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# Existence and fractional inequalities for hybrid $\rho$-Caputo fractional integro-differential equations with non-local conditions

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**Abstract.** This paper focuses on investigating the existence of solutions and fractional inequalities for a hybrid $\rho$-Caputo fractional differential equation accompanied by a non-local condition. By leveraging the Dhage fixed point theorem, we establish specific criteria that guarantee the existence of solutions for our problem. Furthermore, through the application of the $\rho$-Caputo derivative techniques, we illustrate various fractional inequalities.

## 1. Introduction

The theory of fractional differential equations plays a significant role in modeling systems across a wide range of disciplines, including biology, physics, rheology, electrodynamics, chemistry, and signal and image processing, among others (see references [6, 7, 9, 10, 13, 15]). As a result of their diverse applications, fractional differential equations have garnered extensive research attention, yielding noteworthy theoretical, solution-oriented, and applicative outcomes (see references [2, 11, 12]). In this context, Almeida introduced the $\rho$-Caputo derivative as a generalization of approaches like Caputo and Caputo-Hadamard derivatives [14]. For further insights and recent advancements in problems involving the $\rho$-Caputo derivative, we refer readers to [1, 4, 5]. Hybrid differential equations find their relevance in modeling several non-homogeneous physical processes, encompassing various dynamical systems as specific cases. This area has seen substantial interest due to its broad applications, with recent attention in works such as [3, 8, 11].

In 2010, Dhage and Lakschmikantham extended the concept of hybrid differential equations to the integer domain with the following equation[3]:

$$\begin{cases} \frac{d}{dt}\left(\frac{\upsilon(s)}{h(s,\upsilon(s))}\right) = g(s,\upsilon(s)), \quad s \in [0,T], \\ \upsilon(0) = \upsilon_0. \end{cases}$$

Under mixed Lipschitz and Carathéodory conditions, they established existence results and differential inequalities. In 2015, Hilal and Kajouni extended this problem to the fractional domain [8]. They provided sufficient conditions for the existence of solutions to the following problem:

$$\begin{cases} {}^{C}D^{\gamma}\left(\frac{\upsilon(s)}{h(s,\upsilon(s))}\right) = g(s,\upsilon(s)), \quad s \in [0,T], \\ a\upsilon(0) + b\upsilon(T) = c. \end{cases}$$

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where ${}^{C}D^{\gamma}$ denotes the Caputo fractional derivative, $f$ and $g$ are appropriately defined functions. In 2020, Hannabou and Kajouni utilized Dhage's fixed point theorem to investigate a hybrid fractional integro-differential equation [11]. This equation, given by

$$\begin{cases} {}^{C}D^{\gamma}\Big[\frac{v(s)-\sum_{j=1}^{m} I^{\beta_i}\vartheta_i(s,v(s))}{h(s,v(s))}\Big] = g(s,v(t)), \;\; s \in J = [0,T], \\ a\frac{v(0)}{h(0,v(0))} + b\frac{v(T)}{h(T,v(T))} = c, \end{cases}$$

expanded the scope of study in this area.

Building upon these works, this paper extends the results obtained in [11] to the $\rho$-Caputo fractional setting. Specifically, our focus lies on the hybrid $\rho$-Caputo fractional integro-differential boundary value problem:

$$\begin{cases} {}^{C}D^{\gamma,\rho}\Big[\frac{v(t)-\sum_{j=1}^{m} I^{\beta_i,\rho}\vartheta_j(t,v(t))}{h(t,v(t))}\Big] = g(t,v(t)), \;\; t \in J = [0,T], 0 < \rho < 1, \\ a\frac{v(0)}{h(0,v(0))} + b\frac{v(T)}{h(T,v(T))} = c. \end{cases} \tag{1}$$

where ${}^{C}D^{\gamma,\rho}$ represents the $\rho$-Caputo fractional derivative of order $\gamma$, $\vartheta_j$ are appropriately defined functions, and $a$, $b$, and $c$ are real constants with $a + b \neq 0$.

The rest of this paper is structured as follows: In Section 2, we furnish fundamental insights into fractional calculus and fixed-point theorems that hold pertinence to our investigation. Building upon Dhage's fixed-point theorem, Section 3 is dedicated to establishing the existence of solutions for the hybrid $\rho$-Caputo fractional integro-differential boundary value problem. Section 4 delves into the fractional inequalities that are associated with our derived solution. Concluding our study, the final section succinctly summarizes the outcomes and contributions of this work.

## 2. Preliminaries

In this section, we lay the groundwork by introducing fundamental definitions and key findings that will be utilized subsequently.

Consider $X = C(J, \mathbb{R})$, which represents a Banach space consisting of continuous functions mapping from $J = [0, T]$ to $\mathbb{R}$. The norm in this space is defined as:

$$\| y \| = \sup_{t\in J} | y(t) | .$$

Moreover, we define multiplication in $X$ as:

$$(xy)(t) = x(t)y(t)$$

Evidently, X functions as a Banach algebra in accordance with the aforementioned supremum norm and multiplication operation. We denote by $C(J \times \mathbb{R}, \mathbb{R})$ the collection of functions $y : J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following conditions:

(i) For each $x \in \mathbb{R}$, the mapping $s \to g(s, x)$ is measurable.

ii) For each $t \in J$, the mapping $x \to g(s, x)$ is continuous.

The set $C(J \times \mathbb{R}, \mathbb{R})$ is referred to as the Caratheodory class of functions defined on $J \times \mathbb{R}$. These functions possess the property of being Lebesgue integrable when they are bounded by another Lebesgue integrable function defined on the interval $J$.

The space of real-valued Lebesgue functions on the interval $J$ is denoted as $L^1(J, \mathbb{R})$. This space is equipped with a norm $\| \cdot \|_{L^1}$ defined as:

$$\| v \|_{L^1} = \int_0^T | v(s) | \, ds.$$

**Definition 2.1.** *[14](ρ-Riemann-Liouiville fractional integral) Consider $\rho > 0$, where $f$ belongs to the space $L^1([J, \mathbb{R})$, and $\rho \in C^1([a, b])$ is an increasing function with the property that $\rho'(s) \neq 0$ for all $s \in [a, b]$. The ρ-Riemann-Liouville fractional integral of order ρ acting on the function $f$ is defined as follows:*

$$(I^{\gamma,\rho} f)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (\rho(t) - \rho(s))^{\gamma-1} \rho'(s) f(s) ds. \tag{2}$$

It's evident that when $\rho(t) = t$, the expression given in (2) corresponds to the classical Riemann-Liouville fractional integral.

**Definition 2.2.** *[14](ρ-Caputo fractional derivative) Assume that $n - 1 < \rho \leq n$, and $f$ belongs to the function space $C^n([a, b])$, while ρ is an increasing function in the space $C^n([a, b])$, satisfying the condition that $\rho'(s) \neq 0$ for all $s \in [a, b]$. The ρ-Caputo fractional derivative of order γ acting on the function $f$ is defined as follows:*

$$\begin{aligned} ({}^C D^{\gamma,\rho} f)(s) &= (I^{n-\gamma,\rho} f^{[n]})(s) \\ &= \frac{1}{\Gamma(n-\gamma)} \int_0^s (\rho(s) - \rho(u))^{n-\gamma-1} \rho'(u) f^{[n]}(u) du, \end{aligned} \tag{3}$$

*where $f^{[n]}(u) := \left(\frac{1}{\rho'(u)} \frac{d}{du}\right)^n f(u)$ defined over the interval $[a, b]$.*

By virtue of this definition, it becomes evident that when $\gamma = n \in \mathbb{N}$, the expression $(D^{\gamma,\rho} f)(s)$ simplifies to $f^{[n]}(s)$.

**Remark 2.3.** *[14] In particular, when $\gamma \in (0, 1)$, we have*

$$({}^C D^{\gamma,\rho} f)(s) = \frac{1}{\Gamma(\gamma)} \int_0^s (\rho(s) - \rho(u))^{\gamma-1} f'(u) du. \tag{4}$$

**Lemma 2.4.** *[4] Let $\gamma > 0$, if $f \in C^{n-1}([a, b])$, then we have*

$$I^{\gamma,\rho}({}^C D^{\gamma,\rho} f(s)) = f(s) - \sum_{k=0}^{n-1} \frac{f^{[k]}(0)}{k!} (\rho(s) - \rho(0))^k. \tag{5}$$

*In particular, when $\gamma \in (0, 1)$, then*

$$I^{\gamma,\rho}({}^C D^{\gamma,\rho} f(s)) = f(s) - f(0) \tag{6}$$

**Lemma 2.5.** *[4] Let $\gamma > 0$ and $f \in C^1([a, b]$, we have*

$${}^C D^{\gamma,\rho} I^{\gamma,\rho} f(s) = f(s)$$

.

**Lemma 2.6.** *[11](Dhage fixed point theorem)*

*Consider a Banach algebra $X$ and let $\mathcal{S}$ be a non-empty, closed, convex, and bounded subset of $X$. Within this context, three operators $\mathcal{A}$, $\mathcal{G}$: $X \to X$, and $\mathcal{B}$: $\mathcal{S} \to X$ are introduced and satisfy the following conditions:*

- *(a) The operators $\mathcal{A}$ and $\mathcal{G}$ possess Lipschitz properties with Lipschitz constants δ and φ, respectively.*
- *(b) The operator $\mathcal{B}$ is both compact and continuous.*
- *(c) For any $x \in \mathcal{S}$, if $\upsilon = \mathcal{A}\upsilon\mathcal{B}x + \mathcal{G}\upsilon$, then $\upsilon$ belongs to the set $\mathcal{S}$.*
- *(d) The constants δ and φ are chosen such that $\delta M + \varphi < 1$, where $M = \| \mathcal{B}(\mathcal{S}) \|$.*

*Under these conditions, the equation $\upsilon = \mathcal{A}\upsilon\mathcal{B}\upsilon + \mathcal{G}\upsilon$ possesses at least one solution.*

## 3. Existence results

Prior to delving into the existence result, we establish a crucial auxiliary lemma. This lemma assumes a pivotal role in the process of converting the presented problem into a fixed-point formulation.

**Lemma 3.1.** *Given* $0 < \gamma < 1$ *and real constants a, b, and c with* $a + b \neq 0$*, consider an* $\chi \in L^1(J, \mathbb{R})$*. The* $\rho$*-Caputo fractional integro-differential boundary value problem, denoted as:*

$$\begin{cases} {}^{C}D^{\gamma,\rho}\Big[\frac{\upsilon(t)-\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(t,\upsilon(t))}{h(t,\upsilon(t))}\Big] = \chi(t), \quad t \in J = [0,T], 0 < \gamma < 1, \\ a\frac{\upsilon(0)}{h(0,\upsilon(0))} + b\frac{\upsilon(T)}{h(T,\upsilon(T))} = c. \end{cases} \tag{7}$$

*can be equivalently expressed as the integral equation*

$$\begin{aligned} \upsilon(t) &= h(t,\upsilon(t))\Big[\int_0^t \frac{\big(\rho(t)-\rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\chi(s)ds - \frac{1}{a+b}\Big(b\int_0^T \frac{(\rho(T)-\rho(s))^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\chi(s)ds \\ &\quad -c + \frac{b\sum\limits_{j=1}^{m} I^{\beta_i,\rho}\vartheta_i(T,\upsilon(T))}{h(T,\upsilon(T))}\Big)\Big] + \sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(t,\upsilon(t)), \quad t \in J. \end{aligned} \tag{8}$$

*Proof.* Let's consider an $\chi \in L^1(J, \mathbb{R})$. We want to demonstrate the equivalence between the $\rho$-Caputo fractional integro-differential boundary value problem given in (7) and the integral equation (8).

- **First Direction: From Problem to Equation** Let $\upsilon$ be a solution of the problem stated in (7). Applying the $\gamma$-Riemann-Liouville fractional integral operator $I^{\gamma,\rho}$ to both sides of the first equation in (7), we arrive at

$$\frac{\upsilon(t)}{h(t,\upsilon(t))} = I^{\gamma,\rho}\chi(t) + \frac{\upsilon(0)}{h(0,\upsilon(0))} + \frac{\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(t,\upsilon(t))}{h(t,\upsilon(t))}. \tag{9}$$

Consequently,

$$a\frac{\upsilon(0)}{h(0,\upsilon(0))} + b\frac{\upsilon(T)}{h(T,\upsilon(T))} = bI^{\gamma,\rho}\chi(T) + (a+b)\frac{\upsilon(0)}{h(0,\upsilon(0))} + \frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}.$$

Utilizing the nonlocal condition, we can deduce that

$$\frac{\upsilon(0)}{h(0,\upsilon(0))} = \frac{1}{a+b}\Big(c - bI^{\gamma,\rho}\chi(T) - \frac{b\sum\limits_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}\Big).$$

Substituting the value of $\frac{\upsilon(0)}{h(0,\upsilon(0))}$ into (8), we obtain

$$\begin{aligned} \frac{\upsilon(t)}{h(t,\upsilon(t))} &= \int_0^t \frac{\big(\rho(t)-\rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\chi(s)ds - \frac{1}{a+b}\Big(b\int_0^T \frac{(\rho(T)-\rho(s))^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\chi(s)ds \\ &\quad -c + \frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}\Big)\Big] + \frac{\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(t,\upsilon(t))}{h(t,\upsilon(t))}. \end{aligned}$$

This implies that $\upsilon(t)$ satisfies the integral equation (8).

- **Second Direction: From Equation to Problem**

  Now, let $\upsilon(t)$ be a solution of the integral equation (8). Upon applying the $\rho$-Caputo fractional derivative ${}^{C}D^{\gamma,\rho}$ to both sides of equation (8), we obtain the first equation of (7).

  Substituting $t = 0$ and $t = T$ into (8), we obtain:

$$\frac{\upsilon(0)}{h(0,\upsilon(0))} = \frac{-1}{a+b}\Big(\frac{b}{\Gamma(\gamma)}\int_0^T \Big(\rho(t)-\rho(s)\Big)^{\gamma-1}\rho'(s)\chi(s)ds - c + \frac{b\sum\limits_{j=1}^{m} I^{\beta_j,\rho}\vartheta_i(T,\upsilon(T))}{h(T,\upsilon(T))}\Big) + \frac{\sum\limits_{j=1}^{m} I^{\beta_j,\rho}\vartheta_i(0,\upsilon(0))}{h(0,y(0))},$$

and

$$\frac{\upsilon(T)}{h(T,\upsilon(T))} = \int_0^T \frac{\Big(\rho(t)-\rho(s)\Big)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\chi(s)ds - \frac{1}{a+b}\Big(\frac{b}{\Gamma(\gamma)}\int_0^T \Big(\rho(T)-\rho(s)\Big)^{\gamma-1}\rho'(s)\chi(s)ds - c + \frac{b\sum\limits_{j=1}^{m} I^{\beta_j,\rho}\vartheta_i(T,\upsilon(T))}{h(T,\upsilon(T))}\Big) + \frac{\sum\limits_{j=1}^{m} I^{\beta_j,\rho}\vartheta_i(T,\upsilon(T))}{h(T,\upsilon(T))}.$$

  Through straightforward calculations, we find that

$$a\frac{\upsilon(0)}{h(0,\upsilon(0))} + b\frac{\upsilon(T)}{h(T,\upsilon(T))} = c.$$

  Thus, the proof is complete.

□

We present our main result regarding the existence of solutions for problem (1). To proceed, we introduce the following assumptions:

**(H1)** TThe function $h : J \times \mathbb{R} \to \mathbb{R} \setminus 0$ and $\vartheta_j : J \times \mathbb{R} \to \mathbb{R}$, where $\vartheta_j(0,\upsilon(0)) = 0$ for $j = 1,2,...,m$, are continuous. Additionally, there exist two positive functions $\rho$ and $\varphi_j$ for $j = 1,2,..,m$, with bounds $\|\rho\|$ and $\|\varphi_j\|$, such that:

$$|h(t,\upsilon_1(t)) - h(t,\upsilon_2(t))| \leq \phi(t)|\upsilon_1(t) - \upsilon_2(t)|, \tag{10}$$

and

$$|\vartheta_j(t,\upsilon_1(t)) - \vartheta_j(t,\upsilon_2(t))| \leq \varphi_j(t)|\upsilon_1(t) - \upsilon_2(t)|, \quad j = 1,2,...,m, \tag{11}$$

hold for all $t \in J$ and $\upsilon_1, \upsilon_2 \in \mathbb{R}$.

**(H2)** A function $\theta \in L^1(J,\mathbb{R})$ exists such that:

$$|g(t,\upsilon(t))| \leq \theta(t) \quad \text{a.e. } t \in J, \tag{12}$$

for all $\upsilon \in \mathbb{R}$.

**(H3)** There exists a number $\varsigma > 1$ such that:

$$\varsigma \geq \frac{H_0\Delta + \dfrac{\sum_{j=1}^{m}\Theta_{0,j}(\rho(T)-\rho(0))^{\beta_j}}{\Gamma(\beta_j+1)}}{1 - \|\phi\|\Delta - \dfrac{\sum_{j=1}^{m}\|\varphi_j\|\Big(\rho(T)-\rho(0)\Big)^{\beta_j}}{\Gamma(\beta_j+1)}}. \tag{13}$$

Where:

$$\Delta = \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1}\frac{(\rho(T)-\rho(0))^{\gamma}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|} + \left|\frac{b\sum\limits_{j=1}^{m}\|\varphi_j\|\Big(\rho(T)-\rho(0)\Big)^{\beta_j}}{\Gamma(\beta_j+1)h(T,\upsilon(T))}\right|,$$

$H_0 = \sup_{t\in J}|h(t,0)|$ and $\Theta_{0,j} = \sup_{t\in J}|\vartheta_j(t,0)|,\ j=1,2,...,m.$

**Theorem 3.2.** *Under the assumptions* ***(H1)-(H3)****, If the inequality:*

$$\|\phi\|\Bigg[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1}\frac{(\rho(T)-\rho(0))^{\rho}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|}\left|\frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}\right|\Bigg] + \frac{\sum\limits_{j=1}^{m}\|\varphi_j\|\Big(\rho(T)-\rho(0)\Big)^{\beta_j}}{\Gamma(\beta_j+1)} < 1. \tag{14}$$

*holds, then the hybrid ρ-Caputo fractional integro-differential boundary value problem* (1) *possesses at least one solution on the interval J.*

*Proof.* We consider the following set

$$\mathcal{S} = \{\upsilon \in X;\ \|\upsilon\| \le \varsigma\},$$

where $\varsigma$ satisfies the inequality (13).

It's evident that $\mathcal{S}$ constitutes a nonempty, closed, convex, and bounded subset of the space $X$. In order to establish our result, we introduce the operators $\mathcal{A},\mathcal{G} : X \to X$, and $\mathcal{B} : \mathcal{S} \to X$ as defined below:

$$\mathcal{A}\upsilon(t) = h(t,\upsilon(t));\quad t\in J,$$

$$\begin{aligned}\mathcal{B}\upsilon(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t \Big(\rho(t)-\rho(s)\Big)^{\gamma-1}\rho'(s)g(s,\upsilon(s))ds \\ &-\frac{1}{a+b}\Big(\frac{b}{\Gamma(\gamma)}\int_0^T \Big(\rho(T)-\rho(s)\Big)^{\gamma-1}\rho'(s)g(s,\upsilon(s))ds - c + \frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}\Big);\quad t\in J,\end{aligned}$$

$$\mathcal{G}\upsilon(t) = \sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(t,\upsilon(t)) = \sum_{j=1}^{m}\int_0^t \frac{\Big(\rho(t)-\rho(s)\Big)^{\beta_j-1}}{\Gamma(\beta_j)}\rho'(s)\vartheta_j(s,\upsilon(s))ds;\quad t\in J.$$

Now, we will demonstrate that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{G}$ meet the conditions of Lemma 2.6. To achieve this, we will break down the proof into several steps.

(i) *$\mathcal{A}$ and $\mathcal{G}$ are Lipschitzian on X.*
Let $\upsilon_1,\upsilon_2\in X$. According to assumption (H1), we have:

$$\begin{aligned}|\mathcal{A}\upsilon_1(t) - \mathcal{A}\upsilon_2(t)| &= |h(t,\upsilon_1(t)) - h(t,\upsilon_2(t))| \\ &\le |\phi(t)||y_1(t) - y_2(t)| \\ &\le \|\phi\|\|\upsilon_1-\upsilon_2\|,\end{aligned}$$

which, upon taking the norm for $t\in J$, leads to:

$$\|\mathcal{A}\upsilon_1 - \mathcal{A}\upsilon_2\| \le \|\phi\|\|\upsilon_1-\upsilon_2\|,\ \text{for all}\ \upsilon_1,\upsilon_2\in X.$$

Consequently, $\mathcal{A}$ is Lipschitz continuous on $X$ with a Lipschitz constant of $\|\phi\|$.

Continuing in a similar manner, consider $\upsilon_1$ and $\upsilon_2$ in $X$. We observe:

$$\begin{aligned}|\mathcal{G}\upsilon_1(t)-\mathcal{G}\upsilon_2(t)| &= \left|\sum_{j=1}^{m} I^{\beta_j,\rho}h_i(t,\upsilon(t))-\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_i(t,\upsilon(t))\right| \\ &\leq \sum_{j=1}^{m}\int_0^t \left|\frac{\left(\rho(t)-\rho(s)\right)^{\beta_j-1}}{\Gamma(\beta_j)}\rho'(s)\right| |\varphi_j(s)||\upsilon_1(s)-\upsilon_2(s)|ds \\ &\leq \sum_{j=1}^{m}\frac{\|\varphi_j\|}{\Gamma(\beta_j+1)}\left(\rho(T)-\rho(0)\right)^{\beta_j}\|\upsilon_1-\upsilon_2\|,\end{aligned}$$

which implies:

$$\|\mathcal{G}\upsilon_1-\mathcal{G}\upsilon_2\| \leq \sum_{j=1}^{m}\frac{\|\varphi_{\|}\left(\rho(T)-\rho(0)\right)^{\beta_j}}{\Gamma(\beta_j+1)}\|\upsilon_1-\upsilon_2\|.$$

Hence, we conclude that $\mathcal{G}$ is Lipschitz continuous with a Lipschitz constant of $\sum_{j=1}^{m}\frac{\|\varphi_j\|\left(\rho(T)-\rho(0)\right)^{\beta_j}}{\Gamma(\beta_j+1)}$.

(ii) *The operator $\mathcal{B}$ continuous on $X$.*
We first show that $\mathcal{G}$ is continuous. Let $\upsilon_n$ be a sequence in $\mathcal{S}$ converging to $\upsilon \in \mathcal{S}$. Due to the continuity of $g$, it's easy to see that $g(s,\upsilon_n(s)) \to g(s,\upsilon(s))$ as $n \to +\infty$. By using **(H2)**, we can conclude that the expression

$$\left|\frac{\left(\rho(t)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon_n(s))\right| \leq \Delta\theta(s),\ s\in J,$$

where $\Delta = \sup_{s\in J}\left|\frac{\left(\rho(t)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)\right|$.
Then, by the Lebesgue dominated convergence theorem, we have:

$$\lim_{n\to+\infty}\int_0^t \frac{\left(\rho(t)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon_n(s))ds = \int_0^t \frac{\left(\rho(t)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon(s))ds,$$

and similarly:

$$\lim_{n\to+\infty}\int_0^T \frac{b\left(\rho(T)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon_n(s))ds = \int_0^T \frac{b\left(\rho(T)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon(s))ds.$$

Thus, we conclude that: $\lim_{n\to+\infty}\mathcal{B}\upsilon_n = \mathcal{B}\upsilon$. which implies the continuity of $\mathcal{B}$. Next, let's prove that $\mathcal{B}(\mathcal{S})$ is uniformly bounded on $X$. Take $\upsilon \in \mathcal{S}$:

$$\begin{aligned}&|\mathcal{B}\upsilon(t)| \\ &= \left|\int_0^t \frac{\left(\rho(t)-\rho(s)\right)^{\gamma-1}}{\Gamma(\gamma)}\rho'(s)g(s,\upsilon(s))ds\right. \\ &\left.-\frac{1}{a+b}\Big(\frac{b}{\Gamma(\gamma)}\int_0^T\left(\rho(T)-\rho(s)\right)^{\gamma-1}\rho'(s)g(s,\upsilon(s))ds - c + \frac{b\sum_{j=1}^{m} I^{\beta_j,\gamma}\vartheta_j(T,\upsilon(T))}{h(T,\upsilon(T))}\Big)\right| \\ &\leq \|\theta\|_{L^1}\frac{\left(\rho(T)-\rho(0)\right)^{\gamma}}{\Gamma(\gamma+1)}\Big(1+\frac{|b|}{|a+b|}\Big)+\frac{|c|}{|a+b|}+\left|\frac{b\sum_{j=1}^{m} I^{\beta_j,\gamma}\vartheta_j(T,\upsilon(T))}{(a+b)h(T,\upsilon(T))}\right|,\end{aligned}$$

for all $t \in J$.
Let's define:

$$M = \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1} \frac{\big(\rho(T) - \rho(0)\big)^{\gamma}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|} + \left|\frac{b\sum_{j=1}^{m} I^{\beta_i,\rho} h_(T, v(T))}{f(T, v(T))}\right|,$$

which ensures that $\|\mathcal{B}v\| \leq M$ for $v \in \mathcal{S}$. This shows that $\mathcal{B}(\mathcal{S})$ is uniformly bounded. Now, we show that $\mathcal{B}(S)$ is equicontinuous on $X$.
Consider $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $v \in S$. We can analyze the difference:

$$\begin{aligned}
| \quad & \mathcal{B}v(\tau_2) - \mathcal{B}v(\tau_1)| \\
&= \left|\int_0^{\tau_2} \frac{\big(\rho(\tau_2) - \rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)} \rho'(s) g(s, v(s)) ds - \int_0^{\tau_1} \frac{\big(\rho(\tau_1) - \rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)} \rho'(s) g(s, v(s)) ds\right| \\
&\leq \|\theta\|_{L^1} \int_0^{\tau_1} \left|\frac{\big(\rho(\tau_2) - \rho(s)\big)^{\gamma-1} - \big(\rho(\tau_1) - \rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)} \rho'(s)\right| ds \\
&+ \|\theta\|_{L^1} \int_0^{\tau_2} \left|\frac{\big(\rho(\tau_2) - \rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)} \rho'(s) ds\right|, \\
&\leq \|\theta\|_{L^1} \frac{\big(\rho(\tau_2) - \rho(\tau_1)\big)^{\gamma}}{\Gamma(\gamma+1)}.
\end{aligned}$$

As this is independent of $v \in S$ and since $\tau_1$ approaches $\tau_2$, the right-hand side tends to zero. This result, coupled with the Arzelà-Ascoli theorem, confirms that $\mathcal{S}$ is completely continuous on $\mathcal{S}$.

(iii) *The hypothesis* (*c*) *of lemma 2.6 is satisfied.*
Assume $v \in X$ and $x \in S$ are such that $v = \mathcal{A}v\mathcal{B}x + \mathcal{G}v$. Then, we have:

$$\begin{aligned}
|v(t)| \quad & \leq |h(t, v(t))| \Bigg| \Big[ \int_0^t \frac{\big(\rho(t) - \rho(s)\big)^{\gamma-1}}{\Gamma(\gamma)} \rho'(s) g(s, x(s)) ds \\
& + \frac{1}{a+b}\Big(\frac{b}{\Gamma(\gamma)} \int_0^T \big(\rho(T) - \rho(s)\big)^{\gamma-1} \rho'(s) g(s, x(s)) ds + c\Big)\Big]\Bigg| + \left|\sum_{j=1}^{m} I^{\beta_j,\rho} \vartheta_i(T, v(T))\right|.
\end{aligned}$$

Since

$$|h(t, v(t))| \leq |h(t, v(t)) - h(t, 0)| + |h(t, 0)|,$$

and

$$\left|\vartheta_j(t, v(t))\right| \leq \left|\vartheta_j(t, v(t)) - \vartheta_j(t, 0)\right| + \left|\vartheta_j(t, 0)\right|.$$

Thus,

$$\begin{aligned}
|v(t)| \quad & \leq \big(\varsigma\|\phi\| + H_0\big)\Big[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1} \frac{\big(\rho(T) - \rho(0)\big)^{\gamma}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|} \\
& + \left|\frac{b\sum_{j=1}^{m} I^{\beta_j,\rho} \vartheta_j(T, v(T))}{h(T, v(T))}\right|\Big] + \sum_{j=1}^{m} \big(\varsigma\|\varphi_j\| + \theta_{0,j}\big)\Big(\frac{\big(\rho(T) - \rho(s)\big)^{\beta_j}}{\Gamma(\beta_j+1)}\Big).
\end{aligned}$$

Referring to (13), it's clear that $\|v\| \leq \varsigma$. Hence, $v \in \mathcal{S}$, and hypothesis (*c*) is satisfied.

(iv) $\delta M + \rho < 1$. We have

$$|\mathcal{B}v(t)| \leq \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1}\frac{\big(\rho(T)-\rho(0)\big)^{\gamma}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|} + \left|\frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,v(T))}{h(T,v(T))}\right|.$$

So

$$\|\mathcal{B}v\| \leq \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|\theta\|_{L^1}\frac{\big(\rho(T)-\rho(0)\big)^{\gamma}}{\Gamma(\gamma+1)}\Big) + \frac{|c|}{|a+b|} + \left|\frac{b\sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(T,v(T))}{h(T,v(T))}\right|.$$

Using the condition (14), it's possible to deduce that $\delta M + \rho < 1$, where $\rho = \sum_{j=1}^{m}\frac{\big(\rho(T)-\rho(0)\big)^{\beta_j}}{\Gamma(\beta_j+1)}\|\varphi_j\|$ and $\delta = \|\phi\|$.
With all these assumptions satisfied, the conditions of Dhage's fixed point theorem are met. Therefore, by the conclusion of Dhage's fixed point theorem, there exists a point $v \in S$ such that $v = \mathcal{A}v\mathcal{B}v + \mathcal{G}v$, which corresponds to a solution of problem (1) on $J$. This concludes the proof.

□

## 4. Fractional hybrid integro-differential inequalities

This section is dedicated to examining a crucial outcome related to strict inequalities within the framework of our fractional hybrid integro-differential problem. Subsequently, we initiate our discussion with the following definition:

**Definition 4.1.** *[5] We define a function $r : [0,T] \to \mathbb{R}$ to be an element of the set $C_a([0,T],\mathbb{R})$ if it is continuous and adheres to the condition $s^a r(s) \in C([0,T],\mathbb{R})$.*

Functions in the class $C_a([0,T],\mathbb{R})$ exhibit the property described in the following lemma.

**Lemma 4.2.** *[5] Consider $r \in C_a([0,T],\mathbb{R})$ and suppose that for any $s_1 \in (0,+\infty)$, the conditions $r(s_1) = 0$ and $r(s) \leq 0$ for any $s \in (0,s_1)$ hold. Then, we have*

$$^{C}D^{\gamma,\rho} r(s_1) \geq 0.$$

In what follows, we will require the following assumption.

**(C1)** The mapping $x \mapsto \dfrac{x}{h(s,x)}$ is almost everywhere increasing in $\mathbb{R}$ for $\in [0,T]$.

**(C2)** The mapping $x \mapsto \dfrac{\sum_{j=1}^{m}\vartheta_j(s,x)}{h(s,x)}$ is almost everywhere decreasing in $\mathbb{R}$ for $\in [0,T]$.

Now, we present our initial outcome regarding fractional inequalities for the problem (1).

**Theorem 4.3.** *Given that the conditions **(C1)-(C2)** are satisfied, and there exist functions $\kappa_1$ and $\kappa_2$ from $C_a([0,T],\mathbb{R})$ such that:*

$$(i)\ {}^{C}D^{\gamma,\rho}\Big[\frac{\kappa_1(s) - \sum_{j=1}^{m} I^{\beta_i,\rho}\vartheta_j(s,\kappa_1(s))}{h(s,\kappa_1(s))}\Big] \leq g(s,\kappa_1(s)) \ \ s \in J = [0,T],$$
$$(ii)\ {}^{C}D^{\gamma,\rho}\Big[\frac{\kappa_2(s) - \sum_{j=1}^{m} I^{\beta_i,\rho}\vartheta_j(s,\kappa_2(s))}{h(s,\kappa_2(s))}\Big] \geq g(s,\kappa_2(s)) \ \ s \in J = [0,T].$$

*With one of the inequality being strict. Additionally, $\kappa_1^0 < \kappa_2^0$ implies that*

$$\kappa_1(s) < \kappa_2(s), \textit{ for all } s \in [0,T],$$

*where $\kappa_i^0 = s^{1-a}\kappa_i(s)|_{s=0}$, $i = 1,2$.*

*Proof.* Assuming the conclusion is false and the inequality (*ii*) is strict, let's explore this situation further. Since $\kappa_1$ and $\kappa_2$ belong to $C_a([0,T],\mathbb{R})$ and $\kappa_1^0 < \kappa_2^0$, there exists a point $s_1$ within the interval $[0,T]$ where the following conditions hold:

$$\kappa_1(s_1) - \kappa_2(s_1) = 0 \text{ and } \kappa_1(s) - \kappa_2(s) < 0,\ \forall s \in [0,s_1].$$

Consider the functions:

$$Z_i(s) = \frac{\kappa_i(s) - \sum_{j=1}^m I^{\beta_i,\rho}\vartheta_j(s,\kappa_i(s))}{h(s,\kappa_i(s))},\ i = 1,2.$$

These functions satisfy $Z_1(s_1) = Z_2(s_1)$ and due to conditions **(C1)-(C2)** , we have $Z_1(s) < Z_2(s)$ for any $s \in [0,s_1)$. Now, let's define $r(s) = Z_1(s) - Z_2(s)$ for $s \in [0,s_1]$. This function satisfies $r(s_1) = 0$ and $r(s) < 0$ for all $s \in [0,s_1)$, and it also belongs to $C_a([0,T],\mathbb{R})$. By applying lemma 4.2, we can deduce that ${}^C D^{\gamma,\rho} r(s_1) \geq 0$. Consequently, we have:

$$g(s_1,\kappa_1(s_1)) \geq^C D^{\gamma,\rho} Z_1(s_1) \geq^C D^{\gamma,\rho} Z_2(s_1) > g(s_1,\kappa_2(s_1)).$$

This contradicts the assumption that $\kappa_1(s_1) = \kappa_2(s_1)$. Therefore, the conclusion is true. □

**Theorem 4.4.** *Given the conditions **(C1)-(C2)** and taking into consideration the real numbers a, b, and c such that $a + b \neq 0$, let's assume the existence of functions $\kappa_1$ and $\kappa_2$ belonging to $C_a([0,T],\mathbb{R})$ satisfying the following conditions:*

$$(i)\ {}^C D^{\gamma,\rho}\Big[\frac{\kappa_1(s) - \sum_{j=1}^m I^{\beta_i,\rho}\vartheta_j(s,\kappa_1(s))}{h(s,\kappa_1(s))}\Big] \leq g(s,\kappa_1(s))\ \ s \in J = [0,T],$$
$$(ii)\ {}^C D^{\gamma,\rho}\Big[\frac{\kappa_2(s) - \sum_{j=1}^m I^{\beta_i,\rho}\vartheta_j(s,\kappa_2(s))}{h(s,\kappa_2(s))}\Big] \geq g(s,\kappa_2(s))\ \ s \in J = [0,T].$$

*With one of the inequality being strict. Moreover, if $a > 0$, $b < 0$, and $\kappa_1(T) < \kappa_2(T)$, then the inequality:*

$$a\frac{\kappa_1(0)}{h(0,\kappa_1(0))} + b\frac{\kappa_1(T)}{h(T,\kappa_1(T))} < a\frac{\kappa_2(0)}{h(0,\kappa_2(0))} + b\frac{\kappa_2(T)}{h(T,\kappa_2(T))}$$

*implies that*

$$\kappa_1(s) < \kappa_2(s), \text{ for any } s \in [0,T].$$

*Proof.* Starting from the inequality:

$$a\frac{\kappa_1(0)}{h(0,\kappa_1(0))} + b\frac{\kappa_1(T)}{h(T,\kappa_1(T))} < a\frac{\kappa_2(0)}{h(0,\kappa_2(0))} + b\frac{\kappa_2(T)}{h(T,\kappa_2(T))},$$

we derive

$$a\Big(\frac{\kappa_1(0)}{h(0,\kappa_1(0))} - \frac{\kappa_2(0)}{h(0,\kappa_2(0))}\Big) < b\Big(\frac{\kappa_2(T)}{h(T,\kappa_2(T))} - \frac{\kappa_1(T)}{h(T,\kappa_1(T))}\Big).$$

Since $b < 0$ and $\kappa_1(T) < \kappa_2(T)$, in accordance with the condition **(C1)**, we deduce that: $\Big(\frac{\kappa_2(T)}{h(T,\kappa_2(T))} - \frac{\kappa_1(T)}{h(T,\kappa_1(T))}\Big) > 0$. This leads to the conclusion that $\kappa_1(0) < \kappa_2(0)$, supported by the fact that $a > 0$ and the condition **(C)**. Subsequently, by applying Theorem 4.4, we attain the desired result. □

**Theorem 4.5.** *Assuming that the conditions of Theorem 4.4 are met and there exists a positive constant $\Lambda$ satisfying:*

$$\begin{aligned} g(s,u) - g(s,v) \ &\leq \frac{\Lambda}{1 + (\rho(s) - \rho(0))^{\gamma+1}}\Big(\frac{u(s) - \sum_{j=1}^m I^{\beta_i,\rho}\vartheta_j(s,u(s))}{h(s,u(s))} \\ &- \frac{v(s) - \sum_{j=1}^m I^{\beta_i,\rho}\vartheta_j(s,v(s))}{h(s,v(s))}\Big)\ a.\ e.\ s \in J = [0,T]. \end{aligned} \tag{15}$$

*For all $u, v \in \mathbb{R}$ such that $u \geq v$. If the following inequality holds:*

$$a\frac{\kappa_1(0)}{h(0,\kappa_1(0))} + b\frac{\kappa_1(T)}{h(T,\kappa_1(T))} < a\frac{\kappa_2(0)}{h(0,\kappa_2(0))} + b\frac{\kappa_2(T)}{h(T,\kappa_2(T))}$$

*provided that $\Lambda \leq \Gamma(1+\gamma)\Big(\rho(s)-\rho(0)\Big)^{2\gamma}$,then it follows that:*

$$\kappa_1(s) < \kappa_2(s) \text{ for any } s \in [0, T].$$

*Proof.* Let us consider a small positive value $\epsilon > 0$ and define:

$$\frac{\kappa_{2,\epsilon}(s)}{h(s,\kappa_{2,\epsilon}(s))} = \frac{\kappa_2(s)}{h(s,\kappa_2(s))} + \epsilon\Big(1 + (\rho(s)-\rho(0))^{\gamma+1}\Big)$$

Additionally, let's introduce:

$$Z_{2,\epsilon}(s) = \frac{\kappa_{2,\epsilon}(s) - \sum_{j=1}^{m} I^{\beta_j,\rho}\vartheta_j(s,\kappa_{2,\epsilon}(s))}{h(s,\kappa_{2,\epsilon}(s))}.$$

Because $\frac{\kappa_{2,\epsilon}(s)}{h(s,\kappa_{2,\epsilon}(s))} > \frac{\kappa_2(s)}{h(s,\kappa_2(s))}$, we can infer that $\kappa_{2,\epsilon}(s) > \kappa_2(s)$. By employing inequality (15) along with the condition:

$$^{C}D^{\gamma,\rho}\Big[\frac{\kappa_2(s) - \sum_{j=1}^{m} I^{\beta_i,\rho}\vartheta_j(s,\kappa_2(s))}{h(s,\kappa_1(s))}\Big] \geq g(s,\kappa_2(s)) \quad s \in J = [0,T],$$

we can derive the following sequence of inequalities:

$$\begin{aligned} {}^{C}D^{\gamma,\rho}Z_{2,\epsilon}(s) &= ^{C} D^{\gamma,\rho}Z_2(s) + \epsilon\Gamma(1+\gamma)\Big(\rho(s)-\rho(0)\Big)^{\gamma+1} \\ &\geq g(s,\kappa_{2,\epsilon}(s)) - \frac{\Lambda}{1+(\rho(s)-\rho(0))^{\gamma+1}}\Big(Z_{2,\epsilon}(s) - Z_2(s)\Big) + \epsilon\Gamma(1+\gamma)\Big(\rho(s)-\rho(0)\Big)^{\gamma+1} \\ &\geq g(s,\kappa_2(s)) + \epsilon\Big(\Gamma(1+\gamma)(\rho(s)-\rho(0))^{2\gamma} - \Lambda\Big) \\ &> g(s,\kappa_2(s)). \end{aligned}$$

Furthermore, since we have $\kappa_{2,\epsilon}(0) > \kappa_2(0) \geq \kappa_1(0)$, we can apply theorem 4.4 to conclude that $\kappa_{2,\epsilon}(s) > \kappa_1(s)$ for any $s \in [0,T]$. Taking the limit as $\epsilon$ approaches 0, we deduce that $\kappa_2(s) \geq \kappa_1(s)$ for all $s \in [0,T]$, which completes the proof. □

## 5. conclusion

This study not only advances our comprehension of hybrid $\rho$-Caputo fractional integro-differential equations but also lays the groundwork for future investigations and applications in diverse areas of mathematics and its applications.

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