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Multiplicity of weak solutions for the Steklov systems involving the $p(x)$-Laplacian operator

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Abstract. The existence of multiple weak solutions for a Steklov system involving the $p(x)$-Laplacian operator is studied. Our approach is based on variational methods.

1. Introduction

Elliptic problems including $p(x)$-Laplacian are nonlinear problem which classical methods are not applicable for the existence of solutions under different boundary conditions (see [6, 12, 16, 17, 20–22, 24–26] and references therein).
The Steklov problem is an eigenvalue problem with the spectral parameter in the boundary conditions, which has various applications. Its spectrum coincides with that of the Dirichlet-to-Neumann operator. Over the past years, there has been a growing interest in the Steklov problem from the viewpoint of spectral geometry. While this problem shares some common properties with its more familiar Dirichlet and Neumann cousins, its eigenvalues and eigenfunctions have a number of distinctive geometric features, which makes the subject especially appealing.
Steklov conditions are considered a more realistic description of the interactions at the boundary of a physical system. For example, the heat flow through the surface of a body generally depends on the value of the temperature at the surface itself (see [2, 4, 9] and the references therein for some kinds of Steklov problems).
The existence of multiple solutions to the following Steklov problem involving $p(x)$-Laplacian operator have been verified [1]

$$\begin{cases} -\Delta_{p(x)}u = a(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded smooth domain, λ is a positive parameter, $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with a growth condition and $a \in L^\infty(\Omega)$. Also, the existence of at least one positive radial solution belongs to the space $W_0^{1,p(x)}(B) \cap L_a^{q(x)}(B) \cap L_b^{r(x)}(B)$ for the problem

$$\begin{cases} -\Delta_{p(x)}u + R(x)u^{p(x)-2}u = a(x)|u|^{q(x)-2}u - b(x)|u|^{r(x)-2}u & x \in B, \\ u > 0 & x \in B, \\ u = 0 & x \in \partial B, \end{cases}$$

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has been proved [18], where B is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$, $p, q, r \in C_+(B)$, R is a positive radial function that satisfies the suitable conditions and

$$a(x) = \theta(|x|) \quad \text{and} \quad b(x) = \xi(|x|),$$

in which $\theta, \xi \in L^\infty(0,1)$ such that θ is a positive non-constant radially non-decreasing function and ξ is a non-negative radially non-increasing function.
The existence and multiplicity of weak solutions to the following Steklov $p(x)$-Laplacian problem

$$\begin{cases} -\Delta_{p(x)}u + c(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

has been proved [15], where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded smooth domain, for $p \in C(\overline{\Omega})$, $c \in L^\infty(\Omega)$ with $\inf_\Omega c(x) > 0$. $f, g : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the Carathéodory functions with the suitable conditions.
Here, motivated by their works, we are interested in finding enough conditions for the multiplicity of weak solutions to the following Steklov $p(x)$-Laplacian system

$$\begin{cases} -\Delta_{p(x)}u_i = a(x)|u_i|^{p(x)-2}u_i & \text{in } \Omega, \\ |\nabla u_i|^{p(x)-2}\frac{\partial u_i}{\partial \eta} = F_{u_i}(x, u_1, \cdots, u_n) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $i = 1, \cdots, n$, where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary and $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator for $p \in C(\overline{\Omega})$ with $N < p^- < p^+ < +\infty$. We assume that the nonnegative function a belongs to $L^\infty(\Omega)$ with $\inf_\Omega a(x) > 0$, λ is a positive parameter and the function

$$F : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

is a measurable function with respect to $x \in \Omega$ for each $(t_1, \cdots, t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1, \cdots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$; F_{u_i} denotes the partial derivative of F with respect to u_i.

We mean by a weak solution to the problem (1) is as follows:

Definition 1.1. *We say that $u = (u_1, \cdots, u_n) \in X$ is a weak solution of the problem* (1) *if, for each $1 \leq i \leq n$,*

$$\sum_{i=1}^n \int_\Omega |\nabla u_i|^{p(x)} \nabla u_i \nabla v_i dx + \sum_{i=1}^n \int_\Omega a(x)|u_i|^{p(x)-2}u_i - \lambda \sum_{i=1}^n \int_{\partial\Omega} F_{u_i}(x, u_1, \cdots, u_n) v_i dx = 0$$

for every $v = (v_1, \cdots, v_n) \in X$.

The structure of this paper is the following: In Section 2, we present preliminaries and some basic facts. We also introduce a suitable function space for the solution and we prove some remarks which we need for the last section. In Section 3, the existence of multiple weak solutions for Problem (1) is prove by variational methods and three critical points result.

2. Preliminaries

During the note, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary. We suppose that $p \in C(\overline{\Omega})$, satisfy the following condition

$$N < p^- := \inf_{x \in \Omega} p(x) \leqslant p(x) \leqslant p^+ := \sup_{x \in \Omega} p(x) < +\infty. \tag{2}$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$, by

$$L^{p(x)}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} : \text{ u is measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} := \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leqslant 1\}.$$

Also, for $u \in L^{p(x)}(\partial\Omega)$, we put

$$|u|_{p(x),\partial} = \int_{\partial\Omega} |u|^{p(x)}d\sigma.$$

For each $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, the Hölder type inequality

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}$$

holds true. Following the authors of [18], for any $\kappa > 0$, we put

$$\kappa^{\check{r}} := \begin{cases} \kappa^{r^+} & \kappa < 1, \\ \kappa^{r^-} & \kappa \geq 1 \end{cases}$$

and

$$\kappa^{\hat{r}} := \begin{cases} \kappa^{r^-} & \kappa < 1, \\ \kappa^{r^+} & \kappa \geq 1 \end{cases}$$

for $r \in \{u_i : i = 1, \cdots, n\}$. Then the well-known Proposition 2.7 of [14] will be rewritten as follows:

Proposition 2.1. *For each $u \in L^{p(x)}(\Omega)$, we have*

$$|u|_{p(x)}^{\check{p}} \leq \int_\Omega |u(x)|^{p(x)}dx \leq |u|_{p(x)}^{\hat{p}}.$$

We denote the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

As pointed out in [8, 13], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, $W^{1,p(x)}(\Omega)$ is compactly embedded in $\left(C^0(\overline{\Omega}), \|\cdot\|_{C^0(\overline{\Omega})}\right)$. So, in particular, there exists a positive constant $L > 0$ such that

$$\|u\|_{C^0(\overline{\Omega})} \leqslant L\|u\|_{p(x)} \tag{3}$$

for each $u \in W^{1,p(x)}(\Omega)$. In what follows, we set

$$X := \prod_{i=1}^{n} \left(W_i^{1,p(x)}(\Omega)\right),$$

endowed with the norm

$$\|u\| = \sum_{i=1}^{n} \|u_i\|_{p(x)} = \sum_{i=1}^{n} \Big(|\nabla u_i|_{p(x)} + |u_i|_{p(x)}\Big)$$

for $u = (u_1, \cdots, u_n) \in X$. According to the above matters, we conclude that the embedding

$$X \hookrightarrow C^0(\overline{\Omega}) \times \cdots \times C^0(\overline{\Omega})$$

is compact. Now, we introduce the functional $\Phi : X \longrightarrow \mathbb{R}$ as follows

$$\Phi(u_1, \cdots, u_n) := \sum_{i=1}^{n} \int_{\Omega} \frac{1}{p(x)} \Big(|\nabla u_i|^{p(x)} + a(x)|u_i|^{p(x)}\Big) dx.$$

Φ is sequentially weakly lower semicontinuous and it is known that Φ is continuously Gâteaux differentiable functional whose derivative given by

$$\Phi'(u_1, \cdots, u_n)(v_1, \cdots, v_n) = \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i|^{p(x)} \nabla u_i \nabla v_i dx + \sum_{i=1}^{n} \int_{\Omega} a(x)|u_i|^{p(x)-2} u_i$$

for each $(v_1, \cdots, v_n) \in X$.

Remark 2.2. *For every $u_i \in W_i^{1,p(x)}(\Omega)$, $i = 1, 2, \cdots, n$, there exist $k, K > 0$ such that*

$$k\|u_i\|_{p(x)}^{\check{p}} \le \int_{\Omega} (|\nabla u_i|^{p(x)} + a(x)|u_i|^{p(x)}) dx \le K\|u_i\|_{p(x)}^{\hat{p}}.$$

Proof. Since $\inf_{\Omega} a > 0$, so there exists $0 < \delta < 1$ such that $\delta < a(x)$ a.e. $x \in \Omega$. Using Proposition 2.1 and hypothesis $a \in L^{\infty}(\Omega)$ for $i = 1, 2, \cdots, n$, we gain

$$\delta|u_i|_{p(x)}^{\check{p}} \le \int_{\Omega} a(x)|u_i(x)|^{p(x)} dx \le \|a\|_{\infty}|u_i|_{p(x)}^{\hat{p}},$$

and

$$\delta|\nabla u_i|_{p(x)}^{\check{p}} \le |\nabla u_i|_{p(x)}^{\check{p}} \le \int_{\Omega} |\nabla u_i(x)|^{p(x)} dx \le |\nabla u_i|_{p(x)}^{\hat{p}}.$$

Bearing in mind the following elementary inequality due to J. A. Clarkson: for all $q > 0$, there exists $C_q > 0$ such that

$$|a + b|^q \le C_q(|a|^q + |b|^q)$$

for all $a, b \in \mathbb{R}$. Then for $i = 1, 2, \cdots, n$, we imply

$$\frac{\delta}{C_{\check{p}}}\|u_i\|_{p(x)}^{\check{p}} \le \int_{\Omega} (|\nabla u_i|^{p(x)} + a(x)|u_i|^{p(x)}) dx \le (1 + \|a\|_{\infty})\|u_i\|_{p(x)}^{\hat{p}}.$$

It is enough to put $k = \frac{\delta}{C_{\check{p}}}, K = 1 + \|a\|_{\infty}$. □

Remark 2.3. *Due to Remark 2.2 Φ is coercive.*

Proof. Let $u = (u_1, \cdots, u_n) \in X$ and $\|u\| \to \infty$. By definition of $\|\cdot\|$, there exists $1 \le i_0 \le n$ such that $|\nabla u_{i_0}|_{p(x)} \to \infty$. Then Remark 2.2 result in $\Phi(u) \to \infty$. □

Now, imagine that the function

$$F : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

is a measurable function with respect to $x \in \partial\Omega$ for each $(t_1, \cdots, t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1, \cdots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \partial\Omega$; and, F_{u_i} denotes the partial derivative of F with respect to u_i. We define $\Psi : \mathbb{R}^n \to \mathbb{R}$ with

$$\Psi(u_1, \cdots, u_n) := \int_{\partial\Omega} F(x, u_1, \cdots, u_n)dx.$$

The functional Ψ is well defined, continuously Gâteaux differentiable with compact derivative, whose Gâteaux derivative at point $u = (u_1, \cdots, u_n) \in X$ is as follows

$$\Psi'(u_1, \cdots, u_n)(v_1, \cdots, v_n) = \sum_{i=1}^{n} \int_{\partial\Omega} F_{u_i}(x, u_1(x), \cdots, u_n(x))v_i(x)dx$$

for every $(v_1, \cdots, v_n) \in X$. Define

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for each $u = (u_1, \cdots, u_n)$; if $I'_\lambda(u) = 0$, we have

$$\sum_{i=1}^{n} \int_{\Omega} |\nabla u_i|^{p(x)} \nabla u_i \nabla v_i dx + \sum_{i=1}^{n} \int_{\Omega} a(x)|u_i|^{p(x)-2}u_i - \lambda \sum_{i=1}^{n} \int_{\partial\Omega} F_{u_i}(x, u_1, \cdots, u_n)v_i dx = 0,$$

then, the critical points of I_λ are exactly the weak solutions of the problem (1). We set

$$\delta(x) := sup\{\delta > 0 : B(x, \delta) \subseteq \Omega\},$$

and we define

$$R := sup_{x\in\Omega}\delta(x). \tag{4}$$

Obviously, there exists $x^0 = (x_1^0, \cdots, x_N^0) \in \Omega$ such that

$$B(x^0, R) \subseteq \Omega.$$

In the next theorem, we bring our main tool proved in [5] :

Theorem 2.4. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(i) $\dfrac{sup_{\Phi(x)<r}\Psi(x)}{r} < \dfrac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(ii) For each

$$\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{sup_{\Phi(x)<r}\Psi(x)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X.

3. Main result

We formulate our main result as follows:

Theorem 3.1. *Assume that* $F : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies the following conditions*

($F1$) $F(x, 0, \cdots, 0) = 0$, *for a.e.* $x \in \overline{\Omega}$;

($F2$) *There exist* $\eta \in L^1(\partial\Omega)$ *and positive continuous function* γ, *with* $\gamma(x) < p(x)$ *a.e in* $\overline{\Omega}$ *such that*

$$0 \le F(x, u_1, ...u_n) \le \eta(x)(1 + \sum_{i=1}^{n} |u_i|^{\gamma(x)});$$

($F3$) *There exist* $r > 0$, $\delta > 0$ *and* $1 \le i_* \le n$ *such that*

$$\frac{1}{p_*^+}(\frac{2\delta}{R})^{\check{p_*}} m(R^N - (\frac{R}{2})^N) > r,$$

where $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ *is the measure of unit ball of* $\mathbb{R}^N$ *and* Γ *is the Gamma function.*

Such that

$$G_r < H_\delta, \tag{5}$$

where

$$G_r := \frac{|\eta|_{1,\partial}}{r}(1 + L^{\hat{\gamma}}||u||^{\hat{\gamma}}),$$

and

$$H_\delta := \frac{\sum_{i=1}^{n} inf_{x\in\overline{\Omega}} F(x, \delta, \cdots, \delta)}{K(\frac{2}{R})^{\hat{p}-1}\Big(2^N - 1\Big)\delta^{\hat{p}}}.$$

Then, for each

$$\lambda \in \Lambda_{r,\delta} := (\frac{1}{H_\delta}, \frac{1}{G_r}),$$

the problem (1) *admits at least three distinct weak solutions in* X.

Proof. According to previous section, the space X and the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as above satisfy the regularity assumptions of Theorem 2.4. By condition ($F1$) and definition of Φ, Ψ, it is clear that

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Fix $\delta > 0$ and R defined as in (4). By w, we consider the function of the space X, defined by

$$w(x) := \begin{cases} 0 & x \in \overline{\Omega} \setminus B(x^0, R), \\ \delta & x \in \overline{B(x^0, \frac{R}{2})}, \\ \frac{2\delta}{R}(R - |x_i - x_i^0|) & x \in B(x^0, R) \setminus \overline{B(x^0, \frac{R}{2})}, \end{cases}$$

where $x = (x_1, \cdots, x_N) \in \Omega$. Then by Remark 2.2, for $1 \le i_* \le n$, one has

$$\begin{aligned} &\frac{1}{p_*^+}(\frac{2\delta}{R})^{\check{p_*}} m(R^N - (\frac{R}{2})^N) \\ &\qquad < \Phi(w, \cdots, w) \\ &\qquad \le KN(\frac{2\delta}{R})^{\hat{p}} m\Big(R^N - (\frac{R}{2})^N\Big), \end{aligned}$$

then by assumption ($F3$), we gain $\Phi(w, \cdots, w) > r$. On the other hand, we have

$$\begin{aligned}\Psi(w,\cdots,w) &\geq \sum_{i=1}^{n}\int_{\overline{B(x^0,\frac{R}{2})}} F(x,w,\cdots,w)dx\\ &\geq \sum_{i=1}^{n} inf_{x\in\overline{\Omega}}F(x,\delta,\cdots,\delta)\mathfrak{m}(\frac{R}{2})^{N-1},\end{aligned}$$

where $\mathfrak{m} = \frac{N\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ is the hypervolume of the $(N-1)$-dimensional unit sphere (see [28]); and so,

$$\begin{aligned}\frac{\Psi(w,\cdots,w)}{\Phi(w,\cdots,w)} &\geq \frac{\sum_{i=1}^{n} inf_{x\in\overline{\Omega}}F(x,\delta,\cdots,\delta)\mathfrak{m}(\frac{R}{2})^{N-1}}{KN(\frac{2\delta}{R})^{\hat{p}}m\Big(R^N-(\frac{R}{2})^N\Big)}\\ &= \frac{\sum_{i=1}^{n} inf_{x\in\overline{\Omega}}F(x,\delta,\cdots,\delta)}{K(\frac{2}{R})^{\hat{p}-1}\Big(2^N-1\Big)\delta^{\hat{p}}} = H_\delta. \qquad (6)\end{aligned}$$

Now, let $u = (u_1, \cdots, u_n) \in \Phi^{-1}(-\infty, r)$, from Remark 2.2, we have

$$\|u\| \leq \Big(p^+\frac{\Phi(u_1,\cdots,u_n)}{k}\Big)^{\frac{1}{\hat{p}}} \leq (\frac{p^+r}{k})^{\frac{1}{\hat{p}}}. \qquad (7)$$

Then for every $u = (u_1, \cdots, u_n) \in X$, from condition ($F2$), Hölder inequality and (3), we gain

$$\begin{aligned}\int_{\partial\Omega} F(x,u_1,\cdots,u_n)dx &\leq \int_{\partial\Omega} sup_{u\in\Phi^{-1}(-\infty,r)}F(x,u_1,\cdots,u_n)dx\\ &\leq \int_{\partial\Omega}\eta(x)(1+\sum_{i=1}^{n}|u_i|^{\gamma(x)})dx\\ &\leq |\eta|_{1,\partial}(1+\sum_{i=1}^{n}|u_i|_\infty^{\hat{\gamma}})\\ &\leq |\eta|_{1,\partial}(1+L^{\hat{\gamma}}\sum_{i=1}^{n}|\nabla u_i|_{p(x)}^{\hat{\gamma}}).\end{aligned}$$

We know that $\sum_{i=1}^{n} |\nabla u_i|_{p(x)}^{\hat{\gamma}} \leq (\sum_{i=1}^{n} |\nabla u_i|_{p(x)})^{\hat{\gamma}} \leq \|u\|^{\hat{\gamma}}$; Thus,

$$\int_{\partial\Omega} F(x,u_1,\cdots,u_n)dx \leq |\eta|_{1,\partial}(1+L^{\hat{\gamma}}\|u\|^{\hat{\gamma}}).$$

Therefore,

$$\begin{aligned}\frac{1}{r}sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) &= \frac{1}{r}sup_{u\in\Phi^{-1}(-\infty,r)}\int_{\partial\Omega}F(x,u_1,...u_n)dx\\ &\leq \frac{|\eta|_{1,\partial}}{r}(1+L^{\hat{\gamma}}\|u\|^{\hat{\gamma}}) = G_r. \qquad (8)\end{aligned}$$

From assumption (5), relations (6) and (8), one has

$$\frac{1}{r}sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u_i) < \frac{\Psi(w,\cdots,w)}{\Phi(w,\cdots,w)}.$$

Therefore, the assumption (i) of Theorem 2.4 is satisfied. Now, we prove that the functional I_λ for all $\lambda > 0$ is coercive.

With the same arguments as used before, we have

$$\Psi(u) = \int_{\partial\Omega} F(x, u_1, \cdots, u_n)dx \leq |\eta|_{1,\partial}(1 + L^{\hat{\gamma}}\|u\|^{\hat{\gamma}}).$$

The last inequality and Remark 2.2 lead to

$$I_\lambda(u) \geq \frac{1}{p^+}|\nabla u_i|^{\check{p}}_{p(x)} - \lambda|\eta|_{1,\partial}(1 + L^{\hat{\gamma}}\|u\|^{\hat{\gamma}})$$

for each $i = 1, \cdots, n$. Now, suppose that $u \in X$ and $\|u\| \to \infty$. So, there exists $1 \leq i_0 \leq n$ such that $|\nabla u_{i_0}|_{p(x)} \to \infty$. Since $\gamma(x) < p(x)$ a.e. in Ω, so I_λ is coercive. Then condition (ii) holds. So for each

$$\Lambda_{\delta,r} := (\frac{1}{H_\delta}, \frac{1}{G_r}) \subseteq \Big(\frac{\Phi(w, \cdots, w)}{\Psi(w, \cdots, w)}, \frac{r}{sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u_i)}\Big),$$

Theorem 2.4 ensures that for each $\lambda \in \Lambda_{r,\delta}$, the functional I_λ admits at least three critical points in X that are weak solutions of the problem (1). □

Remark 3.2. *An interesting problem is to probe the existence and multiplicity of solutions of this system in the Heisenberg Sobolev spaces and Orlicz Sobolev spaces. Interested reader can see details of these spaces in [10, 12, 17, 19, 23, 27, 29–31] and references therein.*

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