C} \setminus \sigma_w(T)$, $\rho_{SF_+}(T) = \mathbb{C} \setminus \sigma_{SF_+}(T)$, $\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{SF}(T)$, $\rho_{ab}(T) = \mathbb{C} \setminus \sigma_{ab}(T)$ and $\rho_b(T) = \mathbb{C} \setminus \sigma_b(T)$. $\sigma_0(T)$ is denoted by the set of all normal eigenvalues of T, that is $\sigma_0(T) = \sigma(T) \setminus \sigma_b(T)$. For a set $E \subseteq \mathbb{C}$, we write ∂E, int$E$, iso$E$ and acc$E$ as the set of boundary points, interior point, isolated points and accumulation points of E.

For a Cauchy domain ([1]) Ω, if all the curves of $\partial\Omega$ are regular analytic Jordan curves, we say that Ω is an analytic Cauchy domain. For $T \in \mathcal{B}(\mathcal{H})$, if σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, where $\overline{\Omega}$ is the closure of Ω. We denote by $E(\sigma; T)$ the Riesz idempotent of corresponding to σ, i.e.,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we have $\mathcal{H}(\sigma; T) = \mathcal{R}(E(\sigma; T))$. Clearly, if $\lambda \in \text{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$. We write $\mathcal{H}(\sigma; T) = \mathcal{R}(E(\sigma; T))$. We write $\mathcal{H}(\lambda; T)$ instead of $\mathcal{H}(\{\lambda\}; T)$; if in addition, $\dim\mathcal{H}(\lambda; T) < \infty$, then $\lambda \in \sigma_0(T)$.

The single-valued property (SVEP) plays an important role for bounded operators on complex Hilbert spaces. $T \in \mathcal{B}(\mathcal{H})$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for short) if for any open disc $\mathbb{D}_{\lambda_0}$ centered at λ_0, the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ satisfying the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}_{\lambda_0}$ is the function $f \equiv 0$ ([2]). Moreover, $T \in \mathcal{B}(\mathcal{H})$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

It is evident that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T)$ and T has SVEP at every point of the bounded $\partial\sigma(T)$ of the spectrum $\sigma(T)$ according to the identity theorem for analytic functions. Especially, T has SVEP at every isolated point of the spectrum $\sigma(T)$. Besides, if $\text{asc}(T) < \infty$, then T has SVEP at 0 and $n(T) \leq d(T)$ ([3]).

The variants of Weyl's theorem have been explored in lots of papers [5-6] since Weyl's theorem was discovered by Weyl ([4]) in 1909. Property (R) is one of these variants that has been introduced by Aiena,P. in 2011, and was discussed by many authors ([8, 9]). $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy property (R) ([7, Definition 2.3]), if

$$\sigma_a(T) \backslash \sigma_{ab}(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < n(T - \lambda I) < \infty\}$. In the following, we will define a new spectrum stemmed from the single-valued extension property to continue to study the property R.

The new spectrum set is defined as follows. Let

$$\rho_1(T) = \{\lambda \in \mathbb{C} : \ n(T - \lambda I) < \infty, \text{there exists } \epsilon > 0 \text{ such that } T \text{ and } T^* \text{ both have SVEP at } \mu \text{ if } 0 < |\mu - \lambda| < \epsilon\},$$

and let $\sigma_1(T) = \mathbb{C} \setminus \rho_1(T)$. Obviously, $\sigma_1(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$.

Remark 1.1. *(i) $\sigma_1(T)$ may be an emptyset.*

For instance, let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then T is a quasinilpotent operator with $n(T) = 0$ and $\sigma(T) = \sigma(T^*) = \{0\}$. Thus T and T^* both have SVEP at every $\lambda \in B^0(0)$, where $B^0(0)$ is a deleted neighbourhood of 0. Hence $\sigma_1(T) = \emptyset$ by the definition of $\rho_1(T)$.

(ii) $\sigma_1(T)$ is a clopen set.

(a) If $int\sigma(T) = \emptyset$ and $n(T - \lambda_0 I) < \infty$ for some $\lambda_0 \in \sigma(T) \cap acc\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$, then $\sigma_1(T)$ is not a closed set.

For example, let $A, B \in \mathcal{B}(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), B(x_1, x_2, x_3, \cdots) = (0, x_1, 0, \frac{x_3}{3}, 0, \frac{x_5}{5}, \cdots)$$

and $T \in \mathcal{B}(\ell^2 \oplus \ell^2)$ be defined by $T = \begin{pmatrix} A & 0 & 0 & \cdots \\ 0 & B+I & 0 & \cdots \\ 0 & 0 & B+\frac{I}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$*, then $0 \in \sigma(T)$, $n(T) = 0 < \infty$ and $0 \in acc\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \subseteq acc\sigma_1(T)$, but $0 \notin \sigma_1(T)$. Therefore $\sigma_1(T)$ is not a closed set.*

(b) $\sigma_1(T)$ is a closed set when $n(T - \lambda I) < \infty$ or $d(T - \lambda I) < \infty$ for any $\lambda \in int\sigma(T)$.

Suppose that $n(T - \lambda I) < \infty$ for any $\lambda \in int\sigma(T)$, then we claim that $\sigma_1(T)$ is a closed set. In fact, if not, then there exists a point $\lambda_0 \in \partial\sigma_1(T) \cap \rho_1(T)$. We can get there exists a deleted neighbourhood $B^0(\lambda_0)$ of λ_0 such that T and T^* have SVEP at every $\lambda \in B^0(\lambda_0)$ by the definition of $\rho_1(T)$. Take $\lambda_1 \in B^0(\lambda_0) \cap \sigma_1(T)$, then there exists a neighbourhood $B(\lambda_1) \subseteq B^0(\lambda_0)$ of λ_1 such that $B(\lambda_1) \subseteq \sigma_1(T)$. That is to say that $\lambda_1 \in int\sigma_1(T) \subseteq int\sigma(T)$. Then $n(T - \lambda_1 I) < \infty$ and T and T^* both have SVEP at every $\lambda \in B^0(\lambda_1) \subseteq B^0(\lambda_0)$, hence $\lambda_1 \in \rho_1(T)$, which is a contradiction.

Assume that $d(T - \lambda I) < \infty$ for any $\lambda \in int\sigma(T)$. Similar to the proof of the above, we can take $\lambda_1 \in B^0(\lambda_0) \cap \sigma_1(T)$, then we know $\lambda_1 \in int\sigma_1(T) \subseteq int\sigma(T)$. Thus $d(T - \lambda_1 I) < \infty$. So then, $T - \lambda_1 I$ is Browder ([10, Lemma 3.4]) according to T and T^* both have SVEP at λ_1, which is a contradiction to $\lambda_1 \in int\sigma(T)$.

2. Property (*R*) for bounded linear operators and their operator functions

In this section, we will give some characterizations for bounded linear operators and their function calculus that satisfy property (R) by way of the new spectrum set $\sigma_1(T)$. Let $\sigma_d(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I)$ is not closed $\}$. Then we have the following inclusions.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*

(1) T satisfies the property (R);

(2) $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

Proof. (1) $\Rightarrow$ (2). The inclusion " $\supseteq$ " is obvious. For the opposite inclusion, take arbitrarily $\lambda_0 \notin [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$, without loss of generality, suppose that $\lambda_0 \in \sigma(T)$, then $n(T - \lambda_0 I) > 0$.

Case1 Suppose that $\lambda_0 \notin \sigma_1(T)$, then $0 < n(T - \lambda_0 I) < \infty$ and there exists $\epsilon_1 > 0$ such that T and T^* both have SVEP at every $\lambda \in B^0(\lambda_0, \epsilon_1)$, where $B^0(\lambda_0, \epsilon_1)$ is a deleted neighbourhood of λ_0. If $\lambda_0 \notin \mathrm{acc}\sigma(T)$, then $\lambda_0 \in \pi_{00}(T)$. Since T satisfies property (R), we can get $\lambda_0 \notin \sigma_b(T)$. If $\lambda_0 \notin \sigma_d(T)$, then $T - \lambda_0 I$ is an upper semi-Fredholm operator. By the punctured neighborhood theorem of semi-Fredholm operators, there exists $\epsilon < \epsilon_1$ such that $T - \lambda I$ is upper semi-Fredholm and $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n)]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Noting that T and T^* both have SVEP at λ, we know $T - \lambda I$ is a Browder operator ([10, Lemma 3.4]). From $N(T - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda I)^n)]$, we see $T - \lambda I$ is invertible. Namely, $\lambda_0 \in \mathrm{iso}\sigma(T)$. Therefore $\lambda_0 \notin \sigma_b(T)$ combining with the fact that $T - \lambda_0 I$ is an upper semi-Fredholm operator.

Case2 Suppose that $\lambda_0 \notin \sigma_{ab}(T)$, then $\lambda_0 \in \sigma_a(T) \setminus \sigma_{ab}(T)$. Since T satisfies property (R), we can get $\lambda_0 \notin \sigma_b(T)$.

(2) ⇒ (1). It is obvious that $\{[\sigma_a(T)\backslash\sigma_{ab}(T)] \cup \pi_{00}(T)\} \cap [\sigma_1(T) \cap \sigma_{ab}(T)] = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ab}(T)] \cup \pi_{00}(T)\} \cap [\mathrm{acc}\sigma(T)\cap\sigma_d(T)] = \emptyset$, and $\{[\sigma_a(T)\backslash\sigma_{ab}(T)]\cup\pi_{00}(T)\}\cap\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} = \emptyset$. Accordingly, $[\sigma_a(T)\backslash\sigma_{ab}(T)]\cup \pi_{00}(T) = \sigma_0(T)$. It follows that $T \in (R)$. □

Remark 2.2. *In Theorem 2.1, suppose $T \in \mathcal{B}(\mathcal{H})$ satisfies property (R), then each part of the decomposition of $\sigma_b(T)$ can not be deleted.*

(a) Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots),$$

then we have $\sigma_a(T) = \{0, 1\}$, $\sigma_{ab}(T) = \{1\}$ and $\pi_{00}(T) = \{0\}$. So $T \in (R)$. But $\sigma_b(T) \neq [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. That is $\sigma_1(T) \cap \sigma_{ab}(T)$ can not deleted.

(b) Let $A, B \in \mathcal{B}(\ell^2)$ be defined by

$$A = (a_{ij}), a_{ij} = \begin{cases} 1, |i-j| = 1 \\ 0, |i-j| \neq 1 \end{cases}, B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and put $T \in \mathcal{B}(\ell^2 \oplus \ell^2)$ be $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then we have $\sigma_a(T) = \sigma_{ab}(T) = [-2, 2]$ and $\pi_{00}(T) = \emptyset$. Clearly, $T \in (R)$. However, $\sigma_b(T) \neq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. So $acc\sigma(T) \cap \sigma_d(T)$ can not deleted.

(c) Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

then $\sigma_a(T) = \sigma_{ab}(T) = \partial\mathbb{D}$ and $\pi_{00}(T) = \emptyset$. It follows that $T \in (R)$. But $\sigma_b(T) \neq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)]$, we know $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ can not deleted.

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*

(1) $T \in (R)$;

(2) $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup acc\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$;

(3) $\sigma_b(T) = \partial\sigma_1(T) \cup [int\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup acc\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$;

(4) $\sigma_b(T) = \partial\sigma_1(T) \cup [acc\sigma(T) \cap \sigma_{ab}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

Proof. (1) ⇒ (2) By Theorem 2.1 we know that $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Since $[\sigma_1(T)\cap\sigma_{ab}(T)] = [\sigma_1(T)\cap\sigma_{ab}(T)\cap\sigma_{SF_+}(T)]\cup[\sigma_1(T)\cap\sigma_{ab}(T)\cap\rho_{SF_+}(T)] \subseteq [\sigma_1(T)\cap\sigma_{SF+}(T)]\cup\mathrm{acc}\sigma_a(T)$, and $\mathrm{acc}\sigma(T) \cap \sigma_d(T) = [\mathrm{acc}\sigma(T) \cap \sigma_d(T) \cap \mathrm{acc}\sigma_a(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T) \cap \mathrm{iso}\sigma_a(T)] \subseteq \mathrm{acc}\sigma_a(T) \cup [\sigma_1(T) \cap \sigma_{SF_+}(T)]$, we can get $\sigma_b(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. The opposite inclusion is clear, then we have $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(2) ⇒ (1) We only need to prove that $\sigma_b(T) = [\sigma_1(T)\cap\sigma_{ab}(T)]\cup[\mathrm{acc}\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$ by Theorem 2.1. The " ⊇ " is clear. Next we prove the opposite inclusion. $\mathrm{acc}\sigma_a(T) = [\mathrm{acc}\sigma_a(T) \cap \sigma_d(T)] \cup [\mathrm{acc}\sigma_a(T) \cap \rho_d(T)] \subseteq [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup [\sigma_1(T) \cap \sigma_{ab}(T)]$, thus $\sigma_b(T) \subseteq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. It follows that $T \in (R)$ by Theorem 2.1.

(2) ⇒ (3) Noting that $\sigma_1(T)\cap\sigma_{SF_+}(T) = [\sigma_1(T)\cap\sigma_{SF_+}(T)\cap\partial\sigma_1(T)]\cup[\mathrm{int}\sigma_1(T)\cap\sigma_{SF_+}(T)] \subseteq \partial\sigma_1(T)\cup[\mathrm{int}\sigma_1(T)\cap \sigma_{SF_+}(T)]$, then $\sigma_b(T) \subseteq \partial\sigma_1(T) \cup [\mathrm{int}\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Also, the opposite inclusion is obvious. Hence $\sigma_b(T) = \partial\sigma_1(T) \cup [\mathrm{int}\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(3) ⇒ (2) Since $\partial\sigma_1(T) = [\partial\sigma_1(T) \cap \sigma_1(T)] \cup [\partial\sigma_1(T) \cap \rho_1(T)] \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T)$ and $\mathrm{int}\sigma_1(T) \cap \sigma_{SF_+}(T)] \subseteq \sigma_1(T) \cap \sigma_{SF_+}(T)$, $\sigma_b(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$, we have $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(1) ⇒ (4) By Theorem 2.1 we have $\sigma_b(T) = [\sigma_1(T)\cap\sigma_{ab}(T)]\cup [\mathrm{acc}\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$. It is evident that $\mathrm{acc}\sigma(T)\cap\sigma_d(T) \subseteq acc\sigma(T)\cap\sigma_{ab}(T)$. Moreover, $[\sigma_1(T)\cap\sigma_{ab}(T)] = [\sigma_1(T)\cap\sigma_{ab}(T)\cap\partial\sigma_1(T)]\cup[\mathrm{int}\sigma_1(T)\cap \sigma_{ab}(T)] \subseteq \partial\sigma_1(T) \cup [\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T)]$, thus $\sigma_b(T) = \partial\sigma_1(T) \cup [\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(4) ⇒ (1) Observing that $\partial\sigma_1(T) = [\partial\sigma_1(T) \cap \sigma_1(T)] \cup [\partial\sigma_1(T) \cap \rho_1(T)]$, $[\partial\sigma_1(T) \cap \sigma_1(T)] = [\partial\sigma_1(T) \cap \sigma_1(T) \cap \sigma_{ab}(T)] \cup [\partial\sigma_1(T) \cap \sigma_1(T) \cap \rho_{ab}(T)] \subseteq \sigma_1(T) \cap \sigma_{ab}(T)$, and $\partial\sigma_1(T) \cap \rho_1(T) = [\partial\sigma_1(T) \cap \rho_1(T) \cap \sigma_d(T)] \cup [\partial\sigma_1(T) \cap$

$\rho_1(T) \cap \rho_d(T)] \subseteq [\mathrm{acc}\sigma(T) \cap \sigma_d(T)]$, we can acquire $\partial\sigma_1(T) \subseteq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)]$. Besides, $\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T) = [\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T) \cap \sigma_d(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T) \cap \rho_d(T)] \subseteq [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup [\sigma_1(T) \cap \sigma_{ab}(T)]$, then $\partial\sigma_1(T) \cup [\mathrm{acc}\sigma(T) \cap \sigma_{ab}(T)] \subseteq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)]$. Combining with the fact that the opposite inclusion is clear, we know that $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Therefore $T \in (R)$ according to Theorem 2.1. □

It is easy to see that if $\sigma_a(T) \backslash \sigma_{ab}(T) \subseteq \rho_w(T) = \sigma(T) \backslash \sigma_w(T)$ and $\pi_{00}(T) \subseteq \rho_w(T)$, then T satisfies property (R). Therefore the property (R) is closely related to $\sigma_w(T)$, then we get the following inclusions.

Corollary 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*
(1) $T \in (R)$;
(2) $\sigma_w(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in acc\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

Proof. (1) ⇒ (2) The " ⊇ " is evident. By Corollary 2.3 we know that $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. According to $\sigma_w(T) \subseteq \sigma_b(T)$, we have $\sigma_w(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) = d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) = d(T - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) < \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$, and $\{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) = d(T - \lambda I) = \infty\} \subseteq \sigma_1(T) \cap \sigma_{SF_+}(T)$. Hence $\sigma_w(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. It follows that $\sigma_w(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in \mathrm{acc}\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(2) ⇒ (1) Noting that $\sigma_b(T) = \sigma_w(T) \cup [\sigma_b(T) \cap \rho_w(T)] \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\sigma_b(T) \cap \rho_w(T)]$, and $\sigma_b(T) \cap \rho_w(T) \subseteq \mathrm{acc}\sigma_a(T)$, we see that $\sigma_b(T) \subseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Also, $\sigma_b(T) \supseteq [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ is evident. Thus $T \in (R)$ by Corollary 2.3. □

Similarly, in Corollary 2.4, we can also get each part of the decomposition of $\sigma_w(T)$ can not be deleted when $T \in \mathcal{B}(\mathcal{H})$ satisfies property (R) and we can get the following fact from Corollary 2.3.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*
(1) $T \in (R)$;
(2) $\sigma_w(T) = \partial\sigma_1(T) \cup [int\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in acc\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$;
(3) $\sigma_w(T) = \partial\sigma_1(T) \cup [acc\sigma(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in acc\sigma_a(T) : n(T - \lambda I) \neq d(T - \lambda I)\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

For $T \in \mathcal{B}(\mathcal{H})$, $\mathrm{Hol}(\sigma(T))$ denotes the set of all functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$. Given $f \in Hol(\sigma(T))$, we let $f(T)$ denote the Riesz-Dounford functional calculus of T with respect to f ([12]). Before giving the results that $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$, we pay attention to the following fact firstly.

Remark 2.6. *(i) $T \in \mathcal{B}(\mathcal{H})$ satisfies property (R) does not imply $f(T)$ satisfies property (R) for all $f \in Hol(\sigma(T))$.*
Let $A, B \in \mathcal{B}(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots), B(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots),$$

and put $T = \begin{pmatrix} A + I & 0 \\ 0 & B - I \end{pmatrix}$, then $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \cup \{-1\}$, $\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$ and $\pi_{00}(T) = \{-1\}$. So $T \in (R)$. Set $f_1(z) = (z + 1)(z - 1)$, then $0 \in \sigma_a(f_1(T)) \backslash \sigma_{ab}(f_1(T))$, but $0 \notin \pi_{00}(f_1(T))$. That is $f_1(T) \notin (R)$.
(ii) We can not get $T \in (R)$ if there exists some $f \in Hol(\sigma(T))$ such that $f(T)$ satisfies property (R).
Let $A \in \mathcal{B}(\ell^2)$ be defined by (1) and let $B \in B(\ell^2)$ be defined by $B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$, and put $T = \begin{pmatrix} A + I & 0 \\ 0 & B - I \end{pmatrix}$, then $\sigma_a(T^2) = \sigma_{ab}(T^2) = \{re^{i\theta} : r = 2(1 + cos\theta), -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\} \cup \{1\}$ and $\pi_{00}(T^2) = \emptyset$. Hence $T^2 \in (R)$. However, $\sigma_a(T) = \sigma_{ab}(T) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \cup \{-1\}$ and $\pi_{00}(T) = \{-1\}$. It follows that $T \notin (R)$.

From the above Remark, T and $f(T)$ satisfy property (R) are not directly connected. In the following, we will give the sufficient and necessary conditions such that $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$ by means of $\sigma_1(T)$.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$, then $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$ if and only if the following conditions hold:*
(1) $T \in (R)$;
(2) if $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)]$.

Proof. " $\Rightarrow$ " (1) holds evidently.

For (2), " $\supseteq$ " is clear. For the converse, we take $\lambda_1 \in \sigma_0(T)$ when $\sigma_0(T) \neq \emptyset$. Then we firstly claim that $\sigma(T) = \sigma_a(T)$ and $\text{iso}\sigma(T) \subseteq \sigma_p(T)$, where $\sigma_p(T) = \{\lambda \in \mathbb{C} : n(T - \lambda I) > 0\}$. In fact, take $\lambda_2 \in \rho_a(T)$ and put $f(z) = (z - \lambda_1)(z - \lambda_2)$, then $0 \in \sigma_a(f(T)) \setminus \sigma_{ab}(f(T))$. Since $f(T) \in (R)$, we have $f(T)$ is Browder and so is $T - \lambda_2 I$. Accordingly, $T - \lambda_2 I$ is invertible. So $\sigma(T) = \sigma_a(T)$.

Next, we prove $\text{iso}\sigma(T) \subseteq \sigma_p(T)$. Take $\lambda_3 \in \text{iso}\sigma(T)$ with $n(T - \lambda I) = 0$ and set $\sigma_1 = \{\lambda_1\}$, $\sigma_2 = \{\lambda_3\}$ and $\sigma_3 = \sigma(T) \setminus \{\lambda_1, \lambda_3\}$. Then T can be represented as $T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} \begin{matrix} H(\sigma_1; T) \\ H(\sigma_2; T) \\ H(\sigma_3; T), \end{matrix}$ by [11, Theorem 2.10], where $\sigma(T_i) = \sigma_i$, $i = 1, 2, 3$. Let $f(z) = (z - \lambda_1)(z - \lambda_3)$, then $f(T) = \begin{pmatrix} f(T_1) & 0 & 0 \\ 0 & f(T_2) & 0 \\ 0 & 0 & f(T_3) \end{pmatrix} \begin{matrix} H(\sigma_1; T) \\ H(\sigma_2; T) \\ H(\sigma_3; T) \end{matrix}$. We see that $0 \in \text{iso}\sigma(f(T))$ and $0 < n(f(T)) < \infty$. So, $0 \in \pi_{00}(f(T))$. It follows that $f(T)$ is Browder and so is $T - \lambda_3 I$ according to $f(T) \in (R)$. Then we get $T - \lambda_3 I$ is invertible which is a contradiction. Hence $\text{iso}\sigma(T) \subseteq \sigma_p(T)$. We can obtain $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ by Theorem 2.1. What's more, $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \subseteq \text{acc}\sigma(T) \cap \sigma_d(T)$ according to $\sigma(T) = \sigma_a(T)$ and $\text{iso}\sigma(T) \subseteq \sigma_p(T)$. Consequently, $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)]$.

" $\Leftarrow$ " Take $\mu_0 \in \sigma_a(f(T)) \setminus \sigma_{ab}(f(T))$ and assume that

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_t I)^{n_t} g(T), \qquad (*)$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $g(T)$ is invertible. Since $\sigma_{ab}(T)$ satisfies the spectral mapping theorem, then we have $\lambda_i \in \rho_a(T) \cup [\sigma_a(T) \setminus \sigma_{ab}(T)]$ and there must exist some $j (1 \leq j \leq t)$ such that $\lambda_j \in \sigma_a(T) \setminus \sigma_{ab}(T)$. Combining (1) we have $\lambda_j \in \sigma_0(T)$, hence $\sigma_0(T) \neq \emptyset$. From (2), noting that $\{\lambda \in \text{iso}\sigma(T) : n(T - \lambda I) = 0\} \cap \{[\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)]\} = \emptyset$ and $\rho_a(T) \cap \{[\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)]\} = \emptyset$, thus $\text{iso}\sigma(T) \subseteq \sigma_p(T)$ and $\sigma_a(T) = \sigma(T)$. Then $T - \lambda_i I$ is Browder and so is $f(T)$. Hence $\mu_0 \in \pi_{00}(f(T))$. For the converse, take arbitrarily $\mu_0 \in \pi_{00}(f(T))$ and supposed that $f(T) - \mu_0 I$ has the same decomposition as above (*). Then $\lambda_i \in \text{iso}\sigma(T) \cup \rho(T)$ and $n(T - \lambda_i I) < \infty$ for $1 \leq i \leq t$ and there must exist some $j (1 \leq j \leq t)$ such that $\lambda_j \in \text{iso}\sigma(T)$ with $n(T - \lambda_j I) > 0$. Combining (1) we have $\lambda_j \in \sigma_0(T)$, hence $\sigma(T) \neq \emptyset$. Then we can get $\lambda_j \in \pi_{00}(T)$ according to $\text{iso}\sigma(T) \subseteq \sigma_p(T)$. Due to $T \in (R)$, then $\lambda_j \in \sigma_0(T)$. Accordingly, $\lambda_i \notin \sigma_b(T)$ for $1 \leq i \leq t$. It follows that $f(T) - \mu_0 I$ is Browder. It suggests that $\pi_{00}(f(T)) \in \sigma_a(f(T)) \setminus \sigma_{ab}(f(T))$. Therefore, $f(T) \in (R)$. □

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$ if and only if one of the following conditions hold:*
(1) $\sigma(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$;
(2) $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [acc\sigma(T) \cap \sigma_d(T)]$.

Proof. We firstly prove the sufficiency. If (1) holds, then $\sigma_0(T) = \emptyset$ from $\sigma_0(T) \cap \{[\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}\} = \emptyset$. Hence we have $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\text{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Then $T \in (R)$ by Theorem 2.1. Also, $\sigma_a(T) = \sigma_{ab}(T)$ and $\pi_{00}(T) = \emptyset$ due to the fact that $\sigma_0(T) = \emptyset$. In this case, $\sigma_a(f(T)) = f(\sigma_a(T)) = f(\sigma_{ab}(T)) = \sigma_{ab}(f(T))$ due to both $\sigma_a(T)$ and $\sigma_{ab}(T)$ satisfy the spectral mapping theorem. It is easy see that $\pi_{00}(f(T)) \subseteq f(\pi_{00}(T))$. Ultimately, $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$.

If (2) holds, then $T \in (R)$ is clear by Theorem 2.1. Suppose that $\sigma_0(T) = \emptyset$, we have $f(T) \in (R)$ combining with the fact that $T \in (R)$. Furthermore, we can get $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$ by Theorem 2.7 when $\sigma_0(T) \neq \emptyset$.

Next, we will prove the necessity. We know $T \in (R)$ is evident, so $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$ by Theorem 2.1;

Case 1 We can conclude $\sigma(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$ under the condition $\sigma_0(T) = \emptyset$. Namely, (1) holds.

Case 2 Under the condition $\sigma_0(T) \neq \emptyset$, we can get (2) holds by Theorem 2.7. □

From Corollary 2.3 and Corollary 2.4 we can describe the property (R) for operator functions through $\sigma_w(T)$.

Corollary 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$, then $f(T) \in (R)$ for all $f \in Hol(\sigma(T))$ if and only if one of the following conditions hold:*

(1) $\sigma(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup acc\sigma_a(T) \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$;

(2) $\sigma_w(T) = [\sigma_1(T) \cap \sigma_{SF_+}(T)] \cup \{\lambda \in acc\sigma_a(T) : n(T-\lambda I) \neq d(T-\lambda I)\}$.

3. Property (*R*) and Hypercyclic Operators

For an operator $T \in \mathcal{B}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the orbit of x under T is the set of images of x under successive iterates of T:

$$Orb(T, x) = \{x, Tx, T^2x, T^3x, \cdots\}.$$

A vector $x \in \mathcal{H}$ is said to be hypercyclic if the set $Orb(T, x)$ is norm dense in the whole space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called hypercyclic if it has a hypercyclic vector. $\overline{HC(\mathcal{H})}$ denotes the norm-closure of all hypercyclic operators in $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ is said to have hypercyclic property if $T \in \overline{HC(\mathcal{H})}$. Hypercyclic property was proposed by Hilden and Wallen ([14]) in 1974. Kitai has studied many fundamental results regarding the theory of hypercyclic property in her thesis [15]. Also, the relationship between hypercyclic property and Weyl type theorem was explored by Cao ([16]). Then we will continue the work. The following lemma gives the simple description of hypercyclic property due to Herrero ([17]).

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$, then $T \in \overline{HC(\mathcal{H})}$ if and only if the following statements hold:*

(1) $\sigma_w(T) \cup \partial\mathbb{D}$ is connected;

(2) $\sigma_0(T) = \sigma(T) \backslash \sigma_b(T) = \emptyset$;

(3) $\forall \lambda \in \rho_{SF}(T)$, $ind(T-\lambda I) \geq 0$.

To begin with, we give examples which indicate that there is no direct relationship between property (R) and hypercyclic property for $T \in \mathcal{B}(\mathcal{H})$ firstly.

Example 3.2. *For instance: (i) Let $A \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and put $T = A + I$, then $T \in \overline{HC(\mathcal{H})}$ by Lemma 3.1. But since $\sigma_a(T) = \sigma_{ab}(T) = \pi_{00}(T) = \{1\}$, we know $T \notin (R)$. Therefore $T \in \overline{HC(\mathcal{H})} \nRightarrow T \in (R)$.

(ii) Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

then $T \in (R)$. However, since $\forall \lambda \in \rho_{SF}(T)$, $ind(T-\lambda I) \leq 0$, we have $T \notin \overline{HC(H)}$. Therefore $T \in (R) \nRightarrow T \in \overline{HC(\mathcal{H})}$.

(iii) Let $T \in \mathcal{B}(\ell^2)$ be defined by $A, B \in \mathcal{B}(\ell^2)$:

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots), B(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, x_4, \cdots),$$

and let $T \in \mathcal{B}(\ell^2 \oplus \ell^2)$ be defined by $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then we have $\sigma_a(T) = \partial\mathbb{D} \cup \{0\}$, $\sigma_a(T) = \partial\mathbb{D}$ and $\pi_{00}(T) = \emptyset$. Also, $\forall \lambda \in \rho_{SF}(T)$, $ind(T-\lambda I) \leq 0$. So $T \notin (R)$ and $T \notin \overline{HC(H)}$. Thus there exists $T \in \mathcal{B}(\mathcal{H})$ such that $T \notin (R)$ and $T \notin \overline{HC(\mathcal{H})}$.

(iv) Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots),$$

then it is easy to see that $T \in \overline{HC(H)}$ by Lemma 3.1 and $T \in (R)$. It follows that there exists $T \in \mathcal{B}(\mathcal{H})$ such that $T \in (R)$ and $T \in \overline{HC(\mathcal{H})}$.

In the following, we will give the conditions such that $T \in (R)$ and $T \in \overline{HC(\mathcal{H})}$.

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [acc\sigma(T) \cap \sigma_d(T)]$ and $\sigma_w(T) \cup \partial\mathbb{D}$ is connected, then $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$.*

Proof. We can acquire $\sigma_0(T) = \emptyset$ from $\sigma_0(T) \cap \{[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] = \emptyset$. Also, $\{\lambda \in \rho_{SF}(T) : \mathrm{ind}(T - \lambda I) > 0\} \cap \{[\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] = \emptyset$, so $\forall \lambda \in \rho_{SF}(T)$, $\mathrm{ind}(T - \lambda I) \geq 0$. Therefore $T \in \overline{HC(H)}$ combining the fact that $\sigma_w(T) \cup \partial\mathbb{D}$ is connected by Lemma 3.1.

Next, we will prove $T \in (R)$. It follows from (2) and (3) of Lemma 3.1 that $\sigma(T) = \sigma_a(T) = \sigma_{ab}(T) = \sigma_b(T)$. Observing that $\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\} \subseteq \sigma_1(T) = \sigma_1(T) \cap \sigma(T) = \sigma_1(T) \cap \sigma_{ab}(T)$, thus $\sigma_b(T) \subseteq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \subseteq [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Moreover, $[\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \subseteq \sigma_b(T)$ is evident. Hence $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. It follows that $T \in (R)$ by Theorem 2.1. □

Remark 3.4. *If $T \in (R)$ and $T \in \overline{HC(\mathcal{H})}$, we can not get $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [acc\sigma(T) \cap \sigma_d(T)]$.*

For example: Let $A \in \mathcal{B}(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and put $T = A + I$, it is easy to see that $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$. But since $\sigma(T) = \{1\}, \sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\} = acc\sigma(T) \cap \sigma_d(T) = \emptyset$, $\sigma(T) \neq [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [acc\sigma(T) \cap \sigma_d(T)]$. Then we will give the necessary and sufficient conditions for which $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$.

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$, then $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$ if and only if $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$ and $\sigma_w(T) \cup \partial\mathbb{D}$ is connected.*

Proof. " $\Rightarrow$ " We only need to prove $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$ by Lemma 3.1. The inclusion " $\supseteq$ " is clear. For the opposite inclusion, we know that $\sigma_b(T) = [\sigma_1(T) \cap \sigma_{ab}(T)] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ by Theorem 2.1. It follows from $T \in \overline{HC(\mathcal{H})}$ that $\sigma(T) = \sigma_1(T) \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$. Meanwhile, noting that $\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\} \subseteq \mathrm{acc}\sigma(T) \cap \sigma_d(T)$, thus $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$.

" $\Leftarrow$ " Using the same way from the proof of Theorem 3.1, we can conclude that $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$. □

Remark 3.6. *In Corollary 3.5, each part of the decomposition of $\sigma_b(T)$ can not be deleted when $T \in (R)$ and $T \in \overline{HC(\mathcal{H})}$.*

(i) "$\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) \geq d(T - \lambda I)\}$" can not deleted.

Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots),$$

then $T \in \overline{HC(\mathcal{H})}$ and $T \in (R)$ by Lemma 3.1. But since $\sigma(T) = \mathbb{D}$ and $acc\sigma(T) \cap \sigma_d(T) = \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} = \partial\mathbb{D}$, we have $\sigma(T) \neq [acc\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$.

(ii) "$acc\sigma(T)\cap\sigma_d(T)$" cannot deleted.

Let $A, B \in \mathcal{B}(\ell^2)$ be defined by

$$A = (a_{ij}), a_{ij} = \begin{cases} 1, |i-j| = 1 \\ 0, |i-j| \neq 1 \end{cases}, B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and set $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $T \in \overline{HC(\mathcal{H})}$ by 3.1 and $T \in (R)$. But since $\sigma(T) = [-2,2]$, $\sigma_1(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\} = \emptyset$, and $\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} = [-2,0)\cup(0,2]$, we have $\sigma(T) \neq [\sigma_1(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}]\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$.

(iii) "$\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$" can not deleted.

Let $A \in \mathcal{B}(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and let $T = A+I$. Then $T \in \overline{HC(\mathcal{H})}$ by Lemma 3.1 and $T \in (R)$. But since $\sigma(T) = \{1\}$ and $\sigma_1(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\} = acc\sigma(T)\cap\sigma_d(T) = \emptyset$, we have $\sigma(T) \neq [\sigma_1(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}]\cup[acc\sigma(T)\cap\sigma_d(T)]$.

From Corollary 2.3, we can acquire the following results.

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:*

(1) $T \in (R)$ and $T \in \overline{HC(\mathcal{H})}$;

(2) $\sigma(T) = \partial\sigma_1(T)\cup[acc\sigma(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}]\cup[acc\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T)\cup\partial\mathbb{D}$ is connected;

(3) $\sigma(T) = \partial\sigma_1(T)\cup[acc\sigma(T)\cap\sigma_{SF_+}(T)]\cup\{\lambda \in acc\sigma_a(T) : n(T-\lambda I) \neq d(T-\lambda I)\}\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T)\cup\partial\mathbb{D}$ is connected.

In the following, suppose that $T \in \overline{HC(\mathcal{H})}$, then the condition of equivalence that $T \in (R)$ will change. We get the following results.

Theorem 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $T \in \overline{HC(\mathcal{H})}$, then $T \in (R)$ if and only if $\sigma(T) = \sigma_1(T)\cup[acc\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$.*

Proof. " $\Rightarrow$ " We have $\sigma_b(T) = [\sigma_1(T)\cap\sigma_{ab}(T)]\cup$ $[\text{acc}\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$ according to $T \in (R)$ by Theorem 2.1. So, from $T \in \overline{HC(\mathcal{H})}$, we conclude that $\sigma_1(T)\cap\sigma_{ab}(T) = \sigma_1(T)\cap\sigma(T) = \sigma_1(T)$. Therefore $\sigma(T) = \sigma_1(T)\cup$ $[\text{acc}\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$.

" $\Leftarrow$ " we only need prove that $\pi_{00}(T) = \emptyset$ based on $\sigma_a(T) = \sigma_{ab}(T)$. Observing that $\pi_{00}(T)\cap\{\sigma_1(T)\cup[\text{acc}\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}\} = \emptyset$ and $\pi_{00}(T) \subseteq \sigma(T)$. Thus $\pi_{00}(T) = \emptyset$. □

Corollary 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $T \in \overline{HC(\mathcal{H})}$, then the following statements are equivalent:*

(1) $T \in (R)$;

(2) $\sigma(T) = \partial\sigma_1(T)\cup acc\sigma(T)\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$;

(3) $\sigma_w(T) = [\sigma_1(T)\cap\sigma_{ea}(T)]\cup[acc\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$;

(4) $f(T) \in (R)$.

According to Corollary 2.3 and Corollary 3.7, we can get the following corollary.

Corollary 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $T \in (R)$, then the following statements are equivalent:*

(1) $T \in \overline{HC(\mathcal{H})}$;

(2) $\sigma(T) = [\sigma_1(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}]\cup[acc\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T)\cup\partial\mathbb{D}$ is connected;

(3) $\sigma(T) = \partial\sigma_1(T)\cup[acc\sigma(T)\cap\{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}]\cup[acc\sigma(T)\cap\sigma_d(T)]\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T)\cup\partial\mathbb{D}$ is connected;

(4) $\sigma(T) = \partial\sigma_1(T)\cup[acc\sigma(T)\cap\sigma_{SF_+}(T)]\cup\{\lambda \in acc\sigma_a(T) : n(T-\lambda I) \neq d(T-\lambda I)\}\cup\{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T)\cup\partial\mathbb{D}$ is connected.

Let $T \in \mathcal{B}(\mathcal{H})$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots), B(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots),$$

and let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $\sigma_1(T) = \mathbb{D}$, $\sigma(T) = [\sigma_1(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) \geq d(T-\lambda I)\}] \cup [\mathrm{acc}\sigma(T) \cap \sigma_d(T)] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$ and $\sigma_w(T) \cup \partial\mathbb{D}$ is connected. The results are true in above conclusions.

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