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New discrete operators from generalized Baskakov operators

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Abstract. In this paper, we propose three new operators, which are obtained from composition of generalized Baskakov-Szász operators with Szász, Lupaş and operators based on Laguerre polynomials. It is observed here that the new operators are expressed in the discrete form. We also provide their moment generating functions, which may be useful in finding several other convergence results in different settings.

1. Introduction

For $a \geq 0$, Mihesan in [9] considered the following operators

$$(W_m^a f)(x) = \sum_{\nu=0}^{\infty} w_{m,\nu}^a(x)\, f\left(\frac{\nu}{m}\right), \tag{1}$$

where $w_{m,\nu}^a(x) = e^{-\frac{ax}{1+x}} \frac{p_\nu(m,a)}{\nu!} \frac{x^\nu}{(1+x)^{m+\nu}}$, $p_\nu(m,a) = \sum_{i=0}^{\nu} \binom{\nu}{i} (m)_i\, a^{\nu-i}$ and $(m)_i = \prod_{\eta=0}^{i-1}(m+\eta)$, $(m)_0 = 1$. It was obtained by the generating function

$$(1-t)^{-m} e^{at} = \sum_{k=0}^{\infty} p_k(m,a) \frac{t^k}{k!}. \tag{2}$$

Particularly, if $a = 0$, then the operators (1) becomes Baskakov operators.

The generalized Baskakov-Szász operator (see [3]) is defined by

$$(\overline{V}_m^a f)(x) = m \sum_{\nu=0}^{\infty} w_{m,\nu}^a(x) \int_0^\infty s_\nu(mt) f(t)dt, \tag{3}$$

where $s_\nu(r) = e^{-r} r^\nu / \nu!$ is Szász-Basis function.

In the present article, we introduce three new discretely defined operators by composition of (3) with three operators namely Szász, Lupaş and the operators based on Laguerre polynomials. We consider different parameters m and n, in case $m = n$ we get immediately the approximation operator. We obtain moments and basic convergence theorems. The lot of work may be done on such new operators.

Throughout the paper, we denote $\exp_A(t) = e^{At}$ and $B_C[0,\infty)$ denotes the class of bounded continuous functions on positive real axis.

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2. Composition with Szász operators

We can take a composition of $\overline{V}_m^a$ with the Szász-Mirakyan operators S_n, to obtain a new approximation operator $C_{m,n}^a$ as follows:

$$(C_{m,n}^a f)(x) \quad := \quad (\overline{V}_m^a \circ S_n f)(x).$$

For $m = n$, we get the approximation operator $C_{n,n}^a = C_n^a$.

Theorem 2.1. *A concise form of $C_{m,n}^a$ can be given by*

$$\left(C_{m,n}^a f\right)(x) = \sum_{k=0}^{\infty} c_{k,m,n}(x)\, f\left(\frac{k}{n}\right),$$

where

$$c_{k,m,n}^a(x) \quad = \quad \frac{me^{-\frac{ax}{1+x}}}{(1+x)^m}\frac{n^k}{(m+n)^{k+1}}\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu}\frac{a^m}{\nu!}\left(\frac{amx}{(m+n)(1+x)}\right)^{\nu} U(m, 1+m+\nu, a),$$

$U(a, b, z)$ is the Tricomi's confluent hypergeometric series. In particular if a approaches to zero, then

$$c_{k,m,n}^{a=0}(x) = \frac{mn^k}{(m+n)^{k+1}(1+x)^m}\ {}_2F_1\left(k+1, m; 1; \frac{mx}{(m+n)(1+x)}\right).$$

Proof. We have

$$\begin{aligned}
(C_{m,n} f)(x) \quad &= \quad m\sum_{\nu=0}^{\infty} w_{m,\nu}^a(x)\int_0^{\infty} s_\nu(mt)\sum_{k=0}^{\infty} s_k(nt) f\left(\frac{k}{n}\right)dt \\
&:= \quad \sum_{k=0}^{\infty} c_{k,m,n}(x)\, f\left(\frac{k}{n}\right),
\end{aligned}$$

where

$$\begin{aligned}
c_{k,m,n}(x) \quad &= \quad \frac{mn^k}{(m+n)^{k+1}}\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu} w_{m,\nu}^a(x)\left(\frac{m}{m+n}\right)^{\nu} \\
&= \quad \frac{m}{(1+x)^m}\frac{n^k}{(m+n)^{k+1}}\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu} e^{-\frac{ax}{1+x}}\frac{p_\nu(m,a)}{\nu!}\left(\frac{mx}{(m+n)(1+x)}\right)^{\nu} \\
&= \quad \frac{me^{-\frac{ax}{1+x}}}{(1+x)^m}\frac{n^k}{(m+n)^{k+1}}\sum_{\nu=0}^{\infty}\binom{k+\nu}{\nu}\frac{a^m}{\nu!}\left(\frac{amx}{(m+n)(1+x)}\right)^{\nu} U(m, 1+m+\nu, a).
\end{aligned}$$

□

Remark 2.2. *For non-negative constant $a \geq 0$, we have*

$$(W_m^a \exp_A)(x) = e^{\frac{ax}{1+x}\left(e^{\frac{A}{m}}-1\right)}\left(1 + x - xe^{\frac{A}{m}}\right)^{-m}.$$

Also, for Szász–Mirakyan operators [6, Eq. (8)], we have

$$\left(S_n \exp_A\right)(x) = \exp\left(nx\left(e^{A/n} - 1\right)\right).$$

Proposition 2.3. *For the operators $\overline{V}_m^a$, there hold*

$$(\overline{V}_m^a \exp_A)(x) \quad = \quad \frac{m(m-A)^{m-1}}{(m-A-Ax)^m}\exp\left(\frac{axA}{(1+x)(m-A)}\right).$$

Proof. Simple computation after applying (2) leads us to

$$\begin{aligned}
(\overline{V}_m^a \exp_A)(x) &= m\sum_{\nu=0}^{\infty} w_{m,\nu}(x)\int_0^{\infty}\frac{e^{-mt}(mt)^{\nu}}{\nu!}e^{At}dt\\
&= \frac{m}{m-A}e^{-\frac{ax}{1+x}}\frac{1}{(1+x)^m}\sum_{\nu=0}^{\infty}\frac{p_\nu(m,a)}{\nu!}\frac{x^\nu}{(1+x)^\nu}\frac{m^\nu}{(m-A)^\nu}\\
&= \frac{m}{m-A}e^{-\frac{ax}{1+x}}\frac{1}{(1+x)^m}\left[1-\frac{xm}{(1+x)(m-A)}\right]^{-m}\exp\left(\frac{axm}{(1+x)(m-A)}\right)\\
&= \frac{m(m-A)^{m-1}}{(m-A-Ax)^m}\exp\left(\frac{axA}{(1+x)(m-A)}\right).
\end{aligned}$$

□

Proposition 2.4. *For the operator $C^a_{n,m}$, there holds*

$$(C^a_{m,n}\exp_A)(x) = \left(\frac{m(m+n-ne^{A/n})^{m-1}}{(m+n(1+x)(1-e^{A/n}))^m}\right)\exp\left(\frac{axn(e^{A/n}-1)}{(1+x)(m+n-ne^{A/n})}\right).$$

Proof. By applying Remark 2.2, we have

$$\begin{aligned}
(C^a_{m,n}\exp_A)(x) &= (\overline{V}_m^a \circ S_n \exp_A)(x)\\
&= (\overline{V}_m^a \exp_{n(e^{A/n}-1)})(x).
\end{aligned}$$

The result immediately follows by applying Proposition 2.3. □

Theorem 2.5. *If m is a natural number then for $f\in B_C[0,\infty)$, we have*

$$\begin{aligned}
\lim_{m\to\infty}(C^a_{m,n}f)(x) &= (S_nf)(x),\\
\lim_{n\to\infty}\left(C^a_{r,n}f\left(\frac{t}{m}\right)\right)(mx) &= (P_rf)(x),
\end{aligned}$$

where S_n and P_r are respectively Szász-Mirakyan and Post-Widder operators given by

$$(S_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f(k/n),\ \ (P_rf)(x)=\frac{r^r}{x^r}\frac{1}{\Gamma(r)}\int_0^{\infty}e^{-rt/x}t^{r-1}f(t)dt.$$

Proof. By Proposition 2.4 and by [1, Th. 1.1], we have

$$\lim_{n\to\infty}(C^a_{m,n}\exp_{is})(x) = \exp\left((-1+e^{is/n})nx\right)=(S_ne^{ist})(x),$$

leads us to

$$\lim_{n\to\infty}(C^a_{m,n}f)(x) = (S_nf)(x).$$

Finally

$$\lim_{n\to\infty}\left(C^a_{r,n}\exp_{is/m}\right)(mx) = r^r(r-isx)^{-r}=(P_r\exp_{is})(x),$$

thus we get

$$\lim_{n\to\infty}\left(C^a_{r,n}f\left(\frac{t}{m}\right)\right)(mx) = (P_rf)(x).$$

□

3. Composition with Lupaş operators

A. Lupaş [7] introduced the following important operator, which is defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(nx)_k}{k! 2^{k+nx}} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

Later Agratini [2] studied these operators in details.

We can take a composition of $\overline{V}_n^a$ with the Lupas operators L_m, to obtain another new approximation operator $D^a_{n,m}$ as follows:

$$(D^a_{m,n} f)(x) \quad := \quad (\overline{V}_m^a \circ L_n f)(x).$$

The approximation operator can be obtained in above $m = n$.

Theorem 3.1. *A concise form of $D^a_{m,n}$ is given by*

$$(D^a_{m,n} f)(x) = \sum_{k=0}^{\infty} d^a_{k,m,n}(x) f\left(\frac{k}{n}\right),$$

where

$$d^a_{k,m,n}(x) \quad = \quad \frac{1}{k! 2^k} \sum_{\nu=0}^{\infty} \frac{w^a_{m,\nu}(x) m^{\nu+1}}{\nu! (m + n \log 2)^{\nu+1}} \sum_{i=0}^{k} (-1)^{k-i} s_{k,i} \frac{n^i (i+\nu)!}{(m + n \log 2)^i}.$$

In particular if $m = n$, then

$$d^a_{k,n,n}(x) \quad = \quad \frac{1}{k! 2^k} \sum_{\nu=0}^{\infty} \frac{w^a_{n,\nu}(x)}{\nu! (1 + \log 2)^{\nu+1}} \sum_{i=0}^{k} (-1)^{k-i} s_{k,i} \frac{(i+\nu)!}{(1 + \log 2)^i}.$$

Proof. Thus using $(u)_k = \sum_{i=0}^{k} (-1)^{k-i} s_{k,i}.u^i$, where $s_{k,i}$ is Stirling number of first kind, we can write

$$\begin{aligned}(D^a_{m,n} f)(x) \quad &= \quad m \sum_{\nu=0}^{\infty} w^a_{m,\nu}(x) \int_0^{\infty} s_\nu(mt) 2^{-nt} \sum_{k=0}^{\infty} \frac{(nt)_k}{k! 2^k} f\left(\frac{k}{n}\right) dt \\ &:= \quad \sum_{k=0}^{\infty} d^a_{k,n,n}(x) f\left(\frac{k}{n}\right),\end{aligned}$$

where

$$\begin{aligned}d^a_{k,n,n}(x) \quad &= \quad \frac{m}{k! 2^k} \sum_{\nu=0}^{\infty} w^a_{m,\nu}(x) \int_0^{\infty} e^{-mt} \frac{(mt)^\nu}{\nu!} 2^{-nt} (nt)_k dt \\ &= \quad \frac{m}{k! 2^k} \sum_{\nu=0}^{\infty} w^a_{m,\nu}(x) \int_0^{\infty} e^{-mt - nt \log 2} \frac{(mt)^\nu}{\nu!} \sum_{i=0}^{k} (-1)^{k-i} s_{k,i}.(nt)^i dt \\ &= \quad \frac{1}{k! 2^k} \sum_{\nu=0}^{\infty} \frac{w^a_{m,\nu}(x)}{\nu!} \sum_{i=0}^{k} (-1)^{k-i} s_{k,i} \frac{m^{\nu+1} n^i (i+\nu)!}{(m + n \log 2)^{i+\nu+1}}.\end{aligned}$$

□

Proposition 3.2. *The operator $D^a_{m,n}$ satisfies*

$$(D^a_{m,n} \exp_A)(x) \quad = \quad \frac{m(m + n \log(2 - e^{A/n}))^{m-1}}{[(m + n(1+x) \log(2 - e^{A/n})]^m} \exp\left(\frac{-axn \log(2 - e^{A/n})}{(1+x)(m + n \log(2 - e^{A/n}))}\right).$$

Proof. Using $\sum_{k=0}^{\infty} \frac{(c)_k}{k!} z^k = \frac{1}{(1-z)^c}$, $|z| < 1$, and (2), we can write

$$\begin{aligned}
(D^a_{m,n} \exp_A)(x) &= m \sum_{\nu=0}^{\infty} w^a_{m,\nu}(x) \int_0^{\infty} s_\nu(mt) 2^{-nt} \sum_{k=0}^{\infty} \frac{(nt)_k}{k! 2^k} e^{Ak/n} dt \\
&= m \sum_{\nu=0}^{\infty} w^a_{m,\nu}(x) \int_0^{\infty} e^{-mt} \frac{(mt)^\nu}{\nu!} (2 - e^{A/n})^{-nt} dt \\
&= \frac{m}{m + n \log(2 - e^{A/n})} \frac{1}{(1+x)^m} \left[1 - \frac{xm}{(1+x)(m + n\log(2 - e^{A/n}))}\right]^{-m} \\
&\quad \exp\left(\frac{axm}{(1+x)(m + n\log(2 - e^{A/n}))} - \frac{ax}{1+x}\right) \\
&= \frac{m(m + n\log(2 - e^{A/n}))^{m-1}}{[(m + n(1+x)\log(2 - e^{A/n})]^m} \exp\left(\frac{-axn\log(2 - e^{A/n})}{(1+x)(m + n\log(2 - e^{A/n}))}\right).
\end{aligned}$$

□

Theorem 3.3. *For $f \in B_C[0,\infty)$, we have*

$$\begin{aligned}
\lim_{n\to\infty} (D^a_{n,n} f)(x) &= f(x), \\
\lim_{m\to\infty} (D^a_{m,n} f)(x) &= (L_n f)(x), \\
\lim_{n\to\infty} (D^a_{r,m} \left(\frac{t}{n}\right))(nx) &= (P_r f)(x),
\end{aligned}$$

where L_n and P_r are respectively Lupaş and Post-Widder operators.

Proof. By Proposition 3.2, and by [1], we have

$$\lim_{n\to\infty} (D^a_{n,n} \exp_{is})(x) = e^{isx},$$

implying

$$\lim_{n\to\infty} (D^a_{n,n} f)(x) = f(x).$$

Next

$$\lim_{m\to\infty} (D^a_{m,n} \exp_{is})(x) = (2 - e^{is/n})^{-nx} = (L_n \exp_{is})(x),$$

leads us to

$$\lim_{m\to\infty} (D^a_{m,n} f)(x) = (L_n f)(x).$$

Finally

$$\lim_{n\to\infty} \left(D^a_{r,m} \exp_{is/n}\right)(nx) = r^r (r - isx)^{-r} = (P_r \exp_{is})(x),$$

thus we get

$$\lim_{n\to\infty} \left(D^a_{r,m} f\left(\frac{t}{n}\right)\right)(nx) = (P_r f)(x).$$

where L_m and P_r are respectively Szász-Mirakyan and Post-Widder operators □

4. Composition with operators based on Laguerre polynomials

Sucu et al [10] introduced by means of the Laguerre polynomials, the following operators defined for $x \in [0, \infty)$, $\beta > -1$ and $n \in N$ as

$$(G_n^{\beta} f)(x) = e^{-nx/2} 2^{-\beta-1} \sum_{k=0}^{\infty} 2^{-k} L_k^{\beta}\left(\frac{-nx}{2}\right) f\left(\frac{k}{n}\right), \tag{4}$$

where $L_k^{\beta}(-x)$ are the modified Laguerre polynomials defined in terms of confluent hypergeometric series by

$$L_k^{\beta}(-x) := \frac{(\beta+1)_k}{k!} \ {}_1F_1(-k; \beta+1; -x), \ \ \alpha > -1,$$

alternatively $L_k^{\beta}(-x)$ in the equivalent form is expressed as $\sum_{s=0}^{k} \frac{(\beta+k)!}{(k-s)!(\beta+s)!s!} x^s$.

We can take a composition of $\overline{V}_n^{a}$ with the Lupas operators G_m^{β}, to obtain another new approximation operator $E_{n,m}^{a,\beta}$ as follows:

$$(E_{m,n}^{a,\beta} f)(x) := (\overline{V}_m^{a} \circ G_n^{\beta} f)(x).$$

If $m = n$ in above we get approximation operator.

Theorem 4.1. *A concise form of $E_{m,n}^{a,\beta}$ is given by*

$$(E_{m,n}^{a,\beta} f)(x) = \sum_{k=0}^{\infty} e_{k,m,n}^{a,\beta}(x) \, f\left(\frac{k}{n}\right),$$

where

$$\begin{aligned} e_{k,m,n}^{a,\beta}(x) &= \sum_{k=0}^{\infty} \frac{m e^{-\frac{ax}{1+x}}}{(1+x)^m 2^{\beta+k}} f\left(\frac{k}{n}\right) \sum_{s=0}^{k} \frac{(\beta+k)! n^s}{(k-s)!(\beta+s)!(2m+n)^{s+1}} \\ &\quad \sum_{\nu=0}^{\infty} \binom{s+\nu}{\nu} \frac{a^{m+\nu}}{\nu!} U(m, 1+m+\nu, a) \frac{(2m)^{\nu}}{[(2m+n)(1+x)]^{\nu}}, \end{aligned}$$

where $U(a, b, z)$ is Tricomi's confluent hypergeometric function. In particular if $a = 0$, then

$$\begin{aligned} e_{k,m,n}^{0,\beta}(x) &= \sum_{k=0}^{\infty} \frac{m e^{-\frac{ax}{1+x}}}{(1+x)^m 2^{\beta+k}} f\left(\frac{k}{n}\right) \sum_{s=0}^{k} \frac{(\beta+k)! n^s}{(k-s)!(\beta+s)!(2m+n)^{s+1}} \\ &\quad {}_2F_1\left(m, 1+s; 1; \frac{2m}{[(2m+n)(1+x)]}\right). \end{aligned}$$

Proof. We can write

$$\begin{aligned} (E_{m,n}^{a,\beta} f)(x) &= m \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \int_0^{\infty} s_{\nu}(mt) \sum_{k=0}^{\infty} \frac{1}{2^{\beta+k+1}} f\left(\frac{k}{n}\right) e^{-nt/2} L_k^{\beta}\left(\frac{-nt}{2}\right) dt \\ &= \sum_{k=0}^{\infty} e_{k,m,n}^{a,\beta}(x) \, f\left(\frac{k}{n}\right), \end{aligned}$$

where

$$e_{k,m,n}^{a,\beta}(x) = \frac{m}{2^{\beta+k+1}} \sum_{s=0}^{k} \frac{(\beta+k)! n^s}{(k-s)!(\beta+s)!s!2^s} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \frac{m^{\nu}}{\nu!} \int_0^{\infty} e^{-\left(m+\frac{n}{2}\right)t} t^{s+\nu} dt$$

$$= \frac{m}{2^{\beta+k+1}} \sum_{s=0}^{k} \frac{(\beta+k)!n^s}{(k-s)!(\beta+s)!s!2^s} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \frac{m^\nu}{\nu!} \frac{2^{s+\nu+1}}{(2m+n)^{s+\nu+1}} (s+\nu)!$$

$$= \frac{me^{-\frac{ax}{1+x}}}{(1+x)^m 2^{\beta+k}} \sum_{s=0}^{k} \frac{(\beta+k)!n^s}{(k-s)!(\beta+s)!(2m+n)^{s+1}}$$

$$\sum_{\nu=0}^{\infty} \binom{s+\nu}{\nu} \frac{p_\nu(m,a)}{\nu!} \frac{(2m)^\nu}{[(2m+n)(1+x)]^\nu}$$

$$= \frac{me^{-\frac{ax}{1+x}}}{(1+x)^m 2^{\beta+k}} \sum_{s=0}^{k} \frac{(\beta+k)!n^s}{(k-s)!(\beta+s)!(2m+n)^{s+1}}$$

$$\sum_{\nu=0}^{\infty} \binom{s+\nu}{\nu} \frac{a^{m+\nu}}{\nu!} U(m, 1+m+\nu, a) \frac{(2m)^\nu}{[(2m+n)(1+x)]^\nu}.$$

□

Proposition 4.2. *For the operators $E_{m,n}^{a,\beta}$, one can see that*

$$(E_{m,n}^{a,\beta} \exp_A)(x) = \frac{m}{(2-e^{A/n})^{\beta+1}\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]} \left(1 + x - \frac{xm}{\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]}\right)^{-m}$$

$$\exp\left(\frac{ax}{1+x}\left(\frac{m}{\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]} - 1\right)\right).$$

Proof. Using the generating function

$$\left(1-\frac{z}{2}\right)^{-\beta-1} \exp\left(\frac{xz}{2(2-z)}\right) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k L_k^\beta\left(\frac{-x}{2}\right) \tag{5}$$

we can write

$$(E_{m,n}^{a,\beta} \exp_A)(x) = \frac{m}{2^{\beta+1}} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \int_0^\infty \frac{(mt)^\nu}{\nu!} e^{-\left(m+\frac{n}{2}\right)t} \sum_{k=0}^{\infty} \left(\frac{e^{A/n}}{2}\right)^k L_k^\beta\left(\frac{-nt}{2}\right) dt$$

$$= \frac{m}{2^{\beta+1}} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \int_0^\infty \frac{(mt)^\nu}{\nu!} e^{-\left(m+\frac{n}{2}\right)t} \left(1-\frac{e^{A/n}}{2}\right)^{-\beta-1} \exp\left(\frac{nte^{A/n}}{2(2-e^{A/n})}\right) dt$$

$$= \frac{m}{(2-e^{A/n})^{\beta+1}} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \frac{m^\nu}{\nu!} \int_0^\infty t^\nu \exp\left[\frac{nte^{A/n}}{2(2-e^{A/n})} - \left(m+\frac{n}{2}\right)t\right] dt$$

$$= \frac{m}{(2-e^{A/n})^{\beta+1}\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]} \sum_{\nu=0}^{\infty} w_{m,\nu}^{a}(x) \frac{m^\nu}{\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]^\nu}$$

$$= \frac{m}{(2-e^{A/n})^{\beta+1}\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]} \left(1 + x - \frac{xm}{\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]}\right)^{-m}$$

$$\exp\left(\frac{ax}{1+x}\left(\frac{m}{\left[\left(m+\frac{n}{2}\right) - \frac{ne^{A/n}}{2(2-e^{A/n})}\right]} - 1\right)\right).$$

□

In particular, if $m = n$ then

$$(E_{n,n}^{a,\beta}\exp_A)(x) = \frac{\left(1+x-\frac{x}{\left[\frac{3}{2}-\frac{e^{A/n}}{2(2-e^{A/n})}\right]}\right)^{-n}}{(2-e^{A/n})^{\beta+1}\left[\frac{3}{2}-\frac{e^{A/n}}{2(2-e^{A/n})}\right]}\exp\left(\frac{ax}{1+x}\left(\frac{1}{\left[\frac{3}{2}-\frac{e^{A/n}}{2(2-e^{A/n})}\right]}-1\right)\right).$$

Theorem 4.3. *For $f \in B_C[0,\infty)$, we have*

$$\lim_{n\to\infty}(E_{n,n}^{a,\beta}f)(x) = f(x),$$
$$\lim_{m\to\infty}(E_{m,n}^{a,\beta}f)(x) = (G_n^{\beta}f)(x),$$
$$\lim_{r\to\infty}(E_{m,n}^{a,\beta}\left(\frac{t}{r}\right))(rx) = (P_m f)(x),$$

where G_n^{β} is operator based on Laguerre polynomial and P_m is Post-Widder operator.

Proof. By Proposition 4.2, and by [1], we have

$$\lim_{n\to\infty}(E_{n,n}^{a,\beta}\exp_{is})(x) = e^{isx},$$

implying

$$\lim_{n\to\infty}(E_{n,n}^{a,\beta}f)(x) = f(x).$$

Next

$$\lim_{m\to\infty}(E_{m,n}^{a,\beta}\exp_{is})(x) = \frac{1}{(2-e^{A/n})^{\beta+1}}\exp\left(\frac{nx(e^{A/n}-1)}{(2-e^{A/n})}\right) = (G_n^{\beta}\exp_{is})(x),$$

leads us to

$$\lim_{m\to\infty}(E_{m,n}^{a,\beta}f)(x) = (G_n^{\beta}f)(x).$$

Finally

$$\lim_{r\to\infty}\left(E_{m,n}^{a,\beta}\exp_{is/r}\right)(rx) = m^m(m-isx)^{-m} = (P_m\exp_{is})(x),$$

thus we get

$$\lim_{r\to\infty}\left(E_{m,n}^{a,\beta}f\left(\frac{t}{r}\right)\right)(rx) = (P_m f)(x).$$

where G_n^{β} is defined in (4) and P_m is the Post-Widder operators. □

Remark 4.4. *For instance if we consider composition of operators namely Baskakov, Szász-Mirakyan and Szász-Mirakyan operators in following way*

$$(W_n^0 \circ S_n f)(x),$$

then the moment generating function after simple computation will be given by

$$(W_n^0 \circ S_n\exp_A)(x) = (1+x-xe^{(e^{A/n}-1)})^{-n},$$

thus the moments of the composition operator satisfy

$$\sum_{s\geq 0} a_s(W_n^0 \circ S_n e_s)(x)$$

$$= \quad a_0 + a_1 x + a_2\left(x^2 + \frac{x(x+2)}{n}\right) + a_3\left(x^3 + \frac{5x + 6x^2 + 6nx^2 + 2x^3 + 3nx^3}{n^2}\right) + \ldots\ldots$$

Further, in this way we can write

$$(W_n^0 \circ S_n \circ S_n f)(x),$$

and the moment generating function after simple computation will be given by

$$(W_n^0 \circ S_n \circ S_n \exp_A)(x) = (1 + x - xe^{(e^{(e^{A/n}-1)}-1)})^{-n},$$

the moments of the composition operator satisfy

$$\sum_{s\geq 0} b_s (W_n^0 \circ S_n \circ S_n e_s)(x)$$
$$= \quad b_0 + b_1 x + b_2\left(x^2 + \frac{x(x+3)}{n}\right) + b_3\left(x^3 + \frac{12x + 9x^2 + 9nx^2 + 2x^3 + 3nx^3}{n^2}\right) + \ldots\ldots$$

There may be some other new operators by composition of Baskakov type Pólya-Durrmeyer operators [5] with Beta operators of second kind [4] and of q Szász-Kantorovich operators (see [8]) with other q operators. The analysis is different we will discuss them elsewhere.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. M. Acu, V. Gupta, I. Rasa and F. Sofonea, Convergence of special sequences of semi-exponential operators, Mathematics 10 (16)(2022), 2978: https://doi.org/10.3390/math10162978

[2] O. Agratini, On a sequence of linear and positive operators, Facta Univ. (Nis) Ser. Math. Inform. 14 (1999), 41–48.

[3] P.N. Agrawal, V. Gupta, A. Sathish Kumar and A. Kajla, Generalized Baskakov-Szász type operators, Appl. Math. Comput. 236 (2014), 311–324.

[4] A. Aral and V. Gupta, On the q analogue of Stancu-Beta operators, Applied Mathematics Letters 25 (1)(2012), 67-71.

[5] V. Gupta, A. M. Acu and D.F. Sofonea, Approximation of Baskakov type Pólya Durrmeyer operators, Appl. Math. Computat. 294 (2017), 318-331.

[6] V. Gupta, N. Malik and Th. M. Rassias, Moment generating functions and moments of linear positive operators, Modern Discrete Mathematics and Analysis (Edited by N. J. Daras and Th. M. Rassias), Springer 2017. DOI: https://doi.org/10.1007/978-3-319-74325-7_8

[7] A. Lupaş, The approximation by some positive linear operators. In: Proceedings of the International Dortmund Meeting on Approximation Theory (M.W. Müller et al.,eds.), Akademie Verlag, Berlin, 1995, pp. 201–229.

[8] N. Mahmudov and V. Gupta, On certain q-analogue of Szász Kantorovich operators. J. Appl. Math. Comput. 37 (2011), 407-419 https://doi.org/10.1007/s12190-010-0441-4

[9] V. Miheşan, Uniform approximation with positive linear operators generated by generalized Baskakov method, Automat. Comput. Appl. Math. **7** (1) (1998), 34 – 37.

[10] S. Sucu, G. Icoz and S. Varma, On Some Extensions of Szász operators including Boas-Buck-type polynomials, Abstract and Applied Analysis Vol. 2012, (2012), Art. 680340, 15 pages doi:10.1155/2012/680340