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τ-metric spaces and convergence

D. Georgiou[a], I. Kougias[b], A. Megaritis[a,*], F. Sereti[c]

[a]*University of Patras, Department of Mathematics, 26504 Patra, Greece*
[b]*University of Peloponnese, Department of Electrical and Computer Engineering, 26334 Patra, Greece*
[c]*University of Western Macedonia, Department of Mathematics, 52100 Kastoria, Greece*

Abstract. In this paper, based on the meaning of τ-metric space we study the notion of convergence and ideal convergence on this field of spaces and investigate their properties, comparing it also with the usual notions of convergence and ideal convergence on metric spaces. Especially, we study the meaning of convergence on τ-metric spaces, giving new characterizations for this notion, new results for complete τ-metric spaces and new notions of compactness and totally boundedness on such spaces. We also prove theorems that enrich a related theory. Finally, we insert and study the meaning of ideal convergence on τ-metric spaces, giving also new characterizations and investigate its behavior under the view of classical meaning of ideal convergence.

1. Introduction

The notion of convergence wins an essential part in the branches of Mathematical Analysis and Topology (see for example [1, 8, 19, 20]). Several topological notions and properties have been studied in the view of convergence of sequences and nets.

Simultaneously, the importance of this research issue; the meaning of convergence, led to define new types of convergences; namely, statistical convergence and ideal convergence for various fields; metric spaces, fuzzy metric spaces, topological spaces, fuzzy topological spaces, partially ordered sets, fuzzy ordered sets (see for example [5–7, 9–18, 21–23, 26–30]).

On the other hand, since metric and topological spaces have their own significant role in the field of Mathematics, it is absolutely natural to have new related realms of spaces. Especially, in [2–4] the author studies the notion of τ-metric space, where τ is an arbitrary cardinal number, as a generalization of the usual metric space. Especially, the class of all τ-metric spaces as τ runs through the cardinal numbers contains all ordinary metric spaces (for $\tau = 1$). In these works we can also find an initial study of convergence on τ-metric spaces as a tool for the study of the so-called sequentially complete τ-metric spaces. A related study on τ-metrizable spaces can be found on [24].

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* Corresponding author: A. Megaritis
Email addresses: georgiou@math.upatras.gr (D. Georgiou), kougias@uop.gr (I. Kougias), acmegaritis@upatras.gr (A. Megaritis), seretifot@gmail.com (F. Sereti)

2. Preliminary notes

In the following sections $\mathbb{R}_+^\tau$ denotes the topological product of τ copies of the space $\mathbb{R}_+ = [0, +\infty)$ with the usual topology.

We remind that if X_i, $i \in I$, are topological spaces, then the open sets in the product topology of the space $X = \prod_{i \in I} X_i$ are arbitrary unions (finite or infinite) of sets of the form $\prod_{i \in I} U_i$, where each U_i is open in the space X_i and $U_i \neq X_i$ for only finitely many i. Also, on the space $\mathbb{R}_+^\tau$ the operations of addition, multiplication, and multiplication by a scalar, as well as a partial ordering, are defined in a natural way (coordinatewise).

If τ denotes an arbitrary non zero cardinal number, then the meaning of τ-metric space is given as follows.

Definition 2.1. ([3]) Let X be a non empty set. A mapping $\rho_\tau : X \times X \to \mathbb{R}_+^\tau$ is called a *τ-metric* on X if the following axioms are satisfied:

(1) $\rho_\tau(x, y) = \theta$ if and only if $x = y$, where θ is the point of the space $\mathbb{R}_+^\tau$ whose all coordinates are zeros.

(2) $\rho_\tau(x, y) = \rho_\tau(y, x)$, for all $x, y \in X$.

(3) $\rho_\tau(x, z) \leqslant \rho_\tau(x, y) + \rho_\tau(y, z)$, for all $x, y, z \in X$.

The pair (X, ρ_τ) is called a *τ-metric space* and the elements of X are called *points*.

The following are examples of τ-metric spaces (see [3]):
(1) If (X, ρ) is a metric space, I is a set such that $|I| = \tau$ and $\rho_i = \rho$, for every $i \in I$, then the mapping $\rho_\tau : X \times X \to \mathbb{R}_+^\tau$ defined by

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i \in I},$$

for every $x, y \in X$, is a τ-metric on X.
(2) If $\{(X_i, \rho_i) : i \in I\}$ is a family of metric spaces, where $|I| = \tau$, then the mapping $\rho_\tau : X \times X \to \mathbb{R}_+^\tau$ defined by

$$\rho_\tau(x, y) = \{\rho_i(x_i, y_i)\}_{i \in I},$$

is a τ-metric on X, where $X = \prod_{i \in I} X_i$, $x = \{x_i\}_{i \in I}$ and $y = \{y_i\}_{i \in I}$.

Also, we state that every τ-metric space (X, ρ_τ) generates a Tychonoff (that is, a completely regular and Hausdorff) topological space (X, T_{ρ_τ}). The topology T_{ρ_τ} on X defined by the local basis consisting of the sets of the form

$$B(x, O(\theta)) = \{y \in X : \rho_\tau(x, y) \in O(\theta)\},$$

of each point $x \in X$, where $O(\theta)$ runs through all open neighborhoods of the point θ in the space $\mathbb{R}_+^\tau$, is called *the topology induced by the τ-metric ρ_τ*. If $\mathcal{O}(\theta)$ denotes the family of all open neighborhoods of the point θ in $\mathbb{R}_+^\tau$, then the family

$$\mathcal{B} = \{B(x, O(\theta)) : x \in X,\ O(\theta) \in \mathcal{O}(\theta)\}$$

is a base for this topology T_{ρ_τ}.

3. The notion of convergence on τ-metric spaces

In this section we study the notion of convergence of nets on τ-metric spaces and present new facts and properties. Firstly, the usual meaning of convergence of sequences for an arbitrary metric space is reminded as follows.

Definition 3.1. ([8]) A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,ρ) *converges* to a point $x \in X$ if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $x_n \in B(x,\varepsilon)$ for every $n \geqslant n_0$, where $B(x,\varepsilon) = \{y \in X : \rho(x,y) < \varepsilon\}$.

Equivalently, we say that a sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,ρ) *converges* to a point $x \in X$ if $\lim\limits_{n\to+\infty} \rho(x_n,x) = 0$, that is, for every $\varepsilon > 0$ there exists a positive integer n_0 such that $\rho(x_n,x) < \varepsilon$, for every $n \geqslant n_0$. In each case we write $\lim\limits_{n\to+\infty} x_n = x$ and the point x is called the *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$.

However, in a classical topological point of view, the convergence of sequences are not able to describe essential topological properties because the convergence of sequences in a space does not uniquely determine its topology (see for example [8, 20]). But, the convergence of nets managed to successfully address the above problem. The nets give a different view in order to study convergences in topological spaces. We have the so-called Moore-Smith sequences by Moore-Smith [25] and their study, in the view of convergence, in topological spaces by Birkhoff [1]. We state that J. Kelley [19, 20] provides the right description of convergence in topological spaces under the prism of nets.

We remind that a non empty set Λ with a reflexive and transitive binary relation $\leqslant$ is said to be *directed* if every pair of elements of Λ has an upper bound that is, for every $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\lambda_1 \leqslant \lambda$ and $\lambda_2 \leqslant \lambda$. A mapping $s : \Lambda \to X$ from a directed set Λ into a set X is called a *net* on X and is denoted by $s = (x_\lambda)_{\lambda\in\Lambda}$, where $x_\lambda = s(\lambda)$. A net $(y_\mu)_{\mu\in M}$ in X is said to be a *subnet* of the net $(x_\lambda)_{\lambda\in\Lambda}$ in X if there exists a function $\varphi : M \to \Lambda$ with the following properties:

(1) $y_\mu = x_{\varphi(\mu)}$ for every $\mu \in M$.

(2) For every $\lambda \in \Lambda$ there exists $\mu_0 \in M$ such that $\varphi(\mu) \geqslant \lambda$ for every $\mu \in M$ with $\mu \geqslant \mu_0$.

A net is considered to be a generalization of the notion of a sequence and a subnet a generalization of the concept of subsequence to the case of nets.

Definition 3.2. ([8]) If (X,τ) is a topological space, then we say that a net $(x_\lambda)_{\lambda\in\Lambda}$ on X *converges* to a point $x \in X$ if for every open neighborhood U of x there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geqslant \lambda_0$. In this case we write $\lim\limits_{\lambda\in\Lambda} x_\lambda = x$. The point x is called the *limit* of the net $(x_\lambda)_{\lambda\in\Lambda}$.

A natural question that arises is the investigation of the notion of convergence on τ-metric spaces. The papers [2–4] give the initial study on this topic.

Definition 3.3. ([3]) A net $(x_\lambda)_{\lambda\in\Lambda}$ in a τ-metric space (X,ρ_τ) *converges* to a point $x \in X$ if for every open neighborhood $O(\theta)$ of the point $\theta \in \mathbb{R}^\tau_+$ there exists an index $\lambda_{O(\theta)} \in \Lambda$ such that

$$\rho_\tau(x_\lambda,x) \in O(\theta) \text{ for every } \lambda \geqslant \lambda_{O(\theta)}.$$

In this case we write $\rho_\tau\text{-}\lim\limits_{\lambda\in\Lambda} x_\lambda = x$ and the point x is called the *ρ_τ-limit* of $(x_\lambda)_{\lambda\in\Lambda}$. We simply write that the net $(x_\lambda)_{\lambda\in\Lambda}$ *converges to x with respect to ρ_τ* or *ρ_τ-converges to x*.

Remark 3.4. By Definition 3.3 we observe that for every net $(x_\lambda)_{\lambda\in\Lambda}$ in any τ-metric space (X,ρ_τ) we have that

$$\rho_\tau\text{-}\lim_{\lambda\in\Lambda} x_\lambda = x \text{ if and only if } \lim_{\lambda\in\Lambda} \rho_\tau(x_\lambda,x) = \theta.$$

Remark 3.5. A net $(x_\lambda)_{\lambda\in\Lambda}$ in any τ-metric space (X,ρ_τ) converges to a point $x \in X$ with respect to ρ_τ if and only if $(x_\lambda)_{\lambda\in\Lambda}$ converges to x in the topological space (X,T_{ρ_τ}). That is, according to Definitions 3.2 and 3.3, $\rho_\tau(x_\lambda,x) \in O(\theta)$ holds eventually if and only if $x_\lambda \in B(x,O(\theta))$ holds eventually.

Therefore, we can have the following result that follows from Remark 3.5.

Proposition 3.6. *Let (X,ρ_τ) be a τ-metric space and $(x_\lambda)_{\lambda\in\Lambda}$ be a net on X. Then:*

(1) *The ρ_τ-limit of $(x_\lambda)_{\lambda\in\Lambda}$ is unique.*

(2) *If $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in X$ with respect to ρ_τ, then each subnet of it converges to the same point with respect to ρ_τ.*

As we have seen above the sequences are a special case of nets; the set $\mathbb{N}$ of natural numbers is a directed set (whenever it is necessary we refer to the set of positive integers). Thus, in the following example we apply Definition 3.3 for sequences.

Example 3.7. Let I be a set with $|I| = \tau$ and $\rho_i(x, y)$ be the usual metric on $\mathbb{R}$ for every $i \in I$. We also consider the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ and the τ-metric space $(\mathbb{R}, \rho_\tau)$, where $\rho_\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^\tau_+$ is the τ-metric defined by

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i\in I},$$

for every $x, y \in \mathbb{R}$. Then this sequence ρ_τ-converges to zero.

Indeed, let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}^\tau_+$. According to the product topology on $\mathbb{R}^\tau_+$, there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. We state that for every $i = i_0, \ldots, i_k$, there exists $\varepsilon_i > 0$, such that $[0, \varepsilon_i) \subseteq U_i$. In addition, since the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to zero, in the usual sense, for every $i = i_0, \ldots, i_k$, there exists a positive integer $n_i \in \mathbb{N}$ such that

$$\rho_i\left(\frac{1}{n}, 0\right) < \varepsilon_i, \text{ for every } n \geqslant n_i.$$

Let $n_0 = \max\{n_{i_0}, \ldots, n_{i_k}\}$. Then for every $i \in I$,

$$\rho_i\left(\frac{1}{n}, 0\right) \in U_i, \text{ for every } n \geqslant n_0.$$

Especially, for every $i = i_0, \ldots, i_k$,

$$\rho_i\left(\frac{1}{n}, 0\right) \in U_i, \text{ for every } n \geqslant n_0$$

and trivially, for every $i \in I \setminus \{i_0, \ldots, i_k\}$ (for which $U_i = \mathbb{R}_+$),

$$\rho_i\left(\frac{1}{n}, 0\right) \in U_i, \text{ for every } n \in \mathbb{N}.$$

Then we have

$$\left\{\rho_i\left(\frac{1}{n}, 0\right)\right\}_{i\in I} \in \prod_{i\in I} U_i, \text{ for every } n \geqslant n_0,$$

that is,

$$\rho_\tau\left(\frac{1}{n}, 0\right) \in V, \text{ for every } n \geqslant n_0$$

and therefore,

$$\rho_\tau\left(\frac{1}{n}, 0\right) \in O(\theta), \text{ for every } n \geqslant n_0,$$

proving that this sequence ρ_τ-converges to 0.

Generalizing Example 3.7 we can have the following proposition.

Proposition 3.8. *Let (X, ρ) be a metric space, I be a set such that $|I| = \tau$ and $\rho_i = \rho$, for every $i \in I$. On X we consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i\in I},$$

for every $x, y \in X$. A net $(x_\lambda)_{\lambda\in\Lambda}$ on X converges to a point $x \in X$ with respect to the metric ρ if and only if it converges to the same point x with respect to ρ_τ.

Proof. Firstly, we suppose that a net $(x_\lambda)_{\lambda\in\Lambda}$ on X converges to a point $x \in X$ with respect to the metric ρ and we prove that $(x_\lambda)_{\lambda\in\Lambda}$ converges to the same point x with respect to ρ_τ. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}_+^\tau$. According to the product topology on $\mathbb{R}_+^\tau$, there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. Since the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to x in the metric space (X, ρ_i), for every $i = i_0, \ldots, i_k$, there exists $\lambda_i \in \Lambda$ such that

$$\rho_i(x_\lambda, x) \in U_i, \text{ for every } \lambda \geqslant \lambda_i.$$

We choose λ_0 an element of Λ greater than or equal to $\lambda_{i_0}, \ldots, \lambda_{i_k}$. Then for every $i \in I$,

$$\rho_i(x_\lambda, x) \in U_i, \text{ for every } \lambda \geqslant \lambda_0.$$

Especially, for every $i = i_0, \ldots, i_k$,

$$\rho_i(x_\lambda, x) \in U_i, \text{ for every } \lambda \geqslant \lambda_0$$

and trivially, for every $i \in I \setminus \{i_0, \ldots, i_k\}$ (for which $U_i = \mathbb{R}_+$),

$$\rho_i(x_\lambda, x) \in U_i, \text{ for every } \lambda \in \Lambda.$$

Then we have

$$\{\rho_i(x_\lambda, x)\}_{i\in I} \in \prod_{i\in I} U_i, \text{ for every } \lambda \geqslant \lambda_0$$

that is,

$$\{\rho_i(x_\lambda, x)\}_{i\in I} \in V, \text{ for every } \lambda \geqslant \lambda_0$$

and therefore,

$$\rho_\tau(x_\lambda, x) \in O(\theta), \text{ for every } \lambda \geqslant \lambda_0,$$

proving that the net $(x_\lambda)_{\lambda\in\Lambda}$ ρ_τ-converges to x.

Conversely, we suppose that a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in X$ with respect to ρ_τ and we prove that $(x_\lambda)_{\lambda\in\Lambda}$ converges to x with respect to the metric ρ. Let $\varepsilon > 0$ and $i_0 \in I$. We consider the open neighborhood $O(\theta) = \prod_{i\in I} U_i$ of θ in $\mathbb{R}_+^\tau$, where $U_{i_0} = [0, \varepsilon)$ and $U_i = \mathbb{R}_+$ for every $i \in I \setminus \{i_0\}$. Since the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to x with respect to ρ_τ, there exists $\lambda_0 \in \Lambda$ such that

$$\rho_\tau(x_\lambda, x) \in O(\theta), \text{ for every } \lambda \geqslant \lambda_0.$$

Then

$$\{\rho_i(x_\lambda, x)\}_{i\in I} \in \prod_{i\in I} U_i, \text{ for every } \lambda \geqslant \lambda_0$$

and thus,

$$\rho_{i_0}(x_\lambda, x) \in U_{i_0}, \text{ for every } \lambda \geqslant \lambda_0.$$

Hence,

$$\rho(x_\lambda, x) < \varepsilon, \text{ for every } \lambda \geqslant \lambda_0.$$

Therefore, the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to x with respect to the metric ρ. □

We state that if $\{X_i : i \in I\}$ is a family of sets, $X = \prod_{i\in I} X_i$ and $s_i = (x_\lambda^i)_{\lambda\in\Lambda}$ are nets on X_i, $i \in I$, then we consider the net $s : \Lambda \to X$ on X defined by $s(\lambda) = \{x_\lambda^i\}_{i\in I}$, for every $\lambda \in \Lambda$. The net s will be called *the net generated by the nets s_i, $i \in I$*. If s_i, $i \in I$, are sequences, then we refer to s as *the sequence generated by the sequences s_i, $i \in I$*.

Proposition 3.9. *Let I be a set with $|I| = \tau$ and $\{(X_i, \rho_i) : i \in I\}$ be a family of metric spaces. We consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x_i, y_i)\}_{i\in I},$$

where $X = \prod_{i\in I} X_i$, $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. Whenever $i \in I$, $s_i = (x^i_\lambda)_{\lambda\in\Lambda}$ on X_i converges to a point $z_i \in X_i$ with respect to the metric ρ_i if and only if the net s generated by the nets s_i, $i \in I$, on X ρ_τ-converges to the point $z = \{z_i\}_{i\in I} \in X$.

Proof. We suppose that a net $s_i = (x^i_\lambda)_{\lambda\in\Lambda}$ on X_i converges to a point $z_i \in X_i$ with respect to the metric ρ_i for each $i \in I$ and we prove that the net s on X ρ_τ-converges to the point $z = \{z_i\}_{i\in I} \in X$. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}^\tau_+$. According to the product topology on $\mathbb{R}^+_\tau$, there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. Since the net $(x^i_\lambda)_{\lambda\in\Lambda}$, whenever $i = i_0, \ldots, i_k$, converges to $z_i \in X_i$ in the metric space (X_i, ρ_i), there exists $\lambda_i \in \Lambda$ such that

$$\rho_i(x^i_\lambda, z_i) \in U_i, \text{ for every } \lambda \geqslant \lambda_i.$$

We choose λ_0 an element of Λ greater than or equal to $\lambda_{i_0}, \ldots, \lambda_{i_k}$. Then for every $i \in I$,

$$\{\rho_i(x^i_\lambda, z_i)\}_{i\in I} \in \prod_{i\in I} U_i, \text{ for every } \lambda \geqslant \lambda_0$$

that is,

$$\{\rho_i(x^i_\lambda, z_i)\}_{i\in I} \in V, \text{ for every } \lambda \geqslant \lambda_0$$

and therefore,

$$\rho_\tau(s, z) \in O(\theta), \text{ for every } \lambda \geqslant \lambda_0,$$

proving that the net s ρ_τ-converges to z.

Conversely, we suppose that the net s on X ρ_τ-converges to the point $z = \{z_i\}_{i\in I} \in X$ and we prove that the net $s_i = (x^i_\lambda)_{\lambda\in\Lambda}$ on X_i converges to $z_i \in X_i$ with respect to the metric ρ_i, whenever $i \in I$. Let $\varepsilon > 0$ and $i_0 \in I$. We consider the open neighborhood $O(\theta) = \prod_{i\in I} U_i$ of θ in $\mathbb{R}^\tau_+$, where $U_{i_0} = [0, \varepsilon)$ and $U_i = \mathbb{R}_+$ for every $i \in I \setminus \{i_0\}$. Since the net s converges to z with respect to ρ_τ, there exists $\lambda_0 \in \Lambda$ such that

$$\rho_\tau(s, z) \in O(\theta), \text{ for every } \lambda \geqslant \lambda_0.$$

Then

$$\{\rho_i(x^i_\lambda, z_i)\}_{i\in I} \in \prod_{i\in I} U_i, \text{ for every } \lambda \geqslant \lambda_0$$

and thus,

$$\rho_{i_0}(x^{i_0}_\lambda, z_{i_0}) \in U_{i_0}, \text{ for every } \lambda \geqslant \lambda_0.$$

Hence,

$$\rho_{i_0}(x^{i_0}_\lambda, z_{i_0}) < \varepsilon, \text{ for every } \lambda \geqslant \lambda_0,$$

proving that the net $(x^{i_0}_\lambda)_{\lambda\in\Lambda}$ converges to $z_{i_0} \in X_{i_0}$ with respect to the metric ρ_{i_0}. Since the index i_0 is chosen to be arbitrary and fixed we have completed the proof. □

Undoubtedly, the meaning of Cauchy sequences plays an essential role in the field of convergences. We remind that a sequence $(x_n)_{n\in\mathbb{N}}$ on a metric space (X, ρ) is said to be *Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\rho(x_n, x_m) < \varepsilon, \text{ for every } n, m \geqslant n_0.$$

A metric space (X, ρ) is called *complete* if each Cauchy sequence converges to a point $x \in X$.

In the paper [3] the author studies the notion of sequentially completeness for τ-metric spaces in order to obtain a generalization of Banach fixed point theorem.

Definition 3.10. ([3]) A net $(x_\lambda)_{\lambda\in\Lambda}$ is said to be *Cauchy* in a τ-metric space (X, ρ_τ) if for every open neighborhood $O(\theta)$ of the point θ of the space $\mathbb{R}^\tau_+$ there exists an index $\lambda_{O(\theta)} \in \Lambda$ such that

$$\rho_\tau(x_\nu, x_\mu) \in O(\theta), \text{ for every } \nu, \mu \geqslant \lambda_{O(\theta)}.$$

We state that if a net is a sequence, then Definition 3.10 is referred to the notion of Cauchy sequence.

Definition 3.11. ([2–4]) A τ-metric space (X, ρ_τ) is said to be *sequentially complete* if every Cauchy sequence converges in (X, ρ_τ).

Similar to Propositions 3.8 and 3.9 we can prove the following results.

Proposition 3.12. *Let (X, ρ) be a metric space, I be a set such that $|I| = \tau$ and $\rho_i = \rho$, for every $i \in I$. On X we consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i\in I},$$

for every $x, y \in X$. A net $(x_\lambda)_{\lambda\in\Lambda}$ on X is Cauchy with respect to the metric ρ if and only if $(x_\lambda)_{\lambda\in\Lambda}$ is Cauchy with respect to ρ_τ.

Proposition 3.13. *Let (X, ρ) be a metric space, I be a set such that $|I| = \tau$ and $\rho_i = \rho$, for every $i \in I$. On X we consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i\in I},$$

for every $x, y \in X$. Then (X, ρ) is a complete metric space if and only if (X, ρ_τ) is sequentially complete.

Proposition 3.14. *Let I be a set with $|I| = \tau$ and $\{(X_i, \rho_i) : i \in I\}$ be a family of metric spaces. We consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x_i, y_i)\}_{i\in I},$$

where $X = \prod_{i\in I} X_i$, $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. Whenever $i \in I$, $s_i = (x^i_\lambda)_{\lambda\in\Lambda}$ on X_i is Cauchy with respect to the metric ρ_i if and only if the net s generated by the nets s_i, $i \in I$, on X is Cauchy with respect to the τ-metric ρ_τ.

Proposition 3.15. *Let I be a set with $|I| = \tau$ and $\{(X_i, \rho_i) : i \in I\}$ be a family of metric spaces. We consider the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x_i, y_i)\}_{i\in I},$$

where $X = \prod_{i\in I} X_i$, $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. Then (X, ρ_τ) is sequentially complete if and only if (X_i, ρ_i) is complete, whenever $i \in I$.

Proof. Let (X, ρ_τ) be sequentially complete, $i_0 \in I$ be fixed and $(x^{i_0}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X_{i_0}. We shall prove that this sequence converges to a point of X_{i_0}. For each $i \in I \setminus \{i_0\}$ we consider a fixed point of X_i; let it be c_i. Then we consider the sequence $s_i = (x^i_n)_{n\in\mathbb{N}}$, where $x^i_n = c_i$ for each $n \in \mathbb{N}$. Also, we set $s_{i_0} = (x^{i_0}_n)_{n\in\mathbb{N}}$. Then the sequence s, where $s(n) = (x^i_n)_{i\in I}$, for each $n \in \mathbb{N}$, is generated by the sequences s_i, $i \in I$. By Proposition 3.14, the sequence s of $X = \prod_{i\in I} X_i$ is Cauchy and thus it ρ_τ-converges to a point $\{z_i\}_{i\in I}$ of X. By Proposition 3.9, the sequence $(x^{i_0}_n)_{n\in\mathbb{N}}$ converges to z_{i_0} of X_{i_0}.

Conversely, we suppose that each (X_i, ρ_i), $i \in I$, is complete and we shall prove that (X, ρ_τ) is sequentially complete. Let s, where $s(n) = (x^i_n)_{i\in I}$, for each $n \in \mathbb{N}$, be a Cauchy sequence of $X = \prod_{i\in I} X_i$. For each $i \in I$, we consider the sequence $s_i = (x^i_n)_{n\in\mathbb{N}}$ of X_i. Then s is generated by s_i, $i \in I$. By Proposition 3.14, each s_i, $i \in I$, is Cauchy, and thus, it converges to a point z_i of X_i. By Proposition 3.9, the sequence s ρ_τ-converges to $\{z_i\}_{i\in I}$ of X. □

In this section, we enrich the study of completeness, proving a corresponding Baire theorem for τ-metric spaces. The following lemmas will be useful for the proof of this theorem.

Lemma 3.16. *For each $x \in X$ and $U \in T_{\rho_\tau}$ with $x \in U$ there exists $O(\theta) \in \mathcal{O}(\theta)$ such that*

$$x \in B(x, O(\theta)) \subseteq \mathrm{Cl}(B(x, O(\theta))) \subseteq U.$$

Proof. It follows immediately, since the family $\mathcal{B}$ is a base for the regular space (X, T_{ρ_τ}). □

Lemma 3.17. *Let X be a τ-metric space and $A \subseteq X$.*

(1) *If A is closed in (X, T_{ρ_τ}), then any sequence in A, which converges in X with respect to ρ_τ, converges in A.*

(2) *If τ is finite and any sequence in A, which converges in X with respect to ρ_τ, converges in A, then A is closed in (X, T_{ρ_τ}).*

Proof. (1) It follows by Remark 3.5, since it is well known that the statement (1) of the lemma is true for every topological space if "sequence" replaced by "net".

(2) Suppose that A is not closed. Then $X \setminus A$ is not open. So, there exists an element $x \in X \setminus A$ such that

$$B(x, O(\theta)) \cap A \neq \emptyset,$$

for every $O(\theta) \in \mathcal{O}(\theta)$. For $n = 1, 2, \dots$ we put $O_n(\theta) = \left[0, \frac{1}{n}\right)^\tau$ and choose a point $x_n \in B(x, O_n(\theta)) \cap A$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in A with a ρ_τ-limit x that is not in A, which is a contradiction. □

We recall that for a topological space X the following statements are equivalent:

(1) X is a Baire space.

(2) The intersection of any countable collection of open dense subsets of X is dense.

(3) The union of any countable collection of closed nowhere dense subsets of X is nowhere dense.

Theorem 3.18. (Baire Category Theorem for τ-metric spaces) *Let (X, ρ_τ) be a sequentially complete τ-metric space, where τ is finite. Then (X, T_{ρ_τ}) is a Baire space.*

Proof. Let $\{U_n : n = 1, 2, \dots\}$ be a countable family of open dense subsets of X. We prove that the intersection $\bigcap_{n=1}^{\infty} U_n$ is dense. Let V be a non empty open subset of X. Since U_1 is dense, $V \cap U_1 \neq \emptyset$. By Lemma 3.16 there exist $x_1 \in X$ and $O_1(\theta) \in \mathcal{O}(\theta)$ such that

$$x_1 \in B(x_1, O_1(\theta)) \subseteq \mathrm{Cl}(B(x_1, O_1(\theta))) \subseteq V \cap U_1.$$

Without loss of generality we can suppose that $O_1(\theta) = \prod_{i=1}^{\tau} U_{1i}$, where $U_{1i} \subseteq [0, 1)$ for every $i = 1, \dots, \tau$. In a recursive manner, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ in X and a subfamily $\{O_n(\theta) : n = 1, 2, \dots\}$ of $\mathcal{O}(\theta)$ such that:

(1) $\mathrm{Cl}(B(x_1, O_1(\theta))) \subseteq V \cap U_1$.

(2) $\mathrm{Cl}(B(x_{n+1}, O_{n+1}(\theta))) \subseteq B(x_n, O_n(\theta)) \cap U_{n+1}$, $n = 1, 2, \dots$.

(3) $O_n(\theta) = \prod_{i=1}^{\tau} U_{ni}$, where $U_{ni} \subseteq \left[0, \frac{1}{n}\right)$ for each $i = 1, \dots, \tau$ and $n = 1, 2, \dots$.

We prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $O(\theta)$ be an open neighborhood of the point θ in the space $\mathbb{R}_+^\tau$. Then there exists a positive integer n_0 such that

$$\left[0, \frac{1}{n_0}\right)^\tau \subseteq O(\theta).$$

For every $\nu, \mu \geqslant n_0$ with $\nu > \mu$ we have $x_\nu \in B(x_\mu, O_\mu(\theta))$. Therefore,

$$\rho_\tau(x_\nu, x_\mu) \in O_\mu(\theta) \subseteq \left[0, \frac{1}{\mu}\right)^\tau \subseteq \left[0, \frac{1}{n_0}\right)^\tau$$

and hence, $\rho_\tau(x_\nu, x_\mu) \in O(\theta)$.

Since the τ-metric space (X, ρ_τ) is sequentially complete, the sequence $(x_n)_{n\in\mathbb{N}}$ ρ_τ-converges to some point x. Since

$$x_n \in \mathrm{Cl}(B(x_n, O_n(\theta))) \subseteq B(x_1, O_1(\theta)) \text{ for every } n = 1, 2, \ldots,$$

by Lemma 3.17, $x \in \mathrm{Cl}(B(x_1, O_1(\theta)))$. Since

$$x_n \in \mathrm{Cl}(B(x_n, O_n(\theta))) \subseteq B(x_2, O_2(\theta)) \text{ for every } n = 2, 3, \ldots,$$

by Lemma 3.17, $x \in \mathrm{Cl}(B(x_2, O_2(\theta)))$. Continuing in the same way,

$$x \in \bigcap_{n=1}^{\infty} \mathrm{Cl}(B(x_n, O_n(\theta))) \subseteq \bigcap_{n=1}^{\infty} (V \cap U_n) = V \cap \bigcap_{n=1}^{\infty} U_n.$$

Thus $V \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset$, proving the theorem. □

In what follows, we continue such a study introducing the notions of sequentially compactness and totally boudedness for τ-metric spaces and investigating their relations with the notion of sequentially completeness.

Definition 3.19. A τ-metric space (X, ρ_τ) is said to be *sequentially compact* if each sequence has a convergent subsequence with respect to ρ_τ.

Definition 3.20. A τ-metric space (X, ρ_τ) is said to be *totally bounded* if for every open neighborhood $O(\theta)$ of θ in $\mathbb{R}_+^\tau$, there exists a finite subset A of X such that $X = \bigcup_{x\in A} B(x, O(\theta))$.

Similar to the classical theory of metric spaces, if (X, ρ_τ) is a τ-metric space and $Y \subseteq X$, then Y is totally bounded, if the condition of Definition 3.20 is satisfied, restricting the τ-metric ρ_τ on $Y \times Y$.

Remark 3.21. By Definition 3.19 we have that if the topological space (X, T_{ρ_τ}) is compact, then the τ-metric space (X, ρ_τ) is sequentially compact.

Theorem 3.22. *Every sequentially compact τ-metric space is sequentially complete.*

Proof. Let (X, ρ_τ) be a sequentially compact τ-metric space and $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. We shall prove that $(x_n)_{n\in\mathbb{N}}$ ρ_τ-converges to a point $x \in X$. Since X is sequentially compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $\rho_\tau\text{-}\lim_{k\to+\infty} x_{n_k} = x$. We prove that $\rho_\tau\text{-}\lim_{n\to+\infty} x_n = x$. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}_+^\tau$. According to the product topology on $\mathbb{R}_\tau^+$, there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. Without loss of generality we suppose that $U_i = [0, \varepsilon_i)$ for every $i = i_0, \ldots, i_k$. We consider the set $W = \prod_{i\in I} U_i'$, where $U_i' = \left[0, \frac{\varepsilon_i}{2}\right)$ for every $i = i_0, \ldots, i_k$ and $U_i' = \mathbb{R}_+$ for every $i \in I \setminus \{i_0, \ldots, i_k\}$. Then $W \subseteq V$. Since $\rho_\tau\text{-}\lim_{k\to+\infty} x_{n_k} = x$, there exists $k_1 \in \mathbb{N}$ such that

$$\rho_\tau(x_{n_k}, x) \in W, \text{ for every } k \geqslant k_1.$$

Also, since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $k_2 \in \mathbb{N}$ such that

$$\rho_\tau(x_n, x_{n_k}) \in W, \text{ for every } n, n_k \geqslant k_2.$$

Let $k_0 = \max\{k_1, k_2\}$. Then from the definition of the subsequence we have that $n_{k_0} \geqslant k_0 \geqslant k_1$ and $n_{k_0} \geqslant k_0 \geqslant k_2$. Since

$$\rho_\tau(x_n, x) \leqslant \rho_\tau(x_n, x_{n_{k_0}}) + \rho_\tau(x_{n_{k_0}}, x),$$

we have that $\rho_\tau(x_n, x) \in V$ and thus, $\rho_\tau(x_n, x) \in O(\theta)$, for every $n \geqslant k_0$, proving that $(x_n)_{n\in\mathbb{N}}$ ρ_τ-converges to x. □

Theorem 3.23. *Every sequentially compact τ-metric space is totally bounded.*

Proof. Let (X, ρ_τ) be a sequentially compact τ-metric space. We prove that it is totally bounded. For that, we suppose in contrast that it is not totally bounded. Then there is an open neighborhood $O_1(\theta)$ of θ in $\mathbb{R}_+^\tau$ such that for every finite subset A of X we have

$$X \neq \bigcup_{x\in A} B(x, O_1(\theta)).$$

Similar to the proof of Theorem 3.22 there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O_1(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. Without loss of generality we suppose that $U_i = [0, \varepsilon_i)$ for every $i = i_0, \ldots, i_k$. We consider the set $W = \prod_{i\in I} U_i'$, where $U_i' = \left[0, \frac{\varepsilon_i}{2}\right)$ for every $i = i_0, \ldots, i_k$ and $U_i' = \mathbb{R}_+$ for every $i \in I \setminus \{i_0, \ldots, i_k\}$. Then $W \subseteq V$.

Let $x_1 \in X$. Then there exists $x_2 \in X$ such that $x_2 \notin B(x_1, O_1(\theta))$. That is, $\rho_\tau(x_1, x_2) \notin O_1(\theta)$. Also, there exists $x_3 \in X$ such that $x_3 \notin B(x_1, O_1(\theta)) \cup B(x_2, O_1(\theta))$. That is, $\rho_\tau(x_1, x_3) \notin O_1(\theta)$ and $\rho_\tau(x_2, x_3) \notin O_1(\theta)$. Continuing with the same arguments, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$x_n \notin B(x_m, O_1(\theta)), \text{ for every } n \neq m,$$

that is,

$$\rho_\tau(x_n, x_m) \notin O_1(\theta), \text{ for every } n \neq m.$$

We shall prove that $(x_n)_{n\in\mathbb{N}}$ has not a ρ_τ-convergence subsequence. Indeed, if $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to a point $x \in X$ with respect to ρ_τ, then there exists $k_0 \in \mathbb{N}$ such that

$$\rho_\tau(x_{n_k}, x) \in W, \text{ for every } k \geqslant k_0.$$

Then for every $k_1, k_2 \geqslant k_0$ we have

$$\rho_\tau(x_{n_{k_1}}, x) \in W \text{ and } \rho_\tau(x_{n_{k_2}}, x) \in W.$$

Since $\rho_\tau(x_{n_{k_1}}, x_{n_{k_2}}) \leqslant \rho_\tau(x_{n_{k_1}}, x) + \rho_\tau(x_{n_{k_2}}, x)$, we have that $\rho_\tau(x_{n_{k_1}}, x_{n_{k_2}}) \in V$, that is $\rho_\tau(x_{n_{k_1}}, x_{n_{k_2}}) \in O_1(\theta)$ which is a contradiction. Thus, X is totally bounded. □

Proposition 3.24. *If (X, ρ_τ) is totally bounded and $Y \subseteq X$, then Y is also totally bounded.*

Proof. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}_+^\tau$. Similar to the proof of Theorem 3.22 there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \ldots, i_k$. Without loss of generality we suppose that $U_i = [0, \varepsilon_i)$ for every $i = i_0, \ldots, i_k$. We consider the set $W = \prod_{i\in I} U_i'$, where $U_i' = \left[0, \frac{\varepsilon_i}{2}\right)$ for every $i = i_0, \ldots, i_k$ and $U_i' = \mathbb{R}_+$ for every $i \in I \setminus \{i_0, \ldots, i_k\}$. Then $W \subseteq V$. Since X is totally bounded, there exists a finite subset A of X such that

$$X = \bigcup_{x\in A} B(x, W).$$

Let D be the set of all $x \in A$ such that $B(x, W) \cap Y \neq \emptyset$. For each $x \in A$ we denote by y_x an element of Y for which $y_x \in B(x, W) \cap Y$. Then the set $C = \{y_x : x \in D\}$ is a finite subset of Y such that

$$Y \subseteq \bigcup_{y \in C} B(y, O(\theta)).$$

Indeed, if $y \in Y$, then there exists $x \in A$ such that $y \in B(x, W)$ (or equivalently, $\rho_\tau(y, x) \in W$) and $\rho_\tau(x, y_x) \in W$. Since $\rho_\tau(y, y_x) \leqslant \rho_\tau(y, x) + \rho_\tau(x, y_x)$, we have that $\rho_\tau(y, y_x) \in V$ and hence, $\rho_\tau(y, y_x) \in O(\theta)$. Thus, if $B_Y(y, O(\theta)) = \{y' \in Y : \rho_\tau(y, y') \in O(\theta)\}$, then $Y = \bigcup_{y \in C} B_Y(y, O(\theta))$, as it holds that $B_Y(y, O(\theta)) = Y \cap B(y, O(\theta))$. Therefore, Y is totally bounded. □

Theorem 3.25. *Let (X, ρ_τ) be a τ-metric space, where τ is finite. If X is sequentially complete and totally bounded, then it is sequentially compact.*

Proof. We suppose that X is not sequentially compact. Then by Remark 3.21, (X, T_{ρ_τ}) is not compact. There exists an open cover $c = \{U_j : j \in J\}$ of X for which there is not a finite subcover. Since X is totally bounded, there exists a finite subset A_1 of X such that

$$X = \bigcup_{x \in A_1} B(x, O_1(\theta)),$$

where $O_1(\theta) = \prod_{i=1}^{\tau} U_{1i}$ with $U_{1i} = \left[0, \frac{1}{2}\right)$ for every $i = 1, \ldots, \tau$. Thus, there exists $x_1 \in A_1$ for which the set $B(x_1, O_1(\theta))$ can not be covered by a finite number of elements of c. By Proposition 3.24 the set $Y_1 = B(x_1, O_1(\theta))$ is totally bounded and similarly, there exists a finite subset A_2 of $B(x_1, O_1(\theta))$ such that

$$B(x_1, O_1(\theta)) = \bigcup_{x \in A_2} B_{Y_1}(x, O_2(\theta)),$$

where $O_2(\theta) = \prod_{i=1}^{\tau} U_{2i}$ with $U_{2i} = \left[0, \frac{1}{2^2}\right)$ for every $i = 1, \ldots, \tau$ and $B_{Y_1}(x, O_2(\theta)) = Y_1 \cap B(x, O_2(\theta))$, for every $x \in A_2$.

There exists $x_2 \in A_2$ for which the set $Y_2 = B_{Y_1}(x_2, O_2(\theta))$ can not be covered by a finite number of elements of c. Continuing with the same arguments we can construct a sequence of open sets

$$Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_n \supseteq \ldots,$$

where $O_n(\theta) = \prod_{i=1}^{\tau} U_{ni}$ with $U_{ni} = \left[0, \frac{1}{2^n}\right)$ for every $i = 1, \ldots, \tau$, such that

$$O_1(\theta) \supseteq O_2(\theta) \supseteq \ldots \supseteq O_n(\theta) \supseteq \ldots$$

and a sequence $(x_n)_{n \in \mathbb{N}}$, which is Cauchy.

Indeed, let $O(\theta)$ be an open neighborhood of the point θ in the space $\mathbb{R}^\tau_+$. Then there exists a positive integer n_0 such that

$$\left[0, \frac{1}{2^{n_0}}\right)^\tau \subseteq O(\theta).$$

For every $\nu, \mu \geqslant n_0$ with $\mu > \nu$ we have $Y_\mu \subseteq Y_\nu$ and so $x_\mu \in B_{Y_{\nu-1}}(x_\nu, O_\nu(\theta))$. Therefore,

$$\rho_\tau(x_\mu, x_\nu) \in O_\nu(\theta) = \left[0, \frac{1}{2^\nu}\right)^\tau \subseteq \left[0, \frac{1}{2^{n_0}}\right)^\tau \subseteq O(\theta)$$

and hence, $\rho_\tau(x_\mu, x_\nu) \in O(\theta)$.

Since X is sequentially complete, this sequence ρ_τ-converges to a point $x \in X$. Since c is a cover of X, there exists $j_0 \in J$ such that $x \in U_{j_0}$. Moreover, since U_{j_0} is open in the topology T_{ρ_τ}, there exists $n \in \mathbb{N}$ such that

$$B(x, O_n(\theta)) \subseteq U_{j_0}.$$

Since the sequence $(x_n)_{n\in\mathbb{N}}$ ρ_τ-converges to x, there exists $m \in \mathbb{N}$, $m \geqslant n+1$, such that $x_m \in B(x, O_{n+1}(\theta))$. Then

$$Y_m \subseteq B(x, O_n(\theta)) \subseteq U_{j_0}.$$

Indeed, if $y \in Y_m$, then $\rho_\tau(y, x_m) \in O_m(\theta)$, where $O_m(\theta) = \prod_{i=1}^{\tau} U_{mi}$ with $U_{mi} = \left[0, \frac{1}{2^m}\right)$ for every $i = 1, \dots, \tau$. We state that

$$U_{mi} = \left[0, \frac{1}{2^m}\right) \leqslant \left[0, \frac{1}{2^{n+1}}\right) = \left[0, \frac{1}{2} \cdot \frac{1}{2^n}\right)$$

for every $i = 1, \dots, \tau$. Also, $\rho_\tau(x_m, x) \in O_{n+1}(\theta)$, where $O_{n+1}(\theta) = \prod_{i=1}^{\tau} U_{(n+1)i}$ with $U_{(n+1)i} = \left[0, \frac{1}{2^{n+1}}\right)$ for every $i = 1, \dots, \tau$. We state that

$$U_{(n+1)i} = \left[0, \frac{1}{2^{n+1}}\right) = \left[0, \frac{1}{2} \cdot \frac{1}{2^n}\right)$$

for every $i = 1, \dots, \tau$. By the axiomatic properties of τ-metric, $\rho_\tau(y, x) \leqslant \rho_\tau(y, x_m) + \rho_\tau(x_m, x)$. Thus, $\rho_\tau(y, x) \in O_n(\theta)$. Then Y_m is covered by U_{j_0}, which is a contradiction. □

Corollary 3.26. *Let (X, ρ_τ) be a τ-metric space, where τ is finite. Then X is sequentially complete and totally bounded if and only if it is sequentially compact.*

4. Ideal convergence on τ-metric spaces

The ideal convergence on τ-metric spaces give us a different approach to the notion of convergence. The ideal convergence of sequences is based on the notion of an ideal on $\mathbb{N}$ [21, 22].

Let D be a non empty set. A family $\mathcal{I}$ of subsets of D is called *ideal* if $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$.

(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$ and *non trivial* if $\mathcal{I} \neq \{\emptyset\}$ and $D \notin \mathcal{I}$.

Definition 4.1. (see, for example, [21, 30]) Let $\mathcal{I}$ be a non trivial ideal on $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X, ρ) is said that *$\mathcal{I}$-converges* to a point $x \in X$ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \rho(x_n, x) \geqslant \varepsilon\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}\lim_{n\to+\infty} x_n = x$ and the point x is said to be the *$\mathcal{I}$-limit* of the sequence $(x_n)_{n\in\mathbb{N}}$.

Definition 4.2. (see, for example, [23]) Let (X, τ) be a topological space. A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is said that *$\mathcal{I}$-converges* to a point $x \in X$, where $\mathcal{I}$ is a non trivial ideal of Λ, if for every open neighborhood U of x,

$$\{\lambda \in \Lambda : x_\lambda \notin U\} \in \mathcal{I}.$$

It is known that in every metric space (X, ρ), any sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}_f$-converges to a point $x \in X$, where $\mathcal{I}_f$ denotes the ideal of all finite subsets of $\mathbb{N}$, if and only if it converges (with the usual notion) to the same point x (see for example [23]).

In the next we insert the notion of ideal convergence on τ-metric spaces.

Definition 4.3. Let $\mathcal{I}$ be a non trivial ideal on $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of a τ-metric space (X, ρ_τ) is said that $\mathcal{I}$-ρ_τ-*converges* to a point $x \in X$ if for every open neighborhood $O(\theta)$ of the point θ in the space $\mathbb{R}^\tau_+$, we have

$$\{n \in \mathbb{N} : \rho_\tau(x_n, x) \notin O(\theta)\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}\text{-}\rho_\tau\text{-}\lim\limits_{n\to+\infty} x_n = x$ and the point x is said to be the $\mathcal{I}$-ρ_τ-*limit* of the sequence $(x_n)_{n\in\mathbb{N}}$.

Remark 4.4. The sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to x with respect to ρ_τ if and only if $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to x in the topological space (X, T_{ρ_τ}).

Based on Remark 4.4 we can have the following facts.

Proposition 4.5. *Let (X, ρ_τ) be a τ-metric space, $\mathcal{I}$ a non trivial ideal on $\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence on X. Then:*

(1) *The $\mathcal{I}$-ρ_τ-limit of $(x_n)_{n\in\mathbb{N}}$ is unique.*

(2) *$(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}_f$-ρ_τ-converges to a point $x \in X$ if and only if it ρ_τ-converges to the same point x.*

Proposition 4.6. *Let (X, ρ) be a metric space, $\mathcal{I}$ be a non trivial ideal on $\mathbb{N}$, I be a set such that $|I| = \tau$, $\rho_i = \rho$, for every $i \in I$, and the τ-metric $\rho_\tau : X \times X \to \mathbb{R}^\tau_+$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x, y)\}_{i\in I},$$

for every $x, y \in X$. A sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to a point $x \in X$ with respect to the metric ρ if and only if it $\mathcal{I}$-ρ_τ-converges to the same point x.

Proof. We suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to a point $x \in X$ with respect to the metric ρ and we shall prove that it $\mathcal{I}$-ρ_τ-converges to the same point x. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}^\tau_+$. According to the product topology on $\mathbb{R}^\tau_+$, there exists a set $V = \prod\limits_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \dots, i_k$. Then for every $i = i_0, \dots, i_k$, there exists $\varepsilon_i > 0$ such that $[0, \varepsilon_i) \subseteq U_i$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to x in the metric space (X, ρ_i), where $i = i_0, \dots, i_k$, we have that

$$\{n \in \mathbb{N} : \rho_i(x_n, x) \geqslant \varepsilon_i\}) \in \mathcal{I}$$

and thus,

$$\{n \in \mathbb{N} : \rho_i(x_n, x) \notin U_i\} \in \mathcal{I}.$$

Then by the notion of ideal

$$\bigcup_{i=i_0}^{i_k} \{n \in \mathbb{N} : \rho_i(x_n, x) \notin U_i\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \{\rho_i(x_n, x)\}_{i\in I} \notin V\} \subseteq \bigcup_{i=i_0}^{i_k} \{n \in \mathbb{N} : \rho_i(x_n, x) \notin U_i\},$$

we have that

$$\{n \in \mathbb{N} : \{\rho_i(x_n, x)\}_{i\in I} \notin V\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \rho_\tau(x_n, x) \notin O(\theta)\}) \subseteq \{n \in \mathbb{N} : \{\rho_i(x_n, x)\}_{i\in I} \notin V\},$$

we have

$$\{n \in \mathbb{N} : \rho_\tau(x_n, x) \notin O(\theta)\} \in \mathcal{I},$$

proving that $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-ρ_τ-converges to x.

Conversely, we suppose that a sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-ρ_τ-converges to a point $x \in X$ and we shall prove that it $\mathcal{I}$-converges to the same point x. Let $\varepsilon > 0$ and $i_0 \in I$. We consider the open neighborhood $O(\theta) = \prod_{i\in I} U_i$ of θ in $\mathbb{R}_+^\tau$, where $U_{i_0} = [0, \varepsilon)$ and $U_i = \mathbb{R}_+$ for every $i \in I \setminus \{i_0\}$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-ρ_τ-converges to x, we have that

$$\{n \in \mathbb{N} : \rho_\tau(x_n, x) \notin O(\theta)\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \rho_{i_0}(x_n, x) \geqslant \varepsilon\} \subseteq \{n \in \mathbb{N} : \rho_\tau(x_n, x) \notin O(\theta)\},$$

we have

$$\{n \in \mathbb{N} : \rho_{i_0}(x_n, x) \geqslant \varepsilon\} \in \mathcal{I}$$

or

$$\{n \in \mathbb{N} : \rho(x_n, x) \geqslant \varepsilon\} \in \mathcal{I}.$$

Hence, $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to x. □

Proposition 4.7. *Let $\{(X_i, \rho_i) : i \in I\}$ be a family of metric spaces, where $|I| = \tau$, and $\mathcal{I}$ be a non trivial ideal on $\mathbb{N}$. We consider also the τ-metric $\rho_\tau : X \times X \to \mathbb{R}_+^\tau$ defined by*

$$\rho_\tau(x, y) = \{\rho_i(x_i, y_i)\}_{i\in I},$$

where $X = \prod_{i\in I} X_i$, $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. Whenever $i \in I$, $s_i = (x_n^i)_{n\in\mathbb{N}}$ on X_i $\mathcal{I}$-converges to a point $z_i \in X_i$ with respect to the metric ρ_i if and only if the sequence s generated by the sequences s_i, $i \in I$, $\mathcal{I}$-ρ_τ-converges to the point $z = \{z_i\}_{i\in I} \in X$.

Proof. We suppose that the sequence $s_i = (x_n^i)_{n\in\mathbb{N}}$ on X_i $\mathcal{I}$-converges to a point $z_i \in X_i$ with respect to the metric ρ_i, whenever $i \in I$, and we shall prove that it $\mathcal{I}$-ρ_τ-converges to the point $z = \{z_i\}_{i\in I} \in X$. Let $O(\theta)$ be an open neighborhood of θ in $\mathbb{R}_+^\tau$. According to the product topology on $\mathbb{R}_+^\tau$, there exists a set $V = \prod_{i\in I} U_i$ such that $V \subseteq O(\theta)$, where each U_i is an open neighborhood of 0 in $\mathbb{R}_+$ and $U_i \neq \mathbb{R}_+$ for only finitely many i; let $i_0, \dots, i_k$. Then for every $i = i_0, \dots, i_k$, there exists $\varepsilon_i > 0$ such that $[0, \varepsilon_i) \subseteq U_i$. Since the sequence $(x_n^i)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to z_i in the metric space (X_i, ρ_i), where $i = i_0, \dots, i_k$, we have that

$$\{n \in \mathbb{N} : \rho_i(x_n^i, z_i) \geqslant \varepsilon_i\} \in \mathcal{I}$$

and thus,

$$\{n \in \mathbb{N} : \rho_i(x_n^i, z_i) \notin U_i\} \in \mathcal{I}.$$

Then by the notion of ideal

$$\bigcup_{i=i_0}^{i_k} \{n \in \mathbb{N} : \rho_i(x_n^i, z_i) \notin U_i\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \{\rho_i(x_n^i, z_i)\}_{i\in I} \notin V\} \subseteq \bigcup_{i=i_0}^{i_k} \{n \in \mathbb{N} : \rho_i(x_n^i, z_i) \notin U_i\},$$

we have that

$$\{n \in \mathbb{N} : \{\rho_i(x^i_n, z_i)\}_{i\in I} \notin V\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \rho_\tau(s, z) \notin O(\theta)\}) \subseteq \{n \in \mathbb{N} : \{\rho_i(x^i_n, z_i)\}_{i\in I} \notin V\},$$

we have

$$\{n \in \mathbb{N} : \rho_\tau(s, z) \notin O(\theta)\} \in \mathcal{I},$$

proving that $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-ρ_τ-converges to x.

Conversely, we suppose that the sequence s on X $\mathcal{I}$-ρ_τ-converges to a point $z = \{z_i\}_{i\in I} \in X$ and we prove that the sequence $s_i = (x^i_n)_{n\in\mathbb{N}}$ on X_i $\mathcal{I}$-converges to $z_i \in X_i$ with respect to the metric ρ_i, whenever $i \in I$. Let $\varepsilon > 0$ and $i_0 \in I$. We consider the open neighborhood $O(\theta) = \prod_{i\in I} U_i$ of θ in $\mathbb{R}^\tau_+$, where $U_{i_0} = [0, \varepsilon)$ and $U_i = \mathbb{R}_+$ for every $i \in I \setminus \{i_0\}$. Since the sequence s $\mathcal{I}$-ρ_τ-converges to z, we have that

$$\{n \in \mathbb{N} : \rho_\tau(s, z) \notin O(\theta)\} \in \mathcal{I}.$$

Since

$$\{n \in \mathbb{N} : \rho_{i_0}(x^{i_0}_n, z_{i_0}) \notin U_{i_0}\} \subseteq \{n \in \mathbb{N} : \rho_\tau(s, z) \notin O(\theta)\},$$

we have

$$\{n \in \mathbb{N} : \rho_{i_0}(x^{i_0}_n, z_{i_0}) \notin U_{i_0}\} \in \mathcal{I},$$

proving that the sequence $(x^{i_0}_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $z_{i_0} \in X_{i_0}$ with respect to the metric ρ_{i_0}. Since the index i_0 is chosen to be arbitrary and fixed we have completed the proof. □

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