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On geometry of Lorentzian immersions with non-null hyperelastic curves

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Abstract. We characterize Lorentzian submanifolds by using non-null hyperelastic curves along a Lorentzian immersion defined between suitable two Lorentzian manifolds. We introduce a Lorentzian hyperelastic immersion as a map that carries a hyperelastic curve in the submanifold to a hyperelastic curve in the ambient manifold by the isometric immersion theory. Then, we investigate the characterization of submanifolds by using hyperelastic curves along Lorentzian immersions. Also, we exemplify the findings.

1. Introduction

The geometry of submanifolds is a fascinating and complex field that deals with the study of manifolds embedded in higher dimensions. A particularly important problem within this area of mathematics is the study of izometric immersions between two Riemannian manifolds. An isometric immersion is the imbedding of a manifold, defined as a total manifold into another manifold called the ambient, locally and conserving distance. On the other hand, the geometry of manifolds with certain curves were first studied on Riemannian manifolds in [7]. They presented the idea of a circle and expressed some important characterization about submanifold. Then, Ikawa gave the concept of a helix in a Riemannian manifold and he studied the behavior of helices immersions defined between two Riemannian manifolds, [5].

Lorentzian immersions are particularly important in space-time theory because a function defined in a differentiable way at any point in space-time can provide information about the motion of space-time. Lorentzian immersions are mathematical tools used in modeling the dynamics of space-time. In [2], Graves has studied isometric immersions defined between two Lorentzian space. The main goal of the author in that study is to make classification, up to a proper motion of ambient Lorentzian manifold, of all such immersions.Then, special classes of parallel submanifolds in Lorentzian space R_1^m and Euclidean space of signature (2, $m-2$) R_2^m have investigated. Magid has classified all umbilical submanifolds as well as isometric immersions $R^n \to R_1^{n+k}$, $R_1^n \to R_1^{n+2}$ and $R_1^n \to R_2^{n+2}$ with paralel second fundamental forms, [3]. Also, in [14], Ikawa surveyed curves in an indefinite-Riemannian manifold. The author studied circle and helix with respect to causel character of tangent vectors of curves and gave characterization related with Lorentzian submanifold.

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Besides these studies, the concept of elastic curves and hyperelastic curves has been used recently in the characterization of submanifolds. In [12], the authors presented the role of hyperelastic curves which are general type of classical elastic curves on the theory of immersions on Riemannian manifolds. Also, they gave relations between hyperelastic curves and umbilical submanifolds. Then, in [13], Riemannian maps were studied with the help of hyperelastic curves. Such maps are very important tools for study and comparing the geometry of two manifolds. Recently, biharmonic curves and triharmonic curves have also been studied along Riemannian submersions and Riemannian transformations, [15–17].

In this work, we first give some geometric materials related with Lorentzian immersion and the main problem of the paper. Then, we establish isometric Lorentzian immersions which transport non-null hyperelastic curves in section 3. Given that an arbitrary non-null curve taken from the submanifold carry to a non-null hyperelastic curve in the ambient manifold, we give to the isotropic condition of the Lorentzian submanifold. Also, we define the notation of Lorentzian hyperelastic immersion. By using Lorentzian hyperelastic immersions, we give some characterizations of the Lorentzian submanifolds with respect to mean curvature vector field of the manifold, curvature and torsion of the non-null hyperelastic curve in the submanifold. At last, we obtain some results for constant sectional curvature Lorentzian manifolds. Then, we give an example for our findings.

2. Materials and method

In this section, some geometric concepts of Lorentzian submanifolds and some formulas which are subsequently useful are given.

Let $\mathbb{R}^n_s$ indicate the pseudo-Riemannian $n-$space with the canonical metric of index s defined by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{n} dx_j^2,$$

where $(x_1, ..., x_n)$ is the coordinate system of $\mathbb{R}^n_s$. Let M be an $m-$ dimensional C^∞ manifold endowed with a metric g. If the signature of the metric g is s, then M is known an indefinite-Riemannian manifold of signature s. If $s = 1$, then M is known a Lorentzian manifold. Also, if g is positive definite, then M is called a Riemannian manifold. A vector w in $\mathbb{R}^n_s$ is called spacelike (timelike or lightlike, respectively) if $\langle w, w\rangle > 0$ ($\langle w, w\rangle < 0, \langle w, w\rangle = 0$ and $u \neq 0$, respectively). A curve α in $\mathbb{R}^n_s$ is known spacelike (timelike or lightlike, respectively) if its velocity vector α' is spacelike (timelike or lightlike, respectively) at each point. Let $f : M \to \bar{M}$ be an isometric immersion of an $n-$dimensional Lorentzian manifold M into an $(n + p)$-dimensional Lorentzian manifold $\bar{M}$. With ∇ ($\bar{\nabla}$ respectively), we denote the connection of M ($\bar{M}$ respectively). So, we can give following the Gauss and Weingarten which are important formulas for submanifolds theory.

Assume that h and A stand for the second fundamental form and the shape operator of M, respectively. For $X, Y \in \chi(M)$ and $V \in \chi(M)^\perp$, Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

and

$$\bar{\nabla}_X V = -A_V X + D_X V, \tag{2}$$

where D is the connection in the normal bundle. Also, we have the following relation

$$< A_V(X), Y >=< h(X, Y), V > . \tag{3}$$

On the other hand, from the Gauss and Weingarten formulas, we obtain

$$\bar{R}(X, Y)Z = R(X, Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y + \left(\tilde{\nabla}_X h\right)(Y, Z) - \left(\tilde{\nabla}_Y h\right)(X, Z) \tag{4}$$

where $\bar{R}$ and R are the Riemannian curvature tensor fields of $\bar{M}$ and M, respectively and $\tilde{\nabla}_X h$ is a symmetric $\mathfrak{I}(M)$−bilinear function $\chi(M)\times\chi(M)\longrightarrow\chi(M)^{\perp}$ (see, [9]). Also, for the covariant derivative of h, we have

$$\left(\tilde{\nabla}_X h\right)(Y,Z)=D_X h\,(Y,Z)-h\,(\nabla_X Y,Z)-h\,(Y,\nabla_X Z) \tag{5}$$

and covariant derivative of shape operator A is

$$(\nabla_X A)_V\,Y=\nabla_X\,(A_V Y)-A_{D_X V}Y-A_V\nabla_X Y. \tag{6}$$

In addition, we have the following equality for h and A,

$$<\left(\tilde{\nabla}_X h\right)(Y,Z)\,,V>=<\nabla_X\,(A)_V\,Y,Z>\,. \tag{7}$$

If $h\,(X,Y)$ satisfies

$$h\,(X,Y)=<X,Y>H, \tag{8}$$

then M is known a totally umbilical submanifold and H is the mean curvature vector field. If the second fundamental form h zeros identically on M, then M is called totally geodesic, [5, 8].

Definition 2.1. *Let $\mathbb{R}_v^{n+d}$ be the $(n+d)$-dimensional pseudo-Euclidean space and N_s^n be the connected pseudo-Riemannian manifold. An isometric immersion $\varphi : N_s^n \to \mathbb{R}_v^{n+d}$ is called pseudo-isotropic at $p\in N_s^n$ if*

$$<h(u,u),h(u,u)>=\lambda(p)\in\mathbb{R}$$

does not depen on the choice of the unit tangent vector $u\in T_pN_s^n$ and φ is said to be pseudo-isotropic if φ is pseudo-isotropic at each point of N_s^n. Also, if λ is a constant function, the immersion is called constant pseudo-isotropic, [8, 18].

Definition 2.2. *Hyperelastic curves are known as crucial points of the functional $\mathcal{F}_\gamma^r=\int(\kappa^r+\lambda)ds$ for a natural number $r\geq 2$. Also, these curves are called as free hyperelastic curves if $\lambda=0$, [1].*

Critical points of the bending energy functional $\mathcal{F}_\gamma^r$ are characterized by the Euler-Lagrange equation

$$\nabla_T^2\left(\kappa^{r-2}\nabla_T T\right)+\kappa^{r-2}R(\nabla_T T,T)T+\varepsilon_0\varepsilon_1\nabla_T(\lambda T)=0, \tag{9}$$

for some constant $b\in\mathbb{R}$ and

$$\lambda=\frac{2r-1}{r}\kappa^r+b \tag{10}$$

[10].

Let γ be a unit speed non-null curve in a Lorentzian manifold M. Then, the Frenet apparatus of γ are established as

$$\nabla_T N_i=-\varepsilon_{i-1}\kappa_i N_{i-1}+\varepsilon_{i+1}\kappa_{i+1}N_{i+1},\qquad 0\leq i\leq n-1$$

where $\kappa_1=\kappa\geq 0$, κ_2, κ_3, $\kappa_4,...,\kappa_{n-1}$ are curvatures of γ and $N_o=T$ the unit tangent vector field, $N_1=N$ the unit normal vector field and $N_2=B$ the unit binormal vector field. Also, $\kappa_0 N_{-1}=\kappa_n N_n=0$ and $\varepsilon_i=<N_i,N_i>=\mp 1$, [6].

3. Geometry of the Lorentzian Submanifold by Non-null Hyperelastic Curves

In this section, firstly, some important equations and relations will be obtained for use in the rest of the work.

Consider two Lorentzian manifolds, denoted as M and $\overline{M}$. Let i be an isometric immersion where γ represents a non-null curve on M and $\overline{\gamma}(s) = i \circ \gamma(s)$ denotes corresponding non-null curve with curvature $\overline{\kappa}$ in $\overline{M}$. Additionally, we assume that $\overline{\gamma}(s)$ is a non-null hyperelastic curve in $\bar{M}$. By applying (9), it follows that

$$\overline{\nabla}_T^2\left(\overline{\kappa}^{r-2}\overline{\nabla}_T T\right)+\overline{\kappa}^{r-2}\overline{R}(\overline{\nabla}_T T, T)T + \varepsilon_0\varepsilon_1\overline{\nabla}_T(\lambda T)=0, \tag{11}$$

for some constant $b \in \mathbb{R}$ and

$$\lambda = \frac{2r-1}{r}\overline{\kappa}^r + b. \tag{12}$$

For the first component of (11), from (1) and (2), we find

$$\begin{aligned}\overline{\nabla}_T^2\left(\overline{\kappa}^{r-2}\overline{\nabla}_T T\right) = \;& \nabla_T^2(\xi\nabla_T T) + h(T, \nabla_T(\xi\nabla_T T)) - A_{\xi h(T.\nabla_T T)}T \\ & + D_T\xi h(T,\nabla_T T) - \nabla_T\left(A_{\xi h(T,T)}T\right) - h\left(A_{\xi h(T,T)}T, T\right) \\ & - A_{D_T\xi h(T,T)}T + D_T^2\xi h(T,T),\end{aligned} \tag{13}$$

where $\xi = (\kappa^2 + \|h(T,T)\|^2)^{\frac{r-2}{2}}$ and κ is the curvature of γ in M. For the second component of (11), if (4) and (1) are used and the necessary calculations are made, we obtain

$$\begin{aligned}\overline{\kappa}^{r-2}\bar{R}(\bar{\nabla}_T T, T)T = \;& \xi R(\nabla_T T, T)T - \xi A_{h(T,T)}\nabla_T T + \xi A_{h(\nabla_T T,T)}T \\ & + \xi\left(\tilde{\nabla}_{\nabla_T T}h\right)(T,T) - \xi\left(\tilde{\nabla}_T h\right)(\nabla_T T, T).\end{aligned} \tag{14}$$

If we use (13) and (14) in (11), we get

$$\begin{aligned}& \nabla_T^2(\xi\nabla_T T) + h(T,\nabla_T(\xi\nabla_T T)) + D_T\xi h(T,\nabla_T T) - \nabla_T\left(A_{\xi h(T,T)}T\right) \\ & -\xi h\left(A_{h(T,T)}T, T\right) - A_{D_T\xi h(T,T)}T + D_T^2\xi h(T,T) + \xi R(\nabla_T T, T)T \\ & -\xi A_{h(T,T)}\nabla_T T + \xi\left(\tilde{\nabla}_{\nabla_T T}h\right)(T,T) - \xi\left(\tilde{\nabla}_T h\right)(\nabla_T T, T) \\ & +\varepsilon_0\varepsilon_1\nabla_T(\lambda T) + \varepsilon_0\varepsilon_1\lambda h(T,T) = 0,\end{aligned} \tag{15}$$

where

$$\lambda = \frac{2r-1}{r}(\kappa^2 + \|h(T,T\|^2)^{\frac{r}{2}} + b.$$

On the other hand, the tangent part of (15) is

$$\begin{aligned}& \nabla_T^2(\xi\nabla_T T) - \nabla_T\left(A_{\xi h(T,T)}T\right) - A_{D_T\xi h(T,T)}T + \xi R(\nabla_T T, T)T \\ & \qquad - \xi A_{h(T,T)}\nabla_T T + \varepsilon_0\varepsilon_1\nabla_T(\lambda T) = 0.\end{aligned} \tag{16}$$

Making use of (6) and (5), we have

$$\nabla_T\left(A_{\xi h(T,T)}T\right) = \nabla_T(A)_{\xi h(T,T)}T + A_{D_T\xi h(T,T)}T + A_{\xi h(T,T)}\nabla_T T \tag{17}$$

and

$$\xi\left(\tilde{\nabla}_T h\right)(T,T) = \xi D_T h(T,T) - 2\xi h(\nabla_T T, T), \tag{18}$$

respectively. Using (17) and (18) into (16), we obtain

$$\begin{aligned}&\nabla_T^2(\xi\nabla_T T)-\nabla_T A_{\xi h(T,T)}T-2A_{(D_T\xi)h(T,T)}T-2\xi A_{(\tilde{\nabla}_T h)(T,T)}T\\&-4\xi A_{h(\nabla_T T,T)}T-2\xi A_{h(T,T)}\nabla_T T+\xi R(\nabla_T T,T)T+\varepsilon_0\varepsilon_1\nabla_T(\lambda T)=0.\end{aligned}\tag{19}$$

Substituting Frenet equations of γ in (19), we get

$$\begin{aligned}&\varepsilon_0\varepsilon_1\left(\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s\right)T+(\varepsilon_1\xi\kappa_{ss}-\varepsilon_0\kappa^3\xi+2\varepsilon_1\kappa_s\xi_s+\varepsilon_1\xi_{ss}\kappa-\varepsilon_2\kappa\tau^2\xi+\varepsilon_0\lambda\kappa)N\\&+(2\varepsilon_1\kappa_s\tau\xi+\varepsilon_1\varepsilon_2\kappa\tau_s\xi+2\varepsilon_1\varepsilon_2\kappa\tau\xi_s)B-4\xi\kappa A_{h(N,T)}T-2\xi\kappa A_{h(T,T)}N\\&=2A_{(D_T\xi)h(T,T)}T+2\xi A_{(\tilde{\nabla}_T h)(T,T)}T+\nabla_T A_{\xi h(T,T)}T-\xi\kappa R(N,T)T.\end{aligned}\tag{20}$$

Then, the inner product of (20) with T and using (18), we have

$$\begin{aligned}\varepsilon_1(\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s)-6\xi\kappa<h(T,N),h(T,T)>&=2<A_{(D_T\xi h)(T,T)}T,T>\\&-4\kappa\xi<A_{h(T,N)}T,T>+<\nabla_T A_{\xi h(T,T)}T,T>.\end{aligned}$$

With aid of (3), (18) and (7), we can write

$$\begin{aligned}\varepsilon_1(\lambda_s-3\xi\kappa\kappa_s-2\kappa^2\xi_s)&=2<h(T,T),D_T\xi h(T,T)>+<\xi h(T,T),D_T h(T,T)>\\&=2(D_T\xi)\|h(T,T)\|^2+\frac{3}{2}\xi D_T\|h(T,T)\|^2\\&=\xi((r-2)(\kappa^2+\|h(T,T)\|^2)^{-1}\|h(T,T)\|^2+\frac{3}{2})D_T\|h(T,T)\|^2.\end{aligned}\tag{21}$$

Also, take into consideration the normal part of (15), we have

$$\begin{aligned}&h(T,\nabla_T(\xi\nabla_T T))+D_T\xi h(T,\nabla_T T)-\xi h\left(A_{h(T,T)}T,T\right)+D_T^2\xi h(T,T)\\&+\xi(\bar{\nabla}_{\nabla_T T}h)(T,T)-\xi\left(\tilde{\nabla}_T h\right)(\nabla_T T,T)+\varepsilon_0\varepsilon_1\lambda h(T,T)=0.\end{aligned}\tag{22}$$

From (5), we obtain

$$\begin{aligned}\xi D_T^2 h(T,T)&=\xi\left(\tilde{\nabla}_T^2 h\right)(T,T)+4\xi D_T h(T,\nabla_T T)-2\xi h(\nabla_T T,\nabla_T T)\\&-2\xi h\left(T,\nabla_T^2 T\right).\end{aligned}\tag{23}$$

Combining (23) with (22) and using (5), we find

$$\begin{aligned}&h(T,\xi_s\nabla_T T)+4\xi h\left(T,\nabla_T^2 T\right)+5(D_T\xi)h(T,\nabla_T T)+4\xi\left(\tilde{\nabla}_T h\right)(T,\nabla_T T)\\&+3\xi h(\nabla_T T,\nabla_T T)-\xi h\left(A_{h(T,T)}T,T\right)+(D_T^2\xi)h(T,T)+2(D_T\xi)(\bar{\nabla}_T h)(T,T)\\&+\xi\left(\tilde{\nabla}_T^2 h\right)(T,T)+\xi\left(\tilde{\nabla}_{\nabla_T T}h\right)(T,T)-\varepsilon_0\varepsilon_1\lambda h(T,T)=0.\end{aligned}$$

If we use the Frenet equations, we get

$$\begin{aligned}&\varepsilon_1(\kappa\xi_s+4\xi\kappa_s+5(D_T\xi)\kappa)h(T,N)+4\varepsilon_1\varepsilon_2\xi\kappa\tau h(T,B)+4\varepsilon_1\xi\kappa\left(\tilde{\nabla}_T h\right)(T,N)\\&+3\xi\kappa^2 h(N,N)+2(D_T\xi)(\tilde{\nabla}_T h)(T,T)+\varepsilon_1\xi\kappa\left(\tilde{\nabla}_N h\right)(T,T)=\varepsilon_0\varepsilon_1(4\xi\kappa^2-D_T^2\xi)h(T,T)\\&+\xi h\left(A_{h(T,T)}T,T\right)-\varepsilon_0\varepsilon_1\lambda h(T,T)-\left(\tilde{\nabla}_T^2\xi h\right)(T,T).\end{aligned}\tag{24}$$

If we change B with $-B$ in (24) and from (24), we find

$$h(T,B)=0.\tag{25}$$

Thus we have proved the following proposition.

Proposition 3.1. *Assuming i is an isometric Lorentzian immersion from one Lorentzian manifold M to Lorentzian manifold $\bar{M}$ and γ represents a non-null curve with curvature κ on M such that $\bar{\gamma}(s) = i \circ \gamma(s)$ is a non-null hyperelastic curve with $\bar{\kappa}$ in $\bar{M}$. If any of the given statements hold true, then it can be concluded that M is actually an isotropic Lorentzian submanifold.*

(i) $r = 2$, *namely $\overline{\gamma}(s)$ is an non-null elastic curve,*
(ii) *κ is a constant when $\overline{\kappa} \neq 0$.*

Proof. *Let $\overline{\gamma} = \overline{\gamma}(s)$ be a non-null hyperelastic curve and $\gamma(s)$ be a non-null curve with curvatures $\overline{\kappa}$ and κ, respectively.*
*(**i**) If $r = 2$, so, we have from (21)*

$$D_T \|h(T,T)\|^2 = 0. \tag{26}$$

This means that $\|h(T,T)\|$ is constant. So, M is an isotropic Lorentzian submanifold.
*(**ii**) Now, let the curvature κ be a constant. We have following equation from (21),*

$$\xi((r-2)(\kappa^2 + \|h(T,T)\|^2)^{-1}\|h(T,T)\|^2 + \tfrac{3}{2})D_T\|h(T,T)\|^2 = 0.$$

Also, because of $\overline{\kappa} \neq 0$ and $r > 2$, then we obtain (26). □

By the Proposition 3.1, we can give following corollary without proof.

Corollary 3.2. *Assume that $\overline{\gamma}(s) = i \circ \gamma(s)$ with curvature $\overline{\kappa}$ is a non-null hyperelastic curve in $\bar{M}$. M is totally umbilic Lorentzian submanifold, if M is isotropic.*

Besides, we present the notion of Lorentzian hyperelastic immersion.

Definition 3.3. *A Lorentzian hyperelastic immersion is a type of isometric immersion from one Lorentzian manifold to another, such that every non-null hyperelastic curve on the submanifold is mapped to a non-null hyperelastic curve on Lorentzian manifold.*

Now, we give following theorem for Lorentzian hyperelastic immersions.

Theorem 3.4. *If i is a Lorentzian hyperelastic immersion between Lorentzian manifolds M and $\bar{M}$ and γ is a non-null hyperelastic curve with constant curvature κ in M, then M is a totally umbilical and the following condition is satisfied*

$$D_T^2 H = CH, \tag{27}$$

where H is the mean curvature vector field, $C = \varepsilon_0(\varepsilon_1\kappa^2 + \|H\|^2 - \varepsilon_1\frac{\lambda}{\xi}) = const.$ and T is the tangent vector field of γ. Conversely if the isometric immersion between two Lorentzian manifolds is totally geodesic, then it is also a Lorentzian hyperelastic immersion.

Proof. Assuming i is a non-null hyperelastic immersion between two Lorentzian manifolds M and $\bar{M}$, and $\gamma(s)$ is a non-null hyperelastic curve in M with a constant curvature κ and the tangent vector field T,the curve γ satisfies (9) and (10). Therefore, by Proposition 3.1, we can see that M is an isotropic Lorentzian submanifold. Additionally, as $\overline{\gamma}$ is a non-null hyperelastic curve in $\bar{M}$, we have (11) and (12). Using Corollary 3.2, we can conclude that M is a totally umbilical. As a result, equation from (24) can be written as:

$$\begin{aligned} 4\varepsilon_1\xi\kappa\left(\tilde{\nabla}_T h\right)(T,N) + 3\xi\kappa^2 h(N,N) + \varepsilon_1\xi\kappa\left(\tilde{\nabla}_N h\right)(T,T) &= 4\varepsilon_0\varepsilon_1\xi\kappa^2 h(T,T) \\ - \varepsilon_0\varepsilon_1\xi h\left(A_{h(T,T)}T,T\right) + \lambda h(T,T) - \xi\left(\tilde{\nabla}_T^2 h\right)(T,T). \end{aligned} \tag{28}$$

Changing N into $-N$ in (28), we get

$$4\xi\kappa\left(\tilde{\nabla}_T h\right)(T,N) + \xi\kappa\left(\tilde{\nabla}_N h\right)(T,T) = 0. \tag{29}$$

If we use (29) into (28), we have

$$-\varepsilon_1\xi\kappa^2 H = \xi < A_{h(T,T)}T, T > H - \varepsilon_1\lambda H - \xi\left(\tilde{\nabla}_T^2 h\right)(T,T).$$

Hence, by means of (3) and (23), we obtain

$$-\varepsilon_1\xi\kappa^2 H = \xi\,\|H\|^2 H - \varepsilon_1\lambda H - \varepsilon_0\xi D_T^2 H.$$

So, we have

$$D_T^2 H = \varepsilon_0(\varepsilon_1\kappa^2 + \|H\|^2 - \varepsilon_1\frac{\lambda}{\xi})H.$$

On the contrary, if we assume that an isotropic immersion between two Lorentzian manifolds is totally geodesic and γ is a non-null hypereastic curve with κ, then $i \circ \gamma$ satisfies the equation (11) and (12). □

Furthermore, the subsequent result can be presented.

Corollary 3.5. *Assuming i denotes a hyperelastic immersion and γ represents a non-null hyperelastic curve with constant curvature in M, it follows that the torsion of γ remains constant.*

Proof. If i is a non-null hyperelastic immersion and γ is a non-null hyperelastic curve with the constant curvature κ in M, then from (20), we can write

$$(-\varepsilon_0\kappa^2 - \varepsilon_2\tau^2 + \varepsilon_0\tfrac{\lambda}{\xi})N + \varepsilon_1\varepsilon_2\tau_s B = 4A_{h(N,T)}T + 2A_{h(T,T)}N. \tag{30}$$

Also, the inner product of equation (30) with the unit normal vector field N yields

$$\|H\|^2 = \frac{1}{2}\left(-\kappa^2 - \varepsilon_0\varepsilon_2\tau^2 + \frac{\lambda}{\xi}\right). \tag{31}$$

From by assumption $\|h(T,T)\| = const. = \|H\|$ and (31), the torsion τ of non-null hyperelastic curve γ is a constant, too. □

A non-null curve γ in a Lorentzian manifold M is called a non-null elastic curve (or elastica) if it satisfies (9) with (10) for $r = 2$ [10]. A characterization of Lorentzian submanifolds that are totally umbilical can be obtained by analyzing the behavior of non-null elastic curves in relation to a Lorentzian immersion, as per the theorem presented.

Theorem 3.6. *Consider an immersion $i : M \to \bar{M}$ between two Lorentzian manifolds M and $\bar{M}$, such that it maps every non-null elastic curve with its tangent vector T, curvature κ and torsion τ to another non-null elastic curve on $\bar{M}$. Under these conditions, it follows that M is totally umbilical and the mean curvature vector field H satisfies*

$$D_T^2 H = \frac{1}{2}\left(\varepsilon_0(2\varepsilon_1 - 1))\kappa^2 - \varepsilon_2\tau^2 - \varepsilon_0(2\varepsilon_1 - 1)\lambda + \varepsilon_1\frac{\kappa_{ss}}{\kappa}\right)H, \quad \|H\| = const.. \tag{32}$$

Conversely, if the Lorentzian submanifold M is totally umbilical, and its mean curvature vector field H satisfies the equation given in (32) and $D_{\nabla_T T}H = 0$, then any non-null elastic curve on M is mapped to another non-null elastic curve on $\bar{M}$ by the immersion $i : M \to \bar{M}$.

Proof. Assuming that $\gamma(s)$ is a non-null elastic curve in the Lorentzian submanifold M, with unit tangent vector field T, we can derive the following equation

$$(\nabla_T)^3 T + R(\nabla_T T, T)T + \varepsilon_0\varepsilon_1\nabla_T(\lambda T) = 0$$

with aid of (10) for $r = 2$. Because of $\overline{\gamma}$ is a non-null elastic curve with curvature $\bar{\kappa}$ in $\bar{M}$, we obtain (11) with (12) for $r = 2$. From Proposition 3.1 and Corollary 3.2, M is totally umbilical Lorentzian submanifold. For $r = 2$, if we use (30) and taking inner product with N, we obtain

$$\|H\|^2 = \frac{1}{2}\left(\varepsilon_0\varepsilon_1\frac{\kappa_{ss}}{\kappa} - \varepsilon_0\varepsilon_2\tau^2 + \lambda - \kappa^2\right). \tag{33}$$

Also, in case of $r = 2$, by straightforward calculation, the normal part is

$$D_T^2 H = \varepsilon_0\left(-\varepsilon_1\lambda + \varepsilon_1\kappa^2 + \|H\|^2\right)H. \tag{34}$$

Using (33) in (34), we get (32).
Now, we suppose that M is totally umbilical and H satisfies (32). Also, we have

$$\overline{\nabla}_T^3 T = \nabla_T^3 T - \varepsilon_0\nabla_T(\|H\|^2\,T) - \varepsilon_1(\kappa^2 + \|H\|^2)H + \varepsilon_0 D_T^2 H. \tag{35}$$

To continue calculations, if we add $\overline{R}(\overline{\nabla}_T T, T)T + \varepsilon_0\varepsilon_1\overline{\nabla}_T(\lambda T)$ both sides of (35) and using (1), (4), (27), we get

$$\begin{aligned}&\overline{\nabla}_T^3(T) + \overline{R}(\overline{\nabla}_T T, T)T + \overline{\nabla}_T(\lambda T) = \nabla_T^3(T) + \varepsilon_0\varepsilon_1\nabla_T((\lambda - 2\varepsilon_1\|H\|^2)T)\\ &+R(\nabla_T T, T)T - \varepsilon_0(D_{\nabla_T T})H.\end{aligned} \tag{36}$$

Because of γ is a non-null elastic curve in M, the tangent part of (36) is given as

$$(\nabla_T)^3 T + R(\nabla_T T, T)T + \varepsilon_0\varepsilon_1\nabla_T(\lambda T) = 0,$$

where

$$\lambda = \frac{3}{2}\kappa^2 + \overline{b}, \tag{37}$$

for $\overline{b} = b + 2\varepsilon_1\|H\|^2$. Also normal part of (36) is found zero if $D_{\nabla_T T}H = 0$. □

4. On Constant Sectional Curvature Lorentzian Manifolds

In this section, some characterizations will be given for Lorentzian manifolds with constant sectional curvature by utilizing the theorems and results obtained in the previous section. Since the proofs were given in detail earlier, the results obtained here are presented without proof. Assume that M is a Lorentzian manifold with constant sectional curvature C. So, we have

$$R(X, Y)Z = C(< Z, X > Y - < Z, Y > X).$$

So, from (9), a unit speed non-null curve $\gamma = \gamma(s)$ parametrized by arc length s is a non-null hyperelastic curve if it satisfies

$$\nabla_T^2\left(\kappa^{r-2}\nabla_T T\right) + C\kappa^{r-2}\nabla_T T + \varepsilon_0\varepsilon_1\nabla_T(\lambda T) = 0 \tag{38}$$

with (10), ([10]).
On the other hand, let M be a Lorentzian submanifold of Lorentzian manifold $\bar{M}$ with sign ε and $\dim M > 2$. If M is totally umbilic and $\bar{M}$ has constant sectional curvature $\overline{C}$, then the norm of mean curtavure $\|H\|$ is constant and M has constant sectional curvature $\overline{C} + \varepsilon\|H\|^2$, [9]. Then, we have the following corollary as a result of Theorem 3.4 and Corollary 3.2 .

Corollary 4.1. *If an immersion i satisfies the conditions of being a Lorentzian hyperelastic immersion between a submanifold M ($\dim M > 2$) and a Lorentzian manifold $\bar{M}$ with constant sectional curvature $\overline{C}$, then M can be characterized as an isotropic Lorentzian submanifold of $\bar{M}$. Additionally, M has a constant sectional curvature $C = \overline{C} + \varepsilon\|H\|^2$, where H represents the mean curvature vector field.*

In additionally, we give some well know results for non-null elastic curves on the Lorentzian plane and a pseudo-hyperbolic space H_0^2. A non-null elastic curve on Lorentzian plane satisfies the following differential equation

$$\kappa^{''} - \frac{1}{2}\kappa^3 - \frac{1}{2}\lambda\kappa = 0, \tag{39}$$

[4]. The equilibrium equation of the spacelike elastic curve on the pseudo-hyperbolic space is given as follows

$$2\kappa_g'' + \kappa_g^3 - \left(\frac{1}{2r^2} + \sigma\right)\kappa_g = 0, \tag{40}$$

where σ is the tension parameter, κ_g is the geodesic curvature and r is the radius of the pseudo-hyperbolic space, [11].
Now, we present an example of Lorentzian elastic immersion.

Example 4.2. *Assume that x and y be coordinate functions of $\mathbb{R}_1^2$. Then, the immersion i is given as*

$$\begin{array}{rccc} i: & \mathbb{R}_1^2 & \rightarrow & H_0^2 \subset \mathbb{R}_1^3 \\ & (u,v) & \rightarrow & (x^2+y^2, 2xy, 0) \end{array}$$

where $(x^2 - y^2)^2 = 1$. On the other hand, suppose that α is a curve defined as follows

$$\begin{array}{rccc} \alpha: & I \subset \mathbb{R} & \rightarrow & \mathbb{R}_1^2 \\ & t & \rightarrow & (\cosh t, \sinh t) \end{array}$$

Since the curvature κ of α is equal to 1, from (39), we can say that α is an spacelike elastic curve and $(i \circ \alpha)(t) = (\cosh 2t, \sinh 2t, 0)$. By straightforward calculation and using (40), we can see that this curve is an elastica on H_0^2. So, the immersion i is a Lorentzian elastic immersion.

Lastly, we get the following result from Theorem 3.6.

Corollary 4.3. *Suppose we have a Lorentzian manifold M and a Lorentzian manifold $\bar{M}(\bar{C})$ with constant curvature, and let $i : M \rightarrow \bar{M}(\bar{C})$ be a Lorentzian hyperelastic immersion between these two manifolds. Let γ be a non-null elastic curve with curvature κ, torsion τ, and unit vector field T. Under these conditions, M is totally umbilical, and the mean curvature vector field H satisfies the following condition*

$$D_T^2 H = \left(\frac{\kappa_{ss}}{\kappa} - \varepsilon_1\varepsilon_2\tau^2\right)H, \ \ ||H|| = const.. \tag{41}$$

On the other hand, if M is a Lorentzian submanifold of a Lorentzian manifold $\bar{M}$ with constant sectional curvature C, and it is totally umbilical such that the mean curvature vector field satisfies the condition given in (41), then the corresponding image of a non-null elastic curve γ in M is also a non-null elastic curve in $\bar{M}$.

Declarations

Conflict of interest: The author declares no conflict of interest.

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