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Blow up solution of inverse problem for nonlinear hyperbolic equation with variable-exponents

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Abstract. In this work, we study the following inverse problem with variable exponents:

$$u_{tt} - div(|\nabla u|^{r(.)-2}\nabla u) + a|u_t|^{m(.)-2}u_t - b|u|^{p(.)-2}u = f(t)w(x),$$

where $a, b > 0$ are constants and variable exponents $p(.), r(.)$ and $m(.)$ are given functions. We analyzed the finite-time blow-up of the solution using the alternative method proposed by Georgiev and Todorova with negative initial energy.

1. Introduction

In this article, we consider the following inverse problem for the wave equation:

$$u_{tt} - div(|\nabla u|^{r(.)-2}\nabla u) + a|u_t|^{m(.)-2}u_t - b|u|^{p(.)-2}u = f(t)w(x), (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0, (x,t) \in \partial\Omega \times (0,\infty) \tag{2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \tag{3}$$

$$\int_\Omega u(x,t)w(x)dx = \phi(t), t > 0, \tag{4}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$ and a unit outer normal ν. Also, a and b are positive constants and $w(x)$ and $\phi(t)$ are real valued functions with specific conditions that will be determined later. In addition, we suppose that $p(.), r(.)$ and $m(.)$ are given measurable and continuous functions on $\bar{\Omega}$ such that:

$$2 \leq r_1 \leq r(x) \leq r_2 < m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 \leq r_*(x) \tag{5}$$

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with

$$p_1 := essinf_{x\in\bar{\Omega}}p(x), p_2 := esssup_{x\in\bar{\Omega}}p(x),$$
$$r_1 := essinf_{x\in\bar{\Omega}}r(x), r_2 := esssup_{x\in\bar{\Omega}}r(x),$$
$$m_1 := essinf_{x\in\bar{\Omega}}m(x), m_2 := esssup_{x\in\bar{\Omega}}m(x),$$

and

$$r_*(x) = \begin{cases} \frac{Nr(x)}{esssup_{x\in\Omega}(N-r(x))}, & r_2 < N \\ +\infty, & r_2 \geq N \end{cases}.$$

In addition, suppose that the $p(.), r(.)$ and $m(.)$ provide the log-Hölder continuity condition:

$$\left|q(x) - q(y)\right| \leq -\frac{A}{\log\left|x-y\right|}, \quad for \quad a.e. \quad x, y \in \Omega, \quad with \quad \left|x-y\right| < \tau, \tag{6}$$

$A > 0, 0 < \tau < 1$. $\Delta_{r(.)}u = div(|\nabla u|^{r(.)-2}\nabla u)$ is called $r(.) - Laplacian$ term.

Inverse problems are common across various fields, including physics, engineering, geophysics, medical imaging, remote sensing, and more. They often involve solving complex equations, dealing with uncertainty, and using computational methods to find the most likely solutions. In many cases, inverse problems are ill-posed, meaning that there may be multiple solutions or the solutions might be sensitive to small changes in the data [12, 15].

Before proceeding, it is important mentioning some significant prior findings in the field of inverse problems. Kalantarov and Eden worked the following inverse problem [4]:

$$u_t + b(x,t,u,\nabla u) - \Delta u - |u|^p u = F(t)w(x), \quad x \in \Omega, t > 0,$$
$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \Omega,$$
$$\int u(x,t)w(x)dx = \phi(t), t > 0.$$

They established conditions on the data that ensure the global nonexistence of solutions when $\phi(t) \equiv 1$. Additionally, they proved a stability result with an opposite sign for $b(x,t,u,\nabla u) \equiv 0$ and the power type nonlinearity. They used the $v(x,t) = u(x,t)e^{-\lambda t}$ transformation method to reach their blow up conclusion. Gür, Yaman and Yılmaz worked the following inverse problem [11]:

$$u_t - \nabla.\left[\left(k_1 + k_2 |\nabla u|^{m-2}\right)\nabla u\right] + h(u,\nabla u) - |u|^{p-2} u = F(t)w(x), \quad x \in \Omega, t > 0,$$
$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \Omega,$$
$$\int u(x,t)w(x)dx = 1, t > 0.$$

They are studying the potential for blow-up phenomena in finite time for solutions to inverse problems related to nonlinear parabolic equations with k_1, k_2 positive constants and $p > m \geq 2$. They utilized the Ladyzhenskaya-Kalantarov lemma to reach their results [13].

The methods used in the previous studies have led to blow-up results in inverse problems [16–18]. The issue of global nonexistence and blow-up results for nonlinear parabolic equations is explored in [2, 7]. Our goal in this article is to achieve a new contribution to the literature by employing the previously unused method proposed by Georgiev and Todorova [10] to tackle the blow-up of solutions in inverse problems. This study will provide a novel perspective and contribute to the existing body of knowledge in the field.

2. Preliminaries and Main Result

In this chapter, we remind some functionals and notations about the Sobolev and Lebesgue spaces with variable exponents [9, 13]. $\|.\|_q$ shows the norm of L^q- over the region Ω, and more specifically, L^2-norm is denoted $\|.\|$ in Ω. We suppose that the function $w(x)$ satisfies the following conditions:

$$u_0 \in H_0^2(\Omega) \cap L^{r(.)}(\Omega) \cap L^{p(.)}(\Omega), u_1 \in L^2(\Omega) \cap L^{m(.)}(\Omega), \int_\Omega u_0(x)w(x)dx = \phi(0). \tag{7}$$

$$w \in H_0^2(\Omega) \cap L^{r(.)}(\Omega) \cap L^{m(.)}(\Omega) \cap L^{p(.)}(\Omega), \quad \int_\Omega w^2(x)dx = 1. \tag{8}$$

In order to study problems (1)-(4), we need some hypotheses and theories about Lebesgue and Sobolev spaces with variable exponents (for detailed, see, other works [3],[5]-[6],[8],[9]). Let $p(x) \geq 1$ and measurable, we assume that

$$C_+(\bar{\Omega}) = \{h | h \in C(\bar{\Omega}), \quad h(x) > 1 \quad for \quad any \quad x \in \bar{\Omega}\},$$

$$h^+ = \max_{\bar{\Omega}} h(x), \quad h^- = \min_{\bar{\Omega}} h(x) \quad for \quad any \quad h \in C(\bar{\Omega}),$$

$$L^{p(.)}(\Omega) := \left\{ \nu : \Omega \to \mathbb{R}; \quad measurable \quad in \quad \Omega : \varrho_{p(.)}(\lambda\nu) < +\infty, \quad for \quad some \quad \lambda > 0 \right\},$$

where

$$\varrho_{p(.)}(\nu) = \int_\Omega |\nu(x)|^{p(x)} dx.$$

Given by the following Luxembourg-type norm

$$\|\nu\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{\nu(x)}{\lambda} \right| dx \leq 1 \right\}.$$

$L^{p(.)}(\Omega)$ is a Banach space [3]. The Sobolev space $W^{1,p(.)}(\Omega)$ with variable exponent is defined as follows:

$$W^{1,p(.)}(\Omega) = \left\{ \nu \in L^{p(.)}(\Omega) \quad such \quad that \quad \nabla\nu \quad exists \quad and \quad |\nabla\nu| \in L^{p(.)}(\Omega) \right\}.$$

The equality written above is a Banach space with respect to the $\|\nu\|_{W^{1,p(.)}(\Omega)} = \|\nu\|_{p(.)} + \|\nabla\nu\|_{p(.)}$ norm. In addition, let $W_0^{1,p(.)}(\Omega)$ in the space $W^{1,p(.)}(\Omega)$ be given as the closure of $C_0^\infty(\Omega)$. Let us also note that space $W^{1,p(.)}(\Omega)$ has a different definition when it comes to variable exponents. But, under the condition (6), both definitions are the same [13]. The space $W^{-1,p'(.)}(\Omega)$ is a dual space of $W^{1,p(.)}(\Omega)$ and is defined like the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ [3].

Lemma 2.1. *[3] Let $\Omega \subset \mathbb{R}^n$ be a bounded region, and assuming that $p(.)$ provides the inequality* (6), *then*

$$\|\nu\|_{p(.)} \leq C\|\nabla\nu\|_{p(.)}, \quad for \quad all \quad \nu \in W_0^{1,p(.)}(\Omega),$$

where $C > 0$ is a constant that depends only on p_1, p_2 and Ω. Particularly, $\|\nabla\nu\|_{p(.)}$ defines a norm equal to the $W_0^{1,p(.)}(\Omega)$ norm.

Lemma 2.2. *[3] If $r(.) \in C(\bar{\Omega})$ and $q(x) : \Omega \to [1, \infty)$ such that provides the following inequality is a measurable function,*

$$essinf_{x\in\Omega}(r_*(x) - q(x)) > 0 \quad with \quad r_*(x) = \begin{cases} \frac{Nr(x)}{esssup_{x\in\Omega}(N-r(x))}, & if r_2 < N \\ \infty, & if r_2 \geq N \end{cases}.$$

Then the embedding $W_0^{1,r(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.3. *[3] Let us assume that $p, q, s \geq 1$ such that the following equality is satisfied are measurable functions on Ω,*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad y \in \Omega.$$

If $u \in L^{p(.)}(\Omega)$ and $\nu \in L^{q(.)}(\Omega)$, then $u\nu \in L^{s(.)}(\Omega)$, with

$$\|u\nu\|_{s(.)} \leq 2\|u\|_{p(.)}\|\nu\|_{q(.)}.$$

Lemma 2.4. *[3] Assume that p is a measurable function on Ω. Then the following inequality is provided,*

$$\left\|g\right\|_{p(.)} \leq 1 \quad if \quad and \quad only \quad if \quad \varrho_{p(.)}(g) \leq 1.$$

Lemma 2.5. *[3] If p is a measurable function on Ω that provides* (5)*, then the following inequality is provided for $\forall u \in L^{p(.)}(\Omega)$,*

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\}.$$

Let us multiply equation (1) by $w(x)$ and taking $\phi(t) \equiv 1$, integrate over region Ω with (8) and integral over determination condition (4), then the problem (1)-(4) is equivalent to the following direct problem

$$u_{tt} - div(|\nabla u|^{r(.)-2}\nabla u) + a|u_t|^{m(.)-2}u_t - b|u|^{p(.)-2}u = f(t)w(x), (x,t) \in \Omega \times (0,T) \tag{9}$$

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T) \tag{10}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \tag{11}$$

in which the unknown function $f(t)$ is replaced by

$$f(t) = \int_{\Omega} |\nabla u|^{r(.)-1}\nabla w(x)dx + a\int_{\Omega} |u_t|^{m(.)-1}w(x)dx - b\int_{\Omega} |u|^{p(.)-1}w(x)dx. \tag{12}$$

The local existence of solutions for the equations given in (1)-(3) can be established using the Galerkin method as in the study of Antontsev [1].

Theorem 2.6. (*Local Existence*) *Let $u_0 \in W_0^{1,r(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ and suppose that* (5) *and* (8)-(9) *be supplied, then problem* (1)-(3) *have a only weak solution such that*
$u \in L^\infty\left((0,T), W_0^{1,r(.)}(\Omega)\right) \cap L^{p(.)}((0,T),\Omega)$, $\quad u_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}((0,T),\Omega)$,
$u_{tt} \in L^\infty\left((0,T), W_0^{-1,r'(.)}\right)(\Omega)$, *for any $T > 0$ and* $\frac{1}{r(.)} + \frac{1}{r'(.)} = 1$.

3. Blow Up

Our objective in this part is to establish the occurrence of blow up for specific solutions characterized by negative initial energy. In this section, our approach relies on the Georgiev and Todorova Method [10]. First, let's give the energy function of the solution.

$$E(t) := \frac{1}{2}\int_\Omega |u_t|^2 dx + \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} dx - b\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx. \tag{13}$$

Theorem 3.1. *If the conditions of Theorem* (2.6) *be satisfied and suppose that*

$$E(0) < 0 \tag{14}$$

Then the solution to problem (1)-(3) *blow up in finite time.*

In order to demonstrate main result, we initially define several lemmas.

Lemma 3.2. *Assume that the conditions of the Lemma* (2.2) *are satisfied, there exists a constant* $C > 1$*, which depends solely on* Ω*, such that*

$$\varrho_{p(.)}^{\frac{s}{p_1}}(u) \le C(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)), \tag{15}$$

for any $r_1 \le s \le p_1$ *and* $u \in W_0^{1,r(.)}(\Omega)$.

Proof. See [14]. □

For the special case, the following result can be written.

Corollary 3.3. *Under the conditions of Lemma* (3.2)*, for every* $r_1 \le s \le p_1$ *and* $u \in W_0^{1,r(.)}(\Omega)$*, the inequality*

$$\|u\|_{p_1}^{s} \le C(\|\nabla u\|_{r(.)}^{r_1} + \|u\|_{p_1}^{p_1}), \tag{16}$$

is formulated.

Now, we define

$$H(t) := -E(t) \tag{17}$$

and assuming C is a general positive constant dependent solely on Ω, if (13) and (16) are employed simultaneously, the following result is written.

Corollary 3.4. *Under the conditions of Lemma* (3.2)*, we have*

$$\varrho_{p(.)}^{\frac{s}{p_1}}(u) \le C(|H(t)| + \|u_t\|_2^2 + \varrho_{p(.)}(u)), \tag{18}$$

for any $r_1 \le s \le p_1$ *and* $u \in W_0^{1,r(.)}(\Omega)$.

For the special case, the following is derived.

Corollary 3.5. *Under the conditions of Lemma* (3.2)*, for every* $r_1 \le s \le p_1$ *and* $u \in W_0^{1,r(.)}(\Omega)$*, the following inequality is satisfied*

$$\|u\|_{p_1}^{s} \le C(|H(t)| + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}). \tag{19}$$

Lemma 3.6. *[14] If it is assumed that the inequalities* (5) *and* (6) *are provided and* $E(0) < 0$, *the solution of the problem* (1)-(3) *satisfies the following for some* $c > 0$,

$$\varrho_{p(.)}(u) \geq c\,\|u\|_{p_1}^{p_1}. \tag{20}$$

Proof.

$$\varrho_{p(.)}(u) = \int\limits_{\Omega} |u|^{p(x)}dx = \int\limits_{\Omega_+} |u|^{p(x)}dx + \int\limits_{\Omega_-} |u|^{p(x)}dx,$$

where

$$\Omega_+ = \{x \in \Omega / \, |u(x,t)| \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega / \, |u(x,t)| < 1\}.$$

So we comprehend

$$\varrho_{p(.)}(u) \geq \int\limits_{\Omega_+} |u|^{p_1}dx + \int\limits_{\Omega_-} |u|^{p_2}dx \geq \int\limits_{\Omega_+} |u|^{p_1}dx + c_1\left(\int\limits_{\Omega_-} |u|^{p_1}dx\right)^{\frac{p_2}{p_1}}.$$

This yields

$$c_2\Big(\varrho_{p(.)}(u)\Big)^{\frac{p_1}{p_2}} \geq \int\limits_{\Omega_-} |u|^{p_1}dx \quad \text{and} \quad \varrho_{p(.)}(u) \geq \int\limits_{\Omega_+} |u|^{p_1}dx,$$

and, hence,

$$c_2\Big(\varrho_{p(.)}(u)\Big)^{\frac{p_1}{p_2}} + \varrho_{p(.)}(u) \geq \|u\|_{p_1}^{p_1}. \tag{21}$$

Because

$$0 < H(0) < H(t) \leq \frac{b}{p_1}\varrho_{p(.)}(u),$$

then (21) gives rise to

$$\varrho_{p(.)}(u)\left[1 + c_2\Big(\frac{p_1}{b}H(0)\Big)^{\frac{p_1}{p_2}-1}\right] \geq \|u\|_{p_1}^{p_1}.$$

From the obtained inequality, it is seen that (20) is provided, and the proof is completed. □

Lemma 3.7. *Suppose* (5) *is provided, and let u be the solution of the equation* (1)-(3). *Then the following inequality is provided,*

$$\int\limits_{\Omega} |u|^{m(x)}dx \leq C\left(\Big(\varrho_{p(.)}(u)\Big)^{\frac{m_1}{p_1}} + \Big(\varrho_{p(.)}(u)\Big)^{\frac{m_1}{p_1}}\right). \tag{22}$$

Proof. See [14]. □

Lemma 3.8. *Let u be the solution of the equation* (1)-(3), *there is a constant* $c_1 > 0$ *such that the following inequality is provided,*

$$\|\nabla u(.,t)\|_{r(.)} \geq c_1, \quad \forall t \geq 0. \tag{23}$$

Proof. See [14]. □

Proof. (*Theorem* 3.1) We perform the multiplication of (1)-(4) by u_t and then integrate it over the domain Ω to obtain

$$E'(t) = -a \int_\Omega |u_t(x,t)|^{m(x)} dx \leq 0, \tag{24}$$

for almost every $t \in [0, T)$ because E is an absolutely continuous function [10]; as a result, $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1} \varrho_{p(.)}(u), \tag{25}$$

for every $t \in [0, T)$, by referring to (14). Next, we define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t) dx, \tag{26}$$

for a small ϵ which will be determined later, and for

$$0 < \alpha \leq \min\left\{ \frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}, \frac{1}{m_2 - 1}, \frac{1}{r_2 - 1}, \frac{1}{p_2 - 1} \right\}. \tag{27}$$

The derivatives of both sides of (26) are taken using equations (9)-(12), resulting in the following,

$$\begin{aligned} L'(t) = {} & (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_\Omega {u_t}^2 dx + \varepsilon b \varrho_{p(.)}(u) - \varepsilon \int_\Omega |\nabla u|^{r(x)} dx - \varepsilon a \int_\Omega |u_t|^{m(x)-1} u dx \\ & + \varepsilon \int_\Omega |\nabla u|^{r(x)-1} \nabla w(x) dx + \varepsilon a \int_\Omega |u_t|^{m(x)-1} w(x) dx - \varepsilon b \int_\Omega |u|^{p(x)-1} w(x) dx. \end{aligned} \tag{28}$$

For $0 < \eta < 1$, using the equation (17), we have

$$\varepsilon b \varrho_{p(.)}(u) \geq \varepsilon(1-\eta) p_1 H(t) + \frac{\varepsilon(1-\eta)}{2} p_1 \|u_t\|_2^2 + \frac{\varepsilon(1-\eta)}{r_2} p_1 \int_\Omega |\nabla u|^{r(x)} dx + \frac{\varepsilon b \eta}{p_2} p_1 \varrho_{p(.)}(u). \tag{29}$$

Now, if Young's inequality is applied to the last four terms on the right side of (28) respectively, the following estimates are obtained,

$$\int_\Omega |u_t|^{m(x)-1} |u| \, dx \leq \frac{1}{m_1} \int_\Omega \delta^{m(x)} |u|^{m(x)} dx + \frac{m_2 - 1}{m_2} \int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)} dx, \quad \forall \delta > 0, \tag{30}$$

$$\int_\Omega |\nabla u|^{r(x)-1} |\nabla w(x)| dx \leq \frac{1}{r_1} \int_\Omega \delta^{r(x)} |\nabla w(x)|^{r(x)} dx + \frac{r_2 - 1}{r_2} \int_\Omega \delta^{-\frac{r(x)}{r(x)-1}} |\nabla u|^{r(x)} dx, \quad \forall \delta > 0, \tag{31}$$

$$\int_\Omega |u_t|^{m(x)-1} |w(x)| dx \leq \frac{1}{m_1} \int_\Omega \delta^{m(x)} |w(x)|^{m(x)} dx + \frac{m_2 - 1}{m_2} \int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)} dx, \quad \forall \delta > 0, \tag{32}$$

$$\int_\Omega |u|^{p(x)-1} |w(x)| dx \leq \frac{1}{p_1} \int_\Omega \delta^{p(x)} |w(x)|^{p(x)} dx + \frac{p_2 - 1}{p_2} \int_\Omega \delta^{-\frac{p(x)}{p(x)-1}} |u|^{p(x)} dx, \quad \forall \delta > 0. \tag{33}$$

where δ is a real number depending on the time.

Let δ be chosen in such a way as to achieve the following equality,

$$\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t). \tag{34}$$

Here $k > 0$ is specified later, if this equality is written instead of (30) and (32), the desired estimates are obtained, if (30) is considered first, the following inequality is obtained,

$$\int_{\Omega} |u_t|^{m(x)-1} |u| dx \leq \frac{1}{m_1} \int_{\Omega} k^{1-m(x)} |u|^{m(x)} H^{\alpha(m(x)-1)}(t) dx + \frac{(m_2-1)k}{am_2} H^{-\alpha}(t) H'(t). \tag{35}$$

Using the (25) inequality and Lemma (3.7), the following is is acquired,

$$H^{\alpha(m_2-1)}(t) \int_{\Omega} |u|^{m(x)} dx \leq C\left[\left(\varrho_{p(.)}(u)\right)^{\frac{m_1}{p_1}+\alpha(m_2-1)} + \left(\varrho_{p(.)}(u)\right)^{\frac{m_2}{p_1}+\alpha(m_2-1)}\right]. \tag{36}$$

We subsequently utilize (27) and Lemma (3.2), for

$$s = m_2 + \alpha p_1(m_2-1) \leq p_1 \quad \textit{and} \quad s = m_1 + \alpha p_1(m_2-1) \leq p_1,$$

to infer, from (36), that

$$H^{\alpha(m_2-1)}(t) \int_{\Omega} |u|^{m(x)} dx \leq C\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right). \tag{37}$$

When (37) is substituted for (35), (30) is written as follows,

$$\int_{\Omega} |u_t|^{m(x)-1} |u| dx \leq \frac{k^{1-m_1}}{m_1} C\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) + \frac{(m_2-1)k}{am_2} H^{-\alpha}(t) H'(t). \tag{38}$$

Similarly, the (32) inequality is written as follows,

$$\int_{\Omega} |u_t|^{m(x)-1} |w(x)| dx \leq \frac{1}{m_1} \int_{\Omega} k^{1-m(x)} |w(x)|^{m(x)} H^{\alpha(m(x)-1)}(t) dx + \frac{(m_2-1)k}{am_2} H^{-\alpha}(t) H'(t). \tag{39}$$

Using the (25) inequality, the following is obtained,

$$H^{\alpha(m_2-1)}(t) \leq \left(\frac{b}{p_1}\right)^{\alpha(m_2-1)} \left(\varrho_{p(.)}(u)\right)^{\alpha(m_2-1)}. \tag{40}$$

By using inequality (27) and Lemma (3.2), (40) can be written as following, with

$$h = p_1\alpha(m_2-1) \leq p_1$$

being, due to $r_1 < h$,

$$H^{\alpha(m_2-1)}(t) \leq C_1\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right). \tag{41}$$

Also, the following equation can be written,

$$\int_{\Omega} |w(x)|^{m(x)} dx = C_2,$$

where C_2 is a positive constant. By utilizing the inequality (41) and the last written equation, the final form of inequality (32) is expressed as follows,

$$\int_{\Omega} |u_t|^{m(x)-1} |w(x)| dx \leq \frac{k^{1-m_1}}{m_1} C_3 \left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u) \right) + \frac{(m_2-1)k}{a m_2} H^{-\alpha}(t) H'(t). \tag{42}$$

Now let's consider inequality (31), where $n > 0$ is specified later, if δ is chosen to satisfy the following equation,

$$\delta^{-\frac{r(x)}{r(x)-1}} = n H^{-\alpha}(t),$$

then, the equation (31) is written as follows,

$$\int_{\Omega} |\nabla u|^{r(x)-1} |\nabla w(x)| \, dx \leq \frac{1}{r_1} \int_{\Omega} n^{1-r(x)} |\nabla w(x)|^{r(x)} H^{\alpha(r(x)-1)}(t) dx + \frac{(r_2-1)n}{r_2} H^{-\alpha}(t) \int_{\Omega} |\nabla u|^{r(x)} dx. \tag{43}$$

Using the (25) inequality, the following is obtained,

$$H^{\alpha(r_2-1)}(t) \leq \left(\frac{b}{p_1} \right)^{\alpha(r_2-1)} \left(\varrho_{p(.)}(u) \right)^{\alpha(r_2-1)}. \tag{44}$$

By using inequality (27) and Lemma (3.2), (44) can be written as following, with

$$v = \alpha p_1 (r_2 - 1) \leq p_1,$$

being, due to $r_1 < v$,

$$H^{\alpha(r_2-1)}(t) \leq C_4 \left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u) \right). \tag{45}$$

Also, the following equation can be written,

$$\int_{\Omega} |\nabla w(x)|^{r(x)} dx = C_5,$$

where C_5 is a positive constant. By utilizing the inequality (45) and the last written equation, the final form of inequality (31) is expressed as follows,

$$\int_{\Omega} |\nabla u|^{r(x)-1} |\nabla w(x)| \, dx \leq \frac{n^{1-r_1}}{r_1} C_6 \left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u) \right) + \frac{(r_2-1)n}{r_2} H^{-\alpha}(t) \int_{\Omega} |\nabla u|^{r(x)} dx. \tag{46}$$

Now let's consider inequality (33), where $l > 0$ is specified later, if δ is chosen to satisfy the following equation,

$$\delta^{-\frac{p(x)}{p(x)-1}} = l H^{-\alpha}(t),$$

then, the equation (33) is written as follows,

$$\int_{\Omega} |u|^{p(x)-1} |w(x)| \, dx \leq \frac{1}{p_1} \int_{\Omega} l^{1-p(x)} |w(x)|^{p(x)} H^{\alpha(p(x)-1)}(t) dx + \frac{(p_2-1)l}{p_2} H^{-\alpha}(t) \varrho_{p(.)}(u). \tag{47}$$

Using the (25) inequality, the following is obtained,

$$H^{\alpha(p_2-1)}(t) \leq \left(\frac{b}{p_1} \right)^{\alpha(p_2-1)} \left(\varrho_{p(.)}(u) \right)^{\alpha(p_2-1)}. \tag{48}$$

By using inequality (27) and Lemma (3.2), (48) can be written as following, with

$$z = p_1\alpha(p_2 - 1) \leq p_1,$$

being, due to $r_1 < z$,

$$H^{\alpha(r_2-1)}(t) \leq C_4\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right). \tag{49}$$

Also, the following equation can be written,

$$\int_\Omega |w(x)|^{p(x)}dx = C_8,$$

where C_8 is a positive constant. By utilizing the inequality (49) and the last written equation, the final form of inequality (33) is expressed as follows,

$$\int_\Omega |u|^{p(x)-1}|w(x)|\,dx \leq \frac{l^{1-p_1}}{p_1}C_9\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) + \frac{(p_2-1)l}{p_2}H^{-\alpha}(t)\varrho_{p(.)}(u). \tag{50}$$

By utilizing Lemma (2.5) and (3.8), we obtain

$$\varrho_{r(.)}(\nabla u) \geq c_2\|\nabla u\|_{r(.)}^{r_1}. \tag{51}$$

If the inequalities (29), (38), (42), (46), (50) are substituted into (28), the following inequality is obtained,

$$\begin{aligned}
L'(t) \geq\, & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega {u_t}^2dx + \varepsilon(1-\eta)p_1H(t) + \frac{\varepsilon(1-\eta)}{2}p_1\|u_t\|_2^2 \\
& + \frac{\varepsilon(1-\eta)}{r_2}p_1\int_\Omega |\nabla u|^{r(x)}dx + \frac{\varepsilon b\eta}{p_2}p_1\varrho_{p(.)}(u) - \varepsilon\int_\Omega |\nabla u|^{r(x)}dx \\
& - \frac{k^{1-m_1}\varepsilon a}{m_1}C\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) - \frac{\varepsilon(m_2-1)k}{m_2}H^{-\alpha}(t)H'(t) \\
& - \frac{n^{1-r_1}\varepsilon}{r_1}C_6\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) - \frac{\varepsilon(r_2-1)n}{r_2}H^{-\alpha}(t)\int_\Omega |\nabla u|^{r(x)}dx \\
& - \frac{k^{1-m_1}\varepsilon a}{m_1}C_3\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) - \frac{\varepsilon(m_2-1)k}{m_2}H^{-\alpha}(t)H'(t) \\
& - \frac{l^{1-p_1}\varepsilon b}{p_1}C_9\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right) - \frac{(p_2-1)l\varepsilon b}{p_2}H^{-\alpha}(t)\varrho_{p(.)}(u)
\end{aligned} \tag{52}$$

If it is rearranged, the following is written,

$$\begin{aligned}
L'(t) \geq\, & \left[(1-\alpha) - \frac{2\varepsilon(m_2-1)k}{m_2}\right]H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1H(t) \\
& + \varepsilon\left[1 + \frac{(1-\eta)p_1}{2}\right]\int_\Omega {u_t}^2dx \\
& + \varepsilon\left[\frac{(1-\eta)p_1}{r_2} - \frac{(r_2-1)\delta^{-\frac{r_2}{r_2-1}}}{r_2} - 1\right]\int_\Omega |\nabla u|^{r(x)}dx \\
& + \varepsilon\left[\frac{b\eta p_1}{p_2} - \frac{(p_2-1)b\delta^{-\frac{p_2}{p_2-1}}}{p_2}\right]\varrho_{p(.)}(u) \\
& - \varepsilon\left[\frac{k^{1-m_1}a}{m_1}C + \frac{k^{1-m_1}a}{m_1}C_3 + \frac{n^{1-r_1}}{r_1}C_6 + \frac{l^{1-p_1}b}{p_1}C_9\right]\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right)
\end{aligned} \tag{53}$$

When β is written as shown below

$$\beta = \min\left\{(1-\eta)p_1, \frac{b\eta p_1}{p_2} - \frac{(p_2-1)b\delta^{-\frac{p_2}{p_2-1}}}{p_2}, \frac{(1-\eta)p_1}{r_2} - \frac{(r_2-1)\delta^{-\frac{r_2}{r_2-1}}}{r_2} - 1, 1 + \frac{(1-\eta)p_1}{2}\right\} > 0,$$

and η is chosen appropriately and such that $\beta > 0$, (53) is written as follows,

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha) - \frac{2\varepsilon(m_2-1)k}{m_2}\right] H^{-\alpha}(t)H'(t) + \varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \int_\Omega |\nabla u|^{r(x)}dx + \varrho_{p(.)}(u)\right] \\ & - \varepsilon\left[\frac{k^{1-m_1}a}{m_1}C + \frac{k^{1-m_1}a}{m_1}C_3 + \frac{n^{1-r_1}}{r_1}C_6 + \frac{l^{1-p_1}b}{p_1}C_9\right]\left(\|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right). \end{aligned} \tag{54}$$

σ be,

$$\sigma = \left[\frac{k^{1-m_1}a}{m_1}C + \frac{k^{1-m_1}a}{m_1}C_3 + \frac{n^{1-r_1}}{r_1}C_6 + \frac{l^{1-p_1}b}{p_1}C_9\right] > 0.$$

Based on this and using (51), the following inequality is written,

$$L'(t) \geq \left[(1-\alpha) - \frac{2\varepsilon(m_2-1)k}{m_2}\right] H^{-\alpha}(t)H'(t) + \varepsilon(\beta-\sigma)\left[H(t) + \|u_t\|_2^2 + \|\nabla u\|_{r(.)}^{r_1} + \varrho_{p(.)}(u)\right]. \tag{55}$$

The values of k, l, and n are chosen appropriately and sufficiently large to ensure that the condition $\beta - \sigma > 0$ is satisfied. The chosen value of k, along with a sufficiently small ϵ value, is written as follows,

$$(1-\alpha) - \frac{2\varepsilon(m_2-1)k}{m_2} \geq 0 \quad \textit{and} \quad L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)dx > 0.$$

By utilizing the aforementioned information and inequality (20) and (51), (55) is expressed as follows,

$$L'(t) \geq \varepsilon(\beta-\sigma)\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right]. \tag{56}$$

and

$$L(t) \geq L(0) > 0, \quad t > 0.$$

Hence, employing Holder's and Young's inequalities, we obtain the following

$$\|u\|_2 \leq \left[\left(\int_\Omega (|u|^2)^{p_1/2}dx\right)^{\frac{2}{p_1}} .\left(\int_\Omega 1dx\right)^{1-\frac{2}{p_1}}\right]^{\frac{1}{2}} \leq c\|u\|_{p_1}, \tag{57}$$

and

$$\left|\int_\Omega uu_t dx\right| \leq \|u_t\|_2.\|u\|_2 \leq c\|u_t\|_2.\|u\|_{p_1},$$

then

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\,\|u_t\|_2^{1/1-\alpha}.\|u\|_{p_1}^{1/1-\alpha} \leq c\left[\|u_t\|_2^{\theta/1-\alpha} + \|u\|_{p_1}^{\mu/1-\alpha}\right], \tag{58}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. Let us consider $\theta=2(1-\alpha)$, thus

$$\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p_1.$$

Setting $s=\frac{2}{1-2\alpha}\leq p_1$, we obtain

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\leq c\left[\|u\|_{p_1}^s+\|u_t\|_2^2\right].$$

Thus, Corollary (3.5) yields the following

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\leq C\left[H(t)+\|u\|_{p_1}^{p_1}+\|u_t\|_2^2\right],\forall t\geq 0. \tag{59}$$

Subsequently,

$$L^{\frac{1}{1-\alpha}}(t)=\left\{H^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx\right\}^{\frac{1}{1-\alpha}}\leq c\left\{H(t)+\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\right\}\leq c\left\{H(t)+\|u\|_{p_1}^{p_1}+\|u_t\|_2^2\right\}. \tag{60}$$

Based on (56) and (60), considering the case where $\lambda(\varepsilon(\beta-\sigma),c)>0$ is true, the following can be written,

$$L'(t)\geq\lambda L^{\frac{1}{1-\alpha}}(t). \tag{61}$$

By integration of (61), we find

$$L^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{L^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{1-\alpha}t}.$$

Hence, the solution blows up in finite time $T*$, such that

$$T^*=\frac{1-\alpha}{\lambda\alpha L^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

Conclusion

In this study, we have made a novel contribution to the literature through our conducted procedures. Upon reviewing the previous works related to the inverse problem, it is evident that the methods employed in our study differ from those used in the past. Recently, we have applied the Georgiev and Todorova method, commonly used in various blow-up articles, to induce the blow-up of the solution in the inverse problem. This study will assist other researchers in the field of inverse problems who wish to utilize this method in their work.

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