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Arbitrary-order Fréchet derivatives of the exponential and logarithmic functions in real and complex Banach algebras: Applications to stochastic functional differential equations

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Abstract. In this paper we derive the explicit, closed-form, recursion-free formulae for the arbitrary-order Fréchet derivatives of the exponential and logarithmic functions in unital Banach algebras (complex or real). These computations are obtained via the Bochner integrals for the Banach algebra valued functions, with respect to the standard Lebesgue measure. As an application, we utilize our results in the approximation schemes of the solutions to stochastic functional differential equations.

1. Introduction

1.1. Motivation

Recent progress in stochastic analysis requires a further advancement in the study of the higher-order Fréchet derivatives of the elementary functions. In particular, the papers [17], [18], and [33] provide the approximate solutions to certain classes of stochastic differential equations, obtained by the multivariate or infinitely-dimensional Taylor polynomials, and thus demand the Fréchet differentiability of the diffusion and drift coefficients up to certain orders. However, effectively determining the said higher-order derivatives is, in practice, impossible to achieve, as calculating the Fréchet derivatives is a difficult task even for the matrix functions. In that sense, the results from [17] and [18] are in need of an alternative solving strategy.

In this paper we find a way to go around this computational difficulty: we manage to rewrite the problem in terms of the real Hilbert spaces and the corresponding real operator algebras, where we express the unknown higher-order derivatives of the diffusion and drift coefficients via the higher-order Fréchet derivatives of the exponential and logarithmic functions, defined in those operator algebras. Thus, we

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proceed to study the higher-order Fréchet derivatives of the exponential and logarithmic functions in arbitrary unital Banach algebras, complex or real. Consequently, we derive the recursion-free, closed-form integral representations of these derivatives, which are precisely the Bochner integrals for the Banach algebra valued functions, with respect to the standard Lebesgue measure. To our knowledge, no similar results exist in the available literature. By doing so, we simultaneously generalize some results from [1]–[3], [5]–[14], [21], [34], and [43] to the setting of the unital Banach algebras and C^*-algebras.

1.2. A quick revision of Fréchet derivatives

For the given normed spaces V_1 and V_2, let $\overline{\mathbb{D}_1}$ be the closed unit ball in V_1. Assume that U is an open subset of V_1, $u_0 \in U$, and let $f : U \to V_2$ be a function. The function f is Fréchet differentiable at point u_0, if there exists a bounded linear operator $f'_{u_0} \in \mathcal{L}(V_1, V_2)$ such that

$$\lim_{r\to 0} \frac{1}{r}\Big(f(u_0 + rv) - f(u_0)\Big) = f'_{u_0}(v)$$

uniformly for $v \in \overline{\mathbb{D}_1}$. In that sense, the operator f'_{u_0} is the Fréchet derivative of f at point u_0. By definition, it is always assumed that r is a real scalar. Direct verification shows that, if $L \in \mathcal{L}(V_1, V_2)$, then, for every $u_0 \in V_1$, it follows that $L'_{u_0} = L$. The first-order derivatives appear frequently in the perturbation analysis of the matrix and operator functions and equations, see [1]–[3], [6], [7], [12], [13], [15], [19]–[22], [24]–[26], [39] and so on.

The higher-order Fréchet derivatives are defined in the following manner. For a fixed $D \in \mathbb{N}$, observe the space $\mathcal{M}_D(V_1^D, V_2)$ of all bounded (continuous) $D-$linear mappings from V_1^D to V_2 (a mapping is $D-$linear if it is linear in each coordinate). Then, $\mathcal{M}_D(V_1^D, V_2)$ is isometrically isomorphic to the space (see [20])

$$\mathcal{L}\left(V_1, \underbrace{\mathcal{L}\left(V_1, \mathcal{L}\left(\ldots, \mathcal{L}(V_1, V_2)\right)\ldots\right)}_{D-1-\text{ nested parentheses}} \right).$$

Respectively, if the mapping $v \mapsto f'_{u_0}v$ is Fréchet differentiable at u_0, considered as a mapping from U to $\mathcal{L}(V_1, V_2)$, then its Fréchet derivative belongs to the space $\mathcal{L}(V_1, \mathcal{L}(V_1, V_2))$, and the latter is isomorphic to $\mathcal{M}_2(V_1^2, V_2)$. Thus, the second-order Fréchet derivative of f at (u_0, u_0) is $f''_{(u_0,u_0)} \in \mathcal{M}_2(V_1^2, V_2)$. By continuing this process (assuming the higher-order derivatives exist), it follows that the Dth-order Fréchet derivative of $f : U \to V_2$ at point $\underbrace{(u_0, \ldots, u_0)}_{D}$ is $f^{(D)}\underbrace{(u_0, \ldots, u_0)}_{D} \in \mathcal{M}_D(V_1^D, V_2)$.

The higher-order Fréchet derivatives are used when a more sophisticated analysis is required, and when the first-order approximations are not enough. Regardless, effectively computing the higher-order derivatives is a quite complicated task. Often, these calculations are, in one way or another, transfered to the costly recursive procedures, or, to the functional calculus of the square matrix functions ([1]–[6], [12], and [26]), and to the functional calculus on the complex unital Banach algebras ([19] and [22]).

The main advantage of our results, compared to the cited ones, lies in their amenability: they are recursion-free, spectrum-independent, and are valid in both real and complex Banach algebras. The **recursion-free** part paves a clear path for the numerical procedures, which will use significantly less memory and faster algorithms. The **spectrum-independent** part is extremely convenient for Banach algebras, since effectively calculating the spectrum of the given element is in general impossible. Finally, the **real and complex spaces** part is of crucial importance, given that most of the ODEs, PDEs and SDEs occur in the real Banach spaces, and not every real Banach space can be treated as a complex one.

2. Fréchet derivatives of the exponential function

In this section we assume that $\mathcal{A}$ is a fixed unital Banach algebra over the field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ with unity 1.

The exponential function $a \mapsto e^a$, at point $a \in \mathcal{A}$, is defined as the series

$$e^a := \sum_{k=0}^{\infty} \frac{1}{k!} a^k,$$

where $a^0 := 1_{\mathcal{A}}$ for every $a \in \mathcal{A}$. The above series is absolutely convergent by the virtue of the extended triangle inequality:

$$\|e^a\| = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} a^k \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|a\|^k = e^{\|a\|} < \infty,$$

which implies its convergence in $\mathcal{A}$ (the characterization of all Banach spaces, see [35]).

It is well-known that the exponential function has the first-order Fréchet derivative at any point $a \in \mathcal{A}$, and the corresponding integral formula (1) below has been established in several different papers, see e. g. [1]–[3], [5]–[7], and [9]:

$$(e^a)'_a(b) = \int_0^1 e^{a(1-s)} b e^{as} ds, \quad b \in \mathcal{A}. \tag{1}$$

In particular, the results derived in [1]–[3], [5], and [6, pp. 310–320], are obtained for the exponentials of the complex matrices only, with a comment that some of the results can be extended to the infinitely-dimensional spaces. The authors of the paper [7] treat the integral in (1) as a Riemman integral, and assume that $\mathcal{A}$ is a complex unital Banach algebra. Similarly, the book [9, pp. 161–180] mentions the expressions like (1) on several occasions, and treats them as Bochner integrals with respect to an arbitrary absolutely-convergent complex measure, however, it also assumes that $\mathcal{A}$ is a complex unital Banach algebra, and there is no explicit proof that (1) is valid in any unital Banach algebra.

In what follows, we proceed to prove that (1) holds in arbitrary unital Banach algebras $\mathcal{A}$, real or complex, where the integral representation in (1) is meant in the sense of the Bochner integral with respect to the scalar Lebesgue measure. The proof is more technical than intuitive as demonstrated below.

Indeed, we follow the same idea as in [5, pages 167-171]. For a given $a \in \mathcal{A}$, consider the following linear differential equation

$$\frac{dh(r)}{dr} = a\, h(r), \quad h(0) = c \in \mathcal{A}, \tag{2}$$

where $h : [0, R] \to \mathcal{A}$ is a continuously differentiable function and $R > 1$ is arbitrary. By [14, page 8], it follows that (2) has a unique solution in the set of continuously differentiable $\mathcal{A}$-valued functions. Then, by the Fundamental Theorem for the Bochner integral, we get

$$h(r) - h(0) = \int_0^r a\, h(s)\, ds, \quad r \in [0, R].$$

Notice that $f(r) = e^{ar}$ is given by the series which is uniformly convergent on every bounded set, so the differentiation under the infinite sum is allowed. Since $f'(r) = ae^{ar}$ and this is a continuous function, we see that f is the solution to (2) with the initial condition $f(0) = 1 \in \mathcal{A}$. Therefore

$$e^{ar} = 1 + \int_0^r a e^{as} ds, \qquad r \in [0, R].$$

If we consider the inhomogeneous case

$$\frac{dh(r)}{dr} = a\, h(r) + g(r), \quad h(0) = c \in \mathcal{A}, \quad r \in [0, R], \tag{3}$$

where $g : [0, R] \to \mathcal{A}$ is a continuous function, we see that

$$\frac{d}{dr}\left(e^{-ar}h(r)\right) = e^{-ar}\left(\frac{dh(r)}{dr} - a\,h(r)\right) = e^{-ar}g(r),$$

and obtain

$$h(r) = e^{ar}c + \int_0^r e^{a(r-s)}g(s)\,ds, \quad r \in [0, R],$$

where we used that e^{ar} commutes with e^{as}. Now, take $a, b \in \mathcal{A}$. Then $h(r) = e^{(a+b)r}$ is the solution to the differential equation

$$\frac{dh(r)}{dr} = (a+b)h(r), \quad h(0) = 1.$$

We can rewrite this equation in the form

$$\frac{dh(r)}{dr} = a\,h(r) + b\,h(r), \quad h(0) = 1,$$

and consider the last equation as the inhomogeneous in the form (3). Thus, we get

$$h(r) = e^{ar} + \int_0^r e^{a(t-s)}b\,h(s)\,ds, \quad r \in [0, R],$$

implying

$$e^{a+b} - e^a = \int_0^1 e^{a(1-s)}b\,e^{(a+b)s}ds. \tag{4}$$

Notice that in general e^a does not commute with e^b. Hence, the following result is proved.

Theorem 2.1. *For every $a \in \mathcal{A}$, the limit $\lim_{r\to 0} \|e^{a+rb} - e^a\| = 0$ converges uniformly in $b \in K$, where K is a bounded subset of $\mathcal{A}$.*

Proof. For a fixed $r > 0$ we have

$$e^a - e^{a+rb} = \sum_{k=1}^{\infty} \frac{1}{k!}a^k - \sum_{k=1}^{\infty} \frac{1}{k!}(a+rb)^k.$$

By definition, the above series both unconditionally converge, therefore they can be observed term by term. Respectively, for every $k \in \mathbb{N}$ one has

$$\begin{aligned}(a+rb)^k &= \underbrace{(a+rb)\cdot\ldots\cdot(a+rb)}_{k-\text{times}}\\ &= a^k + rba^{k-1} + arba^{k-2} + a^2rba^{k-3} +\\ &\quad + r^2b^2a^{k-2} + ar^2b^2a^{k-3} + \ldots + r^kb^k,\end{aligned}$$

which gives that

$$\frac{1}{k!}\left(a^k - (a+rb)^k\right) = \frac{1}{k!}\left(rba^{k-1} + \ldots + r^kb^k\right)$$

and such expressions are bounded by (for $r < 1$):

$$\frac{1}{k!}\|rba^{k-1} + \ldots + r^kb^k\| \leq \frac{r}{k!}\left(\|b\|\|a\|^{k-1} + \ldots + \|b\|^k\right) < \frac{r}{k!}\left(\|a\| + \|b\|\right)^k.$$

Consequently,

$$\|e^{a+rb} - e^a\| = \left\| \sum_{k=1}^{\infty} \frac{1}{k!} a^k - \sum_{k=1}^{\infty} \frac{1}{k!} (a+rb)^k \right\| \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\| a^k - (a+rb)^k \right\|$$
$$\leq \sum_{k=1}^{\infty} \frac{r}{k!} (\|a\| + \|b\|)^k = r e^{(\|a\|+\|b\|)} = O(r).$$

Thus when $r \to 0+0$ the proposed limit holds. □

Remark 2.2. *If $\mathbb{F} = \mathbb{C}$ then the statement directly follows from the functional calculus for holomorphic functions (as pointed out in [19]). Let γ^* denote the graph of γ, which is a compact subset of $\mathbb{C}$. The mapping $c \mapsto \sigma(c)$ is upper semi-continuous. This means that for every $a \in \mathcal{A}$ and $\epsilon > 0$ there exists some $\delta > 0$, such that if $b \in \mathcal{A}$ and $\|b\| < \delta$, then $\sigma(a+b) \subset \sigma(a) + D(0;\epsilon)$, where $D(0;\epsilon) = \{z \in \mathbb{C} : \ |z| < \epsilon\}$. Since we have $a, b \in \mathcal{A}$ arbitrary, we adjust r by $\|rb\| < \delta$. Thus, if γ is a cycle surrounding $\sigma(a)$, and the distance from γ^* to $\sigma(a)$ is greater then ϵ, then γ surrounds $\sigma(a+rb)$ if $\|rb\| < \delta$. We have*

$$e^a = \frac{1}{2\pi i} \int_{\gamma^*} e^z (z-a)^{-1} dz$$

and

$$e^{a+rb} = \frac{1}{2\pi i} \int_{\gamma^*} e^z (z-a-rb)^{-1} dz.$$

It follows that

$$\|e^{a+rb} - e^a\| \leq \frac{1}{2\pi} \int_{\gamma} \|e^z\| \|(z-a-rb)^{-1} - (z-a)^{-1}\| \cdot |dz|$$

and the last integral is a line integral. Since

$$(z-a-rb)^{-1} - (z-a)^{-1} = r(z-a-rb)^{-1} b (z-a)^{-1}$$

and

$$\lim_{r\to 0} (z-a-rb)^{-1} b (z-a)^{-1} = (z-a)^{-1} b (z-a)^{-1},$$

it follows that

$$\lim_{r\to 0} |r| \|e^z\| \|(z-a-rb)^{-1} b (z-a)^{-1}\| = 0$$

uniformly in $z \in \gamma^*$ and uniformly in $b \in K$. Thus,

$$\lim_{r\to 0} \|e^{a+rb} - e^a\| = 0$$

uniformly in $b \in K$. ♣

Summing up the calculations above, we now prove that (1) holds in $\mathcal{A}$ in the sense of the Bochner integral with respect to the scalar Lebesgue measure.

Theorem 2.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be defined as $f(a) = e^a$ for every $a \in \mathcal{A}$. Then f is Fréchet differentiable at every $a \in \mathcal{A}$ and*

$$f'_a(b) = \int_0^1 e^{a(1-s)} b e^{as} ds \tag{5}$$

holds for every $b \in \mathcal{A}$, where the integral in (5) is interpreted as the Bochner integral of the $\mathcal{A}$−valued expression, with respect to the Lebesgue measure.

Proof. Notice that $b \mapsto \int_0^1 e^{a(1-s)} b e^{as} ds$ is a bounded linear operator in $\mathcal{A}$. For $r \in \mathbb{R}$, $r \neq 0$, from (4), we have

$$\begin{aligned}
&\frac{1}{r}\Big(e^{a+rb} - e^a - r\int_0^1 e^{a(1-s)} b e^{as} ds\Big) \\
&= \frac{1}{r}\Big(r\int_0^1 e^{a(1-s)} b\, e^{(a+rb)s} ds - r\int_0^1 e^{a(1-s)} b e^{as} ds\Big) \\
&= \int_0^1 e^{a(1-s)} b\Big(e^{(a+rb)s} - e^{as}\Big) ds.
\end{aligned}$$

Theorem 2.1 implies that $\lim\limits_{r\to 0}\Big(e^{(a+rb)s} - e^{as}\Big) = 0$ uniformly in $s \in [0,1]$ and $b \in \overline{\mathbb{D}_{\mathcal{A}}}$. Thus,

$$\lim_{r\to 0}\int_0^1 e^{a(1-s)} b\Big(e^{(a+rb)s} - e^{as}\Big) ds = 0$$

uniformly in $b \in \overline{\mathbb{D}_{\mathcal{A}}}$, and the statement follows. □

2.1. *Higher-order derivatives of the exponential function*

Below we proceed to derive a more general integral representation for the higher-order Fréchet derivatives of the exponential function. To our knowledge, there are no such results in the available literature, not even for the matrix exponential function. On that note, we mention that paper [19] also studies the higher-order Fréchet derivatives of the exponential function, but again, observes the exponential function over a complex unital Banach algebra, and uses the holomorphic functional calculus to achieve the desired results. Consequently, the integration paths used in [19] are spectrum-dependent and are not fixed, since they must surround the spectrum of a in $\mathcal{A}$, which is not possible if we allow $\mathbb{F}$ to be $\mathbb{R}$.

Let $m > 1$ be a fixed positive integer, and let S_m denote the set of all permutations of the set $\{1,\ldots,m\}$. One given $j \in S_m$ gives the ordered $m-$tupple $(j(1),\ldots,j(m))$ of numbers $1,\ldots,m$. For a fixed $j \in S_m$ and for a fixed $2 \le k \le m$, observe the following $m+1-k-$tupple $(j(k),\ldots,j(m))$. We define the set $P(j,k)$ in the following manner

$$P(j,k) := \{\, j(p) \big| \text{ if } j(p) = \min\left(j(k),\ldots,j(p)\right),\ k \le p \le m\},$$

and

$$P(j,1) := \{\, j(p) \big| \text{ if } j(p) = \min\left(j(1),\ldots,j(p)\right),\ 1 \le p \le m\}.$$

Notice that $P(j,\ell)$ is nonempty because $j(\ell) \in P(j,\ell)$, for every $1 \le \ell \le m$.

Similarly, observe the $k-1-$tupple $(j(1),\ldots,j(k-1))$. We define the set $Q(j,k)$ as

$$Q(j,k) := \{\, j(q) \big| \text{ if } j(q) = \min\left(j(q),\ldots,j(k-1)\right),\ 1 \le q \le k-1\}$$

and finally

$$Q(j,m+1) := \{\, j(q) \big| \text{ if } j(q) = \min\left(j(q),\ldots,j(m)\right),\ 1 \le q \le m\}.$$

It follows that $j(\ell-1) \in Q(j,\ell)$ for every $2 \le \ell \le m+1$.

Example 2.4. *Let $m = 5$ and let $j \in S_5$ be given as*

$$j : (1,2,3,4,5) \mapsto (3,4,5,1,2).$$

Choose $k = 3$. Then $j(3) = 5$, and $P(j,3) = \{1,5\}$, $P(j,1) = \{1,3\}$, while $Q(j,3) = \{3,4\}$ and $Q(j,6) = \{1,2\}$. ♣

Let $(s_1, \dots, s_m) \in [0,1]^m$ be an arbitrary element of the $m-$dimensional unit cube. For an arbitrary but fixed $j \in S_m$ define the scalar expressions

$$\begin{aligned}\psi_k(j) &:= \prod_{j(p)\in P(j,k)} (1 - s_{j(p)}) \cdot \prod_{j(q)\in Q(j,k)} s_{j(q)}, \quad \text{for every } 2 \le k \le m,\\ \psi_1(j) &:= \prod_{j(p)\in P(j,1)} (1 - s_{j(p)}),\\ \psi_{m+1}(j) &:= \prod_{j(q)\in Q(j,m+1)} s_{j(q)}.\end{aligned} \tag{6}$$

Note that ψ_1, ψ_k and ψ_{m+1} all depend on the chosen $j \in S_m$.

Theorem 2.5. *Let $\mathcal{A}$ be a unital complex Banach algebra and $a \in \mathcal{A}$. The mapping $f : a \mapsto e^a$ has the Fréchet derivative of any order at a. Precisely, for every $m \in \mathbb{N}$, and for every $(b_1, \dots, b_m) \in \mathcal{A}^m$, the mth-order Fréchet derivative of f at point $(a, \dots, a)$, in $(b_1, \dots, b_m)$, is given as*

$$f^{(m)}_{\underbrace{(a,\dots,a)}_{m-times}}(b_1,\dots,b_m) = \sum_{j\in S_m} \underbrace{\int_0^1 \dots \int_0^1}_{m-times} \Psi(a,m,j,b_1,\dots,b_m) ds_m ds_{m-1} \dots ds_1, \tag{7}$$

where

$$\Psi(a,m,j,b_1,\dots,b_m) := e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdot \ldots \cdot e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)},$$

and $\psi_k(j)$ are provided by (6) *for every fixed $j \in S_m$.*

Proof. Notice that for every $m \in \mathbb{N}$ and for every $1 \le k \le m$ the expressions $\psi_k(j)$ are bounded on $[0,1]$ and the functions $e^{a\psi_k(j)}$ are bounded and integrable in both the Lebesgue and Riemman sense. The proof follows from the Dominated convergence theorem and the mathematical induction. When $m = 1$, the statement obviously holds. Assume the theorem is true for m and it will be proved that the statement holds for $m + 1$ as well.

Since, in general, the algebra $\mathcal{A}$ is not commutative, the exponential identity $e^{a+b} = e^a e^b$ does not hold. However, there is a way of going around this and that is by noting that

$$e^{a+rb} = e^a + r f'_a(b) + o(r) = e^a + r \int_0^1 e^{a(1-s)} b e^{as} ds + o(r), \quad r \to 0. \tag{8}$$

Denote by

$$I(j,k,b) = \int_0^1 e^{a(1-s)\psi_k(j)} b e^{as\psi_k(j)} ds,$$

where one $j \in S_m$ is fixed. For that particularly chosen j, observe the integrand $\Psi(a,m,j,b_1,\dots,b_m)$ of the corresponding summand from (7):

$$\Psi(a,m,j,b_1,\dots,b_m) = e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdot \ldots \cdot e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}. \tag{9}$$

This expression is Fréchet differentiable at point a, if there exists the limit

$$\begin{aligned}&\left(e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdots e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\right)'_a (b_{m+1})\\ &= \lim_{r\to 0} r^{-1} \left[\Psi(a + rb_{m+1}, m, j, b_1, \dots, b_m) - \Psi(a,m,j,b_1,\dots,b_m)\right].\end{aligned} \tag{10}$$

Applying the transformation (8) to each exponential $e^{(a+rb_{m+1})\psi_k(j)}$ in (10), we get

$$\begin{aligned}
&\Psi(a + rb_{m+1}, m, j, b_1, \dots, b_m)\\
&\quad = e^{(a+rb_{m+1})\psi_1(j)} b_{j(1)} \cdots b_{j(m)} e^{(a+rb_{m+1})\psi_{m+1}(j)}\\
&\quad = \left(e^{a\psi_1(j)} + rI(j, 1, b_{m+1}) + o(r)\right) b_{j(1)} \left(e^{a\psi_2(j)} + rI(j, 2, b_{m+1}) + o(r)\right)\\
&\qquad \cdot b_{j(2)} \cdots b_{j(m)} \left(e^{a\psi_{m+1}(j)} + rI(j, m+1, b_{m+1}) + o(r)\right)\\
&\quad = e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdots e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\qquad + rI(j, 1, b_{m+1}) b_{j(1)} e^{a\psi_2(j)} b_{j(2)} e^{a\psi_3(j)} b_{j(3)} \cdots e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\qquad + e^{a\psi_1(j)} b_{j(1)} rI(j, 2, b_{m+1}) b_{j(2)} e^{a\psi_3(j)} b_{j(3)} \cdots b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\qquad + e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} rI(j, 3, b_{m+1}) b_{j(3)} e^{a\psi_4(j)} b_{j(4)} \cdots b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\qquad + \cdots\\
&\qquad + e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} e^{a\psi_3(j)} b_{j(3)} \cdots e^{a\psi_m(j)} b_{j(m)} rI(j, m+1, b_{m+1})\\
&\qquad + o(r).
\end{aligned}$$

This shows that the limit in (10) exists and is equal to the following

$$\begin{aligned}
&\left(e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdot \ldots \cdot e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\right)'_a (b_{m+1})\\
&= \left(\int_0^1 e^{a\psi_1(j)(1-s_{m+1})} b_{m+1} e^{a\psi_1(j)s_{m+1}} ds_{m+1}\right) \cdot b_{j(1)} e^{a\psi_2(j)} b_{j(2)} e^{a\psi_3(j)} b_{j(3)} \cdots e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\quad + e^{a\psi_1(j)} b_{j(1)} \left(\int_0^1 e^{a\psi_2(j)(1-s_{m+1})} b_{m+1} e^{a\psi_2(j)s_{m+1}} ds_{m+1}\right) \cdot b_{j(2)} e^{a\psi_3(j)} b_{j(3)} \cdots b_{j(m)} e^{a\psi_{m+1}(j)}\\
&\quad + \cdots\\
&\quad + e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \ldots e^{a\psi_m(j)} b_{j(m)} \cdot \left(\int_0^1 e^{a\psi_{m+1}(j)(1-s_{m+1})} b_{m+1} e^{a\psi_{m+1}(j)s_{m+1}} ds_{m+1}\right).
\end{aligned} \tag{11}$$

Since every entity in (11) is bounded, and does not depend on the set of integration, it follows that

$$\begin{aligned}
&\left(e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \cdot \ldots \cdot e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\right)'_a (b_{m+1}) =\\
&= \int_0^1 \left[e^{a\psi_1(j)(1-s_{m+1})} b_{m+1} e^{a\psi_1(j)s_{m+1}} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} \ldots e^{a\psi_m(j)} b_{j(m)} e^{a\psi_{m+1}(j)}\right] ds_{m+1} +\\
&+ \int_0^1 \left[e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)(1-s_{m+1})} b_{m+1} e^{a\psi_2(j)s_{m+1}} b_{j(2)} \ldots b_{j(m)} \cdot e^{a\psi_{m+1}(j)}\right] ds_{m+1} +\\
&+ \cdots +\\
&+ \int_0^1 \left[e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} b_{j(2)} e^{a\psi_3(j)} \ldots b_{j(m)} e^{a\psi_{m+1}(j)(1-s_{m+1})} \cdot b_{m+1} e^{a\psi_{m+1}(j)s_{m+1}}\right] ds_{m+1}.
\end{aligned} \tag{12}$$

Notice that one integrand of the form (9) gives $m + 1$ summands, and each summand (as in (12)) is an integral of the form

$$\int_0^1 e^{a\psi'_1(j')} b_{j'(1)} e^{a\psi'_2(j')} b_{j'(2)} \cdot \ldots \cdot e^{a\psi'_{m+1}(j')} b_{j'(m+1)} e^{a\psi'_{m+2}(j')} ds_{m+1},$$

where:

- j' is a permutation of the set $\{1, \dots, m+1\}$, such that $j'(i_0) = m + 1$ for one particular index i_0, and $j'(i) = j(i)$ for the remaining indices $i \neq i_0$;

- in that sense, if $j'(i_0) = m+1$ and $j'(i) = j(i)$ for the remaining indices $i \neq i_0$, then $\psi'_{i_0}(j') := \psi_{i_0}(j)(1-s_{m+1})$, $\psi'_{i_0+1}(j') := \psi_{i_0}(j)s_{m+1}$, and $\psi'_i(j') := \psi_i(j)$.

It is not difficult to see that, for every $2 \leq k \leq m+1$, the sets $P(j',1)$, $P(j',k)$, $Q(j',k)$, and $Q(j',m+2)$, which are defined in the analogous way, as were the sets $P(j,1)$, $P(j,k)$, $Q(j,k)$, and $Q(j,m+1)$, respectively, are compatible with the afore-mentioned construction of $\psi'_i(j')$, $1 \leq i \leq m+2$. The above observation holds for every fixed permutation $j \in S_m$, which proves that every integrand (9) of every summand from (7) is Fréchet differentiable, ergo the Dominated convergence theorem can be applied (recall that the Fréchet derivative produces a *bounded* linear operator for b_{m+1}), thus

$$\begin{aligned}
f^{(m+1)}_{\underbrace{(a,\dots,a)}_{m+1-\text{times}}}(b_1,\dots,b_{m+1}) &= \left(f^{(m)}_{\underbrace{(a,\dots,a)}_{m-\text{times}}}(b_1,\dots,b_m) \right)'_a (b_{m+1}) \\
&= \Bigg(\sum_{j\in S_m} \underbrace{\int_0^1 \dots \int_0^1}_{m-\text{times}} \left[e^{a\psi_1(j)} b_{j(1)} e^{a\psi_2(j)} \dots b_{j(m)} e^{a\psi_{m+1}(j)} \right] ds_m ds_{m-1} \dots ds_1 \Bigg)'_a (b_{m+1}) \\
&= \sum_{j\in S_m} \underbrace{\int_0^1 \dots \int_0^1}_{m-\text{times}} \left(\left[e^{a\psi_1(j)} b_{j(1)} \dots b_{j(m)} e^{a\psi_{m+1}(j)} \right] \right)'_a (b_{m+1}) ds_m ds_{m-1} \dots ds_1 \\
&= \sum_{j'\in S_{m+1}} \underbrace{\int_0^1 \dots \int_0^1}_{m+1-\text{times}} \left[e^{a\psi'_1(j')} b_{j'(1)} \dots e^{a\psi'_{m+1}(j')} b_{j'(m+1)} e^{a\psi'_{m+2}(j')} \right] ds_{m+1} ds_m \dots ds_1.
\end{aligned}$$

□

Example 2.6. *Let $m = 2$. Direct calculations give*

$$\begin{aligned}
(e^a)''_{(a,a)}(b_1,b_2) &= \int_0^1 \int_0^1 e^{a(1-s_1)(1-s_2)} b_2 e^{a(1-s_1)s_2} b_1 e^{as_1} ds_2 ds_1 \\
&\quad + \int_0^1 \int_0^1 e^{a(1-s_1)} b_1 e^{as_1(1-s_2)} b_2 e^{as_1 s_2} ds_2 ds_1.
\end{aligned} \tag{13}$$

It can be easily verified that Theorem 2.5 holds: both addends in (13)*, read from left to right, define one permutation of the index-set for $\{b_1, b_2\}$: the first one defines $j_1 : (1,2) \mapsto (2,1)$ while the second one defines $j_2 : (1,2) \mapsto (1,2)$. Recall the sets $P(j,k)$ and $Q(j,k)$, determined by the fixed permutation j and the position k. By applying* (6) *to j_1, the following holds*

$$P(j_1,1) = \{1\},\ P(j_1,2) = \{2\},\ Q(j_1,2) = \{1\},\ Q(j_1,3) = \{1\},$$
$$\psi_1(j_1) = (1-s_1)(1-s_2),\ \psi_2(j_2) = (1-s_1)s_2,\ \psi_3(j_2) = s_1,$$

while, for j_2 we get

$$P(j_2,1) = \{1\},\ P(j_2,2) = \{2\},\ Q(j_2,2) = \{1\},\ Q(j_2,3) = \{1,2\},$$
$$\psi_1(j_2) = (1-s_1),\ \psi_2(j_2) = s_1(1-s_2),\ \psi_3(j_2) = s_1 s_2.$$

Indeed, (13) *has the form*

$$(e^a)''_{(a,a)}(b_1,b_2) = \sum_{j\in S_2} \int_0^1 \int_0^1 e^{a\psi_1(j)} b_{j(1)} e^{\psi_2(j)} b_{j(2)} e^{\psi_3(j)} ds_2 ds_1.$$

♣

Example 2.7. *Let $m = 3$. Then the third-order Fréchet derivative of the function $f : a \mapsto e^a$ at point (a, a, a) is a bounded trilinear operator, which by the virtue of Theorem 2.5 attains the value at point (b_1, b_2, b_3):*

$$
\begin{aligned}
f'''_{(a,a,a)}(b_1, b_2, b_3) = &\int_0^1\int_0^1\int_0^1 e^{a\psi_1} b_1 e^{a\psi_2} b_2 e^{a\psi_3} b_3 e^{a\psi_4} ds_3 ds_2 ds_1\\
&+ \int_0^1\int_0^1\int_0^1 e^{a\psi_5} b_1 e^{a\psi_6} b_3 e^{a\psi_7} b_2 e^{a\psi_8} ds_3 ds_2 ds_1\\
&+ \int_0^1\int_0^1\int_0^1 e^{a\psi_9} b_2 e^{a\psi_{10}} b_1 e^{a\psi_{11}} b_3 e^{a\psi_{12}} ds_3 ds_2 ds_1\\
&+ \int_0^1\int_0^1\int_0^1 e^{a\psi_{13}} b_2 e^{a\psi_{14}} b_3 e^{a\psi_{15}} b_1 e^{a\psi_{16}} ds_3 ds_2 ds_1\\
&+ \int_0^1\int_0^1\int_0^1 e^{a\psi_{17}} b_3 e^{a\psi_{18}} b_1 e^{a\psi_{19}} b_2 e^{a\psi_{20}} ds_3 ds_2 ds_1\\
&+ \int_0^1\int_0^1\int_0^1 e^{a\psi_{21}} b_3 e^{a\psi_{22}} b_2 e^{a\psi_{23}} b_1 e^{a\psi_{24}} ds_3 ds_2 ds_1.
\end{aligned}
$$

Notice that every summand contains one permutation of the set $\{b_1, b_2, b_3\}$. Below we proceed to determine the scalars $\psi_1, \dots, \psi_{24}$ by (6). Recall the sets $P(j,k)$ and $Q(j,k)$, defined via the fixed permutation j and the position k. Let $j_1 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ be the first permutation of the index set. Then

$$P(j_1, 1) = \{1\},\ P(j_1, 2) = \{2\},\ P(j_1, 3) = \{3\},$$

$$Q(j_1, 2) = \{1\},\ Q(j_1, 3) = \{1, 2\},\ Q(j_1, 4) = \{1, 2, 3\}.$$

Applying (6) we get

$$\psi_1 = 1 - s_1,\ \psi_2 = (1 - s_2)s_1,\ \psi_3 = (1 - s_3)s_1 s_2,\ \psi_4 = s_1 s_2 s_3.$$

The remaining parameters are computed accordingly:

Let $j_2 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ be the second permutation of the index set. Then

$$P(j_2, 1) = \{1\},\ P(j_2, 2) = \{3, 2\},\ P(j_2, 3) = \{2\},$$

$$Q(j_2, 2) = \{1\},\ Q(j_2, 3) = \{1, 3\},\ Q(j_2, 4) = \{1, 2\},$$

$$\psi_5 = 1 - s_1,\ \psi_6 = (1 - s_2)(1 - s_3)s_1,\ \psi_7 = (1 - s_2)s_1 s_3,\ \psi_8 = s_1 s_2.$$

Let $j_3 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ be the third permutation of the index set. Then

$$P(j_3, 1) = \{2, 1\},\ P(j_3, 2) = \{1\},\ P(j_3, 3) = \{3\},$$

$$Q(j_3, 2) = \{2\},\ Q(j_3, 3) = \{1\},\ Q(j_3, 4) = \{1, 3\},$$

$$\psi_9 = (1 - s_2)(1 - s_1),\ \psi_{10} = (1 - s_1)s_2,\ \psi_{11} = (1 - s_3)s_1,\ \psi_{12} = s_1 s_3.$$

Let $j_4 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ be the fourth permutation of the index set. Then

$$P(j_4, 1) = \{2, 1\},\ P(j_4, 2) = \{3, 1\},\ P(j_4, 3) = \{1\},$$

$$Q(j_4,2)=\{2\},\ Q(j_4,3)=\{2,3\},\ Q(j_4,4)=\{1\},$$
$$\psi_{13}=(1-s_2)(1-s_1),\ \psi_{14}=s_2(1-s_1)(1-s_3),\ \psi_{15}=(1-s_1)s_2s_3,\ \psi_{16}=s_1.$$

Let $j_5:\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ *be the fifth permutation of the index set. Then*

$$P(j_5,1)=\{3,1\},\ P(j_5,2)=\{1\},\ P(j_5,3)=\{2\},$$
$$Q(j_5,2)=\{3\},\ Q(j_5,3)=\{1\},\ Q(j_5,4)=\{1,2\},$$
$$\psi_{17}=(1-s_3)(1-s_1),\ \psi_{18}=(1-s_1)s_3,\ \psi_{19}=s_1(1-s_2)\ \psi_{20}=s_1s_2.$$

Finally, let $j_6:\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ *be the sixth permutation of the index set. Then*

$$P(j_6,1)=\{3,2,1\},\ P(j_6,2)=\{2,1\},\ P(j_6,3)=\{1\},$$
$$Q(j_6,2)=\{3\},\ Q(j_6,3)=\{2\},\ Q(j_6,4)=\{1\},$$
$$\psi_{21}=(1-s_3)(1-s_2)(1-s_1),\ \psi_{22}=s_3(1-s_2)(1-s_1),\ \psi_{23}=(1-s_1)s_2,\ \psi_{24}=s_1.$$

Combining the previous calculations we obtain

$$\begin{aligned}
&f'''_{(a,a,a)}(b_1,b_2,b_3)\\
&=\int_0^1\int_0^1\int_0^1 e^{a(1-s_1)}b_1e^{as_1(1-s_2)}b_2e^{as_1s_2(1-s_3)}b_3e^{as_1s_2s_3}ds_3ds_2ds_1\\
&\quad+\int_0^1\int_0^1\int_0^1 e^{a(1-s_1)}b_1e^{as_1(1-s_2)(1-s_3)}b_3e^{as_1s_3(1-s_2)}b_2e^{as_1s_2}ds_3ds_2ds_1\\
&\quad+\int_0^1\int_0^1\int_0^1 e^{a(1-s_2)(1-s_1)}b_2e^{as_2(1-s_1)}b_1e^{as_1(1-s_3)}b_3e^{as_1s_3}ds_3ds_2ds_1\\
&\quad+\int_0^1\int_0^1\int_0^1 e^{a(1-s_2)(1-s_1)}b_2e^{as_2(1-s_3)(1-s_1)}b_3e^{as_2s_3(1-s_1)}b_1e^{as_1}ds_3ds_2ds_1\\
&\quad+\int_0^1\int_0^1\int_0^1 e^{a(1-s_3)(1-s_1)}b_3e^{as_3(1-s_1)}b_1e^{as_1(1-s_2)}b_2e^{as_1s_2}ds_3ds_2ds_1\\
&\quad+\int_0^1\int_0^1\int_0^1 e^{a(1-s_3)(1-s_2)(1-s_1)}b_3e^{as_3(1-s_2)(1-s_1)}b_2e^{as_2(1-s_1)}b_1e^{as_1}ds_3ds_2ds_1.
\end{aligned}$$

♣

3. Fréchet derivatives of the logarithmic function

Let $\mathcal{A}$ be a unital Banach algebra over the field $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$. Then $e^{\mathcal{A}}=\{e^a:\ a\in\mathcal{A}\}$ denotes the range of the exponential function in $\mathcal{A}$. Recall that the logarithm is defined at $a\in\mathcal{A}$, if and only if there exists an element $c\in\mathcal{A}$, such that $e^c=a$, that is, if and only if $a\in e^{\mathcal{A}}$. In that case, we write $c\in\ln a$. However, note that even if $\mathcal{A}$ is the set of square matrices, the value $\ln a$ is not uniquely determined, but rather refers to the set $\{c\in\mathcal{A}:\ e^c=a\}$, see [5], [6], [9], and [25].

The matrix and operator logarithms appear quite naturally in the operator theory and matrix analysis, as well as in the applied linear algebra, see e.g. [8], [10], [11], [13], [14], [21], [34], and [43]. Respectively, the perturbation analysis is always required for the function $a\mapsto\ln a$, and is commonly conducted via its first-order Fréchet derivatives, see [3] and [13]. Additionally, it is worth mentioning that the Fréchet derivatives in general subject to the chain rule, thus derivatives of the logarithmic function can be used to study the derivatives of the exponential function, and vice versa.

In the afore-mentioned literature, the primal focus was on the logarithmic function itself, and its first-order Fréchet derivative. Their existence and uniqueness varied from the matrix to the operator setting, and depended on the choice of the field $\mathbb{F}$. In this section, we study the logarithmic function, together with its arbitrary-order Fréchet derivatives, in both real and complex setting.

3.1. The complex case

When $\mathbb{F} = \mathbb{C}$, one can apply the rich spectral theory of complex unital Banach algebras and the corresponding functional calculus. Consequently, the following result emerges.

Theorem 3.1. *[39, Theorem 10.30, p. 264]. Let $\mathcal{A}$ be a unital complex Banach algebra (not necessarily commutative), and let $a \in \mathcal{A}$. Suppose that 0 belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$, where $\sigma_{\mathcal{A}}(a)$ is the spectrum of a in $\mathcal{A}$. Then there exists $c \in \mathcal{A}$ such that $e^c = a$.*

Much like the complex logarithm, $\ln a$ is not uniquely determined, see [5], [6], [9], [11], [19], [25], and [43] . The following therem characterizes the set of logarithms for $a \in e^{\mathcal{A}}$.

Theorem 3.2. *[43, Theorem 2.] Let V be a complex Banach space, and let $\mathcal{A}$ be the (unital complex) Banach algebra of bounded linear operators in V. Assume that $a \in \mathcal{A}$ is any element whose resolvent set contains a curve connecting $\lambda_1 = 0$ and some $\lambda_2 \geq \|a\|$.*

Then, there exists a continuously differentiable function $f : [0, 1] \to \mathcal{A}$, such that $f(0) = 0$, and $f(s) \neq 0$ when $s \neq 0$, $f(1) = 1$, and $g(s) := (f(s) - 1)/f(s)$ lies entirely in the resolvent set of a. Moreover, the equation $a = e^c$ has a solution given as

$$c_f = \ln a = \int_0^1 f'(s)(a - 1_{\mathcal{A}})(f(s)a + (1 - f(s))1_{\mathcal{A}})^{-1} ds. \tag{14}$$

These solutions c_f commute with any operator which commutes with a. Any other solution to $e^c = a$, which commutes with a, differs from c_f by a logarithm of the identity. The integral converges in $\mathcal{A}$. The logarithm is an analytic function of a.

By the uniform boundedness principle, it is not difficult to see that the above integral absolutely converges as well, thus the expression in (14) is Lebesgue-measurable and can be regarded once again as a Bochner integral of an $\mathcal{A}$−valued function with respect to the Lebesgue measure.

Specially, if $\sigma(a) \cap \mathbb{R}_0^- = \emptyset$, then the function $f_1(s) := s$ can be observed and (14) reduces to

$$\ln a = \int_0^1 (a - 1)((a - 1)s + 1)^{-1} ds. \tag{15}$$

It is a known result (see e.g. [13], [25] or [43]), that if a and b are elements in $\mathcal{A}$ such that $\sigma(a + rb) \cap \mathbb{R}_0^- = \emptyset$ for every $0 \leq r \leq 1$, then (15) gives

$$\ln(a + rb) - \ln a = r \int_0^1 ((a - 1)s + 1)^{-1} b((a - 1)s + 1)^{-1} ds + o(r), \quad r \to 0. \tag{16}$$

Then the relation (16) is, for convenience, rewritten as

$$\begin{aligned}
&\int_0^1 ((a - 1)s + 1)^{-1} b((a - 1)s + 1)^{-1} ds \\
&\quad = \int_0^1 s^{-1}((a - 1) + s^{-1})^{-1} bs^{-1}((a - 1) + s^{-1})^{-1} ds \\
&\quad = \int_0^1 s^{-2}((a - 1) + s^{-1})^{-1} b((a - 1) + s^{-1})^{-1} ds \\
&\quad = \int_1^\infty ((a - 1) + s')^{-1} b((a - 1) + s')^{-1} d(s') \\
&\quad = \int_0^\infty (a + s'')^{-1} b((a + s'')^{-1} d(s''),
\end{aligned} \tag{17}$$

where $s' := s^{-1}$ and $s'' := s' - 1$, i.e.

$$(\ln)'_a : b \mapsto \int_0^\infty (a + s)^{-1} b(a + s)^{-1} ds. \tag{18}$$

3.2. The real case

Let $\mathbb{F} = \mathbb{R}$. Following the notation from [30] and [34], the spectrum of a in $\mathcal{A}$ is denoted as:

$$\sigma^*_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : (a - \lambda 1_{\mathcal{A}})(a - \bar{\lambda} 1_{\mathcal{A}}) - \text{is not invertible in } \mathcal{A}\}.$$

The set $\sigma^*_{\mathcal{A}}$ is a non-empty compact in $\mathbb{C}$, see e.g. [30]. The following theorem gives sufficient conditions for the existence of a real logarithm of the given element $a \in \mathcal{A}$:

Theorem 3.3. *[34, Theorem 2.1.] Let $\mathcal{A}$ be a unital real Banach algebra (not necessarily commutative), and let $a \in \mathcal{A}$.*

(1) Suppose that 0 belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma^*_{\mathcal{A}}(a)$. Then there exists $c_1 \in \mathcal{A}$ such that $e^{c_1} = a^2$.

(2) If $(-\infty, 0]$ belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma^*_{\mathcal{A}}(a)$ then there exists $c \in \mathcal{A}$ such that $e^c = a$.

Corollary 3.4. *[34, Corollary 2.2.] Let $\mathcal{A}$ be a unital real Banach algebra (not necessarily commutative), and let $a \in \mathcal{A}$. Suppose that 0 belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma^*_{\mathcal{A}}(a)$. Then*

$$a \in e^{\mathcal{A}} \Leftrightarrow a = e^2, \text{ for some } e \in \mathcal{A}.$$

It will be shown that under these assumptions the integral expression (15) holds in real Banach algebras. The authors of [13] showed this for the space of real square matrices, though their argument can without loss of generality be extended to real Banach algebras, since they had derived their results with the help of the abstract linear differential equations, which are valid in general Banach spaces. Below is presented a slightly modified idea from [13, pp. 10].

Let $\mathcal{A}$ be a real unital Banach algebra, and let $a \in e^{\mathcal{A}}$, i.e. a has a real logarithm in the sense of Theorem 3.3. For every $s \in [0, 1]$, let $c(s) \in \mathcal{A}$ be such that the following parametric equation is satisfied (precisely, this is a C_0 semigroup in $\mathcal{A}$):

$$e^{c(s)} = (a - 1)s + 1, \quad 0 \le s \le 1. \tag{19}$$

By construction it follows that a and $c(s)$ commute, thus $c(s)$ satisfies the ODE

$$\frac{d}{ds} c(s) = (a - 1)e^{-c(s)}, \quad 0 \le s \le 1, \quad c(0) = 0. \tag{20}$$

The eq. (19) implies that $c(s)$ is a real logarithm of $(a - 1)s + 1$ for every $s \in [0, 1]$, therefore

$$a = c(1) = \int_0^1 (a - 1)e^{-c(s)} ds = \int_0^1 (a - 1)((a - 1)s + 1)^{-1} ds, \tag{21}$$

so the relation (15) holds in real unital Banach algebras as well (though it has to be taken into account for which elements of $\mathcal{A}$ the real logarithm exists). Furthermore, the authors of [13] showed that the relation (16) holds in the space of real square matrices, and, by using the same argument, it can be concluded that the realtion (16), and, consequently, (18), hold in the real unital Banach algebras:

$$(\ln)'_a : b \mapsto \int_0^\infty (a + s)^{-1} b (a + s)^{-1} ds, \tag{22}$$

where a and b satisfy conditions of Theorem 3.3.

3.3. Higher-order derivatives of the logarithmic function

Given that the same Bochner integral representations hold for the first-order Fréchet derivative of the logarithmic function in both real and complex Banach algebras, it is convenient to derive the general formula for the arbitrary-order Fréchet derivatives of the logarithmic function simultaneously, in both real and complex Banach algebras. Precisely, the following result is obtained:

Theorem 3.5. *Let $\mathcal{A}$ be a unital Banach algebra over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $a \in e^{\mathcal{A}}$, and let $m \in \mathbb{N}$ be arbitrary. Assume that (1) or (2) holds, where*

(1) $\mathbb{F} = \mathbb{R}$ and $(b_1, \ldots, b_m) \in \mathcal{A}^m$ are given, such that

$$\bigcup_{k=1}^{m} \left(\bigcup_{0 \le r \le 1} \sigma^*_{\mathcal{A}}(a + rb_k) \right) \cap \mathbb{R}_0^- = \emptyset. \tag{23}$$

(2) $\mathbb{F} = \mathbb{C}$ and $(b_1, \ldots, b_m) \in \mathcal{A}^m$ are given, such that

$$\bigcup_{k=1}^{m} \left(\bigcup_{0 \le r \le 1} \sigma_{\mathcal{A}}(a + rb_k) \right) \cap \mathbb{R}_0^- = \emptyset. \tag{24}$$

Then, the function $a \mapsto \ln a$ has its Fréchet derivatives of orders up to m at point a, and produces the $m-$linear operator in $(b_1, \ldots, b_m)$ given as

$$(\ln)^{(m)}_{\underbrace{(a, \ldots, a)}_{m-times}}(b_1, \ldots, b_m) = \sum_{j \in S_m} (-1)^{m+1} \int_0^\infty (a+s)^{-1} b_{j(1)} (a+s)^{-1} b_{j(2)} \cdots (a+s)^{-1} b_{j(m)} (a+s)^{-1} ds, \tag{25}$$

where S_m denotes the set of all permutations of the index set $\{1, \ldots, m\}$, and the integral in (25) is the Bochner integral of the $\mathcal{A}-$valued expression with respect to the Lebesgue measure.

Proof. The proof is virtually the same for both cases and it follows from the resolvent equations, by the principle of the mathematical induction and from the Dominated convergence theorem. When $m = 1$ the statement follows from (18) and (22). Assume the statement is true for $m - 1$ and proceed to prove that it also holds for $m \ge 2$.

Due to the assumptions (1) and (2), the given entities are invertible in the respective algebras, and the resolvent equations produce

$$(a + rb + s)^{-1} - (a+s)^{-1} = (a+s)^{-1}(-rb)(a + rb + s)^{-1},$$

thus the function $a \mapsto (a+s)^{-1}$ is Fréchet differentiable at a and

$$(a + rb + s)^{-1} = (a+s)^{-1} - r(a+s)^{-1} b (a+s)^{-1} + o(r), \quad r \to 0.$$

Let $j \in S_{m-1}$ be fixed. The latter goes to show that the function

$$a \mapsto (a+s)^{-1} b_{j(1)} (a+s)^{-1} b_{j(2)} \ldots (a+s)^{-1} b_{j(m-1)} (a+s)^{-1}$$

is Fréchet differentiable at a and the corresponding bounded linear operator L attains at point b_m the value given below:

$$\begin{aligned}
Lb_m &= \lim_{r\to 0} r^{-1}\Big[(a+rb_m+s)^{-1}b_{j(1)}\cdots b_{j(m-1)}(a+rb_m+s)^{-1}\\
&\qquad -(a+s)^{-1}b_{j(1)}(a+s)^{-1}\cdots(a+s)^{-1}b_{j(m-1)}(a+s)^{-1}\Big]\\
&= \lim_{r\to 0} r^{-1}\Big[\Big((a+s)^{-1}-r(a+s)^{-1}b_m(a+s)^{-1}+o(r)\Big)b_{j(1)}\\
&\qquad \cdot\Big((a+s)^{-1}-r(a+s)^{-1}b_m(a+s)^{-1}+o(r)\Big)b_{j(2)}\\
&\qquad \cdot\ldots\cdot b_{j(m-1)}\Big((a+s)^{-1}-r(a+s)^{-1}b_m(a+s)^{-1}+o(r)\Big)\\
&\qquad -(a+s)^{-1}b_{j(1)}(a+s)^{-1}\ldots(a+s)^{-1}b_{j(m-1)}(a+s)^{-1}\Big]\\
&= \lim_{r\to 0} r^{-1}\Big[(-r)(a+s)^{-1}b_m(a+s)^{-1}b_{j(1)}\cdots b_{j(m-1)}(a+s)^{-1}-\\
&\qquad -r\,(a+s)^{-1}b_{j(1)}(a+s)^{-1}b_m(a+s)^{-1}b_{j(2)}\ldots b_{j(m-1)}(a+s)^{-1}\\
&\qquad -\cdots-r\,(a+s)^{-1}b_{j(1)}(a+s)^{-1}b_{j(2)}\ldots(a+s)^{-1}b_m(a+s)^{-1}+o(r)\Big]\\
&= (-1)\sum_{j'\in S_m}(a+s)^{-1}b_{j'(1)}(a+s)^{-1}\ldots(a+s)^{-1}b_{j'(m)}(a+s)^{-1},
\end{aligned}$$

where S_m denotes the set of all permutations of the set $\{1,\ldots,m\}$. Finally, the Dominated convergence theorem is applied and

$$\begin{aligned}
(\ln)^{(m)}_{\underbrace{(a,\ldots,a)}_{m-\text{times}}}(b_1,\ldots,b_m) &= \left((\ln)^{(m-1)}_{\underbrace{(a,\ldots,a)}_{m-1-\text{times}}}(b_1,\ldots,b_{m-1})\right)'_a(b_m)\\
&= (-1)^{m+1}\sum_{j'\in S_m}\int_0^{+\infty}(a+s)^{-1}b_{j'(1)}(a+s)^{-1}\ldots(a+s)^{-1}b_{j'(m)}(a+s)^{-1}ds.
\end{aligned}$$

□

3.4. Special case: C^*-algebras

Let C be a complex unital C^*−algebra. The spectral mapping theorem states that for every $a \in C^{-1,+}$ (the set of positive invertible elements in C is traditionally denoted as $C^{-1,+}$), there exists a unique self-adjoint $c \in C^h$, such that $\ln a = c$. Additionally, if a and b are invertible positive elements in C, then, for a small enough positive r, the value $a + rb$ is also a positive and invertible element in C. By definition, it follows that $a \mapsto \ln a$ is Fréchet differentiable at a and for any $b \in C^{-1,+}$ produces a bounded linear operator. By the virtue of Gelfand-Naimark-Segal theorem, the following corollary holds:

Corollary 3.6. *Let $a \in C^{-1,+}$ be a positive invertible element in C. Then, the function $a \mapsto \ln a$ has the Fréchet derivative of any order at point a, and for every $m \in \mathbb{N}$, and for every $(b_1,\ldots,b_m) \in \left(C^{-1,+}\right)^m$, the formula* (25) *is valid.*

4. Applications to SFDEs

4.1. Problem setting

In this section the notation is the same as the one used in [18]. Let $N \in \mathbb{N}$ be an arbitrary fixed natural number and let $\tau > 0$ be an arbitrary positive number. It is significant to consider the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^N$, denoted as $C([-\tau, 0], \mathbb{R}^N)$. Recall that, when equipped with the supremum norm $\|\cdot\|_\infty$, this becomes a Banach space, denoted as

$$V_{\mathbb{R}} := \left(C([-\tau, 0], \mathbb{R}^N), \|\cdot\|_\infty \right).$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is non-decreasing, continuous from the right and $\mathcal{F}_0$ contains all the $\mathbb{P}-$null sets. Let η be a $V_{\mathbb{R}}-$valued random variable, which is $\mathcal{F}_{t_0}-$adapted starting from a certain point in time $t_0 \geq 0$. Let $M \in \mathbb{N}$ (in general $M \neq N$), and let

$$w(t) := (w_1(t), \dots, w_M(t)), \quad t \geq 0$$

be a standard $M-$dimensional Wiener process, which is $\mathcal{F}_t-$adapted and independent from $\mathcal{F}_0$.

Let $(y(t))_{t\geq 0}$ be any $\mathcal{F}_t-$adapted $N-$dimensional stochastic process. Recall that $(y(t))_{t\geq 0}$ is said to have a functional delay (i.e., has a time-dependent memory) of length τ, if the value of y at some point t' (i.e., the $N-$dimensional random variable $y(t')$) depends on the values which process had taken during the period $[t'-\tau, t')$:

$$y(t') \sim \{y(t'+\theta) : \theta \in [-\tau, 0)\},$$

where $\sim$ refers to statistical, stochastic, or functional dependence. This delay is portrayed as the process

$$y_t := \{y(t+\theta) : \theta \in [-\tau, 0]\}, \quad t \geq 0.$$

With respect to the previous notation, the functional stochastic differential equation (SFDE for short), with the time-dependent delay of length τ, is governed by the equation

$$dx(t) = \alpha(x_t, t)dt + \beta(x_t, t)dw(t), \quad t \in [t_0, T], \tag{26}$$

where time is measured from the initial moment t_0 until the final moment T, $0 \leq t_0 < T < \infty$. Thus, the SFDE (26) has an initial condition of the form

$$x_{t_0} = \eta, \tag{27}$$

where, in general, η is an $\mathcal{F}_{t_0}-$ adapted process. Under these circumstances, the function η is the experimental input data, which also has the memory of length τ.

The drift and diffusion coefficients α and β are mappings from $V_{\mathbb{R}} \times [t_0, T]$ to $\mathbb{R}^N$ and $\mathbb{R}^{N\times M}$, respectively, which are assumed to be Borel measurable. The appropriate integral form of (26)–(27) is

$$x(t) = \eta(0) + \int_{t_0}^{t} \alpha(x_\ell, \ell)d\ell + \int_{t_0}^{t} \beta(x_\ell, \ell)dw(\ell), \quad t \in [t_0, T]. \tag{28}$$

Such equations model important phenomena with memory, like the various predator-prey models, the population-growth models, the gene expression rates, the particle motion in liquids, the viscoelasticity of fluids, and the controlled movement of the rigid bodies, see [4, 23, 27–29, 31, 32, 36, 40–42].

The process $\{x(t)\}_{t\geq 0}$ defined by (28) has been studied in [18], where it was successfully approximated $\mathbb{P}-$almost everywhere, and in the L^p-sense, by the almost surely continuous processes as demonstrated below. For an arbitrary positive large enough integer n, let $\{t_0, t_1, \dots, t_{n-1}, T\}$ be the equidistant nods in the segment $[t_0, T]$, where $t_k := t_0 + k(T - t_0)/n$ for $0 \leq k \leq n$. Provided that α is Fréchet differentiable up to the order m_1, and that β is Fréchet differentiable up to the order m_2, with respect to their first arguments, at the nod points $x^n_{t_k}$, the following approximate equations emerge for every $k \in \{0, \dots, n-1\}$:

$$x^n(t) = x^n(t_k) + \int_{t_k}^{t} \sum_{i=0}^{m_1} \frac{\alpha^{(i)}_{(x^n_{t_k}, \ell)} \overbrace{(x^n_\ell - x^n_{t_k}, \dots, x^n_\ell - x^n_{t_k})}^{i \text{ times}}}{i!} d\ell + \int_{t_k}^{t} \sum_{i=0}^{m_2} \frac{\beta^{(i)}_{(x^n_{t_k}, \ell)} \overbrace{(x^n_\ell - x^n_{t_k}, \dots, x^n_\ell - x^n_{t_k})}^{i \text{ times}}}{i!} dw(\ell), \quad t \in [t_k, t_{k+1}],$$

(29)

where $\alpha^{(i)}_{(x^{n}+_{t_k},\ell)}$ and $\beta^{(i)}_{(x^n_{t_k},\ell)}$ represent the i-th Fréchet derivatives of the functionals α and β with respect to their first arguments, respectively, at the points $(x^n_{t_k},\ell)$, where $i \geq 1$, and $\alpha^{(0)}_{(x^n_{t_k},\ell)} = \alpha(x^n_{t_k},\ell) \in \mathbb{R}^N$, $\beta^{(0)}_{(x^n_{t_k},\ell)} = \beta(x^n_{t_k},\ell) \in \mathbb{R}^{N\times M}$. In that sense, for each $k \in \{0,1,\dots,n-1\}$, a different initial condition is considered: when $k = 0$, the initial condition is (27), while, for $k > 0$, the initial condition is $x^n_{t_k} = \{x^n(t_k+\theta), \mid \theta \in [-\tau,0]\}$, which is derived from the solution of the previous equation. The approximate solution $x^n = \{x^n(t) \mid t \in [t_0-\tau,T]\}$, constructed by the successive connection of the initial condition (27) and the processes $\{x^n(t) \mid t \in [t_k,t_{k+1}]\}$ at the points t_k, $k \in \{0,1,\dots,n-1\}$, is almost surely a continuous process which approximates the solution to the eq. (28) $\mathbb{P}$−almost everywhere and in the L^p−sense. Other than [18], the paper [33] proposes a numerical method for solving the eq. (28) under certain conditions.

As previously mentioned, it is quite difficult to calculate the arbitrary-order Fréchet derivatives of the diffusion and drift coefficients α and β, and one of the main goals of this paper is to find another way of effectively obtaining these derivatives, without the restrictive assumptions or the loss in precision. It is worth mentioning that even in some special cases of SFDEs (which appear when the process has time-constant delays, or even if it does not have delays at all, see [16] and [17], for example), one runs into the problem of computing the higher-order Fréchet derivatives of the matrix functions, which itself is again a computational problem.

4.2. Transferring the problem to real Hilbert spaces

To start, note that the Banach space $V_{\mathbb{R}}$ consists of the continuous N−dimensional vector-valued functions. Respectively, for each $a \in V_{\mathbb{R}}$ there exists an N−tuple of real, scalar, continuous functions $a_1,\dots,a_N \in C([-\tau,0],\mathbb{R})$ such that

$$a = (a_1,\dots,a_N).$$

Recall that each coordinate space $C([-\tau,0],\mathbb{R})$ is dense in $L^2([-\tau,0],\mathbb{R})$ with respect to the $\|\cdot\|_2$−norm, and that the convergence in $\|\cdot\|_\infty$−norm implies the same in $\|\cdot\|_2$−sense, while the converse does not hold. Motivated by this observation, rather than the initial space $V_{\mathbb{R}}$, the space $\mathcal{H}$, which is defined as

$$\mathcal{H} := L^2([\tau,0],\mathbb{R}^N) := \left\{ a : [-\tau,0] \to \mathbb{R}^N : \int_{[-\tau,0]} \left(\|a(\theta)\|_{(2,\mathbb{R}^N)}\right)^2 d\mu(\theta) < \infty \right\},$$

where μ denotes the standard Lebesgue measure on $[\tau,0]$ and $\|\cdot\|_{(2,\mathbb{R}^N)}$ denotes the standard Euclidean norm in $\mathbb{R}^N$, will be the subject of consideration. The scalar product in $\mathcal{H}$ is well-defined as

$$\langle a,b\rangle_{\mathcal{H}} := \int_{[-\tau,0]} \langle a(\theta),b(\theta)\rangle_{\mathbb{R}^N} d\mu(\theta).$$

The space $\mathcal{H}$ becomes a Hilbert space by taking the completion with respect to the $\|\cdot\|_{(2,\mathcal{H})}$−norm, which is generated via the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

Proposition 4.1. *With respect to the previous notation, the space*

$$\left(C([-\tau,0],\mathbb{R}^N), \|\cdot\|_{(2,\mathcal{H})}\right)$$

is dense in $\mathcal{H}$.

Proof. Recall that $\mathcal{H}$ is isomorphic to

$$\mathcal{H} \cong L^2([-\tau,0],\mathbb{R}) \otimes \mathbb{R}^N \cong \underbrace{L^2([-\tau,0],\mathbb{R}) \oplus L^2([-\tau,0],\mathbb{R}) \oplus \ldots \oplus L^2([-\tau,0],\mathbb{R})}_{N-\text{times}}$$

where $\otimes$ denotes the tensor product and $\oplus$ denotes the orthogonal sum. Since $C([-\tau,0],\mathbb{R})$ is dense in $L^2([-\tau,0],\mathbb{R})$ with respect to the $\|\cdot\|_2$−norm and $\mathcal{H}$ is merely the copy of N such spaces, the proof immediately follows. □

It is also straightforward that $\mathbb{R}^N$ can be embedded into $\mathcal{H}$ via the constant functions: for every $(r_1, \dots, r_N) \in \mathbb{R}^N$, let $\mathbf{r}$ be the constant function given as

$$\mathbf{r} : \theta \mapsto (r_1, \dots, r_N), \quad \forall \theta \in [-\tau, 0].$$

Thus, in the further text, the space $(\mathbb{R}^N, \frac{1}{\tau}\|\cdot\|_2,)$ considered as a subspace of $\mathcal{H}$ without any restrictions, will be observed. Respectively, one can observe $\overline{\alpha} : \mathcal{H} \to \mathcal{H}$, where $\overline{\alpha}$ represents the continuous extension of the initial α from $V_{\mathbb{R}}$ to the entire $\mathcal{H}$.

Assumption $\overline{\alpha}$. The functional $\alpha : V_{\mathbb{R}} \times [-\tau, 0] \to \mathbb{R}^N$ allows a continuous extension in the first argument $\overline{\alpha} : \mathcal{H} \times [t_0, T] \to \mathcal{H}$.

Since α is m_1-times Fréchet differentiable in its first argument by assumption, and it allows a continuous extension to the entire space $\mathcal{H}$, it follows that all the Fréchet derivatives of α (which are bounded linear operators in the respective spaces) also allow the continuous linear extensions to $\mathcal{H}$, which are precisely the Fréchet derivatives of the function $\overline{\alpha}$ of the respective order. Furthermore, the considered Fréchet derivatives of α are taken with respect to their first argument,s and are chosen precisely at the nod points $x^n_{t_k}$, while the time argument ℓ runs freely over $[t_k, t_{k+1}]$. Without loss of generality, one can fix $t \in [t_0, T]$, and introduce the function $\bar{\alpha}^t : \mathcal{H} \to \mathcal{H}$, defined as

$$\bar{\alpha}^t(\cdot) := \overline{\alpha}(\cdot, t),$$

which is also Fréchet differentiable m_1-times in $\mathcal{H}$ by the same argument. Denote its first-order derivative at some point $a \in \mathcal{H}$ as $D_a\bar{\alpha}^t$. Then $D_a\bar{\alpha}^t$ is a bounded linear operator in the direction b_1, for some $b_1 \in \mathcal{H}$, by definition. This means that, when computing the higher-order derivatives of $\bar{\alpha}^t$, one computes the respective higher-order derivatives of $D_a\bar{\alpha}^t(b_1)$ with respect to a. In other words, the second-order derivative of $\bar{\alpha}^t$ at a, at point (b_1, b_2), is the first-order derivative of $a \mapsto D_a\bar{\alpha}^t(b_1)$ at point b_2:

$$D^2_{(a,a)}\bar{\alpha}^t(b_1, b_2) = \lim_{r\to 0+0} r^{-1}\left(D_{(a+rb_2)}\bar{\alpha}^t(b_1) - D_{(a)}\bar{\alpha}^t(b_1)\right).$$

In that sense, one can think of $\{D_a\bar{\alpha}^t\}_{a\in I}$ as the family of bounded linear operators in $\mathcal{H}$, which is indexed by some $I \subset \mathcal{H}$, which contains an open ball around a. Thus, when a continuously varies over I, the operators $D_a\bar{\alpha}^t$ continuously vary as well. Precisely, this is a homotopy perturbation method parametrized by $I \subset \mathcal{H}$. The Frećhet derivatives of $D_a\bar{\alpha}^t$ are analyzed as follows.

Recall that any bounded linear operator L on the real Hilbert space $\mathcal{H}$ has its unique adjoint L^*, such that

$$\langle Lx, y\rangle_{\mathcal{H}} = \langle x, L^*y\rangle_{\mathcal{H}}$$

holds for all $x, y \in \mathcal{H}$. This is a consequence of the Riesz representation lemma and is valid for both real and complex Hilbert spaces. Since L^* is uniquely determined and obtained via the continuous linear functionals on $\mathcal{H}$, it follows that $L \mapsto L^*$ is a uniformly continuous mapping.

An operator U is said to be orthogonal if $U^*U = I$. Recall the following result:

Theorem 4.2. *[8, Theorem 4.3.] Every bounded linear operator on a real Hilbert space is a linear combination of orthogonal operators; five operators suffice.*

It is clear from the proof of [8, Theorem 4.3.] that this decomposition depends continuously on the given operator.

On the other hand, consider an important result and observation derived in [37], where the authors had studied real Hilbert spaces and infinite matrices over them. Note that the space $\mathcal{H}$ is separable (with the multivariate Hermite polynomials being an appropriate countable basis, see [38]), therefore, any bounded linear operator on $\mathcal{H}$ can be interpreted as an infinite bounded matrix over $\mathcal{H}$.

Adopting the notation and terminology from [37], an operator $R \in \mathcal{L}(\mathcal{H})$ is said to be a rotation in $\mathcal{H}$, if there exists a skew-symmetric $H \in \mathcal{L}(\mathcal{H})$ (i.e. $H = -H^*$), such that $R = e^H$. Unlike the matrix case, these rotations do not form a group w.r.t. multiplication in infinitely-dimensional Hilbert spaces, as neatly noted in [37]. In that same paper, the authors had proved that every orthogonal operator can be expressed as a product of at most three rotations; combining the previous analysis, for every $L \in \mathcal{L}(\mathcal{H})$ there exist $C_1, \ldots, C_5 \in \mathbb{R}$ and skew-symmetric operators $H_1, \ldots, H_{15} \in \mathcal{L}(\mathcal{H})$, all of which continuously depend on L, such that

$$L = \sum_{k=1}^{5} C_k e^{H_{3k-2}} e^{H_{3k-1}} e^{H_{3k}}. \tag{30}$$

The last expression allows the utilization of the exponential function in $\mathcal{H}$, together with its Fréchet derivatives studied in Section 2: as L continuously varies over $\mathcal{H}$ (or over $I \subset \mathcal{H}$), then so do the summands from the right-hand-side in (30). However, the procedure itself is costly in computation, as the latter consists of products of three exponential terms. Thus L can be simplified even further in the following sense. First observe the following:

$$L = \frac{1}{2}(L - L^*) + \frac{1}{2}(L + L^*), \tag{31}$$

where the first addend is a skew-symmetric operator while the second is a symmetric (self-adjoint) operator. Then there exists a rotation R such that

$$\frac{1}{2}(L - L^*) = \ln R.$$

Respectively (31) gives

$$L = \ln R + \frac{1}{2}(L + L^*), \tag{32}$$

and determining the Fréchet derivatives of L comes down to computing the corresponding derivatives of $\ln R$ and $\frac{1}{2}(L + L^*)$. The first one is achieved by the results obtained in Section 3. For the second addend, one can either apply (30) directly, or, if the circumstances allow it, can proceedas follows. Recall another result from [8]:

Lemma 4.3. *[8, Lemma 3.1.] If H is symmetric in $\mathcal{H}$ then $\mathcal{H}$ can be written as the orthogonal sum of two infinite-dimensional $H-$invariant closed linear subspaces.*

Respectively, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and $\frac{1}{2}(L + L^*)$ can be represented as a block-diagonal operator matrix

$$\frac{1}{2}(L + L^*) = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}. \tag{33}$$

However, note that this decomposition relies on the corresponding eigenspaces of $\frac{1}{2}(L + L^*)$. If the latter is possible, i.e., if $\frac{1}{2}\left(D_a\bar{\alpha}^t + (D_a\bar{\alpha}^t)^*\right)$ and $\frac{1}{2}\left(D_{a+rb_2}\bar{\alpha}^t + (D_{a+rb_2}\bar{\alpha}^t)^*\right)$ have the same eigenspaces for every small enough positive r, then (33) can be applied for both of them. This way, the computation of the Fréchet derivatives of $D_a\bar{\alpha}^t$ is divided into two separate problems: one is computing the appropriate derivatives of the logarithmic function, while the other is first reducing the remaining operators from (32) via (33), and then, decomposing the reduced operators L_1 and L_2 in terms of (30), and then finally computing the derivatives of the corresponding exponential functions defined respectively in $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$.

Basically, the same procedure applies to the function β. The main difference is that $\beta : V_{\mathbb{R}} \times [t_0, T] \to \mathbb{R}^{N\times M}$, so $\mathcal{H}_M$ is defined as

$$\mathcal{H}_M := \oplus_{k=\overline{1,M}} \mathcal{H} \cong \mathcal{H} \otimes \{1, \ldots, M\}.$$

Then $\mathbb{R}^{N\times M}$ can be embedded into $\mathcal{H}_M$ via the constant functions.

Assumption $\overline{\beta}$. The functional $\beta : V_{\mathbb{R}} \times [-\tau, 0] \to \mathbb{R}^{N\times M}$ allows a continuous extension in the first argument $\overline{\beta} : \mathcal{H} \times [t_0, T] \to \mathcal{H}_M$.

Respectively, for every fixed $t \in [t_0, T]$ define $\bar{\beta}^t_M : \mathcal{H}_M \times [t_0, T] \to \mathcal{H}_M$ as

$$\bar{\beta}^t_M = \left[\bar{\beta}(\cdot, t),\ \underbrace{0,\ 0,\ 0,\ \ldots,\ 0}_{M-1-\text{times}}\right] : \underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \ldots \oplus \mathcal{H}}_{M-\text{times}} \to \mathcal{H}_M.$$

It follows that for every $t \in [t_0, T]$ the mapping $\bar{\beta}^t_M$ maps $\mathcal{H}_M$ to $\mathcal{H}_M$. The decompositions (30), (32), and (33) transfer the problem to computing the Fréchet derivatives of the exponential and logarithmic functions.

4.2.1. The complex case

Though the stochastic processes stuided in [16]–[18] are real processes, at this point the results of this paper can be applied even if $\mathbb{F} = \mathbb{C}$. Precisely, rather than $\mathbb{R}^N$, $\mathbb{R}^{N\times M}$, and $V_{\mathbb{R}}$, assume that $\mathbb{C}^N$, $\mathbb{C}^{N\times M}$ and $V_{\mathbb{C}}$ are observed instead. Then, $L^2\left([-\tau, 0], \mathbb{C}^N\right)$ defines the complex Hilbert space, $\mathcal{H}_{\mathbb{C}}$, which is defined in the same manner as $\mathcal{H}$ in the previous section, though the involution portion in the definition of the corresponding scalar product is denoted as:

$$\langle h_1, h_2\rangle_{\mathcal{H}_{\mathbb{C}}} := \int_{[-\tau,0]} \langle h_1(\theta), h_2(\theta)\rangle_{\mathbb{C}^N} d\mu(\theta).$$

The space $\mathcal{H}_M$ for β is defined in the same way. Accordingly, under the assumptions **Assumption** $\bar{\alpha}$ and **Assumption** $\bar{\beta}$, for every fixed $t \in [t_0, T]$ the corresponding Fréchet derivatives at a: $D_a\bar{\alpha}^t$ and $D_a\bar{\beta}^t_M$, which are bounded linear operators in the corresponding complex Hilbert spaces, are observed.

On the other hand, recall that any bounded linear operator on a complex Hilbert space can be represented as a combination of four unitary operators on that space. Precisely, for any $L \in \mathcal{L}(\mathcal{H})$,

$$L = \operatorname{Re} L + i \operatorname{Im} L, \tag{34}$$

where $\operatorname{Re} L$ and $\operatorname{Im} L$ are self-adjoint bounded linear operators,

$$\operatorname{Re} L := 1/2(L + L^*), \quad \operatorname{Im} L := 1/2i(L - L^*).$$

Scaling by the factor of $\|L\|^{-1}$, under the assumption that $\|\operatorname{Re} L\|, \|\operatorname{Im} L\| \leq 1$, the identity

$$S = \frac{1}{2}\left(S + i\sqrt{I - S^2}\right) + \frac{1}{2}\left(S - i\sqrt{I - S^2}\right) \tag{35}$$

is obtained, where $S \in \{\|L\|^{-1} \operatorname{Re} L, \|L\|^{-1} \operatorname{Im} L\}$ (in this case the positive square root is well-defined). Recall that $S \pm i\sqrt{I - S^2}$ are unitary operators, so, for easier notation,

$$U_+(S) := S + i\sqrt{I - S^2}, \quad U_-(S) := S - i\sqrt{I - S^2} \tag{36}$$

are introduced, and, by combining the expressions (34), (35), and (36),

$$\begin{aligned} L = &\frac{\|L\|}{2}\left(U_+(\|L\|^{-1} \operatorname{Re} L) + U_-(\|L\|^{-1} \operatorname{Re} L)\right) + \\ &+ \frac{i\|L\|}{2}\left(U_+(\|L\|^{-1} \operatorname{Im} L) + U_-(\|L\|^{-1} \operatorname{Im} L)\right). \end{aligned} \tag{37}$$

Finally, we recall the lemma (see e.g. [9]) which states that any unitary operator U can be expressed as $U = e^{iH_U}$, where H_U is a bounded self-adjoint linear operator. In other words, for every $L \in \mathcal{L}(\mathcal{H})$ there exist four self-adjoint operators $H_1(L), H_2(L), H_3(L), H_4(L) \in \mathcal{L}(\mathcal{H})$ (which depend on the choice of L) such that

$$L = \frac{\|L\|}{2}\left(e^{iH_1(L)} + e^{iH_2(L)} + ie^{iH_3(L)} + ie^{iH_4(L)}\right). \tag{38}$$

Returning to $D_a\bar{\alpha}^t$ and $D_a\bar{\beta}^t_M$, rather than computing the higher-order Fréchet derivatives of $\bar{\alpha}^t$ and $\bar{\beta}^t_M$ directly, simply decompose $D_a\bar{\alpha}^t$ and $D_a\bar{\beta}^t_M$ into linear combinations of the exponentials as in (38), and proceed to evaluate the Fréchet derivatives of the respective exponential terms. Notice that, as opposed to the real case in the equation (30), there is only a linear combination of the exponential terms and no multiplication, so the computational cost is fairly lower in the complex case.

Declarations

Conflict of interest. The authors declare that there is no conflict of interest in publishing the findings obtained in this article.

Data sharing not applicable for this article as no datasets were generated or analyzed during the study.

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