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Existence of solutions for a class of boundary value problems for weighted $p(t)$-Laplacian impulsive systems

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Abstract. In this paper we investigate a class of boundary value problems for weighted $p(t)$-Laplacian impulsive systems. We prove existence of at least one solution and existence of at least one nonnegative solution. The arguments are based upon of recent theoretical results on the fixed point theory for the sum of operators.

1. Introduction

In this paper, we investigate the weighted $p(t)$-Laplacian system

$$-\left(w(t)|x'(t)|^{p(t)-2}x'(t)\right)' + f\left(t, x(t), (w(t))^{\frac{1}{p(t)-1}}x'(t)\right) = 0, \quad t \in (0,T),\ t \neq t_j, \tag{1}$$

where $x : [0,T] \to \mathbb{R}^N$, $N \geq 1$, with the following impulsive boundary conditions

$$x(t_j^+) - x(t_j) = A_j\left(t_j, x(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}}x'(t_j)\right), \quad j \in \{1, \ldots, k\}, \tag{2}$$

$$\begin{aligned} w(t_j)|x'(t_j^+)|^{p(t_j)-2}x'(t_j^+) &= w(t_j)|x'(t_j)|^{p(t_j)-2}x'(t_j) \\ &\quad + B_j\left(t_j, x(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}}x'(t_j)\right),\ j \in \{1, \ldots, k\}, \end{aligned} \tag{3}$$

$$ax(0) - b(w(0))^{\frac{1}{p(0)-1}}x'(0) = 0, \tag{4}$$

$$cx(T) + dw(T)|x'(T)|^{p(T)-2}x'(T) = 0, \tag{5}$$

where

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(H1) $f \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$,

$$|f(t,x,y)| \leq a_1(t) + a_2(t)|x|^{p_1} + a_3(t)|y|^{p_2}, \quad t \in [0,T], \quad x, y \in \mathbb{R}^N,$$

$a_1, a_2, a_3 \in C([0,T])$, $0 \leq a_1, a_2, a_3 \leq B$ on $[0,T]$ for some constant $B > 1$, $p_1, p_2 \geq 0$.

(H2) $p \in C([0,T])$, $w \in C^1([0,T])$, $p > 1$, $w > 0$ on $[0,T]$,

$$w(t) \leq B, \quad p(t) \leq B, \quad (w(t))^{\frac{1}{p(t)-1}} \leq B, \quad t \in [0,T],$$

$a, b, c, d \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots, t_k < t_{k+1} = T$, $k \in \mathbb{N}$.

(H3) $A_j \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$,

$$|A_j(t,x,y)| \leq a_{1j}(t) + a_{2j}(t)|x|^{p_{1j}} + a_{3j}(t)|y|^{p_{2j}}, \quad j \in \{1, \ldots, k\},$$

$0 \leq a_{1j}, a_{2j}, a_{3j} \leq B$ on $[0,T]$, $p_{1j}, p_{2j} \geq 0$, $j \in \{1, \ldots, k\}$.

(H4) $B_j \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$,

$$|B_j(t,x,y)| \leq b_{1j}(t) + b_{2j}(t)|x|^{q_{1j}} + b_{3j}(t)|y|^{q_{2j}}, \quad j \in \{1, \ldots, k\},$$

$0 \leq b_{1j}, b_{2j}, b_{3j} \leq B$ on $[0,T]$, $q_{1j} \geq 0$, $q_{2j} \geq 0$, $j \in \{1, \ldots, k\}$.

Here, for a function $x : [0,T] \to \mathbb{R}^N$, denote

$$|x(t)| = \max_{i \in \{1,\ldots,N\}} |x_i(t)|, \quad t \in [0,T],$$

$x(t_j^+) = \lim_{t \to t_j^+} x(t)$, $x(t_j^-) = \lim_{t \to t_j^-} x(t)$, $j \in \{1, \ldots, k\}$. For $l \in \mathbb{N}$, define

$$\begin{aligned} PC^l([0,T]) = \Big\{ & g : [0,T] \to \mathbb{R}^N, \quad g \in C^l([0,T] \backslash \{t_j\}_{j=1}^k), \\ & g^{(i)}(t_j^-),\ g^{(i)}(t_j^+) \text{ exist} \quad \text{and} \quad g^{(i)}(t_j^-) = g^{(i)}(t_j), \\ & j \in \{1, \ldots, k\}, \quad i \in \{0, \ldots, l\} \Big\}. \end{aligned}$$

In $X_1 = PC^2([0,T])$, define a norm

$$\|x\|_1 = \max \left\{ \max_{j \in \{0,1,\ldots,k\}} \sup_{t \in (t_j, t_{j+1})} |x(t)|, \quad \max_{j \in \{0,1,\ldots,k\}} \sup_{t \in (t_j, t_{j+1})} |x'(t)|, \max_{j \in \{0,1,\ldots,k\}} \sup_{t \in (t_j, t_{j+1})} |x''(t)| \right\},$$

provided it exists. Note that the space $(X_1, \|\cdot\|_1)$ is a Banach space. We will investigate the problem (1)-(5) for existence of solutions.

In recent years, many researchers in mathematics have been interested in the study of differential equations with nonstandard $p(t)$-growth conditions. This is due to issues originating from nonlinear elasticity theory, electro-rheological fluids, image processing, and so forth (see [1, 6, 8, 13, 15]).
It arises some difficulties with such equations is when $p(t)$ is a general function, the Laplacian problem represents nonhomogeneity and possesses more nonlinearity, leading to additional complication. In fact, the majority of results in the well-known p-Laplacian problem (when p is a constant) cannot be generalized.

Many evolution processes are characterized by the fact that at certain moments of time sudden discontinuous jumps occur. These processes are subject to short-term perturbations whose duration is negligible compared to the process total duration. Thus, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. Recently, there has been a surge in interest in the study of impulsive differential equations, as these equations provide a natural framework for mathematical modeling of a wide range of real-world problems, in medical fields, optimal control models in economics, impact mechanics and inspection process in operations research and so forth (see [3, 10, 16] and the references therein). It is worth to note that $p(t)$-Laplacian impulsive boundary problems have relatively new applications such as ecological competition, respiratory dynamics, and vaccination strategies.

Over the last years, there are many works devoted to the existence of solutions to the Laplacian impulsive differential equation subject to boundary conditions, see for example [14, 17, 18]. The methods include sub-super-solution method, fixed point theorem, monotone iterative method, variational method, critical point theory, coincidence degree, the Leray-Shauder degree and so forth. However, results on the existence of solutions of boundary value problems for p-Laplacian or $p(t)$-Laplacian impulsive differential equations remain rare due to the nonlinearity of $-\Delta_p$ and $-\Delta_{p(t)}$ (see [5, 9, 20, 21]).

In [19] the problem (1)-(5) is investigated under the following conditions

(G1) $w \in PC([0,T])$, $0 < w(t)$, $t \in [0,T]\backslash\{t_j\}_{j=1}^{k}$, $(w(\cdot))^{-\frac{1}{p(\cdot)-1}} \in L^1((0,T))$.

(G2) $f : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory,

$$\lim_{|u|+|v|\to\infty} \frac{f(r,u,v)}{(|u|+|v|)^{\beta(r)-1}} = 0, \quad r \in [0,T],$$

uniformly, $\beta \in C([0,T])$, $1 < \beta^- \le \beta^+ < p^-$.

(G3) $a > 0$, $ad + bc > 0$, $b, c, d \ge 0$.

(G4) $\sum_{i=1}^{k} |A_i(u,v)| \le C_1(|u|+|v|)^{\theta}$ when $|u|+|v|$ is large enough, $0 < \theta < \frac{p^- -1}{p^+ -1}$.

(G5) $\sum_{i=1}^{k} |B_i(u,v)| \le C_2(|u|+|v|)^{\varepsilon}$ when $|u|+|v|$ is large enough, $0 \le \varepsilon < \beta^+ - 1$,

and it is proved existence of at least one solution of the problem (1)-(5). Here

$$z^- = \min_{t\in[0,T]} z(t), \quad z^+ = \max_{t\in[0,T]} z(t).$$

Note that our p_1, p_2, p_{1j}, p_{2j}, q_{1j} and q_{2j} do not depend on the function p, while θ and β in [19] depend on the function p. Moreover, in our paper a, b, c and d admit negative values, while $a > 0$ and $b, c, d \ge 0$ in [19]. Thus, we can consider the results in this paper as complimentary results of the results in [19].
The method used in [19] is the Gaines and Mawhin coincidence degree theory.

In this paper, we investigate, under fairly simple assumptions, two existence criteria of solutions for the nonlinear BVP for weighted $p(t)$-Laplacian impulsive system (1)-(5). A general polynomial growth is assumed for the right-hand side term. Still, it is worth mentioning that the case when the nonlinearity exhibits both a sublinear and superlinear terms seems to be more complex. To prove our main results we propose a new approach based upon recent theoretical results on the fixed point theorem for sum of two operators presented in Section 2. These theoretical results can be used to study other classes of BVPs for impulsive ordinary differential equations as well as for impulsive partial differential equations. The example corroborates the results given by each existence criteria.

This paper is organized as follows. In the next section, we give some auxiliary and preliminary results. In Section 3, we prove existence of solutions and nonnegative solutions for the problem (1)-(5) (Theorems 3.6 and 3.7). In Section 4, we give an example that illustrates our main results.

2. Auxiliary Results

In this section, some definitions and results related to fixed points for sum of two operators will be given. We will start withe following useful definition.

Definition 2.1. *Let (X, d) be a metric space and D be a subset of X. The mapping $T : D \to X$ is said to be expansive if there exists a constant $h > 1$ such that*

$$d(Tx, Ty) \geq h\, d(x, y), \quad \forall\, x, y \in D.$$

Let E be a real Banach space.

Definition 2.2. *Let $K : M \subset E \to E$ be a mapping.*

1. *K is said to be compact if $K(M)$ is contained in a compact subset of E.*
2. *K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

Definition 2.3. *A closed, convex set $\mathcal{P}$ in E is said to be cone if*

1. *$\beta x \in \mathcal{P}$ for any $\beta \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

To prove our first existence result we will use the following fixed point theorem which is a consequence of Leray-Schauder nonlinear alternative [2].

Theorem 2.4. *Let E be a Banach space, Y a closed convex subset of E containing 0 and*

$$U = \{x \in Y \,:\, \|x\| < R\},$$

with $R > 0$. Consider two operators T and $S : \overline{U} \to E$, where

$$Tx = \varepsilon\, x,\ x \in \overline{U},$$

for $\varepsilon > 0$, such that

(i) *$I - S : \overline{U} \to Y$ continuous, compact and*

(ii) $\{x \in Y : x = \lambda(I - S)x, \quad \|x\| = R\} = \emptyset$, *for any* $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Proof. From the hypothesis (i), the operator $\frac{1}{\varepsilon}(I - S) : \overline{U} \to Y$ is continuous and compact.
Suppose that there exist $x_0 \in \partial U$ and $\mu_0 \in (0, 1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - S)x_0,$$

that is

$$x_0 = \lambda_0\, (I - S)x_0,$$

where $\lambda_0 = \frac{\mu_0}{\varepsilon} \in \left(0, \frac{1}{\varepsilon}\right)$. This contradicts the condition (ii). From the Leray-Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{U}$ such that

$$x^* = \frac{1}{\varepsilon}(I - S)x^*,$$

or

$$\varepsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof. □

In the sequel, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, Ω is a subset of $\mathcal{P}$, and U is a bounded open subset of $\mathcal{P}$. Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$.

Assume that $S : \overline{U} \to E$ is a completely continuous mapping and $T : \Omega \to E$ is an expansive one with constant $h > 1$. Thus, the operator $(I - T)^{-1}$ is $(h - 1)^{-1}$-Lipschtzian on $(I - T)(\Omega)$. Suppose that

$$S(\overline{U}) \subset (I - T)(\Omega), \tag{6}$$

and

$$x \neq Tx + Sx, \text{ for all } x \in \partial U \cap \Omega. \tag{7}$$

Then $x \neq (I-T)^{-1}Sx$, for all $x \in \partial U$ and the mapping $(I-T)^{-1}S : \overline{U} \to \mathcal{P}$ is a completely continuous mapping. From [12, Theorem 2.3.1], the fixed point index $i\,((I - T)^{-1}S, U, \mathcal{P})$ is well defined. Thus, we put

$$i_*\,(T + S, U \cap \Omega, \mathcal{P}) = \begin{cases} i\,((I - T)^{-1}S, U, \mathcal{P}), & \text{if } U \cap \Omega \neq \emptyset \\ 0, & \text{if } U \cap \Omega = \emptyset. \end{cases} \tag{8}$$

The basic properties of the index i_* are collected in [7, Theorem 2.3]. For more details of this index see [7] and [11].

The Theorem 2.6 will be used to prove Theorem 3.7. Its proof is based on properties of the index i_* given in the following lemma.

Lemma 2.5. *Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U} \to E$ is a completely continuous mapping and $S(\overline{U}) \subset (I - T)(\Omega)$. Suppose that $T + S$ has no fixed point on $\partial U \cap \Omega$. Then we have the following results:*

(1) *If $0 \in U$ and there exists $\varepsilon > 0$ small enough such that*

$$Sx \neq (I - T)(\lambda x) \text{ for all } \lambda \geq 1 + \varepsilon,\ x \in \partial U \text{ and } \lambda x \in \Omega,$$

then the fixed point index $i_*\,(T + S, U \cap \Omega, \mathcal{P}) = 1$.

(2) *If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Sx \neq (I - T)(x - \lambda u_0), \text{ for all } \lambda > 0 \text{ and } x \in \partial U \cap (\Omega + \lambda u_0),$$

then the fixed point index $i_*\,(T + S, U \cap \Omega, \mathcal{P}) = 0$.

Proof. **(1)** The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{P}$ is completely continuous without fixed point on ∂U and it is readily seen that the following condition is satisfied

$$(I - T)^{-1}Sx \neq \lambda x \quad \text{for all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon.$$

Then, our claim follows from the definition of i_* and [4, Lemma 2.3]

(2) See [7, Proposition 2.16].
□

Theorem 2.6. *Let E be a Banach space, $\mathcal{P} \subset E$ a cone, U_1 and U_2 two open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2$ and $0 \in U_2$. Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_2 \to E$ is a completely continuous mapping and $S(\overline{U}_2) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, and there exist $\varepsilon > 0$ small enough and $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

Then $T + S$ has at least one non-zero fixed point $x^* \in \mathcal{P}$ such that

$$x^* \in (\overline{U}_2 \setminus \overline{U}_1) \cap \Omega.$$

Proof. If $Sx = (I - T)x$ for $x \in \partial U_2 \cap \Omega$, then we get a fixed point $x^* \in \partial U_2 \cap \Omega$ of the operator $T + S$. Suppose that $Sx \neq (I - T)x$ for any $x \in \partial U_2 \cap \Omega$. By Lemma 2.5 (2), we have

$$i_* \left(T + S), U_1 \cap \Omega, \mathcal{P}\right) = 0,$$

and by Lemma 2.5 (1), we have

$$i_* \left(T + S, U_2 \cap \Omega, \mathcal{P}\right) = 1.$$

The additivity property of the index i_* yields

$$i_* \left(T + S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{P}\right) = 1.$$

Consequently, by the existence property of the index, $T + S$ has at least one non-zero fixed point $x^* \in (U_2 \setminus \overline{U}_1) \cap \Omega$. This completes the proof. □

3. Main Results

3.1. Integral representation and some a priori estimates

Let $X = \underbrace{X_1 \times \ldots \times X_1}_{k+3}$ be endowed with the norm

$$\|u\| = \max_{l \in \{0, \ldots, k+2\}} \|u_l\|_1, \quad u = (u_0, \ldots, u_{k+2}).$$

For $u = (u_0, \ldots, u_{k+2}) \in X$, we will write $u \geq 0$ whenever $u_j \geq 0$, $j \in \{0, \ldots, k+2\}$.

For $u \in X$, $u = (u_0, u_1, \ldots, u_{k+2})$, define the operators S_{10}, S_{1j}, $j \in \{1, \ldots, k\}$, S_{1k+1}, S_{1k+2} and S_1 as follows

$$\begin{aligned} S_{10}u(t) &= -w(t)|u_0'(t)|^{p(t)-2}u_0'(t) + \sum_{0 < t_j < t} B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right) \\ &\quad + \int_0^t f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right) ds, \\ S_{1j}u(t) &= u_0(t_j^+) - u_0(t_j) - A_j\left(t, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right), \\ &\quad j \in \{1, \ldots, k\}, \end{aligned}$$

$$S_{1k+1}u(t) = au_0(0) - b(w(0))^{\frac{1}{p(0)-1}} u_0'(0),$$

$$S_{1k+2}u(t) = cu_0(T) + dw(T)|u_0'(T)|^{p(T)-2}u_0'(T),$$

$$S_1u(t) = (S_{10}u(t), S_{11}u(t), \dots, S_{1k+2}u(t)), \quad t \in [0,T].$$

Lemma 3.1. *Suppose* ($H1$)-($H2$). *If* $u \in X$, $u = (u_0, u_1, \dots, u_{k+2})$, *satisfies the equation*

$$S_1u(t) = 0, \quad t \in [0,T], \tag{9}$$

then u_0 *is a solution to the problem* (1)-(5).

Proof. For $t \in [0,T]$, we have

$$S_{10}u(t) = 0, \tag{10}$$
$$S_{1j}u(t) = 0, \quad j \in \{1, \dots, k\} \tag{11}$$
$$S_{1k+1}u(t) = 0, \tag{12}$$
$$S_{1k+2}u(t) = 0. \tag{13}$$

By (11), (12) and (13), it follows that u_0 satisfies (2), (4) and (5).
Consider the equation (10). Then

$$0 = -w(t)|u_0'(t)|^{p(t)-2}u_0'(t) + \sum_{0<t_j<t} B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right) + \int_0^t f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right) ds, \quad t \in [0,T].$$

Hence,

$$0 = -w(t_l^+)|u_0'(t_l^+)|^{p(t_l^+)-2}u_0'(t_l^+) + \sum_{0<t_j<t_l^+} B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right) + \int_0^{t_l} f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right) ds$$

and

$$0 = -w(t_l)|u_0'(t_l)|^{p(t_l)-2}u_0'(t_l) + \sum_{0<t_j<t_l} B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right) + \int_0^{t_l} f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right) ds,$$

we find

$$-w(t_l^+)|u_0'(t_l^+)|^{p(t_l^+)-2}u_0'(t_l^+) + w(t_l)|u_0'(t_l)|^{p(t_l)-2}u_0'(t_l) + B_l\left(t_l, u_0(t_l), (w(t_l))^{\frac{1}{p(t_l)-1}} u_0'(t_l)\right) = 0, \quad l \in \{1, \dots, k\}.$$

Thus, u_0 satisfies (3). Now, we differentiate the equation (10) with respect to t and we find that u_0 satisfies the equation (1). This completes the proof. □

Lemma 3.2. *Suppose* (*H*1)-(*H*4). *Let* $u_0 \in X_1$, $\|u_0\|_1 \leq B$. *Then*

$$\begin{aligned} \left|f\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| &\leq B + B^{p_1+1} + B^{2p_2+1}, \\ \left|A_j\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| &\leq B + B^{p_{1j}+1} + B^{2p_{2j}+1}, \\ \left|B_j\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| &\leq B + B^{q_{1j}+1} + B^{2q_{2j}+1}, \end{aligned}$$

$j \in \{1, \ldots, k\}$, $t \in [0, T]$.

Proof. We have

$$\begin{aligned} &\left|f\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| \\ &\leq a_1(t) + a_2(t)|u_0(t)|^{p_1} + a_3(t)(w(t))^{\frac{p_2}{p(t)-1}} |u_0'(t)|^{p_2} \\ &\leq B + B^{1+p_1} + B^{1+2p_2}, \quad t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} &\left|A_j\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| \\ &\leq a_{1j}(t) + a_{2j}(t)|u_0(t)|^{p_{1j}} + a_{3j}(t)(w(t))^{\frac{p_{2j}}{p(t)-1}} |u_0(t)|^{p_{2j}} \\ &\leq B + B^{1+p_{1j}} + B^{1+2p_{2j}}, \quad j \in \{1, \ldots, k\}, \quad t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} &\left|B_j\left(t, u_0(t), (w(t))^{\frac{1}{p(t)-1}} u_0'(t)\right)\right| \\ &\leq b_{1j}(t) + b_{2j}(t)|u_0(t)|^{q_{1j}} + b_{3j}(t)(w(t))^{\frac{q_{2j}}{p(t)+1}} |u_0(t)|^{q_{2j}} \\ &\leq B + B^{1+q_{1j}} + B^{1+2q_{2j}}, \quad j \in \{1, \ldots, k\}, \quad t \in [0, T]. \end{aligned}$$

This completes the proof. □

Set

$$\begin{aligned} B_1 &= \max_{j \in \{1, \ldots, k\}} \Bigg\{ B^B + \sum_{j=1}^{k} \left(B + B^{q_{1j}+1} + B^{2q_{2j}+1}\right) + T\left(B + B^{p_1+1} + B^{2p_2+1}\right), \\ &\qquad 3B + B^{p_{1j}+1} + B^{2p_{2j}+1}, \quad B(|a| + |b|B), \quad |c|B + |d|B^B \Bigg\}. \end{aligned}$$

Lemma 3.3. *Suppose* (*H*1)-(*H*4). *If* $u \in X$, $\|u\| \leq B$, *then*

$$|S_{1l}u(t)| \leq B_1, \quad t \in [0, T], \quad l \in \{0, 1, \ldots, k+2\}.$$

Proof. We have

$$\begin{aligned} |S_{10}u(t)| &= \Bigg| -w(t)|u_0'(t)|^{p(t)-2} u_0'(t) + \sum_{0 < t_j < t} B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right) \\ &\qquad + \int_0^t f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right) ds \Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq w(t)|u_0'(t)|^{p(t)-1} + \sum_{0<t_j<t} \left| B_j\left(t_j, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right)\right| \\
&\quad + \int_0^t \left| f\left(s, u_0(s), (w(s))^{\frac{1}{p(s)-1}} u_0'(s)\right)\right| ds \\
&\leq B^B + \sum_{j=1}^{k} \left(B + B^{q_{1j}+1} + B^{2q_{2j}+1}\right) + T\left(B + B^{p_1+1} + B^{2p_2+1}\right) \\
&\leq B_1, \quad t \in [0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{1j}u(t)| &= \left| u_0(t_j^+) - u_0(t_j) - A_j\left(t, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right)\right| \\
&\leq |u_0(t_j^+)| + |u_0(t_j)| + \left| A_j\left(t, u_0(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} u_0'(t_j)\right)\right| \\
&\leq 3B + B^{p_{1j}+1} + B^{2p_{2j}+1} \\
&\leq B_1, \quad j \in \{1, \dots, k\}, \quad t \in [0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{1k+1}u(t)| &= \left| au_0(0) - b(w(0))^{\frac{1}{p(0)-1}} u_0'(0)\right| \\
&\leq |a||u_0(0)| + |b|(w(0))^{\frac{1}{p(0)-1}} |u_0'(0)| \\
&\leq |a|B + |b|B^2 \\
&= B(|a| + |b|B) \\
&\leq B_1, \quad t \in [0, T],
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{1k+2}u(t)| &= \left| cu_0(T) + dw(T)|u_0'(T)|^{p(T)-2} u_0'(T)\right| \\
&\leq |c||u_0(T)| + |d|w(T)|u_0'(T)|^{p(T)-2}|u_0'(T)| \\
&\leq |c|B + |d|B^B \\
&\leq B_1, \quad t \in [0, T].
\end{aligned}
$$

This completes the proof. □

Below suppose that $A > 0$ and $C > 0$ are constants such that

(H5) $C(1 + T + T^2) \leq A$.

For $u \in X$, define the operator

$$S_2u(t) = C\int_0^t (t-s)S_1u(s)ds, \quad t \in [0,T].$$

Lemma 3.4. *Suppose* $(H1)$-$(H2)$. *If* $u \in X$, $u = (u_0, \dots, u_{k+2})$, *satisfies the equation*

$$S_2u(t) = (c_0, c_1, \dots, c_{k+2}), \quad t \in [0,T], \tag{14}$$

where c_j, $j \in \{0, \dots, k+2\}$, *are constants, then* u_0 *satisfies the problem* (1)-(5).

Proof. We differentiate the equation (14) two times with respect to t and we get

$$S_1u(t) = 0, \quad t \in [0,T].$$

This completes the proof. □

Lemma 3.5. *Suppose* $(H1)$-$(H5)$. *If* $u \in X$, $\|u\| \le B$, *then*

$$\|S_2u\| \le AB_1.$$

Proof. Using Lemma 3.3 and $(H5)$, we get

$$\begin{aligned} |S_2u(t)| &= C\left|\int_0^t (t-s)S_1u(s)ds\right| \\ &\le C\int_0^t (t-s)|S_1u(s)|ds \\ &\le CT^2B_1 \\ &\le AB_1, \quad t \in [0,T], \end{aligned}$$

and

$$\begin{aligned} \left|\frac{d}{dt}S_2u(t)\right| &= C\left|\int_0^t S_1u(s)ds\right| \\ &\le C\int_0^t |S_1u(s)|ds \\ &\le CTB_1 \\ &\le AB_1, \quad t \in [0,T], \end{aligned}$$

and

$$\begin{aligned} \left|\frac{d^2}{dt^2}S_2u(t)\right| &= C\,|S_1u(t)| \\ &\le CB_1 \\ &\le AB_1, \quad t \in [0,T]. \end{aligned}$$

This completes the proof. □

3.2. Existence of at least one solution

Our first main result for existence of classical solutions of the problem (1)-(5) is as follows.

Theorem 3.6. *Suppose* $(H1)$-$(H5)$ *hold. Then the problem* (1)-(5) *has at least one solution in* $PC^2([0,T])$.

Proof. Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$, i.e., if $\{f_\alpha\}_{\alpha\in I}$ is a family of $\widetilde{Y}$, then the families $\{f_\alpha\}_{\alpha\in I}$, $\{f'_\alpha\}_{\alpha\in I}$, $\{f''_\alpha\}_{\alpha\in I}$ are equi-continuous families in the classical sense. Here I is an index set. Let also, $Y=\overline{\widetilde{Y}}$ be the closure of $\widetilde{Y}$. For $u\in Y$ and $\varepsilon>0$, define the operators

$$\begin{aligned} Tu(t) &= \varepsilon u(t), \\ Su(t) &= u(t)-\varepsilon u(t)-\varepsilon S_2u(t), \quad t\in[0,T]. \end{aligned}$$

For $u\in Y$ with $\|u\|\le B$, we obtain

$$\begin{aligned} \|(I-S)u\| &= \|\varepsilon u+\varepsilon S_2u\| \\ &\le \varepsilon\|u\|+\varepsilon\|S_2u\| \\ &\le \varepsilon B+\varepsilon AB_1. \end{aligned}$$

Let $\{f_\alpha\}_{\alpha\in I}$ be an arbitrary family in Y bounded by the constant B. Then $\{f_\alpha\}_{\alpha\in I}$, $\{f'_\alpha\}_{\alpha\in I}$, $\{f''_\alpha\}_{\alpha\in I}$ are equi-continuous families in Y. Since $S, \frac{d}{dt}S, \frac{d^2}{dt^2}S : Y\to Y$ are continuous operators, we have that $\{Sf_\alpha\}_{\alpha\in I}$, $\left\{\frac{d}{dt}Sf_\alpha\right\}_{\alpha\in I}$, $\left\{\frac{d^2}{dt^2}Sf_\alpha\right\}_{\alpha\in I}$ are equi-continuous families in Y and bounded by AB_1. Now, applying the Arzelà-Ascoli theorem, we conclude that $S:Y\to Y$ is a completely continuous mapping. Now, suppose that there is a $u\in Y$ so that $\|u\|=B$ and

$$u=\lambda_0(I-S)u$$

or

$$u=\lambda_0\varepsilon\,(I+S_2)u, \tag{15}$$

for some $\lambda_0\in\left(0,\frac{1}{\varepsilon}\right)$. Then, using that $S_2u(0)=0$, we get

$$u(0)=\lambda_0\varepsilon(u(0)+S_2u(0))=\lambda_0\varepsilon\,u(0),$$

whereupon $\lambda_0\varepsilon=1$. This is a contradiction. Consequently

$$\{u\in Y: u=\lambda(I-S)u,\ \|u\|=B\}=\emptyset,$$

for any $\lambda\in\left(0,\frac{1}{\varepsilon}\right)$. Then, from Theorem 2.4, it follows that the operator $T+S$ has a fixed point $u^*=(u_0,u_1,\dots,u_{k+2})\in Y$ such that $\|u^*\|\le B$. Therefore

$$\begin{aligned} u^*(t) &= Tu^*(t)+Su^*(t) \\ &= \varepsilon u^*(t)+u^*(t) \\ &\quad -\varepsilon u^*(t)-\varepsilon S_2u^*(t), \quad t\in[0,T], \end{aligned}$$

whereupon

$$S_2u^*(t)=0, \quad t\in[0,T].$$

From here, u_0^* is a solution to the problem (1)-(5). This completes the proof. □

3.3. Existence of at least one nonnegative solution

In the sequel, suppose that the constants B and A which appear in the conditions $(H1)$ and $(H5)$, respectively, satisfy the following inequality

(H6) $AB_1 < \frac{L}{5}$,

where $L \in \mathbb{R}$ is such that $r < L \leq B$ with r a positive constant.

Our second main result for existence of nonnegative solutions of the problem (1)-(5) is as follows.

Theorem 3.7. *Suppose* $(H1)$-$(H6)$. *Then the problem* (1)-(5) *has at least one nonnegative solution in* $PC^2([0,T])$.

Proof. Let X be the space used in the previous section. Let also,

$$\widetilde{\mathcal{P}} = \{u \in X : u \geq 0 \quad \text{on} \quad [0,T]\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{\mathcal{P}}$.

Let $\varepsilon > 0$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1+m\varepsilon)v(t) - \varepsilon\frac{L}{10}, \\ S_3 v(t) &= -\varepsilon S_2 v(t) - m\varepsilon v(t) - \varepsilon\frac{L}{10}, \ t \in [0,T], \end{aligned}$$

where m is a large enough positive constant and $\varepsilon\, m \geq \frac{2}{5}$.

Note that for any fixed point $v \in X$ of the operator $T_1 + S_3$, v_0 is a solution to the problem (1)-(5). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ \Omega &= \overline{\mathcal{P}_{R_1}} = \{v \in \mathcal{P} : \|v\| \leq R_1\}, \ \text{where} \ R_1 = L + \frac{A}{m}B_1 + \frac{L}{5m}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have
$$\|T_1v_1 - T_1v_2\| = (1+m\varepsilon)\|v_1 - v_2\|,$$
whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.
2. For $v \in \overline{\mathcal{P}_L}$, we get
$$\begin{aligned} \|S_3 v\| &\leq \varepsilon\|S_2 v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \\ &\leq \varepsilon\Big(AB_1 + mL + \frac{L}{10}\Big). \end{aligned}$$
Therefore $S_3(\overline{\mathcal{P}_L})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_L} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_L})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}_L} \to X$ is a completely continuous operator.
3. Let $v_1 \in \overline{\mathcal{P}_L}$. Set
$$v_2 = v_1 + \frac{1}{m}S_2 v_1 + \frac{L}{5m}.$$
Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[0,T]$. We have $v_2 \geq 0$ on $[0,T]$ and
$$\|v_2\| \leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m}$$

$$\leq \quad L + \frac{A}{m}B_1 + \frac{L}{5m}$$

$$= \quad R_1.$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon \frac{L}{10} - \varepsilon \frac{L}{10}$$

or

$$\begin{aligned}(I - T_1)v_2 &= -\varepsilon m v_2 + \varepsilon \frac{L}{10} \\ &= S_3 v_1.\end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}_L}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ such that

$$S_3 x = (I - T_1)(x - \lambda u_0).$$

Then

$$-\varepsilon S_2 x - m\varepsilon x - \varepsilon \frac{L}{10} = -m\varepsilon(x - \lambda u_0) + \varepsilon \frac{L}{10}$$

or

$$-S_2 x = \lambda m u_0 + \frac{L}{5}.$$

Hence,

$$\frac{L}{5} > \|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| \geq \frac{L}{5}.$$

This is a contradiction.

5. Assume that there exist $\frac{R_1}{L} \geq \lambda_1 \geq \varepsilon + 1$ and $x_1 \in \partial\mathcal{P}_L$ such that

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{16}$$

Thus,

$$-\varepsilon S_2 x_1 - m\varepsilon x_1 - \varepsilon \frac{L}{10} = -\lambda_1 m\varepsilon x_1 + \varepsilon \frac{L}{10},$$

or

$$S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1)m x_1.$$

From here,

$$\begin{aligned}\|S_2 x_1\| &= \|(\lambda_1 - 1)m x_1 - \tfrac{L}{5}\| \\ &\geq (\lambda_1 - 1)m\|x_1\| - \tfrac{L}{5} \\ &\geq \varepsilon m L - \tfrac{L}{5} \\ &\geq \tfrac{L}{5},\end{aligned}$$

which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Then $T_1 + S_3$ has at least one fixed point $u = (u_0, u_1, \ldots, u_{k+2}) \in \mathcal{P}$ such that $r < \|u\| \leq R_1$, where u_0 is solution of BVP (1)-(5). Consequently, the BVP (1)-(5) has at least one nonnegative solution. This completes the proof. □

4. An Example

Below, we will illustrate our main results.
We consider BVP (1.1)-(1.5) with

$$k = 2, \quad N = 1, \quad T = 1, \quad t_1 = \tfrac{1}{4}, \quad t_2 = \tfrac{1}{2}$$

and

$$B = 10, \quad L = 5, \quad r = 4, \quad A = \tfrac{1}{10B_1}, \quad C = \tfrac{A}{9}.$$

Let

$$w(t) = t^2 + 3, \qquad p(t) = t^2 + 2,$$

$$a_1(t) = a_3(t) = a_{1j}(t) = a_{3j}(t) = b_{1j}(t) = b_{3j}(t) = 0,$$

$$t \in [0,1], \quad j \in \{1,2\},$$

$$a_2(t) = a_{2j}(t) = b_{2j}(t) = \frac{1}{1+t^2}, \qquad t \in [0,1], \quad j \in \{1,2\},$$

$$p_1 = p_{1j} = q_{1j} = 2, \quad p_2 = p_{2j} = q_{2j} = 0, \qquad j \in \{1,2\},$$

and

$$\begin{aligned} f\left(t, x(t), (w(t))^{\frac{1}{p(t)-1}} x'(t)\right) &= \frac{(x(t))^2}{1+t^2}, \\ A_j\left(t_j, x(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} x'(t_j)\right) &= \frac{(x(t_j))^2}{1+t_j^2}, \\ B_j\left(t_j, x(t_j), (w(t_j))^{\frac{1}{p(t_j)-1}} x'(t_j)\right) &= \frac{(x(t_j))^2}{1+t_j^2}, \end{aligned}$$

$$t \in [0,1], \quad j \in \{1,2\}.$$

Then, the problem can be transformed into the following weighted $p(t)$-Laplacian system subject to impulsive boundary conditions

$$\begin{cases} -\left((t^2+3)|x'(t)|^{t^2} x'(t)\right)' + \frac{(x(t))^2}{1+t^2} = 0, & t \in (0,1),\ t \neq t_j, \\ x(t_j^+) - x(t_j) = \frac{(x(t_j))^2}{1+t_j^2}, & \\ ((t_j^+)^2 + 3)|x'(t_j^+)|^{(t_j^+)^2} x'(t_j^+) = ((t_j)^2+3)|x'(t_j)|^{(t_j)^2} x'(t_j) + \frac{(x(t_j))^2}{1+t_j^2}, & j \in \{1,2\}, \\ ax(0) - 3bx'(0) = 0, & \\ cx(1) + 4d\,|x'(1)|\,x'(1) = 0, & \end{cases}$$

where $x : [0,1] \to \mathbb{R}$, and $a, b, c, d \in \mathbb{R}$. We can easily figure out that $(H1)$-$(H4)$ hold, and we have

$$C(1 + T + T^2) = 3C = \frac{A}{3} \le A,$$

i.e., $(H5)$ holds. Next,

$$r < L < B, \qquad AB_1 < \frac{L}{5}.$$

i.e., $(H6)$ holds. Consequently all conditions of Theorem 3.6 and Theorem 3.7 are fulfilled. Hence, the considered BVP has at least one solution and has at least one nonnegative solution.

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