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Singular fractional double-phase problems with variable exponent via Morse's theory

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Abstract. In this manuscript, we deal with a class of fractional non-local problems involving a singular term and vanishing potential of the form:

$$\begin{cases} \mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)} w(x) & = \dfrac{g(x, w(x))}{w(x)^{\xi(x)}} + \mathcal{V}(x)|w(x)|^{\sigma(x)-2}w(x) & \text{in} \quad \mathcal{U}, \\ \qquad\qquad w & > 0 & \text{in} \quad \mathcal{U}, \\ \qquad\qquad w & = 0 & \text{in} \quad \mathbb{R}^N \backslash \mathcal{U}, \end{cases}$$

where $\mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)}$ is a $(p(x,.), q(x,.))-$ fractional double-phase operator with $s_1, s_2 \in (0,1)$, g, and $\mathcal{V}$ are functions that satisfy some conditions. The strategy of the proof for these results is to approach the problem proximatively and calculate the critical groups. Moreover, using Morse's theory to prove our problem has infinitely many solutions.

1. Introduction

Marston Morse, a mathematician, created Morse's hypothesis in the 1920s. He was a member of the university at the Institute for Advanced Study, and Princeton University released Topological Methods in the Theory of Functions of a Complex Variable in 1947 as part of the Annals of Mathematics Studies series. A well-known publication by theoretical physicist Edward Witten that connects Morse's theory to quantum field theory has garnered a lot of attention for this idea during the past two decades. Morse's theory and computation of critical groups are useful tools for studying the multiplicity and existence of solutions to nonlinear problems. As far as we know, this method is rarely used in the study of problems of differential equations. For this, we'll provide a brief overview of the technique.
Let W be a real Banach space, $\phi \in C^1(W, \mathbb{R})$ satisfies the Palais-Smale condition, and $c \in \mathbb{R}$. We consider the following sets:

$$\phi^c = \left\{ u \in W : \phi(u) \leq c \right\},$$

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and

$$K_\phi = \left\{ u \in W : \phi'(u) = 0 \right\}.$$

The critical groups of ϕ at u are defined by

$$C_k(\phi, u) = H_k\Big(\phi^c \cap U, \phi^c \cap U \backslash \{u\}\Big),$$

where $k \in \mathbb{N}$, U is a neighbourhood of u such that $K_\phi \cap U = \{u\}$, and H_k is the singular relative homology with coefficient in an Abelian group G, see [32] for more details. Moreover, the authors in [8] introduced the critical groups of ϕ at infinity by

$$C_k(\phi, \infty) = H_k(W, \phi^a),$$

where a is less than all critical values and $k \in \mathbb{N}$. Concerning the connection between critical groups and critical points of ϕ. Wellem et al. in [34] proved the following statement:

1) If $C_k(\phi, \infty) \not\cong 0$ for some $k \in \mathbb{N}$, then ϕ has critical point u and satisfies $C_k(\phi, u) \not\cong 0$.
2) Let $\theta \in W$ be an isolated critical point of ϕ. If $C_k(\phi, \infty) \not\cong C_k(\phi, \theta)$ for some $k \in \mathbb{N}$, then ϕ must have a non-zero critical point.

Readers may refer to [8, 19, 29, 33, 34] and the references therein for further insights and details on algebraic topology, Morse's theory, and critical groups.

In recent years, double-phase differential operators have garnered significant interest among researchers, owing to their versatile applications across various scientific domains, with a particular focus on their relevance in physical processes. To illustrate, Zhikov [37] proved in the context of elasticity theory that the modulation coefficient $\mu(.)$ plays a pivotal role in shaping the geometry of composites composed of two distinct materials characterized by different curing exponents, namely p and q.

To set the stage for our motivation, we first provide a brief overview of prior research. In his work, Zhikov [36] introduced and examined functionals characterized by integrands that exhibit varying ellipticity depending on the location, thus offering models for strongly anisotropic materials. As an illustrative example, he employed the following function as a prototype:

$$w \mapsto \int_{\mathcal{U}} (|\nabla w|^p + \mu(x)|\nabla w|^q)\, dx. \tag{1}$$

Following this, multiple research endeavours were undertaken in this particular direction, with notable mentions including the influential contributions of Baroni et al. in [6, 7]. For further findings, readers are encouraged to consult the references provided in [3, 15, 24, 31].

The primary focus of our present paper is to investigate the non-local version of double phase function type (1), for variable exponents $p(x,)$ and $q(x,)$ and fractional constant orders $0 < s_1, s_2 < 1$, of the form:

$$\mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)} w(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathcal{U} \backslash \mathfrak{B}_\varepsilon(x)} \left[\frac{|w(x) - w(y)|^{p(x,y)-2}}{|x-y|^{N+s_1 p(x,y)}} + \frac{|w(x) - w(y)|^{q(x,y)-2}}{|x-y|^{N+s_2 q(x,y)}} \right] (w(x) - w(y))dy, \tag{2}$$

where $\mathfrak{B}_\varepsilon(x)$ is the ball of $\mathcal{U}$ of radius ε and center x. By studying the following class of variable-order fractional of double-phase problems driven by $(p(x,.), q(x,.))-$ fractional Laplacian with variable exponents involving a singular term and vanishing potential:

$$\begin{cases} \mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)} w(x) & = \dfrac{g(x, w(x))}{w(x)^{\xi(x)}} + \mathcal{V}(x)|w(x)|^{\sigma(x)-2} w(x) \quad \text{in} \quad \mathcal{U}, \\ \qquad w & > 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \mathcal{U}, \\ \qquad w & = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \mathbb{R}^N \backslash \mathcal{U}. \end{cases} \tag{3}$$

Here $\mathcal{U} \subset \mathbb{R}^N$ an open bounded set, we start by fixing $s_1, s_2 \in (0,1)$, $p, q : \mathcal{U} \times \mathcal{U} \to (1, \infty)$, $\sigma : \mathcal{U} \to (1, \infty)$, and $\xi : \mathcal{U} \to (0, 1]$ are continuous functions that satisfy the following conditions:

$$p(x - z, y - z) = p(x, y), \text{ for all } (x, y, x) \in \mathcal{U} \times \mathcal{U} \times \mathcal{U}, \tag{4}$$

$$p(x,y) = p(y,x), \text{ for all } (x,y) \in \mathcal{U}\times\mathcal{U}, \tag{5}$$

$$1 < \sigma^- < \sigma^+ < q^- < q^+ < p^- < p^+ < +\infty, \tag{6}$$

with $\sigma^- = \min_{x\in\mathcal{U}} \sigma(x)$, $\sigma^+ = \max_{x\in\mathcal{U}} \sigma(x)$, $q^- = \min_{(x,y)\in\mathcal{U}\times\mathcal{U}} q(x,y)$, $q^+ = \max_{(x,y)\in\mathcal{U}\times\mathcal{U}} q(x,y)$, $p^- = \min_{(x,y)\in\mathcal{U}\times\mathcal{U}} p(x,y)$, $p^+ = \max_{(x,y)\in\mathcal{U}\times\mathcal{U}} p(x,y)$, $\mathcal{V}$ vanishing potential satisfies the following assumptions:

(V) $\mathcal{V} : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, there exist $\theta_1 > 0$, and $0 < \eta_1 < 1$ such that

$$\mathcal{V}(x) > \theta_1 > 0 \text{ and } \int_{\mathbb{R}^N} \mathcal{V}(x)|w(x)|^{\sigma(x)}dx \le \eta_1\|w\|_{Y_1},$$

for all $x \in \mathbb{R}^N$, and $w \in Y_1$ with Y_1 is the fractional Sobolev space see section 2.2 for more details.

$g : \mathcal{U}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following condition:
($\mathcal{H}_1$) There exist $\beta \in L^\infty(\mathcal{U})$, and a continuous function $r : \mathcal{U} \to (1,+\infty)$ such that

$$1 < r(x) < p^\star_{s_1}(x) = \frac{Np(x,x)}{N - s_1 p(x,x)},$$

and

$$g(x,y) \le \beta(x)\left(1 + |y|^{r(x)-1}\right), \qquad \text{a.e. } x \in \mathcal{U}, y \in \mathbb{R},$$

$\mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)}$ is the double-phase operator defined by (2). Using Morse's theory, local linking arguments, and variational analysis, more precisely, by computing the critical groups of the energy functional associated with the approximated equations by using some variational method combined with Morse's theory, we prove the existence of infinitely many solutions to problem(3).

For the $p(x,.)$ Laplacian operator, the approaches for ensuring the existence of solutions were addressed in greater depth, we quote, the relevant work of Bahrouni and Radulescu [5] who developed some qualitative properties on the fractional Sobolev space $W^{s,q(x),p(x,y)}(\mathcal{U})$ for $s \in (0,1)$ and $\mathcal{U}$ being a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. Moreover, they studied the existence of solutions to the following problem:

$$\begin{cases} \mathcal{L}w(x) + |w(x)|^{q(x)-1}w(x) & = \lambda|w(x)|^{r(x)-1}w(x) \quad \text{in} \quad \mathcal{U}, \\ w & = 0 \quad \text{in} \quad \partial\mathcal{U}, \end{cases} \tag{7}$$

where

$$\mathcal{L}w(x) = 2\lim_{\varepsilon\to 0^+} \int_{\mathcal{U}\setminus\mathfrak{B}_\varepsilon(x)} \int_{\mathcal{U}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))}{|x-y|^{n+sp(x,y)}}dx,$$

$\lambda > 0$, and $1 < r(x) < p^- = \min_{(x,y)\in\mathcal{U}\times\mathcal{U}} p(x,y)$.
More recently, authors in [11] studied the double phase version of problem(7) with non-linearity logarithmic

$$\mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)}w(x) = \lambda|w(x)|^{r(x)-1}u(x) + \mu(x)|w(x)|^{r(x)-2}\ln(|w(x)|) \text{ in } \mathcal{U}.$$

Readers may refer to [1, 4, 5, 12, 20, 23] and the references therein for more ideas and techniques developed to guarantee the existence of weak solutions for a class of nonlocal fractional problems with variable exponents.
The novelty of our work is to study the existence of infinitely many solutions to a class double phase problems driven by $\mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)}$ double-phase operator involving a singular nonlinearity and vanishing potential with variable exponent, by computing the critical groups of the energy functional associated to the approximated equations by using some variational method combining with Morse's theory.
The structure of this article is as follows. In section 2, we briefly introduce certain homology theory concepts. We also give definitions and basic properties for Lebesgue spaces and fractional Sobolev spaces with variable exponent. In section 3.1, we suggest the approximated problem (8), and use the homological theory to compute critical groups of the energy functional associated with the approximated problem (8). In paragraph 3.2, we will use Morse's relation to show that the approximated problem (8) admits infinitely non-trivial solutions. In the last section, we will prove our fundamental Theorem 4.1.

2. Mathematical background

2.1. Generalized Lebesgue space

We consider the set:

$$C^+(\bar{\mathcal{U}}) = \left\{m : \bar{\mathcal{U}} \to \mathbb{R}^+ : m \text{ is a continuous function and } 1 < m^- < m(y) < m^+ < +\infty\right\},$$

where $m^- = \min_{y\in\bar{\mathcal{U}}} m(y),\ m^+ = \max_{y\in\bar{\mathcal{U}}} m(y).$

Definition 2.1. *(see [20]) Let $m \in C^+(\bar{\mathcal{U}})$. We define the generalized Lebesgue space $L^{m(y)}(\mathcal{U})$ as usual:*

$$L^{m(y)}(\mathcal{U}) = \left\{u : \mathcal{U} \to \mathbb{R} \textit{ is a measurable function } : \exists\lambda > 0 : \int_{\mathcal{U}} |\frac{u(y)}{\lambda}|^{m(y)}dx < \infty\right\}.$$

We equip this space with the so-called Luxemburg norm defined as follows:

$$|w|_{L^{m(y)}(\mathcal{U})} = \inf\left\{\xi > 0 : \int_{\mathcal{U}} |\frac{w(y)}{\xi}|^{m(y)}dy \leq 1\right\}.$$

Lemma 2.2. *(see [20]) For every $w \in L^{m(y)}(\mathbb{R}^N)$, the following properties hold:*

i) If $|w|_{L^{m(y)}(\mathbb{R}^N)} < 1$, *then* $|w|^{m^-}_{L^{m(y)}(\mathbb{R}^N)} \leq \rho_{m(y)}(w) \leq |w|^{m^+}_{L^{m(y)}(\mathbb{R}^N)}$.

ii) If $|w|_{L^{m(y)}(\mathbb{R}^N)} > 1$, *then* $|w|^{m^+}_{L^{m(y)}(\mathbb{R}^N)} \leq \rho_{m(y)}(w) \leq |w|^{m^-}_{L^{q(y)}(\mathbb{R}^N)}$.

iii) $|w|_{L^{m(y)}(\mathbb{R}^N)} < 1, = 1, > 1$ *if only if* $\rho_{m(y)}(w) < 1, = 1, > 1,$

where $\rho_{m(y)} : L^{m(y)}(\mathbb{R}^N) \to \mathbb{R}$ is the mapping defined as follows

$$\rho_{m(y)}(w) = \int_{\mathbb{R}^N} |w(y)|^{m(y)}dy.$$

Proposition 2.3. *(see [20]) For every w and $w_n \in L^{m(y)}(\mathbb{R}^N)$, the following statements are equivalent:*

i) $\lim_{n\to+\infty} |w_n - w|_{L^{m(y)}(\mathbb{R}^N)} = 0,$

ii) $\lim_{n\to+\infty} \rho_{m(y)}(w_n - w) = 0,$

iii) $w_n \to w$ *in measure on* $\mathbb{R}^N$ *and* $\lim_{n\to+\infty} \rho_{m(y)}(w_n) - \rho_{m(y)}w) = 0.$

Lemma 2.4. *(Hölder's inequality, see [20]) For every $m \in C^+(\mathbb{R}^N)$, the following inequality holds:*

$$|\int_{\mathbb{R}^N} v(y)w(y)dy| \leq \left(\frac{1}{m^-} + \frac{1}{m'^-}\right)|v|_{L^{m(y)}(\mathbb{R}^N)}|w|_{L^{m'(y)}(\mathbb{R}^N)},$$

for all $(v, w) \in L^{m(y)}(\mathbb{R}^N) \times L^{m'(y)}(\mathbb{R}^N)$, where $\dfrac{1}{m(y)} + \dfrac{1}{m'(y)} = 1.$

2.2. Generalized fractional Sobolev space

We start by fixing the fractional exponent $s \in (0,1)$. Let $\mathcal{U}$ be an open bounded set of $\mathbb{R}^N$, $m_1 \in C^+(\mathcal{U})$, and $p : \bar{\mathcal{U}} \times \bar{\mathcal{U}} \to (1,\infty)$ is a continuous function that satisfies the conditions (4)- (6). We introduce the generalized fractional Sobolev space $W^{s,m_1(x),p(x,y)}(\mathcal{U})$ as follows

$$W^{s,m_1(x),p(x,y)}(\mathcal{U}) = \left\{ w \in L^{m_1(x)}(\mathcal{U}) : \frac{w(x)-w(y)}{\beta|x-y|^{s+\frac{N}{p(x,y)}}} \in L^{p(x,y)}(\mathcal{U}\times\mathcal{U}) \text{ for some } \beta > 0 \right\}.$$

Let $[w]^{s,p(x,y)} = \inf\left\{\beta > 0 : \int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x)-w(y)|^{p(x,y)}}{\beta^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy < 1\right\}$ be the corresponding variable exponent Gagliardo seminorm. We equip the space $W^{s,m_1(x),p(x,y)}(\mathcal{U})$ with the norm

$$||w||_{W^{s,m_1(x),p(x,y)}(\mathcal{U})} = [w]^{s,p(x,y)} + |w|_{m_1(x)},$$

where $(L^{m_1(x)}(\mathcal{U}), |.|_{m_1(x)})$ is the generalized Lebesgue space.

Lemma 2.5. *(see [5]) Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, $p : \mathcal{U}\times\mathcal{U} \to (1,+\infty)$ be a continuous function that satisfies conditions (4)-(6), and $m_1 \in C^+(\bar{\mathcal{U}})$. Then $W^{s,m_1(x),p(x,y)}(\mathcal{U})$ is a separable, and reflexive Banach space.*

Theorem 2.6. *(see[2, 12, 27]) Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, $p : \mathcal{U}\times\mathcal{U} \to (1,+\infty)$ be a continuous function that satisfies conditions (4)-(6) $m_1 \in C^+(\mathcal{U})$, and*

$$sp(x,y) < N, \ p(x,x) < m_1(x), \ \textit{for all } (x,y) \in \mathcal{U}^2,$$

and $\ell : \overline{\mathcal{U}} \to (1,+\infty)$ is a continuous variable exponent such that

$$p_s^*(x) = \frac{Np(x,x)}{N - sp(x,x)} > \ell(x) \geq \ell^- = \min_{x\in\overline{\mathcal{U}}} \ell(x) > 1.$$

Then the space $W^{s,m_1(x),p(x,y}(\mathcal{U})$ is continuously embedded in $L^{\ell(y)}(\mathcal{U})$. That is, there exists a positive constant $C = C(N, s, p, m_1, \mathcal{U})$ such that

$$|w|_{L^{l(x)}(\mathcal{U})} \leq C||w||_{W^{s,m_1(x),p(x,y)}(\mathcal{U})}, \ \textit{for all} \quad w \in W^{s,m_1(x),p(x,y)}(\mathcal{U}).$$

Moreover, this embedding is compact.

2.3. Homology theory

We now present the fundamental tool that will be used to work with, namely the homology theory.

Definition 2.7. *(see [33]) Given Y is a Banach space, $\psi \in C(Y,\mathbb{R})$, and 0 is an isolated critical point of ψ such that $\psi(0) = 0$. Let $m,n \in \mathbb{N}$. We say that ψ has a local $(m,n)-$ linking near the origin if there exist a neighbourhood U of 0 and non-empty sets F_0, $F \subset U$, and $D \subset Y$ such that $0 \notin F_0 \subset F$, $F \cap D = \emptyset$ and*

1) $\psi\big|_F \leq 0 < \psi\big|_{U\cap D\setminus\{0\}}$,
2) *0 is the only critical point of ψ in $\psi^0 \cap U$, where $\psi^0 = \{w \in Y : \psi(w) = 0\}$,*
3) $\operatorname{Dim} im\,(i^*) - \operatorname{Dim} im\,(j^*) \geq n$, *where*

$$i^* : H_{m-1}(F_0) \to H_{m-1}(Y\setminus D) \textit{ and } j^* : H_{m-1}(F_0) \to H_{m-1}(F)$$

are the homomorphisms induced by the inclusion maps $i : F_0 \to Y\setminus D$ and $j : F_0 \to F$.

Lemma 2.8. *(Morse's relation) (see [32]) If Y is a Banach space, $\psi \in C^1(Y,\mathbb{R}), a, b \in \mathbb{R}\setminus\psi\left(\left\{K_\psi\right)\right., a < b$, $\psi^{-1}((a,b))$ contains a finite number of critical points $\{w_i\}_{i=1}^n$ and ψ satisfies the Palais-Smale condition, then*

1) *for all* $k \in \mathbb{N}_0$, *we have* $\sum_{i=1}^{n} \operatorname{rank} C_k(\psi, u_i) \geqslant \operatorname{rank} H_k\left(\psi^b, \psi^a\right)$;
2) *if the Morse-type numbers* $\sum_{i=1}^{n} \operatorname{rank} C_k(\psi, u_i)$ *are finite for all* $k \in \mathbb{N}_0$ *and vanish for all large* $k \in \mathbb{N}_0$, *then so do the Betti numbers* $\operatorname{rank} H_k\left(\psi^b, \psi^a\right)$ *and we have*

$$\sum_{k\geqslant 0}\sum_{i=1}^{n} \operatorname{rank} C_k(\psi, u_i)\, t^k = \sum_{k\geqslant 0} \operatorname{rank} H_k\left(\psi^b, \psi^a\right) t^k + (1+t)Q(t) \text{ for all } t \in \mathbb{R},$$

where $Q(t)$ *is a polynomial in* $t \in \mathbb{R}$ *with non-negative integer coefficients.*

Theorem 2.9. *(see [30] Let* $\psi \in C^2(Y, \mathbb{R})$ *satisfy the Palais-Smale condition, and let* a *be a regular value of* ψ. *Then,* $H_*(Y, \psi^a) \neq 0$, *implies that* $K_\psi \cap \psi^a \neq \emptyset$.

3. The Approximated Problem

We suggest an approximate problem sequence as

$$\begin{cases} \mathcal{L}^{s_1,s_2}_{p(x,.),q(x,.)} w_n(x) = \dfrac{g_n(x, w_n(x))}{\left(w_n(x) + \frac{1}{n}\right)^{\xi(x)}} + \mathcal{V}(x)|w_n(x) + \dfrac{1}{n}|^{\sigma(x)-2}\left(w_n(x) + \dfrac{1}{n}\right) & \text{in } \mathcal{U}, \\ w_n > 0 & \text{in } \mathcal{U}, \\ w_n = 0 & \text{in } \mathbb{R}^N \backslash \mathcal{U}, \end{cases} \tag{8}$$

because the energy functional linked to our problem is not differentiable due to the inclusion of a singular term. $g_n(x, t)) = \min(n, g(x, t))$, $G_n(x, t) = \displaystyle\int_0^t \frac{g_n(x, s)}{(s + \frac{1}{n})^{\xi(x)}} ds$, and $g_n : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is a sequence of functions that verifies the following conditions.
$(\mathcal{H}_2)$ There exist $\theta > p^+$ and $r > 0$ such that for a.e $x \in \mathcal{U}$ and $|x| \geq r$,

$$0 < \theta G_n(x, t) \leq \frac{t g_n(x, t)}{(t + \frac{1}{n})^{\xi(x)}}.$$

$(\mathcal{H}_3)$ It holds

$$\lim_{t \to +\infty} \frac{g_n(x, t)}{t^{p^+}} = l_1 \text{ uniformly for a.e } x \in \mathcal{U},$$

$(\mathcal{H}_4)$ There exist $\eta > \sigma^-$ and $a_3 > 0$ such that

$$g_n(x, t)t - \eta G_n(x, t) \geq -a_3|t|^{p^-}$$

for all $x \in \mathcal{U}$ and $t \in \mathbb{R}$.

Example 3.1. *Set* $g_n(t) = l(t + \frac{1}{n})^2$, $p(x, y) = p^- = 2$, *and* $\xi(x) = 1$. *A trivial verification shows that* $(\mathcal{H}_1) - (\mathcal{H}_4)$ *are satisfied under a suitable condition on* η, a_3, *and* θ.

Remark 3.2. *If the function* g *satisfies condition* $(\mathcal{H}_1)$. *Then, the sequence of function* g_n *also verifies condition* $(\mathcal{H}_1)$.

3.1. Computation of critical group

For the sake of simplicity, we note $Y_1 := W^{s_1, m_1(x), p(x,y)}(\mathcal{U})$ and $Y_2 := W^{s_2, m_2(x), q(x,y)}(\mathcal{U})$.

Definition 3.3. *We say that* $\{w_n\}_{n\in\mathbb{N}}$ *to be a weak solution of (8) if*

$$\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p(x,y)}}dxdy$$
$$+\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q(x,y)}}dxdy$$
$$=\int_{\mathcal{U}}\left[\frac{g_n(x,w_n(x))}{\left(w_n(x)+\frac{1}{n}\right)^{\xi(x)}}+\mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)-2}\left(w_n(x)+\frac{1}{n}\right)\right]\varphi(x)dx,$$

for all $\varphi\in Y_1^*$, *where* Y_1^* *is the dual space of* Y_1.

Consider the energy functional $\psi: Y_1\to\mathbb{R}$ defined by

$$\psi(w_n)=\psi_1(w_n)-\psi_2(w_n)-\psi_3(w_n),$$

where

$$\psi_1(w_n)=\int_{\mathcal{U}\times\mathcal{U}}\left[\frac{1}{p(x,y)}\frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1p(x,y)}}+\frac{1}{q(x,y)}\frac{|w_n(x)-w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2q(x,y)}}\right]dxdy,$$

$\psi_3(w_n)=\int_{\mathcal{U}}\frac{\mathcal{V}(x)}{\sigma(x)}|w_n(x)+\frac{1}{n}|^{\sigma(x)}dx$, $\psi_2(w_n)=\int_{\mathcal{U}}G_n(x,w_n(x))dx$, and $G_n(x,t)=\int_0^t\frac{g_n(x,s)}{(s+\frac{1}{n})^{\xi(x)}}ds$ is the primitive of $\frac{g_n(x,s)}{(s+\frac{1}{n})^{\xi(x)}}$.

Lemma 3.4. *If g satisfies* ($\mathcal{H}_1$) *condition and the potential* $\mathcal{V}$ *satisfies* (V). *Then* $\psi_2+\psi_3\in C^1(Y_1,\mathbb{R})$ *and*

$$\langle(\psi_2+\psi_3)'(w_n),v_n\rangle=\int_{\mathcal{U}}\left[\frac{g_n\left(x,w_n(x)\right)}{(w_n(x)+\frac{1}{n})^{\xi(x)}}+\mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)-2}\left(w_n(x)+\frac{1}{n}\right)\right]v_n(x)dx,$$

for all $w_n, v_n\in Y_1$.

Proof. (i) ψ_2 is Gateaux differentiable in Y_1.
Let $w_n, v_n\in Y_1$, and $0<t<1$, we have

$$\frac{1}{t}(G_n\left(x,w_n+tv_n\right)-G_n(x,w_n))=\frac{1}{t}\int_0^{w_n+tv_n}\frac{g_n(x,s)}{(s+\frac{1}{n})^{\xi(x)}}ds-\frac{1}{t}\int_0^{w_n}\frac{g_n(x,s)}{(s+\frac{1}{n})^{\xi(x)}}ds$$
$$=\frac{1}{t}\int_{w_n}^{w_n+tv_n}\frac{g_n(x,s)}{(s+\frac{1}{n})^{\xi(x)}}ds.$$

By the mean value theorem, there exists $0<\delta<1$ such that

$$\frac{1}{t}(G_n(x,w_n+tv_n)-G_n(x,w_n))=\frac{g_n(x,w_n+\delta tv_n)}{(w_n+\delta tv_n+\frac{1}{n})^{\xi(x)}}v_n.$$

Combining ($\mathcal{H}_1$) with Young's inequality, we have

$$\begin{aligned}g_n(x,w_n+\delta tv_n)&\le g(x,w_n+\delta tv_n)\\&\le\beta(|v_n|+|w_n+\delta tv_n|^{r(x)}|v_n|)\\&\le\beta 2^{r^+}(1+|w_n|^{r(x)}+|v_n|^{r(x)}).\end{aligned}$$

Since $r(x) \in (1, p^*_{s_1}(x))$, we have $w_n, v_n \in L^{r(x}(\mathcal{U})$. Thanks to the Lebesgue's dominated converge Theorem, we get

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(G_n(x, w_n + tv_n) - G_n(x, w_n)) &= \lim_{t\to 0}\int_{\mathcal{U}} \frac{g_n(x, w_n + \delta t v_n)}{(w_n + \delta t v_n + \frac{1}{n})^{\xi(x)}} v_n dx \\ &= \int_{\mathcal{U}} lim_{t\to 0}\frac{g_n(x, w_n + \delta t v_n)}{(w_n + \delta t v_n + \frac{1}{n})^{\xi(x)}} v_n dx \\ &= \int_{\mathcal{U}} \frac{g_n(x, w_n)}{(w_n + \frac{1}{n})^{\xi(x)}} v_n dx.\end{aligned} \tag{9}$$

$$\begin{aligned}\langle \psi_3'(w_n), v_n\rangle &= \lim_{t\to 0}\frac{\psi_3(w_n + tv_n) - \psi_3(w_n)}{t} \\ &= \lim_{t\to 0}\int_{\mathcal{U}} \frac{\mathcal{V}(x)}{t\sigma(x)}\Big(|w_n + v_n t + \frac{1}{n}|^{\sigma(x)} - |w_n + \frac{1}{n}|^{\sigma(x)}\Big) dx.\end{aligned} \tag{10}$$

Considering the function defined by $L : [0, 1] \to \mathbb{R}$ as $L(z) = \frac{\mathcal{V}(x)}{\sigma(x)}|w_n + z v_n t + \frac{1}{n}|^{\sigma(x)}$. According to the mean value Theorem, there exists $0 < \varepsilon < 1$ such that

$$L'(z)(\varepsilon) = L(1) - L(0). \tag{11}$$

Combining (10) with (11), it follows that $\langle \psi_3'(w_n), v_n\rangle = \int_{\mathcal{U}} \mathcal{V}(x)|w_n(x) + \frac{1}{n}|^{\sigma(x)-2}\Big(w_n(x) + \frac{1}{n}\Big) v_n(x) dx.$ □

(ii) The continuity of Gateaux-derivatives. Let $\{w_{n,k}\}_{k\in\mathbb{N}} \subset Y_1$ such that $w_{n,k} \to w_n$ strongly in Y_1 as $k \to +\infty$. We use Hölder's inequality and condition $(\mathcal{H}_1)$, we have that

$$\begin{aligned}\int_{\mathcal{U}} |\frac{g_n(x, w_{n,k})}{(w_{n,k} + \frac{1}{n})^{\xi(x)}}|^{r'(x)} dx &\le \int_{\mathcal{U}} |g_n(x, w_{n,k})|^{r'(x)} dx \\ &\le \int_{\mathcal{U}} |g(x, w_{n,k})|^{r'(x)} dx \\ &\le 2^{\frac{r^+ +1}{r^+ -1}}\|\beta\|_\infty^{\frac{r^+ +1}{r^+ -1}} \int_{\mathcal{U}} |w_{n,k}|^{r(x)} dx \\ &\le C(\beta, r^+)\int_{\mathcal{U}} |w_{n,k}|^{r(x)} dx \\ &\le C(\|\beta\|_\infty, r^+)\||w_{n,k}|\|_{L^{\frac{p_{s_1^*}(x)}{r(x)}}(\mathcal{U})}\|1\|_{L^{\frac{p_{s_1^*}(x)}{p_{s_1^*}(x)-r(x)}}(\mathcal{U})}.\end{aligned}$$

So, the sequence $\{|\frac{g_n(x, w_{n,k})}{(w_{n,k} + \frac{1}{n})^{\xi(x)}} - \frac{g_n(x, w_n)}{(w_n + \frac{1}{n})^{\xi(x)}}|^{r(x)}\}_{k\in\mathbb{N}}$ is uniformly bounded and equi-integrable in $L^1(\mathcal{U})$. Thanks to Vitali converge theorem implies

$$\lim_{k\to+\infty}\int_{\mathcal{U}} |\frac{g_n(x, w_{n,k})}{(w_{n,k} + \frac{1}{n})^{\xi(x)}} - \frac{g_n(x, w_n)}{(w_n + \frac{1}{n})^{\xi(x)}}|^{r'(x)} dx = 0,$$

where $\dfrac{1}{r'(x)} + \dfrac{1}{r(x)} = 1$. Thus, by Theorem 2.6 and Hölder's inequality, we have

$$\begin{aligned}\|\psi_2'(w_{n,k}) - \psi_2'(w_n)\|_{Y_1^*} &= \sup_{v_n\in Y_1}\|\langle\psi_2'(w_{n,k}) - \psi_2'(w_n), v_n\rangle\|_{Y_1} \\ &\le |\langle\psi_2'(w_{n,k}) - \psi_2'(w_n), v_n\rangle| \\ &\le \|\frac{g_n(x, w_{n,k})}{(w_{n,k} + \frac{1}{n})^{\xi(x)}} - \frac{g_n(x, w_n)}{(w_n + \frac{1}{n})^{\xi(x)}}\|_{L^{q_1'(x)}(\mathcal{U})}\|v_n\|_{L^{q_1(x)}(\mathcal{U})} \to 0 \text{ as } k \to +\infty,\end{aligned}$$

where Y_1^* is the dual space of Y_1. Similarly, we prove that ψ_3' continuous in Y_1. From the Lemma 3.4 and Lemma 4.1 in [12], we have that $\psi \in C^1(Y_1, \mathbb{R})$, and

$$\begin{aligned}\langle \psi'(\mathrm{w}_{n,k}), \mathrm{v}_n\rangle &= \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{w}_{n,k}(\mathrm{x}) - \mathrm{w}_{n,k}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})-2}(\mathrm{w}_{n,k}(\mathrm{x}) - \mathrm{w}_{n,k}(\mathrm{y}))(\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+s_1 p(\mathrm{x},\mathrm{y})}} d\mathrm{x}d\mathrm{y}\\ &+ \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{w}_{n,k}(\mathrm{x}) - \mathrm{w}_{n,k}(\mathrm{y})|^{q(\mathrm{x},\mathrm{y})-2}(\mathrm{w}_{n,k}(\mathrm{x}) - \mathrm{w}_{n,k}(\mathrm{y}))(\mathrm{v}_n(\mathrm{x}) - \mathrm{v}_n(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+s_2 q(\mathrm{x},\mathrm{y})}} d\mathrm{x}d\mathrm{y}\\ &- \int_{\mathcal{U}} \left[\frac{g_n(\mathrm{x}, \mathrm{w}_n(\mathrm{x}))}{(\mathrm{w}_n(\mathrm{x}) + \frac{1}{n})^{\xi(\mathrm{x})}} + \mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x}) + \frac{1}{n}|^{\sigma(\mathrm{x})-2}\left(\mathrm{w}_n(\mathrm{x}) + \frac{1}{n}\right)\right]\mathrm{v}_n(\mathrm{x}) d\mathrm{x},\end{aligned}$$

for all $\mathrm{v}_n \in Y_1^*$.

Theorem 3.5. *The functional ψ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$.*

Proof. Let $\{\mathrm{w}_{n,k}\}_{k\in\mathbb{N}} \subset Y_1$ be a Palais-Smale sequence of ψ at level c. Then, we have

$$\psi(\mathrm{w}_{n,k}) = c + o(1), \text{ and } \psi'(\mathrm{w}_{n,k}) = o(1). \tag{12}$$

Claim 1: The sequence $\{\mathrm{w}_{n,k}\}_{k\in\mathbb{N}}$ is uniformly bounded in Y_1

By using the contradiction approach, we prove Claim 1. We assume the claim 1 does not hold, that is up to a subsequence still denoted by $\{\mathrm{w}_{n,k}\}_{k\in\mathbb{N}}$ such that $\|\mathrm{w}_{n,k}\|_{Y_1} \to +\infty$ as $k \to +\infty$ in Y_1. Let us $\mathrm{v}_{n,k} := \frac{\mathrm{w}_{n,k}}{\|\mathrm{w}_{n,k}\|_{Y_1}}$. Clearly $\{\mathrm{v}_{n,k}\}_{k\in\mathbb{N}}$ is bounded in Y_1. Since Y_1 is a reflexive Banach space, up to a subsequence still denoted by $\{\mathrm{v}_{n,k}\}_{k\in\mathbb{N}}$ such that:

$$\begin{cases} \mathrm{v}_{n,k} \rightharpoonup \mathrm{v}_n \text{ weakly in } Y_1 \text{ as } k \to \infty,\\ \mathrm{v}_{n,k} \to \mathrm{v}_n \text{ strongly } k \to +\infty \text{ in } L^{a(\mathrm{x})}(\mathcal{U}) \text{ for all } 1 < a(\mathrm{x}) < p_{s_1}^*(\mathrm{x}),\\ \mathrm{v}_{n,k} \to \mathrm{v}_n \text{ a.e in } \mathcal{U} \text{ as } k \to \infty.\end{cases} \tag{13}$$

Combining (12) with $\frac{1}{\|\mathrm{w}_{n,k}\|_{Y_1}} = o(1)$, we have

$$\begin{aligned}&\frac{\|\mathrm{v}_{n,k}\|_{Y_1}^{p^+}}{p^-} + \frac{\|\mathrm{w}_{n,k}\|_{Y_1}^{p^+-q^-}\|\mathrm{v}_{n,k}\|_{Y_2}^{q^+}}{q^-} - \|\mathrm{w}_{n,k}\|_{Y_1}^{-p^-}\int_{\mathcal{U}} G_n(\mathrm{x}, \mathrm{w}_{n,k}) d\mathrm{x} - \frac{\|\mathrm{w}_{n,k}\|_{Y_1}^{\sigma^- - p^+}}{\sigma^+}\int_{\mathcal{U}} \mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x}) + \frac{1}{n}|^{\sigma(\mathrm{x})} d\mathrm{x}\\ &= o(1),\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\|\mathrm{v}_{n,k}\|_{Y_1}^{p^+} + \|\mathrm{w}_{n,k}\|_{Y_1}^{q^+-p^-}\|\mathrm{v}_{n,k}\|_{Y_2}^{q^+} - \|\mathrm{w}_{n,k}\|_{Y_1}^{-p^-}\int_{\mathcal{U}} \frac{g_n(\mathrm{x}, \mathrm{w}_{n,k})}{(\mathrm{w}_{n,k} + \frac{1}{n})^{\xi(\mathrm{x})}}\mathrm{w}_{n,k}(\mathrm{x}) d\mathrm{x} - \|\mathrm{w}_{n,k}\|_{Y_1}^{\sigma^- - p^+}\int_{\mathcal{U}} \mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x}) + \frac{1}{n}|^{\sigma(\mathrm{x})} d\mathrm{x}\\ &= o(1).\end{aligned} \tag{15}$$

We use (15) and (14), we have

$$\begin{aligned}&\left(\frac{\eta}{p^-} - 1\right)\|\mathrm{v}_{n,k}\|_{Y_1}^{p^-} + \left(\frac{\eta}{q^-} - 1\right)\|\mathrm{w}_{n,k}\|_{Y_1}^{q^- - p^+}\|\mathrm{v}_{n,k}\|_{Y_2}^{q^+} - \left(\frac{\eta}{\sigma^-} - 1\right)\|\mathrm{w}_{n,k}\|_{Y_1}^{\sigma^- - p^+}\int_{\mathcal{U}} \mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x}) + \frac{1}{n}|^{\sigma(\mathrm{x})} d\mathrm{x}\\ &- \eta\|\mathrm{w}_{n,k}\|_{Y_1}^{-p^-}\int_{\mathcal{U}}\left(G_n(\mathrm{x}, \mathrm{w}_{n,k}) - \frac{g_n(\mathrm{x}, \mathrm{w}_{n,k})}{(\mathrm{w}_{n,k} + \frac{1}{n})^{\xi(\mathrm{x})}}\mathrm{w}_{n,k}(\mathrm{x})\right) d\mathrm{x} = o(1).\end{aligned} \tag{16}$$

We use $(\mathcal{H}_4)$, we can write

$$\begin{aligned}\left(\frac{\eta}{p^-}-1\right)\|v_{n,k}\|_{Y_1}^{p^-} &= \left(1-\frac{\eta}{q^+}\right)\|w_{n,k}\|_{Y_1}^{q^+-p^-}\|v_{n,k}\|_{Y_2}^{q^+}+\eta\|w_{n,k}\|_{Y_1}^{-p^-}\left(\int_{\mathcal{U}}G_n(x,w_{n,k})-\frac{g_n(x,w_{n,k})}{(w_{n,k}+\frac{1}{n})^{\xi(x)}}w_{n,k}(x)dx\right)\\ &+\left(1-\frac{\eta}{\sigma^+}\right)\|w_{n,k}\|_{Y_1}^{\sigma^--p^+}\int_{\mathcal{U}}\mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)}dx+o(1)\\ &\leq\left(1-\frac{\eta}{q^+}\right)\|w_{n,k}\|_{Y_1}^{q^+-p^-}\|v_{n,k}\|_{Y_2}^{q^+}+\left(1-\frac{\eta}{\sigma^+}\right)\|w_{n,k}\|_{Y_1}^{\sigma^--p^+}\int_{\mathcal{U}}\mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)}dx\\ &+a_3\|w_{n,k}\|_{Y_1}^{p^--q^-}\|v_n\|_{L^{p^-}(\mathcal{U})}^{p^-}+o(1)\\ &=o(1).\end{aligned}\tag{17}$$

as $k\to\infty$. This is a contradiction as $\|v_n\|_{Y_1}=1$, and hence Claim 1 follows. Consequently, there exists $w_n\in Y_1$ such that up to a subsequence

$$\begin{cases} w_{n,k}\rightharpoonup w_n \text{ weakly in } Y_1 \text{ as } k\to\infty,\\ w_{n,k}\to w_n \text{ strongly in } L^{a(x)}(\mathcal{U}) \text{ as } k\to+\infty \text{ for all } 1<a(x)<p^*_{s_1}(x),\\ w_{n,k}\to w_n \text{ a.e in } \mathcal{U} \text{ as } k\to\infty.\end{cases}\tag{18}$$

From $(\mathcal{H}_1)$, and (V), we get

$$\int_{\mathcal{U}}\frac{g_n(x,w_{n,k})\,w_{n,k}}{\frac{1}{n}+w_{n,k}}dx=\int_{\mathcal{U}}\frac{g_n(x,w_n)w_n}{\frac{1}{n}+w_n}dx+o(1),\tag{19}$$

$$\int_{\mathcal{U}}G_n(x,w_{n,k})\,dx=\int_{\mathcal{U}}G_n(x,w_n)dx+o(1),\tag{20}$$

$$\int_{\mathcal{U}}\mathcal{V}(x)|w_{n,k}+\frac{1}{n}|^{\sigma(x)}dx=\int_{\mathcal{U}}\mathcal{V}(x)|w_n+\frac{1}{n}|^{\sigma(x)}dx+o(1).\tag{21}$$

We have also $\{w_{n,k}\}_{k\in\mathbb{N}}$ is bounded in Y_2. Since $w_{n,k}\to w_n$ a.e. in $\mathcal{U}$ as $k\to+\infty$, we have that

$$\frac{|w_{n,k}(x)-w_{n,k}(y)|^{p(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)(p(x,y)-1)}}\to\frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)(p(x,y)-1)}}$$

a.e $(x,y)\in\mathcal{U}\times\mathcal{U}$ as $k\to+\infty$. Since $\{w_{n,k}\}_{k\in\mathbb{N}}$ is bounded in Y_1, there exist $c>0$ such that

$$\int_{\mathcal{U}\times\mathcal{U}}\left|\frac{|w_{n,k}(x)-w_{n,k}(y)|^{p(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)(p(x,y)-1)}}\right|^{\frac{p(x,y)}{p(x,y)-1}}dxdy\leq C.$$

So, we have that

$$\frac{|w_{n,k}(x)-w_{n,k}(y)|^{p(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)(p(x,y)-1)}}\rightharpoonup\frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)(p(x,y)-1)}}\text{ as } k\to\infty$$

weakly in $L^{p'(x,y)}(\mathcal{U}\times\mathcal{U})$, where $\frac{1}{p'(x,y)}+\frac{1}{p(x,y)}=1$. Let $w_n\in Y_1$, it is follows that

$$\frac{w_n(x)-w_n(y)}{|x-y|^{\frac{N}{p(x,y)}+s_1}}\in L^{p(x,y)}(\mathcal{U}\times\mathcal{U}),\text{ and }\frac{w_n(x)-w_n(y)}{|x-y|^{\frac{N}{q(x,y)}+s_2}}\in L^{q(x,y)}(\mathcal{U}\times\mathcal{U}).$$

Finally, we get that

$$\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_{n,k}(x)-w_{n,k}(y)|^{p(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)p(x,y)}}dxdy \to \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)p(x,y)}}dxdy \text{ as } k\to\infty$$

and

$$\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_{n,k}(x)-w_{n,k}(y)|^{q(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_2)q(x,y)}}dxdy \to \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)-2}(w_n(x)-w_n(y))}{|x-y|^{(\frac{N}{q(x,y)}+s_2)q(x,y)}}dxdy \text{ as } k\to\infty.$$

Claim 2: $w_{n,k}\to w_n$ *strongly in* Y_1 *as* $k\to\infty$.

Considering the sequence defined as $v_{n,k}=w_{n,k}-w_n$. Since $w_{n,k}\to w_n$ a.e in $\mathcal{U}$ and $\{w_{n,k}\}_{k\in\mathbb{N}}$ is uniformly bounded in Y_1 and Y_2. Thanks to Brezis-Lieb Lemma in [22], we have that

$$\begin{aligned}\int_{\mathcal{U}\times\mathcal{U}} &\frac{|w_{n,k}(x)-w_{n,k}(y)|^{p(x,y)-2}(w_{n,k}(x)-w_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)p(x,y)}}dxdy\\ =\int_{\mathcal{U}\times\mathcal{U}} &\frac{|v_{n,k}(x)-v_{n,k}(y)|^{p(x,y)-2}(v_{n,k}(x)-v_{n,k}(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)p(x,y)}}dxdy\\ +\int_{\mathcal{U}\times\mathcal{U}} &\frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))}{|x-y|^{(\frac{N}{p(x,y)}+s_1)p(x,y)}}dxdy+o(1),\end{aligned} \tag{22}$$

i.e $\|w_{n,k}\|^{p^+}_{Y_1}=\|w_n\|^{p^+}_{Y_1}+\|v_{n,k}\|^{p^+}_{Y_1}+o(1)$. Similarly, we get $\|w_{n,k}\|^{q^+}_{Y_2}=\|w_n\|^{q^+}_{Y2}+\|v_{n,k}\|^{q^+}_{Y_2}+o(1)$. So, we have that

$$\begin{aligned}c+o(1)=\psi(w_{n,k})\le \frac{1}{p^+}\|v_{n,k}\|^{p^+}_{Y_1}+\frac{1}{q^+}\|v_{n,k}\|^{q^+}_{Y_2}+\frac{1}{p^+}\|w_n\|^{p^+}_{Y_1}+\frac{1}{q^+}\|w_n\|^{q^+}_{Y_2}-\int_{\mathcal{U}}G_n(x,w_n(x))dx\\ -\int_{\mathcal{U}}\mathcal{V}(x)|w_{n,k}+\frac{1}{n}|^{\sigma(x)}dx.\end{aligned} \tag{23}$$

On the other hand, using $\psi'(w_{n,k})\to 0$ as $k\to+\infty$, we have that

$$\lim_{k\to+\infty}\|v_{n,k}\|^{p^+}_{Y_1}+\|v_{n,k}\|^{q^+}_{Y_2}=\int_{\mathcal{U}}\frac{g_n(x,w_n)}{(w_n+\frac{1}{n})^{\xi(x)}}w_n(x)dx-\|w_n\|^{p^+}_{Y_1}-\|w_n\|^{q^+}_{Y_2}-\int_{\mathcal{U}}\mathcal{V}(x)|w_n+\frac{1}{n}|^{\sigma(x)}dx. \tag{24}$$

We combine (24) with $\psi(w_n)=0$, we obtain that

$$\lim_{k\to+\infty}\|v_{n,k}\|^{p^+}_{Y_1}+\|v_{n,k}\|^{q^+}_{Y_2}=0.$$

Since $\|v_{n,k}\|^{p^+}_{Y_1}$ and $\|v_{n,k}\|^{q^+}_{Y_2}$ are bounded sequence, we can write $\lim_{k\to+\infty}\|v_{n,k}\|^{p^+}_{Y_1}=a$ and $\lim_{k\to+\infty}\|v_{n,k}\|^{q^+}_{Y_2}=b$. Since $a,b\ge 0$ and $a+b=0$, we get that $a=b=0$. Finally $w_{n,k}\to w_n$ strongly in Y_1 as $k\to+\infty$. □

Now, we will use the notion of the local (m,n) linking for computing $\dim C_k(\psi,0)$.

Theorem 3.6. *The functional* ψ *has a local* $(1,1)-$ *linking at the origin.*

Proof. According to $(\mathcal{H}_3)$ and a direct computation, we have

$$\frac{n^{\xi(x)}l}{2p(x,y)}|w_n(x)|^{p^+ +1} \leq G_n(x, w_n(x)). \tag{25}$$

We define $V = \mathbb{R}$. Clearly V is a one dimensional vector space subspace of Y_1. We choose $r \in (0,1)$ such that $K_\psi \cap \overline{B_r(0)} = \{0\}$, where $B_r(0) = \{w_n \in Y_1 : \|w_n\|_{Y_1} < r\}$ and $K_\psi = \{w_n \in Y_1 : \psi^{'}(w_n) = 0\}$. We consider the set $E = V \cap \overline{B_r(0)}$ for small enough $r \in (0,1)$. Recall that on a finite-dimensional normed space, all norms are equivalent. So, by taking $r \in (0,1)$ even Smaler as necessary, we obtain that

$$\|w_n\|_{Y_1} \leq r \Rightarrow |w_n| \leq \delta \quad \text{for all} \quad w_n \in V = \mathbb{R}.$$

Then for any $w_n \in V \cap \overline{B_r(0)}$, we have

$$\begin{aligned}
\psi(w_n) &= \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(x,y)} \frac{|w_n(x) - w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1 p(x,y)}} dxdy + \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{q(x,y)} \frac{|w_n(x) - w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2 q(x,y)}} dxdy \\
&- \int_{\mathcal{U}} G_n(x, w_n(x))dx - \int_{\mathcal{U}} \frac{\mathcal{V}(x)}{\sigma(x)} |w_n(x) + \frac{1}{n}|^{\sigma(x)} dx \\
&\leq \frac{1}{p^-} \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x) - w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1 p(x,y)}} dxdy + \frac{1}{q^-} \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x) - w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2 q(x,y)}} dxdy \\
&- \frac{n^{\xi^+}l}{2(p^+ + 1)} \int_{\mathcal{U}} |w_n(x)|^{p^+ +1} dx - \frac{1}{\sigma^+} \left(\frac{1}{n}\right)^{\sigma^+} |\mathcal{U}| \\
&\leq 0.
\end{aligned}$$

Further, we consider the set

$$D = \left\{ w_n \in Y_1 : \min(\frac{1}{q^+}, \frac{1}{p^+}) \|w_n\|_{Y_1}^{p(x,y)} > \|\beta\|_\infty C(\mathcal{U}, r, N) \|w_n\|_{Y_1}^{l(x)} \frac{n^{1-\xi^-}}{1-\xi^-} + \frac{1}{\sigma^+} \eta_1 \|w_n\|_{Y_1} \right\},$$

where $l : \mathcal{U} \to (1, \infty)$ is the continuous function such that $l(x) \leq p_s^*(x)$, and $C(\mathcal{U}, r, N)$ is the positive constant. Using condition $(\mathcal{H}_1)$, (V), and Theorem 2.6, we have that for any $w_n \in D$,

$$\begin{aligned}
\psi(w_n) &= \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(x,y)} \frac{|w_n(x) - w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1 p(x,y)}} dxdy + \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{q(x,y)} \frac{|w_n(x) - w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2 q(x,y)}} dxdy \\
&- \int_{\mathcal{U}} G_n(x, w_n(x))dx - \int_{\mathcal{U}} \frac{\mathcal{V}(x)}{\sigma(x)} |w_n(x) + \frac{1}{n}|^{\sigma(x)} dx \\
&\geq \frac{1}{p^+} \|w_n\|_{Y_1}^{p(x,y)} + \frac{1}{q^+} \|w_n\|_{Y_2}^{q(x,y)} - 2\|\beta\|_\infty \|w_n\|_{Y_1}^{l(x)} - \frac{1}{\sigma^+} \eta_1 \|w_n\|_{Y_1} > 0.
\end{aligned}$$

Let $U = \overline{B_r(0)}$, $E_0 = V \cap \partial B_r(0)$, $E = V \cap \overline{B_r(0)}$, and D as above, we have that $0 \notin E_0 \subset E \subset U = \overline{B_r(0)}$ and $E_0 \cap D = \emptyset$. Therefore, we arrive the following

$$\psi_{|E} \leq 0 < \psi_{|D \cap \overline{B_r(0)}}.$$

Let Y be the topological complement of V. We have that $Y_1 = V \bigoplus Y$. So, every $w_n \in Y_1$ can be written in unique way as

$$w_n = v_n + y_n \quad \text{with} \quad v_n \in V, y_n \in Y.$$

We consider the map $h : [0,1] \times Y_1 \backslash D \to Y_1 \backslash D$ defined by

$$h(t, w_n) = (1-t)w_n + tr\frac{v_n}{\|v_n\|}.$$

We have $h(0,\mathrm{w}_n)=\mathrm{w}_n$ and $h(1,\mathrm{w}_n)=r\dfrac{\mathrm{v}_n}{\|\mathrm{v}_n\|}\in V\cap\partial B_r(0)=E_0$. It follows that E_0 is a deformation retract of $Y_1\backslash D$. Hence

$$i^*:H_0(E_0)\to H_0(Y_1)$$

i an isomorphism. Note that $E_0=\{a,-a\}$ for some $a\neq 0$. Therefore, from $\dim H_0(E_0)=2$, since $H_0(E_0)=\mathbb{R}\bigoplus\mathbb{R}$. Thus $\dim im(i^*)=2$.
The set $E=V\cap B_r(0)$ is contractible (it is an interval). By Theorem 11.5 in [19], we have that $H_0(E,E_0)=0$. Thanks to Remark 6.1.26 in [19], we get $\dim im(j^*)=1$. So, finally

$$\dim im(i^*)-\dim im(j^*)=2-1=1.$$

Thus the hypothesis of Definition 2.7 are satisfied. Hence ψ has a local $(1,1)-$ linking at 0. □

Remark 3.7. *For all $k\in\mathbb{N}$, $C_k(\psi,0)\neq 0$.*

Proof. Since ψ has a local $(1,1)$ linking at the origin. By proposition 2.1 in [28], we get that $\dim C_k(\psi,0)\geq 1$. □

Now, we will compute the group critical of ψ at infinitely.

Theorem 3.8. *Suppose that the condition $(\mathcal{H}_3)$ is satisfied. Then, there exists $k\in\mathbb{N}$ such that $C_k(\psi,\infty)=0$.*

Proof. Firstly, we prove that there exists a positive constant A such that ψ^a is homotopic to exists a constant $A>0$ such that ψ^a is homotopic to $S^1=\{\mathrm{w}_n\in Y_1:\|\mathrm{w}_n\|_{Y_1}=1\}$, for all $a<-A$. From the condition $(\mathcal{H}_3)$, it follows that

$$\begin{aligned}\psi(t\mathrm{w}_n)&=\int_{\mathcal{U}\times\mathcal{U}}\frac{t^{p(\mathrm{x},\mathrm{y})}}{p(\mathrm{x},\mathrm{y})}\frac{|\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+s_1p(\mathrm{x},\mathrm{y})}}d\mathrm{x}d\mathrm{y}+\int_{\mathcal{U}\times\mathcal{U}}\frac{t^{q(\mathrm{x},\mathrm{y})}}{q(\mathrm{x},\mathrm{y})}\frac{|\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})|^{q(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+s_2q(\mathrm{x},\mathrm{y})}}d\mathrm{x}d\mathrm{y}\\
&-\int_{\mathcal{U}}\frac{\mathcal{V}(\mathrm{x})}{\sigma(\mathrm{x})}t^{\sigma(\mathrm{x})}|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})}d\mathrm{x}-\int_{\mathcal{U}}G_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))d\mathrm{x}\\
&\leq\frac{t^{p^+}}{p^-}+\frac{t^{q^+}}{q^-}\|\mathrm{w}_n\|_{Y_2}^{q(\mathrm{x},\mathrm{y})}-\frac{lt^{p_{s_1}^{+*}}}{2p_{s_1}^{+*}}\int_{\mathcal{U}}\mathrm{w}_n(\mathrm{x})^{p_{s_1}^{+*}}d\mathrm{x}-\frac{\theta_1t^{\sigma^+}}{\sigma^+}\int_{\mathcal{U}}|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})}d\mathrm{x},\end{aligned}$$

where $p_{s_1}^{+*}=\max_{\mathrm{x}\in\mathcal{U}}p_{s_1}^*(\mathrm{x})$. Since $p_{s_1}^{+*}>p^+>q^+>\sigma^+$, we have that $\psi(t\mathrm{w}_n)\to-\infty$ as $t\to+\infty$. Let $A\in\mathbb{R}$ there exists $t\in\mathbb{R}$ such that $\|t\mathrm{w}_n\|_{Y_1}\geq B$, we have that $\psi(t\mathrm{w}_n)\leq A$. Since $\mathrm{w}_n\in S^1$, we have that

$$\begin{aligned}\frac{d}{dt}\psi(t\mathrm{w}_n)&=\int_{\mathcal{U}\times\mathcal{U}}t^{p(\mathrm{x},\mathrm{y})-1}\frac{|\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+s_1p(\mathrm{x},\mathrm{y})}}d\mathrm{x}d\mathrm{y}-\int_{\mathcal{U}}\mathrm{w}_n(\mathrm{x})\frac{g_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))}{(t\mathrm{w}_n(\mathrm{x})+\frac{1}{n})^{\xi(\mathrm{x})}}d\mathrm{x}\\
&+\int_{\mathcal{U}\times\mathcal{U}}t^{q(\mathrm{x},\mathrm{y})-1}\frac{|\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})|^{q(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+s_2q(\mathrm{x},\mathrm{y})}}d\mathrm{x}d\mathrm{y}-\int_{\mathcal{U}}\mathcal{V}(\mathrm{x})t^{\sigma(\mathrm{x})-1}|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})-1}d\mathrm{x}\\
&\leq t^{p^+-1}+t^{q^+-1}\|\mathrm{w}_n\|_{Y_2}^{q(\mathrm{x},\mathrm{y})}-\int_{\mathcal{U}}\mathrm{w}_n(\mathrm{x})\frac{g_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))}{(t\mathrm{w}_n(\mathrm{x})+\frac{1}{n})^{\xi(\mathrm{x})}}d\mathrm{x}-t^{\sigma^--1}\int_{\mathcal{U}}\mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})-1}d\mathrm{x}\\
&\leq\frac{p^+}{t}\left[A+\int_{\mathcal{U}}G_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))d\mathrm{x}-\frac{p^+}{t}\int_{\mathcal{U}}t\mathrm{w}_n(\mathrm{x})\frac{g_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))}{(t\mathrm{w}_n(\mathrm{x})+\frac{1}{n})^{\xi(\mathrm{x})}}d\mathrm{x}\right]-t^{\sigma^--1}\int_{\mathcal{U}}\mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})-1}d\mathrm{x}\\
&\leq\frac{p^+}{t}\left[A+\left(\frac{1}{\theta}-\frac{1}{p^+}\right)\int_{\mathcal{U}}t\mathrm{w}_n(\mathrm{x})\frac{g_n(\mathrm{x},t\mathrm{w}_n(\mathrm{x}))}{(t\mathrm{w}_n(\mathrm{x})+\frac{1}{n})^{\xi(\mathrm{x})}}d\mathrm{x}\right]\\
&\leq\frac{p^+}{t}\left[A+C_1\left(\frac{1}{\theta}-\frac{1}{p^+}\right)\right]-t^{\sigma^--1}\int_{\mathcal{U}}\mathcal{V}(\mathrm{x})|\mathrm{w}_n(\mathrm{x})+\frac{1}{n}|^{\sigma(\mathrm{x})-1}d\mathrm{x}\\
&<0.\end{aligned}$$

By the implicit function Theorem, there exists an unique $T \in C(S^1, \mathbb{R})$ such that for any $w_n \in S^1$,

$$\psi(T(w_n)w_n) = A.$$

For any $w_n \neq 0$, set $\tau(w_n) = \frac{1}{\|w_n\|}T(\frac{w_n}{\|w_n\|})$. Then $\tau \in C(Y_1\backslash 0, \mathbb{R})$ and for all $w_n \in Y_1$, $\psi(w_n\tau(w_n)) = A$. Moreover, if $\psi(w_n) = A$, then $\tau(w_n) = 1$. We define a function $\tau_1 : Y_1 \to \mathbb{R}$ as

$$\tau_1(w_n) := \begin{cases} \tau(w_n), & \text{if } \psi(w_n) \geq A, \\ 1, & \text{if } \psi(w_n) < A. \end{cases}$$

Since $\psi(w_n) = A$ implies that $\tau(w_n) = 1$, we conclude that $\tau_1 \in C(Y_1\backslash 0, \mathbb{R})$. Finally, we set $H : [0,1] \times Y_1\backslash 0 \to Y_1\backslash 0$ as

$$H(t, w_n) = (1-t)w_n + t\tau_1(w_n)w_n.$$

We have $H(0, w_n) = w_n$, $H(1, w_n) = \tau_1(w_n)w_n \in \psi^A$, and $H(t,.)_{|\psi^A} = id_{|\psi^A}$ for all $t \in [0,1]$. It follows that

$$\psi^A \text{ is a strong deformation retract of } Y_1\backslash 0. \tag{26}$$

We consider the radial retraction $r : Y_1 \to \mathbb{R}$ defined by

$$r(w_n) = \frac{w_n}{\|w_n\|} \text{ for all } w_n \in Y_1.$$

This map is continuous and $r_{|S^1} = id_{|S^1}$. Therefore, S^1 is a retract of $Y_1\backslash 0$. Considering the map defined by

$$h(t, w_n) = (1-t)w_n + tr(w_n) \text{ for all } (t, w_n) \in [0,1] \times Y_1\backslash 0.$$

Then, $h(0, w_n) = w_n$, $h(1, w_n) = r(w_n) \in S^1$, and $h(1,.)_{|S^1} = id_{|S^1}$. Hence, we refer that

$$S^1 \text{ is a deformation retract of } Y_1\backslash 0. \tag{27}$$

Finally, by 26 and 27 it follow that ψ^a and S^1 are homotopie equivalent. We already know that the space Y_1 is an infinite dimensional Banach space. From Remark 6.1.13 in [32], it follows that the sphere unit S^1 is contractible. So, we have that

$$H_k(Y_1, \psi^a) = H_k(Y_1, S^1) = 0 \text{ for all }, k \in \mathbb{N}.$$

Finally, we obtain that

$$C_k(\psi, \infty) = H_k(Y_1, \psi^a) = H_k(Y_1, S^1) = 0, \text{ for all } k \in \mathbb{N}. \tag{28}$$

□

Theorem 3.9. *Suppose that conditions* (V), *and* $(\mathcal{H}_1) - (\mathcal{H}_4)$ *are satisfied. Then, the problem (8) has nontrivial weak solution in* Y_1.

Proof. Since ψ has a local $(1,1)-$ linking near the origin, then $\dim C_1(\psi, 0) \geq 1$, i,e $C_k(\psi, 0) \neq 0$ for some $k \in \mathbb{N}$. Thanks to Theorem 6.2.42 in [32], there exists $w_n \in K_\psi$. □

Theorem 3.10. *Suppose that condition* (V), *and* $(\mathcal{H}_1) - (\mathcal{H}_4)$ *are satisfied. Then, the problem (8) has at least non-trivial weak solution in* Y_1.

Proof. Thanks to Theorem 3.5 ψ satisfies the Palais-Smale condition and is bounded from below and the trivial solution $w_n = 0$ is homological nontrivial and is it a minimizer. The conclusion follows from Theorem 2.1 in [28]. □

3.2. Existence of infinitely non-trivial solutions

Theorem 3.11. *Suppose that conditions* (V), *and* $(\mathcal{H}_1)-(\mathcal{H}_4)$ *are satisfied. Then, the problem (8) has infinitely non-trivial weak solutions in* Y_1.

Proof. We suppose that our problem admits three non-trivial solutions Y_1. That is $K_\psi = \{0, w_n, v_n\}$. From Morse's relation, it follows that

$$C_n(\psi, 0) = \begin{cases} \mathbb{R}, & k = m(0), \\ 0, & \text{otherwise}, \end{cases}$$

where $m(0)$ is a Morse index of 0. We use Morse's relation, we get that

$$\begin{aligned}\sum_{k\geq 0} \operatorname{rank} C_k(\psi, \infty)X^k + (1+X)Q(X) &= \sum_{k\geq 0} \operatorname{rank} C_k(\psi, 0)X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\psi, w_n)X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\psi, v_n)X^k \\ &= X^{m(0)} + 2\sum_{k\geq 0} \beta_k X^k.\end{aligned}$$

From (28), it follows that

$$(1+X)Q(X) = X^{m(0)} + 2\sum_{k\geq 0} \beta_k X^k,$$

where β_k nonnegative integer and Q is a polynomial with nonnegative integer coefficient. In particular, for $X = 1$ we have $2a = 1 + 2\sum_{k\geq 0} \beta_k$. Since $\beta_k \in \mathbb{N}$, we have that $\sum_{k\geq 0} \beta_k = +\infty$ leads to a contradiction. Thus, there exist infinitely solutions to the problem (8). □

4. Fundamental Theorem

Theorem 4.1. *Suppose that conditions* (V), *and* $(\mathcal{H}_1)-(\mathcal{H}_4)$ *are satisfied. Then, problem (3) admits an infinitely weak solutions in* Y_1.

Proof. Let $\{w_n\}_{n\in\mathbb{N}} \subset Y_1$ be the sequence of solutions to problem (8). So, we have that

$$\begin{aligned}&\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p(x,y)}}dxdy \\ &+ \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q(x,y)}}dxdy \\ &= \int_{\mathcal{U}} \frac{g_n(x, w_n(x))}{(w_n(x)+\frac{1}{n})^{\xi(x)}}\varphi(x)dx + \int_{\mathcal{U}} \mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)-2}(w_n(x)+\frac{1}{n})\varphi(x)dx, \text{ for all } \varphi \in Y_1^*.\end{aligned} \tag{29}$$

We take $\varphi = w_n$ in (29), we have that

$$\begin{aligned}&\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1p(x,y)}}dxdy + \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2q(x,y)}}dxdy \\ &= \int_{\mathcal{U}} \frac{g_n(x, w_n(x))}{(w_n(x)+\frac{1}{n})^{\xi(x)}}w_n(x)dx + \int_{\mathcal{U}} \mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)-2}(w_n(x)+\frac{1}{n})w_n(x)dx \\ &\leq \int_{\mathcal{U}} \frac{g_n(x, w_n(x))}{(w_n(x)+\frac{1}{n})^{\xi(x)}}w_n(x)dx + \int_{\mathcal{U}} \mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)}dx.\end{aligned}$$

Combining $(\mathcal{H}_1)$ with (V), it follows that

$$\begin{aligned}
&\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+s_1p(x,y)}}dxdy + \int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)}}{|x-y|^{N+s_2q(x,y)}}dxdy \\
&\leq \int_{\mathcal{U}} \frac{g_n(x,w_n(x))}{(w_n(x)+\frac{1}{n})^{\xi(x)}}w_n(x)dx + \int_{\mathcal{U}} \mathcal{V}(x)|w_n(x)+\frac{1}{n}|^{\sigma(x)}dx \\
&\leq \int_{\mathcal{U}} g(x,w_n(x))|w_n(x))|^{1-\xi(x)}dx + \eta_1\|w_n\|_{Y_1} \\
&\leq \int_{\mathcal{U}} \beta(x)(1+|w_n(x))|^{r(x)-1)})|w_n(x)|^{1-\xi(x)}dx + \eta_1\|w_n\|_{Y_1}.
\end{aligned}$$

Since $r(x)-1 \leq r(x)-\xi(x)$ for all $x \in \mathcal{U}$, we get that

$$\|w_n\|_{Y_1} \leq \frac{\|\beta\|_\infty C(r(x),p(x,y),q_1(x),\xi(x),s_1,\mathcal{U})}{1-\eta_1}.$$

Hence, the sequence $\{w_n\}_{n\in\mathbb{N}}$ is bounded in Y_1. Since Y_1 is a reflexive Banach space, up to a subsequence, still denoted by $\{w_n\}$ such that $w_n \rightharpoonup w$ weakly in Y_1, $w_n \to w$ strongly in $L^{a(x)}(\mathcal{U})$ for $1 \leq a(x) < p^*_{s_1}(x)$, and $w_n \to w$ a.e in $\mathcal{U}$. A similar discussion as in Theorem 3.5 gives that

$$\begin{aligned}
&\lim_{n\to\infty}\Big[\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{p(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p(x,y)}}dxdy \\
&+\int_{\mathcal{U}\times\mathcal{U}} \frac{|w_n(x)-w_n(y)|^{q(x,y)-2}(w_n(x)-w_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q(x,y)}}dxdy\Big] \\
&=\int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p(x,y)}}dxdy \\
&+\int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x)-w(y)|^{q(x,y)-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q(x,y)}}dxdy.
\end{aligned} \tag{30}$$

Since $w_n(x) > 0$, we get that

$$\Big|\frac{g_n(x,w_n(x))\varphi(x)}{(\frac{1}{n}+w_n(x))^{\xi(x)}}\Big| \leq |g(x,w(x))\varphi(x)|.$$

From the dominated converge theorem, it follows that

$$\lim_{n\to+\infty}\int_{\mathcal{U}} \frac{g_n(x,w_n(x))\varphi(x)}{(\frac{1}{n}+w_n(x))^{\xi(x)}}dx = \int_{\mathcal{U}} \frac{g(x,w(x))\varphi(x)}{(w(x))^{\xi(x)}}dx.$$

Similarly, we prove that

$$\lim_{n\to\infty}\int_{\mathcal{U}} \mathcal{V}(x)|w_n+\frac{1}{n}|^{\sigma(x)-2}(w_n+\frac{1}{n})\varphi(x)dx = \int_{\mathcal{U}} \mathcal{V}(x)|w|^{\sigma(x)-2}w(x)\varphi(x)dx. \tag{31}$$

Finally, passing to the limit in (29), we deduce that

$$\begin{aligned}
&\int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x)-w(y)|^{p(x,y)-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p(x,y)}}dxdy \\
&+\int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x)-w(y)|^{q(x,y)-2}(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q(x,y)}}dxdy \\
&=\int_{\mathcal{U}} \frac{g(x,w(x))\varphi(x)}{(w(x))^{\xi(x)}}dx + \int_{\mathcal{U}} \mathcal{V}(x)|w|^{\sigma(x)-2}w(x)\varphi(x)dx, \text{ for all } \varphi \in Y_1^*,
\end{aligned} \tag{32}$$

namely w is a weak solution to (3). □

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