Filomat 38:21 (2024), 7457–7468
https://doi.org/10.2298/FIL2421457D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Impact of quasi-constant curvature in $f(\mathcal{R}, G)$ and $f(\mathcal{R}, \mathcal{T})$-gravity

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Abstract. In this article it is illustrated that a spacetime of quasi-constant curvature is a static spacetime as well as generalized Robertson-Walker spacetime under certain restrictions on the associated scalars. As a consequence, we prove that such a spacetime becomes a Robertson-Walker spacetime and belongs to Petrov classification I, D or O. We investigate this spacetime as a solution of $f(\mathcal{R}, G)$-gravity and $f(\mathcal{R}, \mathcal{T})$-gravity theories and describe the physical explanation of the Friedmann-Robertson-Walker metric. With the models $f(\mathcal{R}, G) = 2\mathcal{R} + \lambda G$ (λ is constant) and $f(\mathcal{R}, \mathcal{T}) = \mathcal{R} + 2\mathcal{T}$, several energy conditions in terms of associated scalars are explored.

1. Introduction

A spacetime is nothing but a Lorentzian manifold M^4 with the signature $(-, +, +, +)$ for the Lorentzian metric g, admitting a globally time-oriented vector. Numerous scholars, including ([4], [17], [18]), have explored spacetimes in different ways.

According to [2], [9], [10], a generalised Robertson-Walker (GRW) spacetime is a Lorentzian manifold M^n $(n \geq 4)$, whose metric can be written as

$$ds^2 = -\left(dx^1\right)^2 + e^q g^*_{v_1 v_2} dx^{v_1} dx^{v_2} \tag{1}$$

in which non-constant $q = q\left(x^1\right)$ being a function dependent on x^1 and $g^*_{v_1 v_2} = g^*_{v_1 v_2}(x^{v_3})$ are only functions of x^{v_3} $(v_1, v_2, v_3 = 2, 3, \ldots, n)$. The equation (1) also be constructed as the warped product $-\mathcal{I} \times_{e^q} \overset{*}{\mathbf{M}}$, the interval $\mathcal{I}$ in $\mathbb{R}$ is open and $\overset{*}{\mathbf{M}}$ indicates $(n-1)$-dimensional Riemannian manifold. The GRW spacetime becomes a Robertson-Walker (RW) spacetime if dim. of the Riemannian manifold $\overset{*}{\mathbf{M}}$ is 3 with constant curvature.

M^4 is described as a perfect fluid spacetime (PFS) if the Ricci tensor $\mathcal{R}_{kl}$ satisfies

$$\mathcal{R}_{kl} = c_1 g_{kl} + d_1 u_k u_l, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53C50; Secondary 53Z05, 83C05, 83C56.
Keywords. Spacetime of quasi-constant curvature; perfect fluid spacetime; energy condition; modified gravity.
Received: 30 November 2023; Accepted: 01 January 2024
Communicated by Ljubica Velimirović
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in which c_1, d_1 are scalars and the flow vector u_l is a unit time-like vector. IN general relativity (GR), the matter field is described by $\mathcal{T}_{kl}$, called the energy-momentum tensor (EMT) and since, heat conduction term is absent, the fluid is named perfect [20]. In a PFS, the EMT [25] is of the form

$$\mathcal{T}_{kl} = \left(\rho + \mu\right) u_k u_l + \rho g_{kl}, \tag{3}$$

where μ and ρ stand for energy density and isotropic pressure. According to the Einstein's field equations (EFE),

$$\mathcal{R}_{kl} - \frac{1}{2} g_{kl}\mathcal{R} = \kappa \mathcal{T}_{kl}, \tag{4}$$

where κ denotes gravitational constant, $R = g^{kl}R_{kl}$ stands for the Ricci scalar.

The conformal curvature tensor C^l_{ijk} for M^4 is described by

$$C^l_{ijk} = \mathcal{R}^l_{ijk} - \frac{1}{2}\left\{g_{ij}\mathcal{R}^l_k - g_{ik}\mathcal{R}^l_j + \delta^l_k\mathcal{R}_{ij} - \delta^l_j\mathcal{R}_{ik}\right\} + \frac{\mathcal{R}}{6}\left\{\delta^l_k g_{ij} - \delta^l_j g_{ik}\right\} \tag{5}$$

in which $\mathcal{R}^l_{ijk}$ denotes the curvature tensor.

Chen and Yano [11] obtain the resulting expression of the curvature tensor in order to examine a conformally flat hypersurface of Euclidean space

$$\mathcal{R}_{lijk} = \alpha\left\{g_{lk}g_{ij} - g_{lj}g_{ik}\right\} + \beta\left\{g_{lk}u_i u_j - g_{lj}u_i u_k + g_{ij}u_l u_k - g_{ik}u_l u_j\right\}, \tag{6}$$

where u_k is a unit vector, often known as the generator, and α, β are scalars. A manifold with quasi-constant curvature, abbreviated by $(\mathbf{QC})_\mathbf{n}$, is an n-dimensional conformally flat manifold obeying (6).

However, it is easily proved that a manifold of quasi-constant curvature becomes conformally flat. Therefore, conformally flatness is not necessary according to the definition. Instead of considering a conformally flat manifold, Vranceanu [30] defined the idea of almost constant curvature using the same equation as (6). Afterwards Mocanu [24] demonstrates that the manifold proposed by Chen and Yano [11] are manifolds of the same type as those Vranceanu introduced. Many authors have studied a manifold with quasi-constant curvature, including ([5], [11], [15], [31]) and many others. If $\mathcal{R}_{lijk}$ obeys (6), then M^4 is called a spacetime of quasi-constant curvature. Here, we suppose that u_k is a unit time-like vector, that is, $u_k u^k = -1$, $u^k = g^{kj}u_j$. If $\beta = 0$, then it becomes spacetime of constant curvature. Throughout the paper we adopt that $\beta \neq 0$ and the term "$(\mathbf{QC})_\mathbf{4}$-spacetime" refers to a 4-dimensional spacetime with quasi-constant curvature. The 4-dimensional Lorentzian manifolds of quasi-constant curvature are conformally flat solutions of EFE for perfect fluid matter [28], and are referred to as infinitesimally spatially isotropic in physical literature.

The scientific world as a whole accepts the idea that our Cosmos is currently going through an accelerated phase. Standard GR cannot describe accelerated expansion without the addition of new concepts or elements, collectively referred to as dark energy. According to GR theory, it is commonly accepted that energy conditions (ECs) are essential resources for studying black holes and worm holes in various modified gravities ([3], [6], [14], [19], [21]). The Raychaudhuri equations [26], which methodically produce the ECs, express the intriguing nature of gravity through the positivity condition $\mathcal{R}_{lk}u^l u^k \geq 0$, u^l is a null vector. In GR theory on matter, through the EFE, this geometric criterion is the same as the null EC (NEC) $\mathcal{T}_{lk}u^l u^k \geq 0$. Certainly, the weak EC (WEC) reflects that $\mathcal{T}_{lk}u^l u^k \geq 0$, for every time-like vector u^l and preserves a positive local energy density. Several modifications and in-depth research have been done on EFE in [7].The "$f(\mathcal{R}, G)$-gravity theory"[14] was one of these modified theories. It was developed by changing the previous Ricci scalar $\mathcal{R}$ by a function of $\mathcal{R}$ and G, the Gauss-Bonnet (GB) invariant. The "$f(\mathcal{R}, \mathcal{T})$-gravity theory, " discovered by Harko et al. [19], was another modified theory. This is an extension of $f(\mathcal{R})$-gravity ([3], [6]) in which the trace $\mathcal{T}$ of the EMT is directly linked to any arbitrary function of $\mathcal{R}$. In this article, we consider two new models, for instance, $f(\mathcal{R}, G) = 2\mathcal{R} + \lambda G$ (λ is constant) and $f(\mathcal{R}, \mathcal{T}) = \mathcal{R} + 2\mathcal{T}$ to explain different ECs.

In [12], it is illustrated that a $(\mathbf{QC})_4$-spacetime is a RW spacetime and a Ricci symmetric ($\nabla_h \mathcal{R}_{lk} = 0$) $(\mathbf{QC})_4$-spacetime belongs to Petrov classification I, D or O. Also, they characterize $(\mathbf{QC})_4$-spacetime in $f(\mathcal{R})$-gravity.

In this article, we illustrate that a $(\mathbf{QC})_4$-spacetime with some restrictions on the associated scalars becomes a GRW spacetime as well as static spacetime and finally we prove that such a spacetime reduces to a RW spacetime. Also, it is shown that a $(\mathbf{QC})_4$-spacetime belongs to Petrov classification I, D or O without assuming Ricci symmetric ($\nabla_h \mathcal{R}_{lk} = 0$). Then we study $(\mathbf{QC})_4$-spacetime solutions in $f(\mathcal{R}, G)$ and $f(\mathcal{R}, \mathcal{T})$-gravity, respectively.

After introduction in Section 2, the analysis of $(\mathbf{QC})_4$-spacetime is presented. Finally, we provide $(\mathbf{QC})_4$-spacetime solutions in $f(\mathcal{R}, G)$ and $f(\mathcal{R}, \mathcal{T})$-gravity in the last two Sections.

2. Spacetime of quasi-constant curvature

Definition 2.1. *[32] A vector v_k on M^4 is called torse-forming if*

$$\nabla_l v_k = \varphi g_{lk} + \Omega_l v_k,$$

being Ω_l a non-vanishing one-form and φ is a scalar function.

If v_k is unit time-like, the aforementioned equation has the form

$$\nabla_l v_k = \varphi \{v_l v_k + g_{lk}\}.$$

Theorem A. [23] A M^4 is a GRW spacetime iff it permits a unit time-like torse-forming vector v_k: $\nabla_l v_k = \varphi \{v_l v_k + g_{lk}\}$ and v_k is an eigen vector of $\mathcal{R}_{lk}$.

Multiplying (6) with g^{ij}, we get

$$\mathcal{R}_{lk} = (3\alpha - \beta) g_{lk} + 2\beta u_l u_k, \tag{7}$$

which is a form of PFS.
Again, multiplying (7) with g^{lk}, we have

$$\mathcal{R} = 6(2\alpha - \beta). \tag{8}$$

If a spacetime is of quasi-constant curvature, then the spacetime becomes conformally flat. Then we have

$$C^l_{ijk} = 0 \tag{9}$$

and

$$\nabla_i \mathcal{L}_{lk} - \nabla_l \mathcal{L}_{ik} = 0, \tag{10}$$

where

$$\mathcal{L}_{lk} = -\frac{\mathcal{R}_{lk}}{2} + \frac{\mathcal{R}}{12} g_{lk}. \tag{11}$$

Using (7) and (8) in (11), we get

$$\mathcal{L}_{lk} = -\left\{\frac{\alpha}{2} g_{lk} + \beta u_l u_k\right\}. \tag{12}$$

Substituting (12) in (10), we obtain

$$\frac{1}{2}\{\alpha_i g_{lk} - \alpha_l g_{ik}\} + \{\beta_i u_l - \beta_l u_i\} u_k + \beta \{u_l (\nabla_i u_k) - u_i (\nabla_l u_k)\} + \beta \{(\nabla_i u_l) - (\nabla_l u_i)\} u_k = 0, \tag{13}$$

where $\alpha_l = \nabla_l \alpha$ and $\beta_l = \nabla_l \beta$.
Multiplying (13) with g^{lk} and u^k, respectively, we have

$$\frac{3\alpha_i}{2} - \beta_i - \beta_l u_i u^l - \beta u_i \left(\nabla_l u^l\right) - \beta u^l \left(\nabla_l u_i\right) = 0 \tag{14}$$

and

$$\frac{1}{2}\{\alpha_i u_l - \alpha_l u_i\} + \{\beta_l u_i - \beta_i u_l\} + \beta\{(\nabla_l u_i) - (\nabla_i u_l)\} = 0. \tag{15}$$

Again, multiplying (15) with u^l, we get

$$\beta_i - \frac{\alpha_i}{2} - \frac{1}{2}\alpha_l u^l u_i + \beta_l u^l u_i + \beta u^l \left(\nabla_l u_i\right) = 0. \tag{16}$$

Adding equations (14) and (16), we have

$$\alpha_i = \left\{\beta\left(\nabla_l u^l\right) + \frac{1}{2}\alpha_l u^l\right\} u_i. \tag{17}$$

If $\alpha \neq$ constant, then (17) becomes

$$\alpha_i = \psi u_i, \tag{18}$$

where $\psi = \beta\left(\nabla_l u^l\right) + \frac{1}{2}\alpha_l u^l$.
From (15) and (18), it follows that

$$\{\beta_l u_i - \beta_i u_l\} + \beta\{(\nabla_l u_i) - (\nabla_i u_l)\} = 0. \tag{19}$$

Using (19) in (13), we obtain

$$\frac{1}{2}\{\alpha_i g_{lk} - \alpha_l g_{ik}\} + \beta\{u_l(\nabla_i u_k) - u_i(\nabla_l u_k)\} = 0. \tag{20}$$

Equations (16) and (18) together imply

$$\beta_i = -\beta_l u^l u_i - \beta u^l \left(\nabla_l u_i\right). \tag{21}$$

If $\beta_i = -\beta_l u^l u_i$, then (21) becomes

$$\beta u^l \left(\nabla_l u_i\right) = 0. \tag{22}$$

Multiplying (20) with u^l, we get

$$\frac{1}{2}\left\{\alpha_i u_k - \alpha_l u^l g_{ik}\right\} - \beta\left\{(\nabla_i u_k) + u_i u^l \left(\nabla_l u_k\right)\right\} = 0. \tag{23}$$

Using (22) in (23), we have

$$\beta\left(\nabla_i u_k\right) = \frac{1}{2}\left\{\alpha_i u_k - \alpha_l u^l g_{ik}\right\}. \tag{24}$$

From (18) and (24), it follows that

$$\nabla_i u_k = \frac{\psi}{2\beta}\{g_{ik} + u_i u_k\}, \tag{25}$$

that is, u_k is a unit torse-forming vector.

Multiplying (7) with u^l, we have

$$\mathcal{R}_{lk}u^l = 3(\alpha - \beta)u_k, \tag{26}$$

that is, u_k is an eigen vector of $\mathcal{R}_{lk}$.

If α = constant, then we have from (24),

$$\nabla_i u_k = 0, \quad \text{since by assumption } \beta \neq 0. \tag{27}$$

A spacetime is said to be stationary if u_k is Killing and static ([27], [29], p. 283) for irrotational vector u_k. A static spacetime is the outcome of $\mathbb{R} \times S$ if it has the metric

$$g[(t, y)] = g_S[y] - \beta(y)dt^2,$$

g_S stands for a Riemannian metric on S.
We define the Lie derivative denoted by £, in a smooth vector v as

$$£_v g_{kl} = \nabla_k v_l + \nabla_l v_k.$$

Since $\nabla_i u_k = 0$, hence $£_u g_{kl} = 0$, which entails that u_l is Killing. Further $\nabla_i u_k = 0$ infers u_l is irrotational. Therefore, the spacetime is static. Thus, we conclude:

Theorem 2.2. *A* $(\mathbf{QC})_4$*-spacetime with* $\beta_i = -\beta_l u^l u_i$ *represents a* GRW *spacetime for* $\alpha \neq$ *constant and static spacetime for* α = *constant.*

It is commonly circulated that every static spacetime belongs to Petrov classification I, D or O. Consequently, under consideration a $(\mathbf{QC})_4$-spacetime is of Petrov classification I, D or O ([13], Section 10.7). Consequently, we state:

Corollary 2.3. *A* $(\mathbf{QC})_4$*-spacetime with* $\beta_i = -\beta_l u^l u_i$ *and* α = *constant belongs to Petrov classification I, D or O.*

From equation (7), we infer that the $(\mathbf{QC})_4$-spacetime represents a PFS. If α is constant, then from (17), we have $\nabla_l u^l = 0$, that is, the velocity vector is divergence-free. Therefore, the acceleration vector and the expansion scalar both vanish. Hence, we provide:

Corollary 2.4. *A* $(\mathbf{QC})_4$*-spacetime with* α = *constant is expansion scalar-free and acceleration vector-free.*

The local components Γ^l_{ij} of the Levi-Civita connection on warped product (1) are [16]

$$\Gamma^l_{11} = \Gamma^1_{11} = \Gamma^1_{1l} = 0,\ \Gamma^l_{ij} = \overset{*}{\mathbf{\Gamma}}{}^l_{ij},\ \Gamma^l_{1i} = \frac{1}{2}\tilde{q}\,\delta^l_i,\ \Gamma^1_{ij} = -\frac{1}{2}\tilde{q}\,e^q\,\overset{*}{\mathbf{g}}_{ij},\ \tilde{q} = \frac{dq}{dx^1}. \tag{28}$$

The local components of the conformal curvature tensor which in general does not vanish identically, are

$$C_{l11i} = \frac{1}{2}\left\{\frac{\overset{*}{\mathbf{R}}}{3}\overset{*}{\mathbf{g}}_{li} - \overset{*}{\mathbf{R}}_{li}\right\} \tag{29}$$

and

$$C_{lijk} = e^q\left\{\overset{*}{\mathbf{R}}_{lijk} - \frac{1}{2}\left(\overset{*}{\mathbf{R}}_{ij}\overset{*}{\mathbf{g}}_{lk} - \overset{*}{\mathbf{R}}_{ik}\overset{*}{\mathbf{g}}_{lj} + \overset{*}{\mathbf{R}}_{lk}\overset{*}{\mathbf{g}}_{ij} - \overset{*}{\mathbf{R}}_{lj}\overset{*}{\mathbf{g}}_{ik}\right) + \frac{\overset{*}{\mathbf{R}}}{6}\left(\overset{*}{\mathbf{g}}_{lk}\overset{*}{\mathbf{g}}_{ij} - \overset{*}{\mathbf{g}}_{lj}\overset{*}{\mathbf{g}}_{ik}\right)\right\}. \tag{30}$$

Since a spacetime of quasi-constant curvature is conformally flat, $-\mathcal{I} \times_{e^q} \overset{*}{\mathbf{M}}$ is conformally flat. Then from (29), we find

$$\overset{*}{\mathbf{R}}_{li} = \frac{\overset{*}{\mathbf{R}}}{3}\overset{*}{\mathbf{g}}_{li}. \tag{31}$$

In light of (30), substituting (31) into $C_{lijk} = 0$, we get

$$\overset{*}{\mathbf{R}}_{lijk} = \frac{\overset{*}{\mathbf{R}}}{6}\left\{\overset{*}{\mathbf{g}}_{ij}\overset{*}{\mathbf{g}}_{lk} - \overset{*}{\mathbf{g}}_{ik}\overset{*}{\mathbf{g}}_{lj}\right\}, \tag{32}$$

that is, $\overset{*}{\mathbf{M}}$ is a spacetime of constant curvature. Thus, we write:

Theorem 2.5. *A* $(\mathbf{QC})_4$*-spacetime with* $\beta_i = -\beta_l u^l u_i$ *and* $\alpha \neq$ *constant becomes a* RW *spacetime.*

3. $f(\mathcal{R}, G)$-gravity

Here, our attention is directed towards a particular class of modified gravity models, $f(\mathcal{R}, G)$ and the gravitational action term is

$$\mathcal{S} = \frac{1}{2\kappa} \int \sqrt{-g} f(\mathcal{R}, G)\, d^4x + \mathcal{S}_{\text{mat}}, \tag{33}$$

$\mathcal{S}_{\text{mat}}$ is the matter action and GB invariant G is represented as

$$G = \mathcal{R}^2 + \mathcal{R}_{lijk}\mathcal{R}^{lijk} - 4\mathcal{R}_{lk}\mathcal{R}^{lk}. \tag{34}$$

The action term equation (33) yields the gravitational field equations of $f(\mathcal{R}, G)$-gravity by

$$\mathcal{R}_{lk} - \frac{\mathcal{R}}{2} g_{lk} = \kappa \mathcal{T}_{lk} + \Sigma_{lk}, \tag{35}$$

where

$$\begin{aligned}\Sigma_{lk} &= \nabla_l \nabla_k f_{\mathcal{R}} + 2\mathcal{R}\nabla_l \nabla_k f_G - g_{lk}\Box f_{\mathcal{R}} - 2g_{lk}\mathcal{R}\Box f_G - 4\mathcal{R}_l^i \nabla_i \nabla_k f_G \\ &+ 4\mathcal{R}_{lk}\Box f_G - 4\mathcal{R}_k^i \nabla_i \nabla_l f_G + 4g_{lk}\mathcal{R}^{ij}\nabla_i \nabla_j f_G + 4\mathcal{R}_{lijk}\nabla^i \nabla^j f_G \\ &+ (1 - f_{\mathcal{R}})\left(\mathcal{R}_{lk} - \frac{\mathcal{R}}{2} g_{lk}\right) - \frac{1}{2}\left(G f_G + \mathcal{R} f_{\mathcal{R}} - f\right) g_{lk}.\end{aligned} \tag{36}$$

Here, $f_{\mathcal{R}} \equiv \dfrac{\partial f}{\partial \mathcal{R}}$, $f_G \equiv \dfrac{\partial f}{\partial G}$ and $\Box$ indicates the d'Alembert operator.

In the framework of $f(\mathcal{R}, G)$ modified gravity, the ECs are derived using the modified gravitational field equations, with the following outcomes

$$\text{NEC} \Longleftrightarrow \mu + \rho \geq 0, \tag{37}$$

$$\text{WEC} \Longleftrightarrow \mu \geq 0 \quad \text{and} \quad \mu + \rho \geq 0, \tag{38}$$

$$\text{DEC} \Longleftrightarrow \mu \geq 0 \quad \text{and} \quad \mu \pm \rho \geq 0, \tag{39}$$

$$\text{SEC} \Longleftrightarrow \mu + 3\rho \geq 0 \quad \text{and} \quad \mu + \rho \geq 0, \tag{40}$$

in which DEC and SEC denote the dominant EC and strong EC, respectively.

Equation (7) infers that

$$\mathcal{R}^{lk} = (3\alpha - \beta) g^{lk} + 2\beta u^l u^k. \tag{41}$$

Equations (7) and (41) together imply

$$\mathcal{R}_{lk}\mathcal{R}^{lk} = 36\alpha^2 - 36\alpha\beta + 12\beta^2. \tag{42}$$

From (6), it follows that

$$\mathcal{R}^{lijk} = \alpha \left\{ g^{lk} g^{ij} - g^{lj} g^{ik} \right\} + \beta \left\{ g^{lk} u^i u^j - g^{lj} u^i u^k + g^{ij} u^l u^k - g^{ik} u^l u^j \right\}. \tag{43}$$

Multiplying (6) and (43), one infers

$$\mathcal{R}_{lijk}\mathcal{R}^{lijk} = 24\alpha^2 - 24\alpha\beta + 12\beta^2. \tag{44}$$

Equations (8), (34), (42) and (43) together give the GB invariant as

$$G = 24\alpha (\alpha - \beta). \tag{45}$$

Consider a flat Friedmann Robertson Walker (FRW) metric

$$ds^2 = a^2(t)\left\{dx^2 + dy^2 + dz^2\right\} - dt^2, \tag{46}$$

in which $a(t)$ indicates the scale factor of the Universe. Given a perfect fluid equation of state for ordinary matter in the FRW background, the field equations for $f(\mathcal{R}, G)$-gravity are as follows:

$$2\dot{\mathcal{H}} f_{\mathcal{R}} + 8\mathcal{H}\dot{\mathcal{H}}\dot{f}_G = \mathcal{H}\dot{f}_{\mathcal{R}} + 4\mathcal{H}^3\dot{f}_G - \ddot{f}_{\mathcal{R}} - 4\mathcal{H}^2\ddot{f}_G, \tag{47}$$

$$24\mathcal{H}^3\dot{f}_G + 6\mathcal{H}^2 f_{\mathcal{R}} = f_{\mathcal{R}}\mathcal{R} - 6\mathcal{H}\dot{f}_{\mathcal{R}} - f(\mathcal{R}, G) + G f_G, \tag{48}$$

where the overdot represents a derivative with respect to the time coordinate t and $\mathcal{H} = \frac{\dot{a}}{a}$ indicates the Hubble parameter. Moreover, we acquire

$$\mathcal{R} = 6\left(2\mathcal{H}^2 + \dot{\mathcal{H}}\right) \tag{49}$$

and

$$G = 24\mathcal{H}^2\left(\mathcal{H}^2 + \dot{\mathcal{H}}\right). \tag{50}$$

From (8), (45), (49) and (50), we acquire

$$\dot{\mathcal{H}} + 2\mathcal{H}^2 = 2\alpha - \beta \tag{51}$$

and

$$\mathcal{H}^2\left(\dot{\mathcal{H}} + \mathcal{H}^2\right) = \alpha(\alpha - \beta). \tag{52}$$

Solving (51) and (52), we obtain either

$$\mathcal{H}^2 = \alpha - \beta \qquad \text{and} \qquad \dot{\mathcal{H}} = \beta \tag{53}$$

or,

$$\mathcal{H}^2 = \alpha \qquad \text{and} \qquad \dot{\mathcal{H}} = -\beta. \tag{54}$$

In a cosmological framework, the deceleration, jerk, and snap parameters can be described as:

$$q = -\frac{1}{\mathcal{H}^2}\frac{\ddot{a}}{a}, \qquad j = \frac{1}{\mathcal{H}^3}\frac{\dddot{a}}{a} \qquad \text{and} \qquad s = \frac{1}{\mathcal{H}^4}\frac{\ddddot{a}}{a}. \tag{55}$$

Since $\mathcal{H} = \frac{\dot{a}}{a}$, with the help of (53) we get

$$\frac{\ddot{a}}{a} = \alpha, \qquad \frac{\dddot{a}}{a} = \dot{\alpha} + \alpha\mathcal{H} \qquad \text{and} \qquad \frac{\ddddot{a}}{a} = \ddot{\alpha} + 2\dot{\alpha}\mathcal{H} + \alpha^2. \tag{56}$$

From (53), (55) and (56), it follows that

$$s = q^2 + 2(j + q) + \frac{\ddot{\alpha}}{(\alpha - \beta)^2}. \tag{57}$$

Also, from $\mathcal{H} = \frac{\dot{a}}{a}$ and equation (54) we find

$$\frac{\ddot{a}}{a} = \alpha - \beta, \ \frac{\dddot{a}}{a} = \dot{\alpha} - \dot{\beta} + (\alpha - \beta)\mathcal{H} \text{ and } \frac{\ddddot{a}}{a} = \ddot{\alpha} - \ddot{\beta} + 2\left(\dot{\alpha} - \dot{\beta}\right)\mathcal{H} + (\alpha - \beta)^2. \tag{58}$$

Equations (54), (55) and (58) reflect that

$$s = q^2 + 2(j+q) + \frac{\ddot{\alpha} - \ddot{\beta}}{\alpha^2}. \tag{59}$$

Thus, in a $(\mathbf{QC})_4$-spacetime obeying $f(\mathcal{R}, G)$-gravity, the deceleration, jerk, and snap parameters are linked by (57) or, (59).
The following subsection deals with the ECs for a $f(\mathcal{R}, G)$-gravity model.

A. $f(\mathcal{R}, G) = 2\mathcal{R} + \lambda G$

For this model, the equation (36) becomes

$$\Sigma_{lk} = \frac{1}{2} g_{lk}\mathcal{R} - \mathcal{R}_{lk}. \tag{60}$$

Using (60) in (35), we infer

$$2\mathcal{R}_{lk} - \mathcal{R}g_{lk} = \kappa\mathcal{T}_{lk}. \tag{61}$$

Here, we consider $(\mathbf{QC})_4$-spacetime solutions in $f(\mathcal{R}, G)$-gravity equation assuming the EMT is of the shape (3). Then equations (3), (7), (8) and (61) reflect that

$$\{\kappa\boldsymbol{\rho} + 6\alpha - 4\beta\}\, g_{lk} + \left\{\kappa\boldsymbol{\rho} + \kappa\boldsymbol{\mu} - 4\beta\right\} u_l u_k = 0. \tag{62}$$

Multiplying (62) with $u^l u^k$, we have

$$\boldsymbol{\mu} = \frac{6\alpha}{\kappa}. \tag{63}$$

Again, multiplying (62) with g^{lk} and using (63), we arrive

$$\boldsymbol{\rho} = \frac{4\beta - 6\alpha}{\kappa}. \tag{64}$$

The ECs for this setup can now be discussed using (63) and (64).

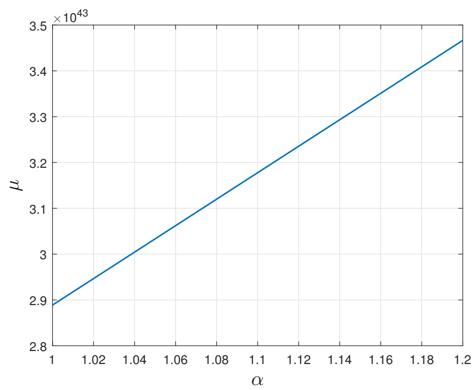

Fig. 1: Development of $\boldsymbol{\mu}$ with respect to α

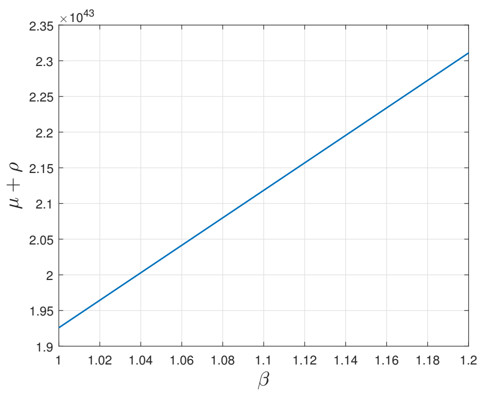

Fig. 2: Development of $\boldsymbol{\mu} + \boldsymbol{\rho}$ with respect to β

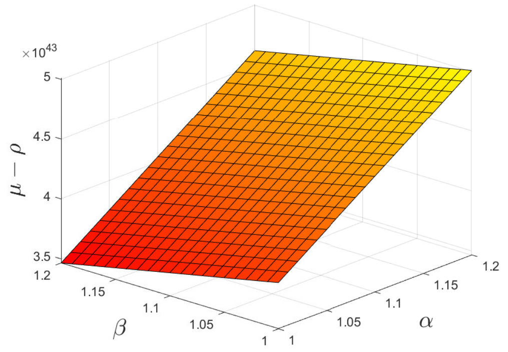

Fig. 3: Development of $\mu - \rho$ with respect to α and β

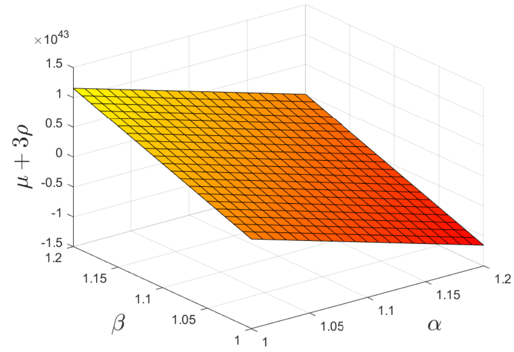

Fig. 4: Development of $\mu + 3\rho$ with respect to α and β

It is seen from Fig. 1 and Fig. 2, the energy density and $\mu + \rho$ can not be negative for the parameters $\alpha, \beta \in [1, 1.2]$ and for higher values of α and β, it is high. NEC and WEC are satisfied because NEC is a part of WEC. Fig. 3 shows the $\mu - \rho$ profile which has a positive range. Utilizing Fig. 1, 2 and 3, we conclude that DEC is satisfied whereas Fig. 4 indicate that SEC is not verified, and this result infers the late-time acceleration of the Cosmos[22]. Moreover, every outcomes compatible with the ΛCDM model [1].

4. $f(\mathcal{R}, \mathcal{T})$-gravity

Now, we study the PFS of quasi-constant curvature obeying $f(\mathcal{R}, \mathcal{T})$-gravity. Our hypothesis states that the action term for the modified theories of gravity has the subsequent shape:

$$\mathcal{S} = \int \sqrt{-g} \left\{ L_m + \frac{f(\mathcal{R}, \mathcal{T})}{16\pi} \right\} d^4x \tag{65}$$

in which L_m is the matter Lagrangian density. The EMT of the matter is described as

$$\mathcal{T}_{lk} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m \sqrt{-g}}{\delta^{lk}}. \tag{66}$$

The field equations of $f(\mathcal{R}, \mathcal{T})$-gravity are

$$\begin{aligned} \{\mathcal{R}_{lk} - \nabla_l \nabla_k + g_{lk}\square\} f_{\mathcal{R}}(\mathcal{R}, \mathcal{T}) + \{\mathcal{T}_{lk} + \Theta_{lk}\} f_{\mathcal{T}}(\mathcal{R}, \mathcal{T}) \\ - \frac{1}{2} f(\mathcal{R}, \mathcal{T}) g_{lk} - 8\pi \mathcal{T}_{lk} = 0 \end{aligned} \tag{67}$$

in which $f_{\mathcal{R}} \equiv \frac{\partial f}{\partial \mathcal{R}}$, $f_{\mathcal{T}} \equiv \frac{\partial f}{\partial \mathcal{T}}$ and

$$\Theta_{lk} = L_m g_{lk} - 2\mathcal{T}_{lk} - 2g^{ij} \frac{\partial^2 L_m}{\partial g^{lk} \partial g^{ij}}. \tag{68}$$

Using equation (3), we acquire the variation of stress energy as

$$\Theta_{lk} = \rho g_{lk} - 2\mathcal{T}_{lk}. \tag{69}$$

In conducting our research, we consider the model outlined below:

$$f(\mathcal{R}, \mathcal{T}) = \mathcal{R} + 2\mathcal{T}. \tag{70}$$

Equations (67) and (70) together produce

$$\mathcal{R}_{lk} + \{2 - 8\pi\}\mathcal{T}_{lk} - \frac{1}{2}(\mathcal{R} + 2\mathcal{T})g_{lk} + 2\Theta_{lk} = 0. \tag{71}$$

When Harko et al. [19] derived the field equations, they did not consider the conservation of the EMT. However, the EMT's conservation was assumed by the author of [8]. In the $(\mathbf{QC})_4$-spacetime solutions to the $f(\mathcal{R}, \mathcal{T})$-gravity equation, we assume that the EMT is conserved.

Making use of (7), (8), (69) and (71) provide us

$$\{8\pi\rho + 3\alpha - 2\beta + \mathcal{T}\}g_{lk} + \left\{(8\pi + 2)\left(\rho + \mu\right) - 2\beta\right\}u_l u_k = 0. \tag{72}$$

Multiplying (72) with g^{lk} and $u^l u^k$, respectively, we have

$$4\{8\pi\rho + 3\alpha - 2\beta + \mathcal{T}\} - (8\pi + 2)\left(\rho + \mu\right) + 2\beta = 0 \tag{73}$$

and

$$-\{8\pi\rho + 3\alpha - 2\beta + \mathcal{T}\} + (8\pi + 2)\left(\rho + \mu\right) - 2\beta = 0. \tag{74}$$

Multiplying (**??**) with g^{lk}, we obtain

$$\mathcal{T} = 3\rho - \mu. \tag{75}$$

Using (75), from (73) and (74) it follows that

$$\rho = \frac{2\beta(8\pi + 3) - 3\alpha(8\pi + 2)}{(8\pi + 2)(8\pi + 4)} \tag{76}$$

and

$$\mu = \frac{3\alpha(8\pi + 2) + 2\beta}{(8\pi + 2)(8\pi + 4)}. \tag{77}$$

Using (76) and (77), the ECs for this configuration can now be discussed.

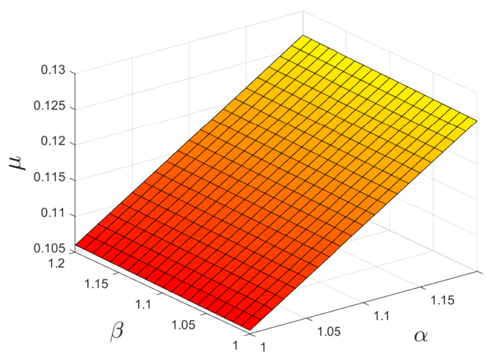

Fig. 5: Development of μ with respect to α and β

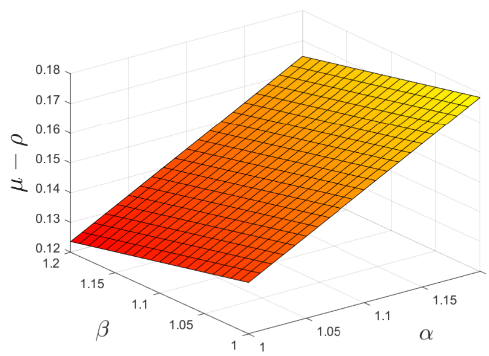

Fig. 6: Development of $\mu - \rho$ with respect to α and β

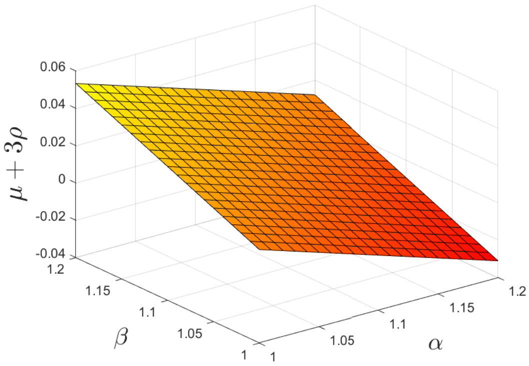

Fig. 7: Development of $\boldsymbol{\mu} + 3\boldsymbol{\rho}$ with respect to α and β

Figs. 5 indicates that $\boldsymbol{\mu}$ is positive for the parameters $\alpha, \beta \in [1, 1.2]$ and consequently $\boldsymbol{\mu} + \boldsymbol{\rho} = \frac{2\beta}{(8\pi + 2)}$ has a positive range. From Figs. 6 and 7, we conclude that DEC is verified for the parameters $\alpha, \beta \in [1, 1.2]$ but the SEC is not valid. Furthermore, both WEC and NEC are met for this construction.

5. Discussion

Exploring $(\mathbf{QC})_4$-spacetime solutions in relation to $f(\mathcal{R}, G)$-gravity and $f(\mathcal{R}, \mathcal{T})$-gravity has been the primary focus of this paper. The deceleration, jerk, and snap parameters in the quasi-constant curvature spacetime obeying $f(\mathcal{R}, G)$-gravity have been demonstrated. Our findings have been assessed both analytically and graphically in this instance. Our formulation was constructed using the analytical technique, and two cosmological models, such as $f(\mathcal{R}, G) = 2\mathcal{R} + \lambda G$ and $f(\mathcal{R}, \mathcal{T}) = \mathcal{R} + 2\mathcal{T}$, were evaluated for stability. The EC profiles for the first model are displayed in Figs. 1, 2, 3, and 4. The evolution of $\boldsymbol{\mu}$ for the parameters $\alpha, \beta \in [1, 1.2]$ has been found to be non negative. NEC, WEC, and DEC were satisfied, but SEC violated the terms of the agreement. These results, however, agree with the ΛCDM model. Figs. 5, 6, and 7 display each energy condition for the second model, just like they did for the first. Our findings for the second model agree with the first model's results.

6. Declarations

6.1. *Funding*

NA.

6.2. *Code availability*

NA.

6.3. *Availability of data*

NA.

6.4. *Conflicts of interest*

The authors have no conflicts to disclose.

References

[1] Aghanim, N., Akrami, Y., Ashdown, M., Aumont, J., Baccigalupi, C., Ballardini, M., Banday, A.J., Barreiro, R.B., Bartolo, N., Basak, S. and Battye, R., Planck 2018 results-VI. Cosmological parameters, Astronomy & Astrophysics, **641**(2020), Article ID A6.

[2] Alías, L., Romero, A. and Sánchez, M., Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, Gen. Relativ. Gravit., **27**(1995), 71–84.

[3] Bertolami, O., Böhmer, C.G., Harko, T. and Lobo, F.S.N., Extra force in $f(\mathcal{R})$ modified theories of gravity, Phys. Rev. D, **75**(2007), Article ID 104016.

[4] Blaga, A.M., Solitons and geometrical structures in a perfect fluid spacetime, Rocky Mt. J. Math., **50**(2020), 41–53.

[5] Boju, V. and Popescu, M., Espaces a courbure quasi-constante, J. Diff. Geom., **13**(1978), 373–383.

[6] Capozziello, S., Nojiri, S. and Odintsov, S.D., The role of energy conditions in $f(\mathcal{R})$ cosmology, Phys. Lett. B, **781**(2018), 99–106.

[7] Carroll, S.M., Duvvuri, V., Trodden, M. and Turner, M.S., Is cosmic speed-up due to new gravitational physics?, Phys. Rev. D, **70**(2004), Article ID 043528.

[8] Chakraborty, S., An alternative $f(\mathcal{R},\mathcal{T})$-gravity theory and the dark energy problem, Gen. Relativ. Gravit., **45**(2013), 2039–2052.

[9] Chen, B.-Y., A simple characterization of generalized Robertson–Walker spacetimes, Gen. Relativ. Gravit., **46**(2014), Article ID 1833.

[10] Chen, B.-Y., Differential Geometry of Warped Product Manifolds and Submanifolds, World Scientific Publishing, Hackensack, 2017.

[11] Chen, B.-Y. and Yano, K., Hypersurfaces of a conformally flat space, Teensor, N.S., **26**(1972), 318–322.

[12] De, U.C., De, K., Zengin, F.Ö. and Demirbağ, S.A., Characterizations of a Spacetime of Quasi-Constant Sectional Curvature and $\mathcal{F}(\mathcal{R})$-Gravity, Fortschr. Phys., **71**(2023), Article ID 2200201.

[13] Ehlers, J. and Kundt, W., Gravitation: An Introduction to Current Research, ed. L. Witten, Wiley, New York, 1962, 49.

[14] Elizalde, E., Myrzakulov, R., Obukhov, V.V. and Sáez-Gómez, D., ΛCDM epoch reconstruction from $F(\mathcal{R},G)$ and modified Gauss–Bonnet gravities, Class. Quantum Gravity, **27**(2010), Article ID 095007.

[15] Ganchev, G. and Mihova, V., Riemannian manifolds of quasi-constant sectional curvatures, J. Reine angew Math. 522 (2000), 119-141.

[16] Gebarowski, A., Nearly conformally symmetric warped product manifolds, Bulletin of the Inst. of Math. Academia Sinica, **20**(1992), 359–371.

[17] Güler, S. and Demirbağ, S.A., A study of generalized quasi-Einstein spacetimes with applications in general relativity, Int. J. Theor. Phys., **55**(2016), 548–562.

[18] Güler, S. and Ünal, B., The Existence of Gradient Yamabe Solitons on Spacetimes, Results Math., **77**(2022), Article ID 206.

[19] Harko, T., Lobo, F.S.N., Nojiri, S. and Odintsov, S.D., $f(R,T)$-gravity, Phys. Rev. D, **84**(2011), Article ID 024020.

[20] Hawking, S.W. and Ellis, G.F.R., The Large Scale Structure of Space-Time, Cambridge University Press, London, 1973.

[21] Laurentis, M.D., Paolella, M. and Capozziello, S., Cosmological inflation in $f(\mathcal{R},G)$ gravity, Phys. Rev. D, **91**(2015), Article ID 083531.

[22] Loo, T.-H., De, A., Mandal, S. and Sahoo, P.K., How a projectively flat geometry regulates $F(\mathcal{R})$-gravity theory?, Phys. Scr., **96**(2021), Article ID 125034.

[23] Mantica, C.A. and Molinari, L.G., Generalized Robertson–Walker spacetimes—A survey, Int. J. Geom. Methods Mod. Phys., **14**(2017), Article ID 1730001.

[24] Mocanu, A.L., Les variétés a curbure quasi-constant de type vranceanu, Luer. conf. Nat. De. Geom. Si Top., Tirgoviste, (1987).

[25] O'Neill, B., Semi-Riemannian Geometry with Applications to the Relativity, Academic Press, New York-London, 1983.

[26] Raychaudhuri, A.K., Banerji, S. and Banerjee, A., General relativity, astrophysics, and cosmology, Springer-Verlag New York, Inc. 1992.

[27] Sánchez, M., On the geometry of static spacetimes, Nonlinear Analysis: Theory, Methods & Applications, **63**(2005), 455–463.

[28] Sen, L.K., Infinitesimal null isotropy and Robertson–Walker metrics, J. Math. Phys., **26**(1985), 407–410.

[29] Stephani, H., Kramer, D., Mac-Callum, M., Hoenselaers, C. and Herlt, E., Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge 2009.

[30] Vranceanu, G., Lecons des Geometric Differential, Ed.de l'Academie, Bucharest, **4**(1968).

[31] Wang, Y., On some properties of Riemannian spaces of quasi-constant curvature, Tensor, N.S., **35**(1981), 173–176.

[32] Yano, K., On the torse-forming directions in Riemannian Spaces, Pro. Imp. Acad. Tokyo, **20**(1944), 340–345.