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Incorporating non-Gaussian stochastic disturbances in panel time series models to account for structural breaks

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Abstract. This paper addresses scenarios in real-valued series, such as exchange rate and gross production rate, where non-normal disturbances are occurred due to structural breaks resulting from events like policy changes, market fluctuations, financial burdens, stock price effects, etc. To justify this, spherically symmetric families of distributions have been explored. We propose a flexible covariate panel autoregressive model that accounts for multiple structural breaks when the error distribution belongs to the family of spherically symmetric distributions. Bayesian estimation and testing methodology are introduced to estimate model parameters and detect significant deviations from normality using a multivariate t-distribution for the analysis. The comparison using symmetric loss functions for parameter estimation and the classical maximum likelihood estimation is discussed, with the Bayes factor being calculated to deliver significant evidence of the error distribution. A simulation study and an analysis of practical economic series are provided to illustrate the performance of the proposed model.

1. Introduction

Researchers provided numerous time series models in the universe, ranging from univariate to multivariate, to identify the significant elements of the data generation process and improve prediction, see [6, 7, 18] and references cited therein. Various linked variables, such as explanatory variables, covariates, and others, are also incorporated into the time series model to increase efficiency and applied to various real-world applications [8, 26, 44]. However, other influence factors, such as external disturbances, irregular dynamic structure, outliers, change point analysis and asymmetric characteristics of the series, are equally important to explain the more realistic presentation of the given process, see [1, 3, 16, 28, 39, 41]. These elements and variables similarly affect the distribution of disturbances from normal to non-normal shape. Various univariate and multivariate time series models based on non-normal errors are already available in the literature and have been validated by numerous practical applications [25, 35, 37]. However, more improvements are required in this area to handle various aspects of real-world applications when disturbances are spherically symmetric, long, or heavy tail shapes. In this paper, a spherically symmetric family of distributions is used in the error component to model panel data time series in the presence of structural

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break and covariate, as well as to characterize its conclusions from a Bayesian perspective. In time series, a structural break is one of the consequences of abnormal conditions in error terms because it breaks the series in different intervals and each interval may or may not follow the normality assumption. Researchers in numerous fields, such as economics, agriculture, environment, finance, and engineering, have explored the structural break with panel data time series models. When the relationship between the variables or parameters changes, a structural break occurs, and it is critical to conduct a thorough analysis/modelling of a given series that contains a single structural break or multiple structural breaks to make stronger inferences. Therefore, several studies consider a single and multiple breaks in level/trend/autoregressive coefficient or/and error variance for making significant inferences in univariate, panel, or vector autoregressive time series models [2, 5, 24, 33, 36, 41]. The change of error distribution from normal to non-normal also happened due to other associated explanatory and covariates series. The associated covariates are variables related to the original time series and depend on their prior data, like an autoregressive process. As a result, a greater impact on the time series due to covariate has been recognized than the explanatory variable. This concept is well discussed by several researchers under unit root hypothesis, estimation purpose, and prediction, see [5, 11, 13].

The references [10, 23] are the first to derive the features of error distribution in terms of spherically symmetric form. In the setting of a regression model with spherically symmetric error, [40] produced maximum likelihood estimators and testing procedures. [20] has discussed Bayesian prediction in a regression model when the error is spherically distributed. [42] has determined the hypothesis testing for a linear regression model when the error follows a class of spherical distributions. [43] has introduced the performance of the positive-rule Stein-type ridge estimator under spherically symmetric error disturbances in a linear regression model. [15] has considered the canonical form of the general linear model with a spherically symmetric error density around a mean vector. [34] has determined the Bayesian estimators of the parameters of the state-space model with disturbances following asymmetric family of distributions. Kock and Eggers [12] proposed Bayesian variable selection for linear parameterizations with error based on a spherically symmetric distribution. [25] has developed the Bayesian procedure in the presence of spherical symmetric error for the covariate autoregressive model and applied it to real-life effective exchange rate series. Therefore, the study addresses spherically symmetric disturbances in a covariate panel autoregressive model with several structural breaks and defines the substantial impact with normal error. Most of the existing literature [9, 31, 38] used the transformation method to change the error structure from non-normal to normal. Therefore, there is a loss of information in the transformation method. As a result, an alternative method can be used, such as a non-normal distribution like chi-square, Cauchy, or uniform distribution, to make the error into a normal or symmetric distribution family. As a result, it is simple for the researcher to deal with a non-normal error without losing information, and it provides a better predictive model for future series.

The rest of the paper is organized as follows. Section 2 provides the model specification based on structural breaks, covariate, and a spherical symmetric error. The basics of Bayesian analysis for estimation and testing are presented in Section 3. Sections 4 and 5 offer simulation results and a practical application based on economic data demonstrating the improved use of the suggested model and approach. Finally, the conclusion is defined in Section 6.

2. Model Specification

Let $\{y_{it},\ i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T\}$ be a panel data time series with cross-sectional units, and each cross-sectional unit has T size of time series. Multiple structural breaks are considered in a covariate panel autoregressive model with breaks at the intercept term, autoregressive coefficients, and covariate series coefficients. Then, the form of the model is:

$$y_{it} = \begin{cases} \alpha_{i1} + \sum_{j=1}^{p_1} \beta_{ij}^{(1)} y_{i,t-j} + \sum_{k=1}^{q_1} \gamma_{ik}^{(1)} w_{i,t-k} + \varepsilon_{it} & 0 < t < T_1 \\ \vdots & \\ \alpha_{ir} + \sum_{j=1}^{p_r} \beta_{ij}^{(r)} y_{i,t-j} + \sum_{k=1}^{q_r} \gamma_{ik}^{(r)} w_{i,t-k} + \varepsilon_{it} & T_{r-1} < t < T_r, \\ \vdots & \\ \alpha_{is} + \sum_{j=1}^{p_s} \beta_{ij}^{(s)} y_{i,t-j} + \sum_{k=1}^{q_s} \gamma_{ik}^{(s)} w_{i,t-k} + \varepsilon_{it} & T_{s-1} < t < T_s \\ \alpha_{is+1} + \sum_{j=1}^{p_{s+1}} \beta_{ij}^{(s+1)} y_{i,t-j} + \sum_{k=1}^{q_{s+1}} \gamma_{ik}^{(s+1)} w_{i,t-k} + \varepsilon_{it} & T_s < t < T \end{cases} \tag{1}$$

where s is the number of structural breaks as $r = 1, 2, 3, \ldots, s$, α_{ir} is the intercept term, $\beta_{ip_r}^{(r)}$ is the autoregressive coefficient at $lag(p_r)$ and $\gamma_{iq_r}^{(r)}$ is the covariate coefficient at $lag(q_r)$ for the i^{th} cross-sectional unit at r^{th} break. It is assumed that the error is normally distributed in most of the existing literature [[44],[41],[5]]. However, this situation is not always satisfied for many practical applications, such as stock market series, gross domestic product rate, and so forth, because extreme values, outliers, or non-asymptotic conditions divert the observation from symmetric to other forms depending on kurtosis. As a result, these applications must be handled by assuming that errors are distributed according to other symmetric distributions like Student's t-distribution, Laplace distribution, and others. One of the symmetric classes is a spherically symmetric distribution where the error distribution is spherically symmetric with the density:

$$f(\varepsilon_{it}) = \int_0^\infty \frac{\delta^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} \psi(\zeta)} \exp\left[-\frac{\delta}{2\psi^2(\zeta)} \varepsilon_{it}^2\right] dG(\zeta), \tag{2}$$

where $\psi(\zeta)$ is a positive measurable function, $\frac{1}{\delta}$ is the scale parameter and $G(\zeta)$ is a distribution function of ζ. [23] has derived the properties of this distribution and [10] has summarized their literature review. In matrix notation, the model defined by (1) is presented as

$$Y = (L \otimes I_n)\alpha + X\beta + W\gamma + \varepsilon, \tag{3}$$

with

$$f(\varepsilon) = \int_0^\infty \frac{\delta^{\frac{nT}{2}}}{(2\pi)^{\frac{nT}{2}} \psi^T(\zeta)} \exp\left[-\frac{\delta}{2\psi^2(\zeta)} \varepsilon' \varepsilon\right] dG(\zeta), \tag{4}$$

where $\otimes$ is the Kronecker product and

$$\begin{array}{ll} y_i = \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{iT} \end{pmatrix}, & Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}', \\ \alpha_i = \begin{pmatrix} \alpha_{i1} & \cdots & \alpha_{ir} & \cdots & \alpha_{is+1} \end{pmatrix}, & \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}', \\ \beta_i^{(r)} = \begin{pmatrix} \beta_{i1}^{(r)} & \beta_{i2}^{(r)} & \cdots & \beta_{ip_r}^{(r)} \end{pmatrix}, & \beta_i = \begin{pmatrix} \beta_i^{(1)} & \cdots & \beta_i^{(r)} & \cdots & \beta_i^{(s+1)} \end{pmatrix}, \\ \beta = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}', & \gamma_i^{(r)} = \begin{pmatrix} \gamma_{i1}^{(r)} & \gamma_{i2}^{(r)} & \cdots & \gamma_{iq_r}^{(r)} \end{pmatrix}, \\ \gamma_i = \begin{pmatrix} \gamma_i^{(1)} & \cdots & \gamma_i^{(r)} & \cdots & \gamma_i^{(s+1)} \end{pmatrix}, & \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}', \end{array}$$

$$X_i^{(r)} = \begin{pmatrix} y_{i,T_{r-1}} & y_{i,T_{r-1}-1} & \cdots & y_{i,T_{r-1}+1-p_r} \\ y_{i,T_{r-1}+1} & y_{i,T_{r-1}} & \cdots & y_{i,T_{r-1}+2-p_r} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i,T_r-1} & y_{i,T_r-2} & \cdots & y_{i,T_r-p_r} \end{pmatrix},$$

$$X_i = \begin{pmatrix} X_i^{(1)} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & X_i^{(r)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & X_i^{(s+1)} \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{n,} \end{pmatrix},$$

$$W_i^{(r)} = \begin{pmatrix} w_{i,T_{r-1}} & w_{i,T_{r-1}-1} & \cdots & w_{i,T_{r-1}+1-q_r} \\ w_{i,T_{r-1}+1} & w_{i,T_{r-1}} & \cdots & w_{i,T_{r-1}+2-q_r} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,T_r-1} & w_{i,T_r-2} & \cdots & w_{i,T_r-q_r} \end{pmatrix},$$

$$W_i = \begin{pmatrix} W_i^{(1)} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & W_i^{(r)} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & W_i^{(s+1)} \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_n \end{pmatrix},$$

$$\varepsilon_i = \begin{pmatrix} \varepsilon_{i1} & \varepsilon_{i2} & \cdots & \varepsilon_{iT} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{pmatrix}',$$

L be a $T \times 1$ vector with all elements 1 and I_n is the identity matrix of order n.

3. Bayesian Inference

Under the Bayesian approach, the observed data information is combined with a prior belief about an unknown parametric function. Given the parameter facts, the data information is defined using the likelihood function. For the proposed model, the likelihood function is given by

$$L(Y \mid \alpha, \beta, \gamma, \delta) = \int_0^\infty \frac{\delta^{\frac{nT}{2}}}{(2\pi)^{\frac{nT}{2}} \psi^{nT}(\zeta)} \exp\begin{bmatrix} -\frac{\delta}{2\psi^2(\zeta)} (Y - (L \otimes I_n)\alpha - X\beta - W\gamma)' \\ (Y - (L \otimes I_n)\alpha - X\beta - W\gamma) \end{bmatrix} dG(\zeta). \tag{5}$$

The knowledge about the parameters known as prior distribution delineated using some functional forms. The following multivariate prior distributions are considered as follows:
$\alpha \sim MN_{n(s+1)}\left(\alpha_0, \frac{1}{\delta}V_1^{-1}\right)$, V_1^{-1} is a symmetric positive definite matrix,
$\beta \sim MN_{nP}\left(\beta_0, \frac{1}{\delta}V_2^{-1}\right)$, V_2^{-1} is a symmetric positive definite matrix,
$\gamma \sim MN_{nQ}\left(\gamma_0, \frac{1}{\delta}V_3^{-1}\right)$, V_3^{-1} is a symmetric positive definite matrix,
$\delta \sim \chi^2(v)$, v is a number of degree of freedom.
Here , $MN_{n(s+1)}$, MN_{nP}, and MN_{nQ} denote multivariate normal distributions with $n(s+1)$, nP, and nQ components, respectively, and their respective mean vectors are α_0, β_0, and γ_0, where $P = \sum_{r=1}^{s+1} p_r$ defined in β prior, $Q = \sum_{r=1}^{s+1} q_r$ defined in γ prior and $V_i = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n}$, $i = 1, 2, 3$.

The joint prior distribution has the expression

$$\pi(\alpha,\beta,\gamma,\delta)=\frac{\delta^{\frac{n(s+P+Q+1)+v}{2}-1}\left|V_1\right|^{\frac{1}{2}}\left|V_2\right|^{\frac{1}{2}}\left|V_3\right|^{\frac{1}{2}}}{(2\pi)^{\frac{n(s+P+Q+1)}{2}}2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\exp\left[-\frac{\delta}{2}\left\{\begin{array}{l}(\alpha-\alpha_0)'\,V_1^{-1}\,(\alpha-\alpha_0)+(\beta-\beta_0)'\,V_2^{-1}\,(\beta-\beta_0)\\+(\gamma-\gamma_0)'\,V_3^{-1}\,(\gamma-\gamma_0)+1\end{array}\right\}\right]. \quad (6)$$

With the help of the likelihood function in (5) and the joint prior distribution in (6), the posterior distribution is

$$\begin{aligned}\pi(\alpha,\beta,\gamma,\delta\mid Y)=&\frac{\delta^{\frac{n(T+s+P+Q+1)+v}{2}-1}\left|V_1\right|^{\frac{1}{2}}\left|V_2\right|^{\frac{1}{2}}\left|V_3\right|^{\frac{1}{2}}}{(2\pi)^{\frac{n(T+s+P+Q+1)}{2}}2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)}\int_0^\infty\frac{1}{\psi^{nT}(\zeta)}\exp\left[-\frac{\delta}{2\psi^2(\zeta)}\left\{(Y-(L\otimes I_n)\,\alpha-X\beta-W\gamma)'\right.\right.\\&(Y-(L\otimes I_n)\,\alpha-X\beta-W\gamma)+\psi^2(\zeta)\,(\alpha-\alpha_0)'\,V_1^{-1}\,(\alpha-\alpha_0)+\psi^2(\zeta)\,(\beta-\beta_0)'\,V_2^{-1}\\&\left.\left.(\beta-\beta_0)+\psi^2(\zeta)\,(\gamma-\gamma_0)'\,V_3^{-1}\,(\gamma-\gamma_0)+\psi^2(\zeta)\right\}\right]dG(\zeta).\end{aligned} \quad (7)$$

3.1. Bayesian Estimation

The Bayes estimator is obtained based on the loss function using the joint posterior distribution given in equation (7). The loss/utility function is a specific risk adopted in a specific context for each parameter in decision theory. Here, we only consider two symmetric loss functions to determine a better optimum result. These loss functions are the squared error loss function (SELF) and the absolute loss function (ALF). Mathematical formulations for deriving Bayes estimators based on these loss functions are extremely complicated because numerous integrations are involved in these expressions. Therefore, the Markov Chain Monte Carlo (MCMC) technique is applied to generate the estimated values of Bayes estimators based on posterior distribution and loss function. For this, the conditional posterior distribution for all model parameters is derived. Since the conditional posterior distribution of each parameter involves integration concerning $dG(\zeta)$. In this situation and for computational purposes, we consider ζ as a parameter with a density function that follows a chi-square distribution with u degree of freedom $\pi(\zeta)\sim\chi^2(u)$ and $\psi^2(\zeta)=(\zeta/u)^{-\frac{1}{2}}$. The chi-square distribution is chosen because it deals with a single unknown parameter (degree of freedom), which depends upon the observations. So, it is better to analyse and make inferences with different degrees of freedom. The conditional posterior density functions for each parameter are then defined as follows:

$$\begin{aligned}\alpha\mid Y,\beta,\gamma,\delta&\sim MN\left(B_1A_1^{-1},\frac{u}{\delta\zeta}A_1^{-1}\right),\\\beta\mid Y,\alpha,\gamma,\delta&\sim MN\left(B_2A_2^{-1},\frac{u}{\delta\zeta}A_2^{-1}\right),\\\gamma\mid Y,\alpha,\beta,\delta&\sim MN\left(B_3A_3^{-1},\frac{u}{\delta\zeta}A_3^{-1}\right),\\\delta\mid Y,\alpha,\beta,\gamma&\sim \text{Gamma}\left(\frac{n(T+s+P+Q+1)+v}{2},\frac{A_4}{2}\right),\end{aligned}$$

where

$$
\begin{aligned}
A_1 &= (L \otimes I_n)' (L \otimes I_n) + \frac{u}{\zeta} V_1^{-1}, \\
B_1 &= (Y - X\beta - W\gamma)' (L \otimes I_n) + \frac{u}{\zeta} \alpha_0' V_1^{-1}, \\
A_2 &= X'X + \frac{u}{\zeta} V_2^{-1}, \\
B_2 &= (Y - (L \otimes I_n)\alpha - W\gamma)' X + \frac{u}{\zeta} \beta_0' V_2^{-1}, \\
A_3 &= W'W + \frac{u}{\zeta} V_3^{-1}, \\
B_3 &= (Y - (L \otimes I_n)\alpha - X\beta)' W + \frac{u}{\zeta} \gamma_0' V_3^{-1}, \\
A_4 &= \frac{\zeta}{u} \left[(Y - (L \otimes I_n)\alpha - X\beta - W\gamma)' (Y - (L \otimes I_n)\alpha - X\beta - W\gamma) \right] + (\alpha - \alpha_0)' V_1^{-1} (\alpha - \alpha_0) \\
&\quad + (\beta - \beta_0)' V_2^{-1} (\beta - \beta_0) + (\gamma - \gamma_0)' V_3^{-1} (\gamma - \gamma_0) + 1.
\end{aligned}
$$

It is observed that the conditional posterior distributions for all model parameters are conditionally in the standard distribution. Thus, the Gibbs sampler method is utilized to generate samples from a closed-form posterior distribution.

3.2. *Bayesian Testing Procedure*

The Bayes factor (BF) is used in decision-making for hypothesis testing problem/model selection procedures from a Bayesian perspective. Bayes factor (BF_{10}) is the likelihood ratio of the marginal probability $P(y|.)$ for the observed series under the alternative (H_1) and the null (H_0) hypothesis. For the proposed study, the hypotheses under null and alternative are:
H_0: Errors are distributed normally ,
H_1: Errors are distributed spherically symmetric.
Thus, the posterior probability under the null hypothesis is

$$
P(H_0 \mid Y) = \frac{|V_1|^{\frac{1}{2}} |V_2|^{\frac{1}{2}} |V_3|^{\frac{1}{2}} \Gamma\left(\frac{nT+v}{2}\right)}{\pi^{\frac{nT}{2}} \Gamma\left(\frac{v}{2}\right) |A_{21}|^{\frac{1}{2}} |C_{21}|^{\frac{1}{2}} |E_{21}|^{\frac{1}{2}} [G_{21}]^{\frac{nT+v}{2}}},
$$

and the alternative hypothesis is

$$
P(H_1 \mid Y) = \frac{|V_1|^{\frac{1}{2}} |V_2|^{\frac{1}{2}} |V_3|^{\frac{1}{2}} \Gamma\left(\frac{nT+v}{2}\right)}{\pi^{\frac{nT}{2}} \Gamma\left(\frac{v}{2}\right)} \int_0^\infty \frac{|A_{11}|^{-\frac{1}{2}} |C_{11}|^{-\frac{1}{2}} |E_{11}|^{-\frac{1}{2}} [G_{11}]^{-\frac{nT+v}{2}}}{\psi^{n(T-s-P-Q-1)}(\zeta)} dG(\zeta),
$$

where

$$
\begin{aligned}
A_{11} &= (L \otimes I_n)' (L \otimes I_n) + \psi^2(\zeta) V_1^{-1}, \\
B_{11} &= (Y - X\beta - W\gamma)' (L \otimes I_n) + \psi^2(\zeta) \alpha_0' V_1^{-1}, \\
C_{11} &= X'X + \psi^2(\zeta) V_2^{-1} - X' (L \otimes I_n) A_{11}^{-1} (L \otimes I_n)' X, \\
D_{11} &= (Y - W\gamma)' X + \psi^2(\zeta) \beta_0' V_2^{-1} - \left((Y - W\gamma)' (L \otimes I_n) + \frac{u}{\zeta} \alpha_0' V_1^{-1} \right)' A_{11}^{-1} (L \otimes I_n)' X, \\
E_{11} &= W'W + \psi^2(\zeta) V_3^{-1} - W' (L \otimes I_n) A_{11}^{-1} (L \otimes I_n)' W - W' \left(I - (L \otimes I_n)' A_{11}^{-1} (L \otimes I_n) \right) X \\
&\quad C_{11}^{-1} W' \left(I - (L \otimes I_n)' A_{11}^{-1} (L \otimes I_n) \right) X,
\end{aligned}
$$

$$
\begin{aligned}
F_{11} &= Y'W + \psi^2(\zeta)\gamma_0' V_3^{-1} - \left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right) A_{11}^{-1}\left(L\otimes I_n\right)' W \\
&- \left(Y'X + \psi^2(\zeta)\beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right) A_{11}^{-1}\left(L\otimes I_n\right)' X\right) C_{11}^{-1} W'\left(I - \left(L\otimes I_n\right)' A_{11}^{-1}\left(L\otimes I_n\right)\right) X, \\
G_{11} &= Y'Y + \psi^2(\zeta)\left(\alpha_0' V_1^{-1}\alpha_0 + \beta_0' V_2^{-1}\beta_0 + \gamma_0' V_3^{-1}\gamma_0 + 1\right) - \left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right) A_{11}^{-1} \\
&\left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right)' - \left(Y'X + \psi^2(\zeta)\beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right) A_{11}^{-1}\left(L\otimes I_n\right)' X\right) C_{11}^{-1} \\
&\left(Y'X + \psi^2(\zeta)\beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \psi^2(\zeta)\alpha_0' V_1^{-1}\right) A_{11}^{-1}\left(L\otimes I_n\right)' X\right)' - F_{11} E_{11}^{-1} F_{11}', \\
A_{21} &= \left(L\otimes I_n\right)'\left(L\otimes I_n\right) + V_1^{-1}, \\
B_{21} &= \left(Y - X\beta - W\gamma\right)'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}, \\
C_{21} &= X'X + V_2^{-1} - X'\left(L\otimes I_n\right) A_{21}^{-1}\left(L\otimes I_n\right)' X, \\
E_{21} &= W'W + V_3^{-1} - W'\left(L\otimes I_n\right) A_{21}^{-1}\left(L\otimes I_n\right)' W - W'\left(I - \left(L\otimes I_n\right)' A_{21}^{-1}\left(L\otimes I_n\right)\right) X \\
&\quad C_{21}^{-1} W'\left(I - \left(L\otimes I_n\right)' A_{21}^{-1}\left(L\otimes I_n\right)\right) X, \\
D_{21} &= \left(Y - W\gamma\right)' X + \beta_0' V_2^{-1} - \left(\left(Y - W\gamma\right)'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(L\otimes I_n\right)' X, \\
F_{21} &= Y'W + \gamma_0' V_3^{-1} - \left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(L\otimes I_n\right)' W \\
&- \left(Y'X + \beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(L\otimes I_n\right)' X\right)' C_{21}^{-1} W'\left(I - \left(L\otimes I_n\right)' A_{21}^{-1}\left(L\otimes I_n\right)\right) X, \\
G_{21} &= Y'Y + \alpha_0' V_1^{-1}\alpha_0 + \beta_0' V_2^{-1}\beta_0 + \gamma_0' V_3^{-1}\gamma_0 + 1 - \left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right)' \\
&- \left(Y'X + \beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(L\otimes I_n\right)' X\right) C_{21}^{-1} \\
&\left(Y'X + \beta_0' V_2^{-1} - \left(Y'\left(L\otimes I_n\right) + \alpha_0' V_1^{-1}\right) A_{21}^{-1}\left(L\otimes I_n\right)' X\right)' - F_{21} E_{21}^{-1} F_{21}'.
\end{aligned}
$$

Since, the null hypothesis error is based on normal distribution. So, we easily get the closed form expression of null hypothesis, whereas the alternative hypothesis depends upon the distribution of spherically symmetric, which is not known in advance. So, the closed form expression is not achieved. For better understanding, one can refer [25] and [2].
For the proposed model, BF_{10} is expressed as

$$
BF_{10} = \frac{P\left(H_1 \mid Y\right)}{P\left(H_0 \mid Y\right)} = \left|A_{21}\right|^{\frac{1}{2}}\left|C_{21}\right|^{\frac{1}{2}}\left|E_{21}\right|^{\frac{1}{2}}\left[G_{21}\right]^{\frac{nT+v}{2}} \int_0^\infty \frac{\left|A_{11}\right|^{-\frac{1}{2}}\left|C_{11}\right|^{-\frac{1}{2}}\left|E_{11}\right|^{-\frac{1}{2}}\left[G_{11}\right]^{-\frac{nT+v}{2}}}{\psi^{n(T-s-P-Q-1)}(\zeta)} dG(\zeta).
$$

The fact says that BF_{10} is more than 20 provide strong evidence for rejecting the null hypothesis. [22] has provided a rule of thumb to interpret the magnitude of the BF using the transformation $2\ln(BF_{10})$. The Monte Carlo methods are employed to handle this integral problem with known information regarding spherical symmetric distribution, number of structural break and location of structural break. In this context, we generate random samples from the spherical symmetric distribution by using random command in R. Then, we use these samples to estimate the integral via Monte Carlo integration techniques.

4. Simulation Study

An artificial series is formed from the model in the simulation research, considering the true value of model parameters and defining the initial value of the remaining variables. The size of the time series (T) is 40 with two cross-sectional units ($n = 2$). In the series, two structural breaks ($s = 2$) are considered and the locations of breakpoints are taken from various sets of combinations $(\frac{T}{4}, \frac{T}{2}), (\frac{T}{2}, \frac{3T}{4})$, and $(\frac{T}{4}, \frac{3T}{4})$ to obtain better outcomes. For making the computation simple, we assume that lag (order) of each observed and covariate series at each break interval is equal to one, that is $p_1 = q_1 = p_2 = q_2 = p_3 = q_3 = 1$. For a series generation, the true parameter values are $(\alpha_{11}, \alpha_{12}) = (1, 3), (\alpha_{21}, \alpha_{22}) = (5, 2), (\alpha_{31}, \alpha_{32}) = (3, 3), (\beta^1_{11}, \beta^1_{21}) = (0.1, 0.5), (\beta^2_{11}, \beta^2_{21}) = (0.1, 0.3), (\beta^3_{11}, \beta^3_{21}) = (0.35, 0.15), (\gamma^1_{11}, \gamma^1_{21}) = (0.3, 0.4), (\gamma^2_{11}, \gamma^2_{21}) = (0.6, 0.2),$

$(\gamma_{11}^3, \gamma_{21}^3) = (0.4, 0.5)$ and $\delta = 5$. To acquire a more generalized inference, various degrees of freedom $u = (5, 10, 20, 30, 50)$ are considered to obtain the estimates and testing of the hypothesis. For Bayesian computation, the 11000 cycles are generated, and 1000 cycles are burn-in period. The complete process is repeated 1000 times and records the average results. The steps for Metropolis and Gibbs algorithm are as follows:
Step 1: Specify the panel time series model, including the specific form of the non-Gaussian stochastic disturbances and the structural break model.
Step 2: Define the prior distributions for the model parameters.
Step 3: Initialize the parameters for the MCMC algorithm.
Step 4: Iterate through the following steps until convergence.
Step 5: Propose new candidate values for the parameters and calculate the acceptance probability based on the target distribution and the proposal distribution.
Step 6: Accept or reject the proposed values based on the acceptance probability.
Step 7: After convergence, analyze the posterior distribution of the parameters to estimate their values and assess their uncertainty.
To examine the convergence of Markov Chain, we summarize the Gelman-Rubin convergence test (GRCT) and the Geweke diagnostic test (GDT) p-value in Table 1 for all parameters. Form Table 1, it is observed that the statistic value of the GRCT is almost equal to one and GDT p-value is greater than 0.05 so the simulated series is converged. The results are shown in Tables 2-4 in terms of average estimate (AE) for various degrees of freedom and breakpoint locations. The mean squared error (MSE) is shown in Figures 1-19 to compare different estimates evaluated in this study.

Table 1: Convergence and stationary test of proposed model parameters .

Parameter (Value)	GRCT	P-value (GDT)	Parameter (Value)	GRCT	P-value (GDT)	Parameter (Value)	GRCT	P-value (GDT)
$\alpha_{11}(1)$	1.0267	0.1651	$\alpha_{21}(5)$	1.0203	0.217	$\alpha_{31}(3)$	1.0126	0.1836
$\alpha_{12}(3)$	1.0101	0.2134	$\alpha_{22}(2)$	0.9989	0.3015	$\alpha_{32}(3)$	1.0325	0.2538
$\beta_{11}^1(0.1)$	1.0311	0.3475	$\beta_{11}^2(0.1)$	1.0238	0.2743	$\beta_{11}^3(0.35)$	1.0261	0.1302
$\beta_{21}^1(0.5)$	1.0245	0.1781	$\beta_{21}^2(0.3)$	1.0191	0.1852	$\beta_{21}^3(0.15)$	1.0221	0.174
$\gamma_{11}^1(0.3)$	1.0091	0.1334	$\gamma_{11}^2(0.6)$	1.0111	0.2032	$\gamma_{11}^3(0.4)$	1.0176	0.2541
$\gamma_{21}^1(0.4)$	1.0167	0.2614	$\gamma_{21}^2(0.2)$	1.0215	0.3278	$\gamma_{21}^3(0.5)$	1.0195	0.2137

From Tables 2-4, we observe that the AEs of the parameters are close to the true parameter values. Moreover, as the degree of freedom increases, the AE values move closer to the true values and thus provide more efficient estimates for all model parameters. Different breakpoint positions are also important when determining the most efficient estimates using various estimators. Figures 1-19 show that for various combinations of breakpoint locations, the MSE of all estimates decreases as the degree of freedom increases. This shows that higher values of degree of freedom result in better-estimated values for all estimators. MSEs of Bayes estimates are lower than MLEs in all parameters. Hence, Bayes estimates perform well when estimating parameters. There is a significant difference in the MSE of MLEs and Bayes estimators for small degrees of freedom, but only a tiny difference in their MSE under the normal level. As a result, this simulation study demonstrates that the degree of freedom, that is, error distribution, plays a vital role in obtaining superior parameter estimates using the derived estimates.

The BF is used to choose the suggested model, and the result is shown in Table 5. Models with spherically symmetric error provide significant BF support when compared to models with normal error. Because the spherically symmetric distribution approximates the normal distribution at larger degrees of freedom, the probability of rejecting the null hypothesis decreases as the degree of freedom increases.

Table 2: MLE and Bayes estimate when T=40 and u=10 .

(T_1, T_2)	(10,20)			(10,30)			(20,30)		
Parameter (Value)	MLE	SELF	ALF	MLE	SELF	ALF	MLE	SELF	ALF
$\alpha_{11}(1)$	1.3342	1.3032	1.3235	1.3280	1.2935	1.3153	1.1531	1.1422	1.1482
$\alpha_{12}(3)$	3.1687	3.1566	3.1624	3.1579	3.1479	3.1552	3.1770	3.1716	3.1734
$\alpha_{21}(5)$	5.2930	5.2848	5.2891	5.2840	5.2266	5.2645	5.2803	5.2279	5.2620
$\alpha_{22}(2)$	2.5725	2.4882	2.4424	2.5837	2.4032	2.4537	2.4339	2.4100	2.4258
$\alpha_{31}(3)$	3.0459	2.9919	3.0274	2.7864	2.7755	2.7821	3.0300	2.9822	3.0120
$\alpha_{32}(3)$	3.2458	3.2353	3.2414	3.1975	3.1539	3.1820	3.1769	3.1353	3.1590
$\beta^1_{11}(0.1)$	0.1150	0.1101	0.1089	0.1094	0.1050	0.1040	0.1069	0.1019	0.1017
$\beta^1_{21}(0.5)$	0.4682	0.4692	0.4688	0.4733	0.4737	0.4737	0.4657	0.4665	0.4660
$\beta^2_{11}(0.1)$	0.1255	0.1263	0.1257	0.1118	0.1125	0.1133	0.1123	0.1137	0.1137
$\beta^2_{21}(0.3)$	0.2610	0.2660	0.2654	0.2586	0.2639	0.2626	0.2776	0.2783	0.2788
$\beta^3_{11}(0.35)$	0.3274	0.3276	0.3276	0.3347	0.3356	0.3352	0.3278	0.3281	0.3280
$\beta^3_{21}(0.15)$	0.1434	0.1442	0.1438	0.1370	0.1390	0.1389	0.1416	0.1449	0.1439
$\gamma^1_{11}(0.3)$	0.3348	0.3299	0.3347	0.3344	0.3297	0.3342	0.3203	0.3190	0.3200
$\gamma^1_{21}(0.4)$	0.4065	0.4081	0.4063	0.4074	0.4067	0.4075	0.4078	0.4069	0.4074
$\gamma^2_{11}(0.6)$	0.6183	0.6184	0.6185	0.6332	0.6290	0.6342	0.6317	0.6274	0.6320
$\gamma^2_{21}(0.2)$	0.2287	0.2209	0.2300	0.2295	0.2240	0.2310	0.2213	0.2181	0.2215
$\gamma^3_{11}(0.4)$	0.4282	0.4212	0.4286	0.4209	0.4204	0.4213	0.4269	0.4241	0.4276
$\gamma^3_{21}(0.5)$	0.5219	0.5214	0.5218	0.5331	0.5263	0.5325	0.5358	0.5351	0.5362
$\delta(5)$	4.4430	4.4840	4.4850	4.4484	4.4883	4.4732	4.4480	4.4742	4.4840

Table 3: MLE and Bayes estimate when T=40 and u=30 .

(T_1, T_2)	(10, 20)			(10, 30)			(20, 30)		
Parameter (Value)	MLE	SELF	ALF	MLE	SELF	ALF	MLE	SELF	ALF
$\alpha_{11}(1)$	1.3137	1.3045	1.3104	1.3168	1.3096	1.3136	1.1399	1.1375	1.1390
$\alpha_{12}(3)$	3.1416	3.1383	3.1401	3.1273	3.1259	3.1263	3.1717	3.1699	3.1705
$\alpha_{21}(5)$	5.2569	5.2532	5.2554	5.2137	5.1961	5.2090	5.2450	5.2284	5.2403
$\alpha_{22}(2)$	2.5660	2.3520	2.4880	2.5060	2.4274	2.4424	2.4261	2.4200	2.4239
$\alpha_{31}(3)$	3.0008	2.9889	2.9961	2.7644	2.7619	2.7630	3.0038	2.9905	2.9986
$\alpha_{32}(3)$	3.2253	3.2213	3.2240	3.4524	3.4390	3.4480	3.4493	3.4360	3.4451
$\beta^1_{11}(0.1)$	0.1173	0.1092	0.1086	0.1157	0.1081	0.1070	0.1144	0.1092	0.1084
$\beta^l_{21}(0.5)$	0.4733	0.4738	0.4736	0.4782	0.4784	0.4783	0.4693	0.4696	0.4694
$\beta^2_{11}(0.1)$	0.1212	0.1113	0.1113	0.1223	0.1222	0.1226	0.1173	0.1172	0.1175
$\beta^2_{21}(0.3)$	0.2655	0.2675	0.2667	0.2666	0.2684	0.2678	0.2804	0.2809	0.2809
$\beta^3_{11}(0.35)$	0.3285	0.3286	0.3285	0.3389	0.3394	0.3391	0.3283	0.3284	0.3284
$\beta^3_{21}(0.15)$	0.1481	0.1481	0.1482	0.1464	0.1465	0.1468	0.1463	0.1467	0.1467
$\gamma^l_{11}(0.3)$	0.3331	0.3325	0.3334	0.3345	0.3347	0.3345	0.3204	0.3200	0.3202
$\gamma^1_{21}(0.4)$	0.4045	0.4051	0.4046	0.4081	0.4082	0.4082	0.4002	0.4004	0.4002
$\gamma^2_{11}(0.6)$	0.6165	0.6161	0.6166	0.6284	0.6260	0.6284	0.6291	0.6265	0.6292
$\gamma^2_{21}(0.2)$	0.2275	0.2266	0.2278	0.2305	0.2283	0.2307	0.2184	0.2177	0.2183
$\gamma^3_{11}(0.4)$	0.4279	0.4277	0.4281	0.4210	0.4212	0.4210	0.4265	0.4253	0.4267
$\gamma^3_{21}(0.5)$	0.5196	0.5189	0.5196	0.5314	0.5300	0.5315	0.5332	0.5316	0.5330
$\delta(5)$	5.2840	5.1299	5.0436	5.2768	5.1284	5.0444	5.2596	5.1237	5.0378

Table 4: MLE and Bayes estimate when T=40 and u=50 .

(T_1, T_2)	$(10, 20)$			$(10, 30)$			$(20, 30)$		
Parameter (Value)	MLE	SELF	ALF	MLE	SELF	ALF	MLE	SELF	ALF
$\alpha_{11}(1)$	1.2985	1.2953	1.2968	1.3126	1.3078	1.3109	1.1508	1.1495	1.1505
$\alpha_{12}(3)$	3.1592	3.1572	3.1582	3.1224	3.1224	3.1222	3.1561	3.1557	3.1556
$\alpha_{21}(5)$	5.2792	5.2773	5.2782	5.2204	5.2116	5.2179	5.1952	5.1886	5.1930
$\alpha_{22}(2)$	2.4410	2.4277	2.4363	2.4706	2.4571	2.4661	2.4287	2.4254	2.4277
$\alpha_{31}(3)$	2.9932	2.9850	2.9901	2.7696	2.7683	2.7689	3.0143	3.0039	3.0111
$\alpha_{32}(3)$	3.2151	3.2137	3.2145	3.4428	3.4368	3.4404	3.4590	3.4502	3.4564
$\beta_{11}^1(0.1)$	0.1227	0.1205	0.1202	0.1182	0.1173	0.1169	0.1102	0.1102	0.1102
$\beta_{21}^1(0.5)$	0.4702	0.4705	0.4703	0.4785	0.4786	0.4785	0.4706	0.4709	0.4706
$\beta_{11}^2(0.1)$	0.1278	0.1179	0.1179	0.1212	0.1202	0.1203	0.1248	0.1151	0.1150
$\beta_{21}^2(0.3)$	0.2697	0.2707	0.2704	0.2613	0.2623	0.2620	0.2792	0.2794	0.2794
$\beta_{11}^3(0.35)$	0.3286	0.3287	0.3287	0.3386	0.3388	0.3386	0.3280	0.3281	0.3281
$\beta_{21}^3(0.15)$	0.1399	0.1411	0.1416	0.1482	0.1488	0.1485	0.1442	0.1442	0.1445
$\gamma_{11}^1(0.3)$	0.3329	0.3331	0.3329	0.3355	0.3346	0.3355	0.3204	0.3203	0.3205
$\gamma_{21}^1(0.4)$	0.4065	0.4065	0.4067	0.4072	0.4075	0.4073	0.4068	0.4074	0.4070
$\gamma_{11}^2(0.6)$	0.6164	0.6160	0.6164	0.6270	0.6262	0.6271	0.6290	0.6287	0.6290
$\gamma_{21}^2(0.2)$	0.2264	0.2254	0.2267	0.2273	0.2264	0.2276	0.2196	0.2190	0.2195
$\gamma_{11}^3(0.4)$	0.4256	0.4250	0.4257	0.4188	0.4193	0.4188	0.4289	0.4277	0.4288
$\gamma_{21}^3(0.5)$	0.5204	0.5202	0.5204	0.5303	0.5305	0.5304	0.5356	0.5350	0.5357
$\delta(5)$	5.4437	5.1400	5.0430	5.4592	5.1443	5.0476	5.4784	5.1487	5.0520

Table 5: BF for the proposed null and alternative hypothesis .

u	$\left(\frac{T}{4}, \frac{T}{2}\right)$	$\left(\frac{T}{2}, \frac{3T}{4}\right)$	$\left(\frac{T}{4}, \frac{3T}{4}\right)$
5	516.9778	498.0591	524.8904
10	318.5348	294.2397	320.1745
20	201.3627	198.8024	199.2185
30	93.9109	91.2651	94.1693
50	32.1709	29.0707	36.5292

5. Real-life Data Analysis

To study the performance of the proposed model and elucidate the methodology, we observe the annual series of gross domestic product (GDP) and real-life effective exchange rate (REER) of the Brazilian and Indian countries from 1961 to 2020. This series is collected from "The World Bank" data source and contains the yearly observations for all countries. It is known that the GDP measures a country's economy and is linked to different factors, such as imports, exports, interest rates, etc. Brazil and India are the cross-sectional units in the proposed research, and their GDP data are the observed panel series. The REER series from both nations are included as covariate variables, demonstrating a significant impact on the GDP series. A normality test is executed on each country's GDP and REER series, and the output is displayed in Table 6, all to ensure the assumption of normality. The Shapiro-Wilk test p-value is highly significant; hence, the GDP and REER series of India and Brazil are not normally distributed. This indicates that the series has switched from a symmetric to an asymmetric shape, as indicated by the skewness value also.

Table 6: Descriptive statistics and normality test for real-life data series .

Series	Country	Mean	Standard deviation	Skewness	Kurtosis	Normality P-value
GDP	Brazil	3874.3605	3696.9298	1.1179	0.1642	< 0.0001
	India	553.1443	561.8869	1.4640	0.9591	< 0.0001
REER	Brazil	88.6897	16.4329	0.3361	-0.5919	0.0002
	India	149.5900	60.8546	0.7108	-0.6056	0.0006

For breakpoint detection, the in-built command "breakpoints" described in "strucchange" package in R-language [45] is used. This used model selection criteria to determine the optimal number and positions of breakpoints. It is easily used to penalize models with more breakpoints or excessive complexity, encouraging the selection of simpler models that adequately explain the structural changes. The methodology presented by [4] and [5] was used to estimate breakpoints in a time series regression model. Table 7 shows the number of breaks and the most acceptable break locations when this method is used.

In both series, two common breakpoints in 1989 and 2008 are established for each country, as shown in Table 7. The reasons behind these years for change point were banking recession and economic crisis. In 1989 especially, a banking recession occurred in many countries, and then a common and similar structural break was identified due to abrupt changes in GDP and REER series. This was well discussed in [17] and [14]. The second common break year is 2008, which occurred due to the global financial and economic crisis, as reported in [19, 21, 32]. The order of the process before and after breaks is calculated using the in-built

Table 7: Summary of break years in the GDP and REER series .

Series	Country	Number of Breaks	T_1	T_2	T_3	T_4
GDP	Brazil	3	1969	1989	2008	NA
	India	4	1989	1997	2008	2014
REER	Brazil	4	1983	1989	1999	2008
	India	4	1976	1981	1989	2008

Table 8: Order (lag) selection for the process based on AIC.

Break interval		1961 – 1989			1990 – 2008			2008 – 2020		
Series	Country	lag 1	lag 2	lag 3	lag 1	lag 2	lag 3	lag 1	lag 2	lag 3
GDP	Brazil	371.10	372.89	374.89	313.02	314.22	316.05	197.06	197.51	199.47
	India	235.87	236.73	237.47	197.57	201.20	205.10	127.57	129.45	129.82
REER	Brazil	203.56	196.02	197.33	154.37	156.18	157.78	85.80	87.16	88.67
	India	239.17	238.64	239.28	118.90	120.67	122.20	77.89	79.88	81.57

command in R software based on the minimal value of the Akaike information criterion (AIC) described in Table 8, utilizing only these breakpoints (1989, 2008) for the investigation.

Table 8 demonstrates that the process order is one for all break intervals in both countries' GDP and REER series, except the break interval (1961-1989) for the REER series, which is of order two. The BF is 39.4601 if the error follows a multivariate Student's t-distribution with u=10 degrees of freedom. This value indicates that the error in real-life economic series belongs to the spherical symmetric family. The suggested model's parameters are then estimated, and the results are recorded in Table 9. Because negative (positive) values are related to the covariate coefficients, Table 9 reveals that the Brazilian GDP and REER series have a negative relationship. In contrast, the Indian series has a positive relationship. Most of the remaining coefficients express a positive sign between the study series. Table 10 presents a comparison of different models fitted to this dataset. Among the considered models, the proposed model stands out as the most favourable choice, as it exhibits the lowest AIC and BIC values, indicating a superior balance between model fit and complexity. Additionally, it boasts the lowest MSE, signifying the highest level of predictive accuracy when compared to the other models. Hence, the proposed model appears to be the most suitable and effective for describing and predicting the dataset. Apart from the compared models, a numerical simulation technique is also available in the literature [29, 30]. Still, it needs to be considered more timely in light of the objectives set forth for the proposed study. In future, we will use various numerical simulation techniques when the objective is towards more application perspective rather than developing models.

6. Conclusion

This paper discusses spherical symmetric disturbances in the panel autoregressive model with covariate and multiple structural breaks. Parameters are estimated using the Bayesian technique under various loss functions, and testing is carried out using the Bayes factor with disturbances considered using a multivariate Student's t-distribution. The proposed model can establish the relationship of non-normal error at each break point interval when the structure of the series is also affected due to covariates. Based on the simulation results, it appears that Bayes estimates outperform MLE and correctly produce from the suggested model. This indicates that the simulated series' residuals have a spherical symmetric pattern. The application to economic series demonstrates the model's potential, demonstrating that the series has a spherically symmetric error and is well suited for this series. For numerical purposes, the chi-square distribution is considered to convert the error as a spherically symmetric distribution. However, other spherically symmetric distributions are also available to convert the non-normal error to a normal error. The future work in this direction is to create a generalized model that includes a mixture of spherically symmetric distributions and introduce the stochastic stability captures the series' change nature. Also, some different simulation methods will be used to compare the developed models based on real-world data applications.

Table 9: Estimated values of the fitted model based on MLE and Bayes estimators.

Parameter	MLE	SELF	ALF
α_{11}	0.1008	0.0890	0.0901
α_{12}	0.0690	0.0517	0.0418
α_{21}	-0.0678	-0.0780	-0.0719
α_{22}	0.6933	0.6270	0.6720
α_{31}	1.0988	0.7913	0.9622
α_{32}	-4.1595	-4.4722	-4.5920
β_{11}^1	0.8900	0.6406	0.6900
β_{21}^1	1.1319	1.2324	1.2666
β_{11}^2	0.8505	0.9004	0.8898
β_{21}^2	0.9639	0.9391	0.9557
β_{11}^3	1.2454	1.1911	1.1979
β_{21}^3	0.1390	0.1168	0.1776
γ_{11}^1	-0.3741	-0.3894	-0.3941
γ_{21}^1	0.0233	-0.4554	-0.4848
γ_{12}^1	-1.0710	-0.6440	-0.5633
γ_{22}^1	2.9973	2.1245	2.7341
γ_{11}^2	-8.6296	-5.4356	-5.3568
γ_{21}^2	-1.4491	-1.9921	-1.4263
γ_{11}^3	-19.8291	-16.3773	-17.8185
γ_{21}^3	148.2338	109.4322	113.3805
δ	4.6166	1.9382	2.0647

Table 10: Model fitting summary for the considered data set .

Model Fitting	AIC	BIC	MSE
Panel AR(1) model with multiple structural breaks, covariate, and spherical symmetric error	326.40	370.38	114.43
Panel AR(1) model with multiple structural breaks [43]	356.10	383.33	245.12
Panel AR(1) model with stationary covariates [44]	372.83	389.59	382.68
Panel AR(1) model with spherically symmetric disturbances [45]	370.95	381.42	409.84
Panel AR(1) model [46]	388.96	399.44	553.40

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Abbreviations: The following abbreviations are used in this manuscript.

MLE	Maximum likelihood estimator
AE	Average estimate
AIC	Akaike information criterion
ALF	Absolute loss function
BF	Bayes factor
GDP	Gross domestic product
GDT	Geweke diagnostic test
GRCT	Gelman-Rubin convergence test
MCMC	Markov chain Monte Carlo
MSE	Mean squared error
REER	Real-life effective exchange rate
SELF	Squared error loss function

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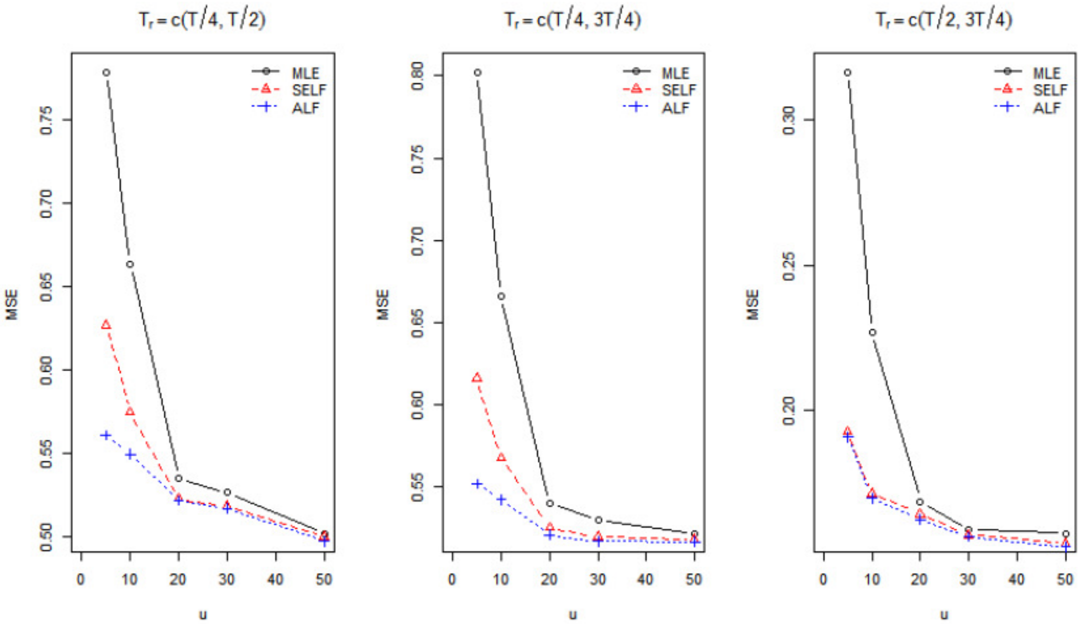

Figure .1: MSE under MLE and Bayes estimator for α_{11} with varying u and T_r .

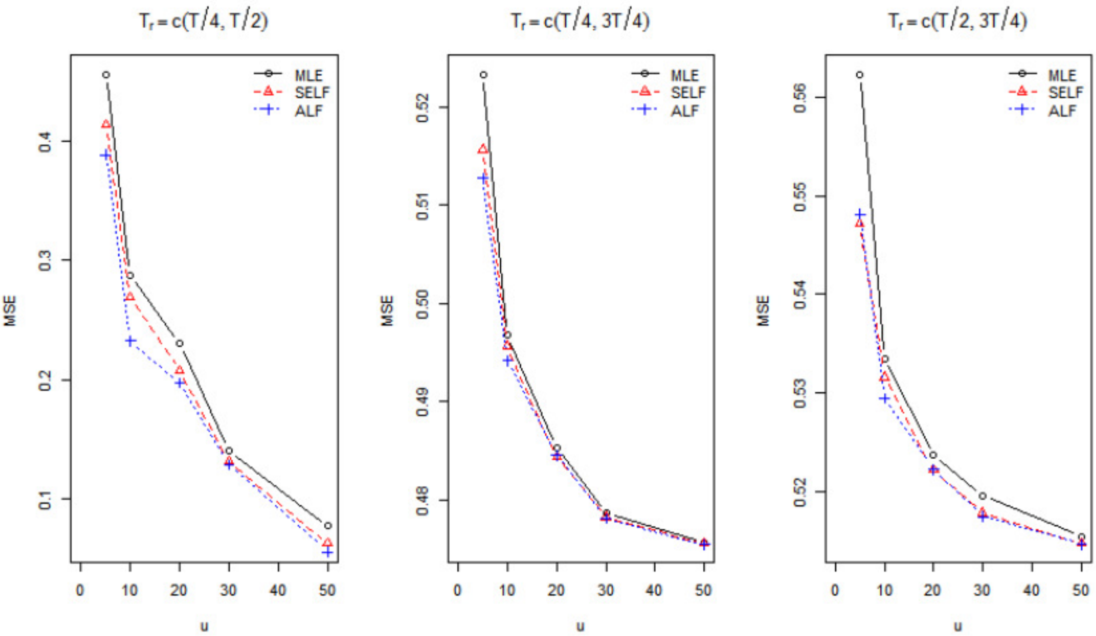

Figure .2: MSE under MLE and Bayes estimator for α_{12} with varying u and T_r .

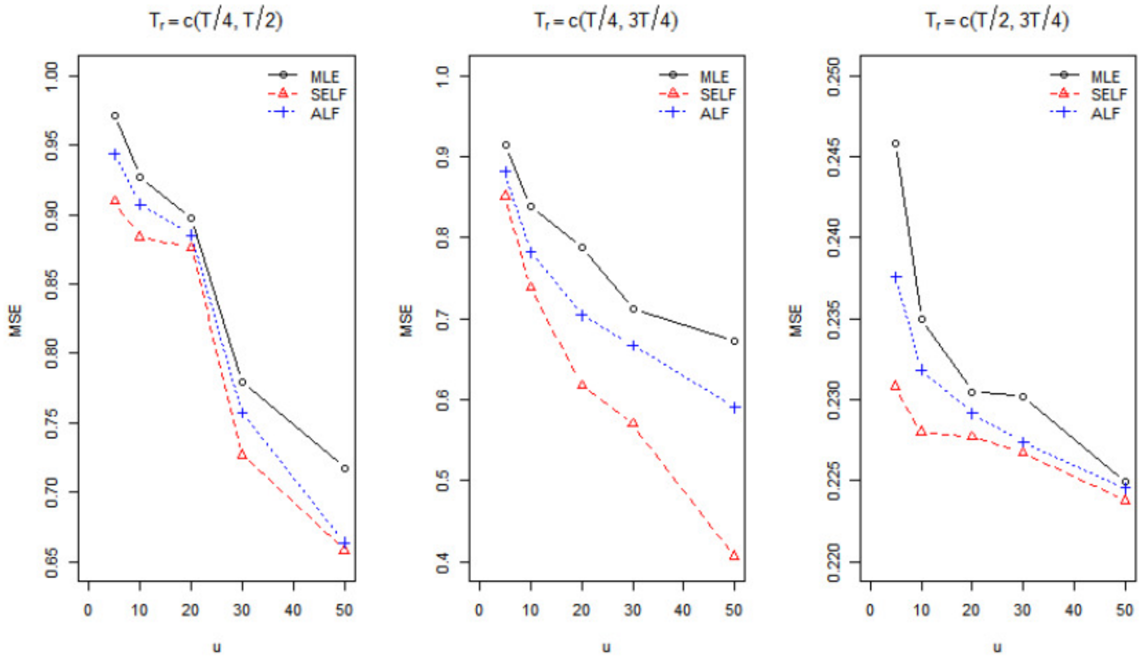

Figure .3: MSE under MLE and Bayes estimator for α_{21} with varying u and T_r .

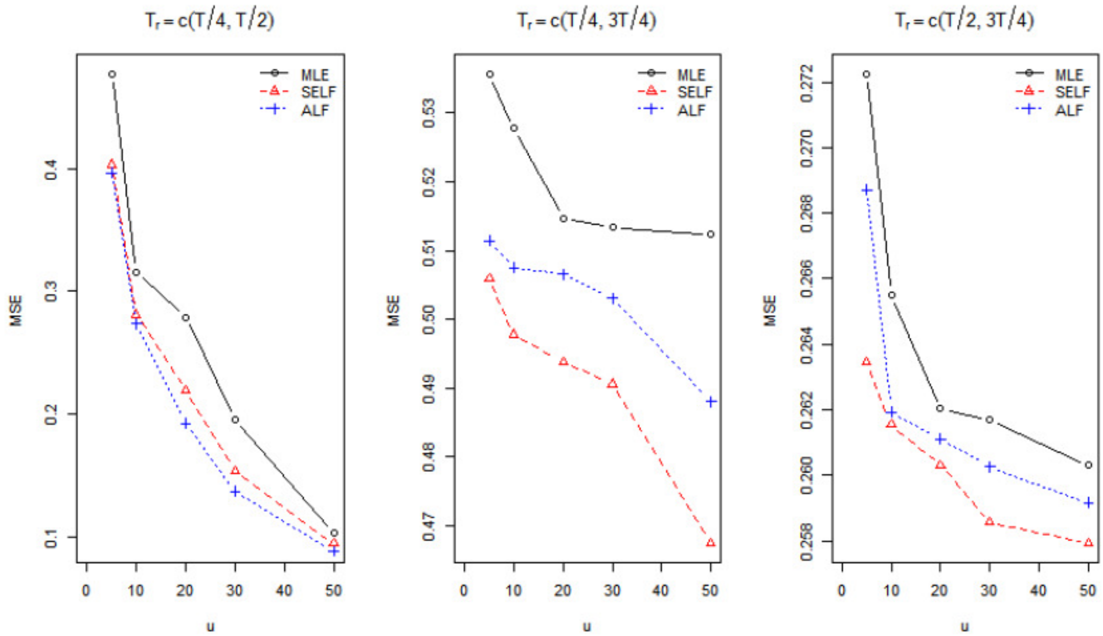

Figure .4: MSE under MLE and Bayes estimator for α_{22} with varying u and T_r .

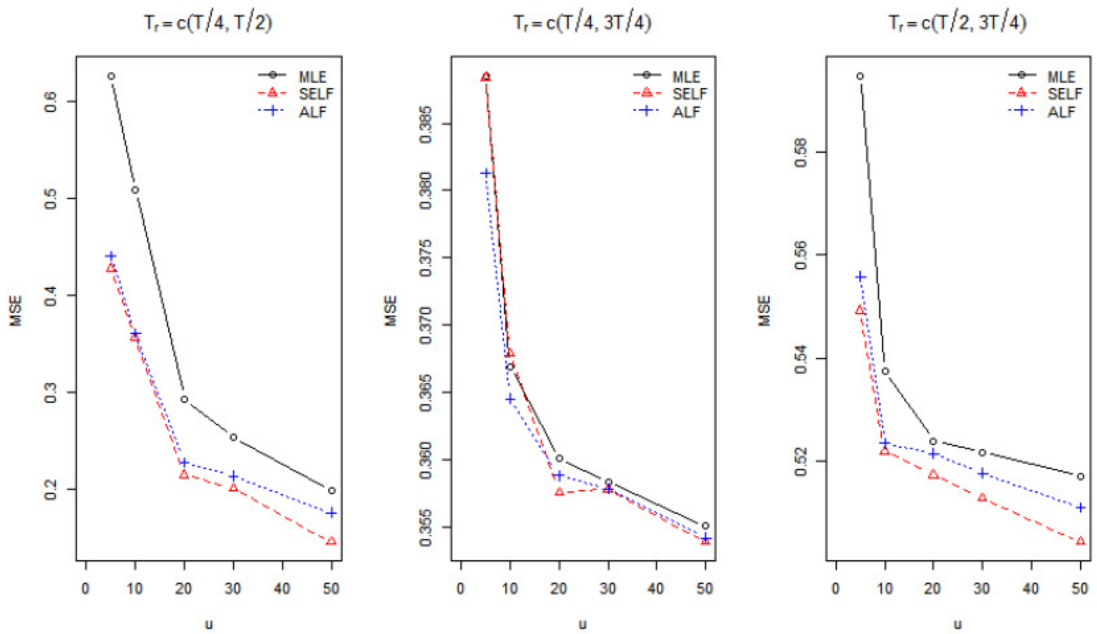

Figure .5: MSE under MLE and Bayes estimator for α_{31} with varying u and T_r .

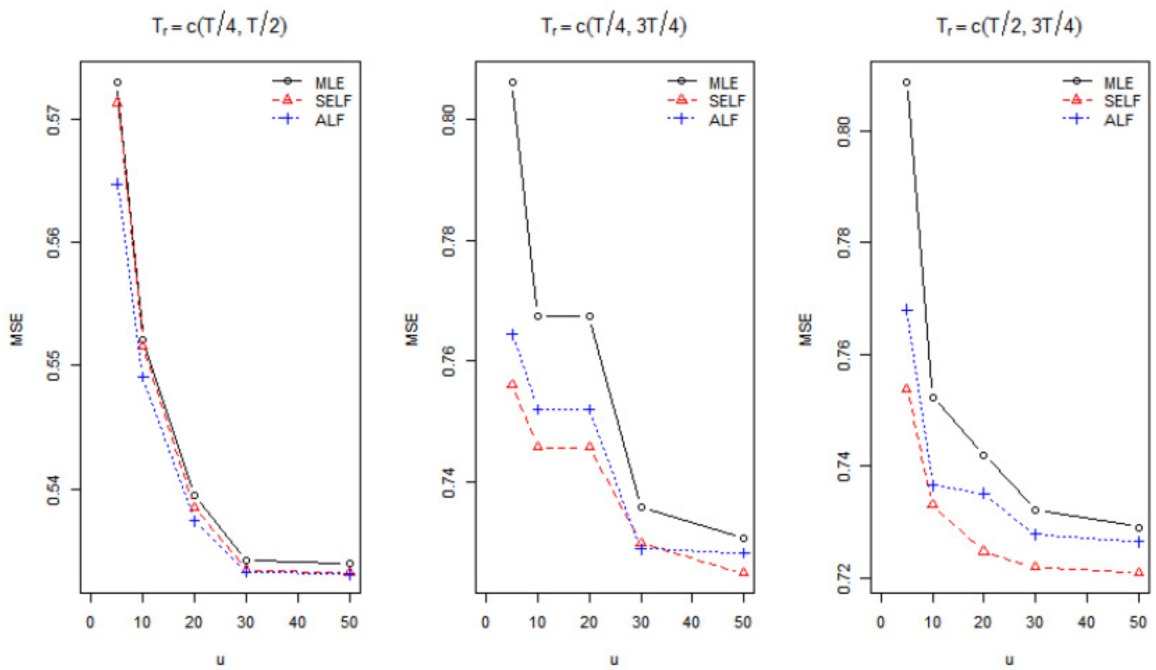

Figure .6: MSE under MLE and Bayes estimator for α_{32} with varying u and T_r .

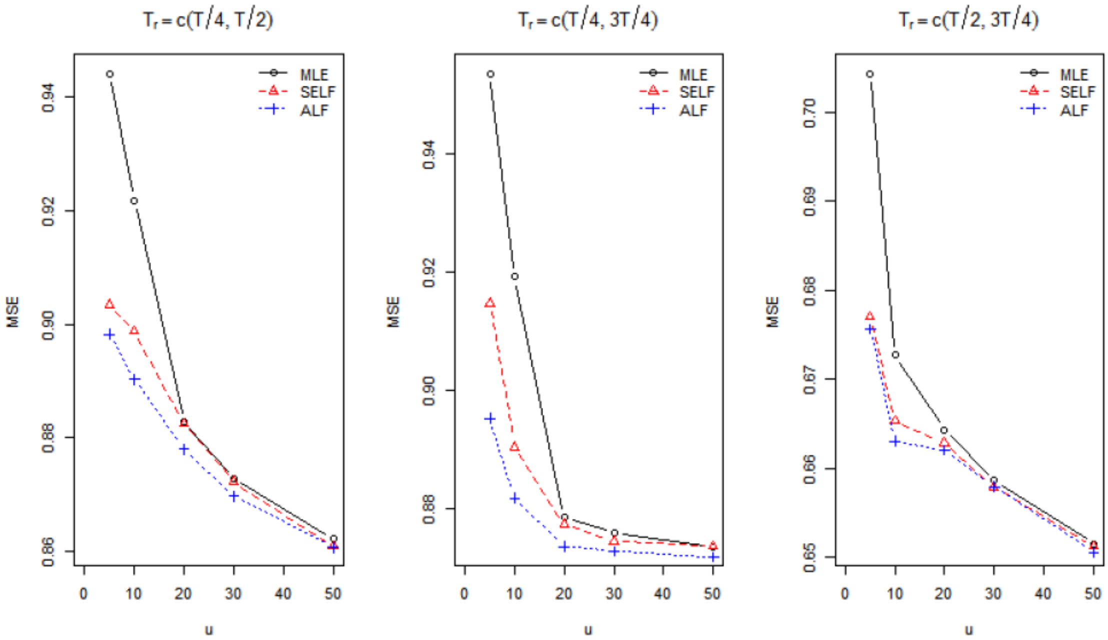

Figure .7: MSE under MLE and Bayes estimator for β_{11}^{1} with varying u and T_r .

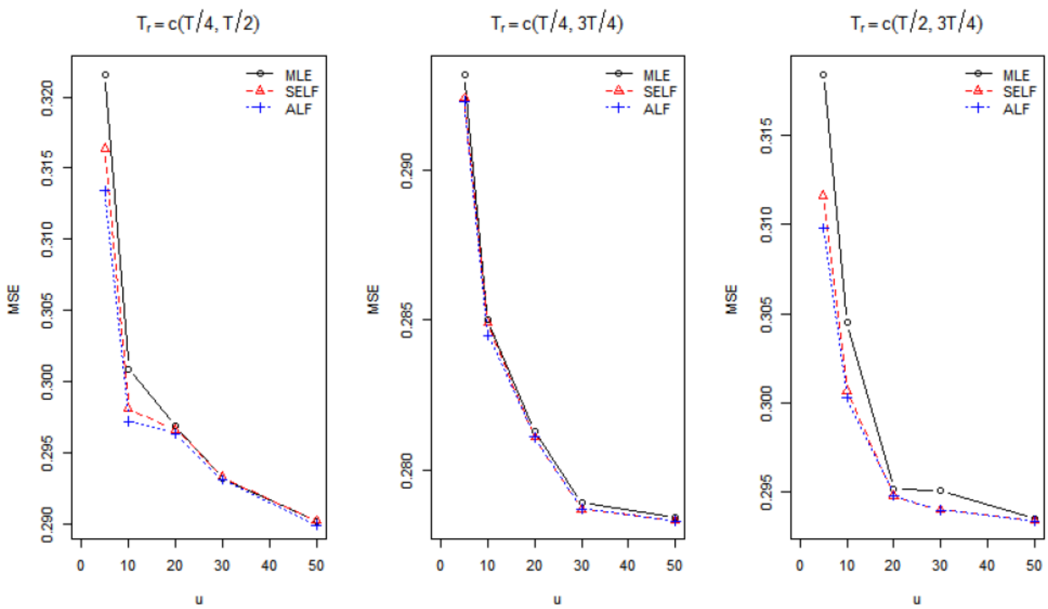

Figure .8: MSE under MLE and Bayes estimator for β_{21}^{1} with varying u and T_r .

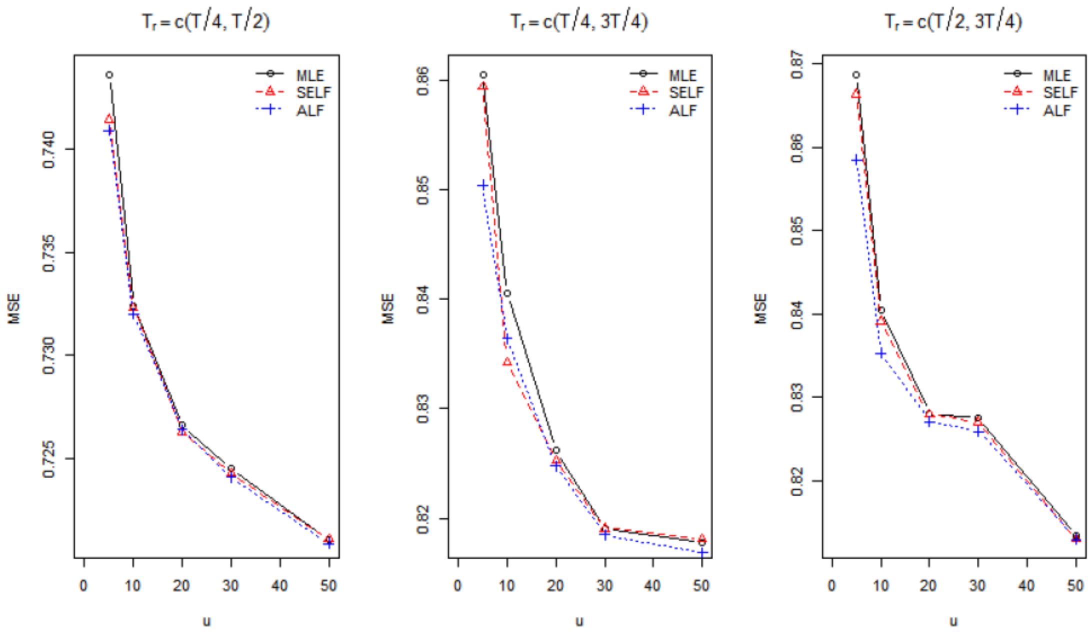

Figure .9: MSE under MLE and Bayes estimator for β^2_{11} with varying u and T_r .

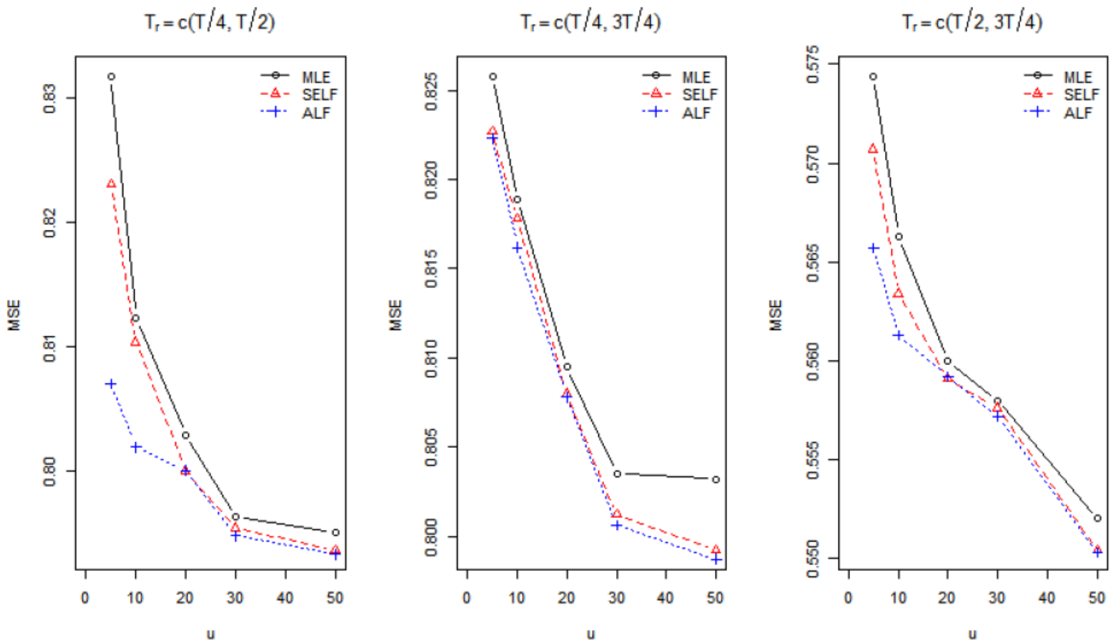

Figure .10: MSE under MLE and Bayes estimator for β^2_{21} with varying u and T_r .

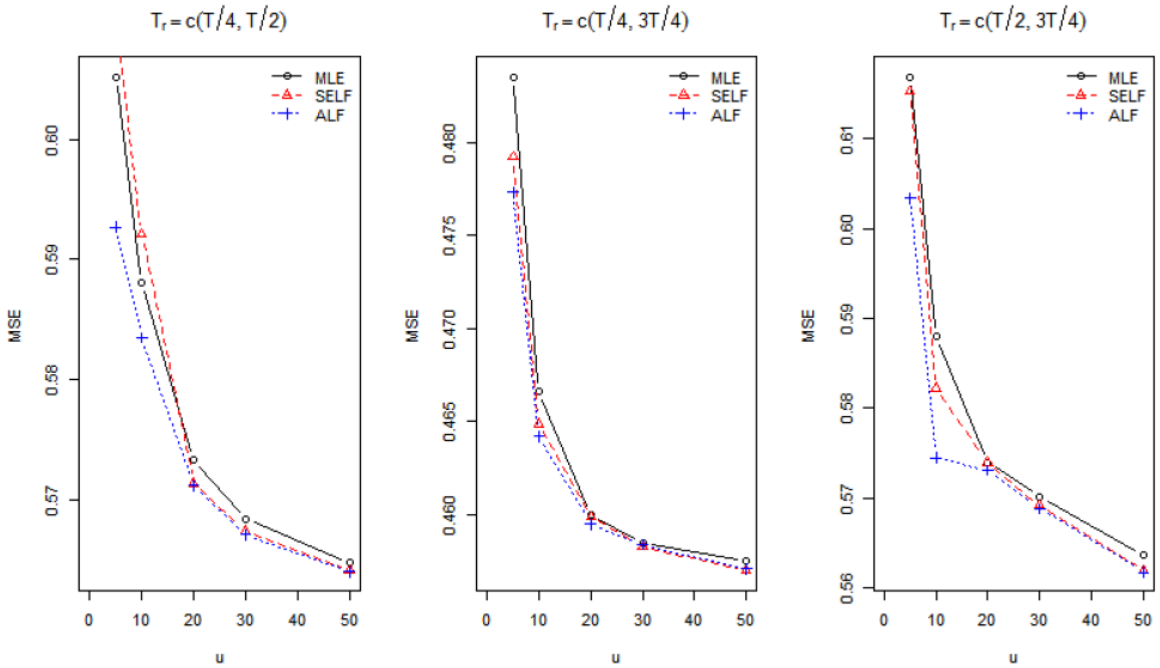

Figure .11: MSE under MLE and Bayes estimator for β_{11}^{3} with varying u and T_r .

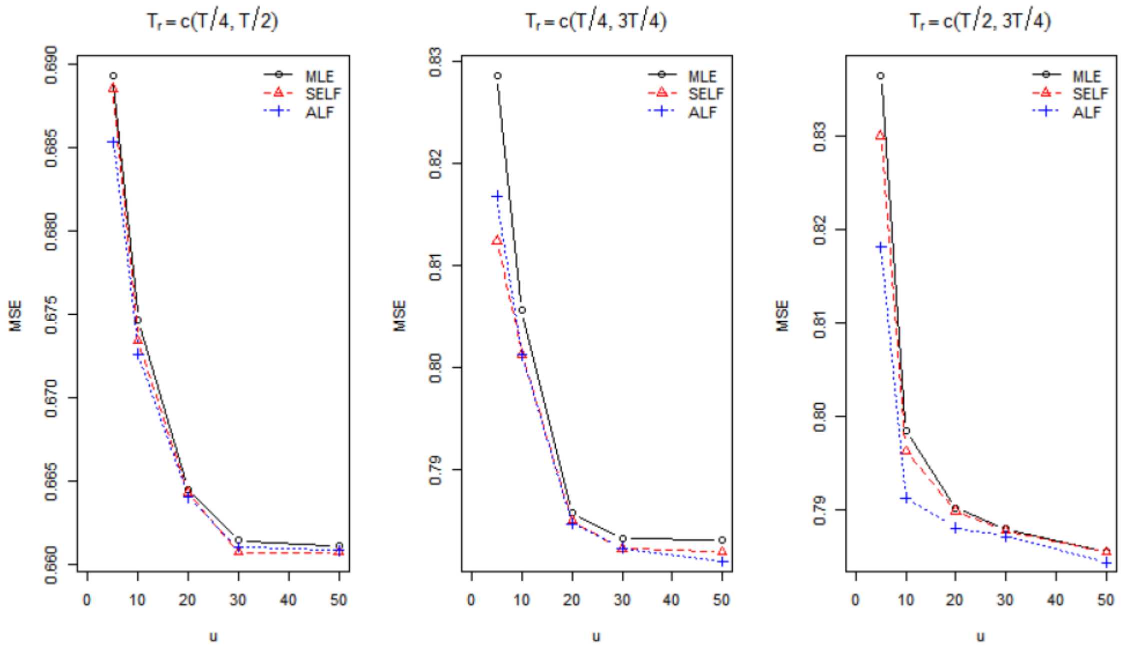

Figure .12: MSE under MLE and Bayes estimator for β_{21}^{3} with varying u and T_r .

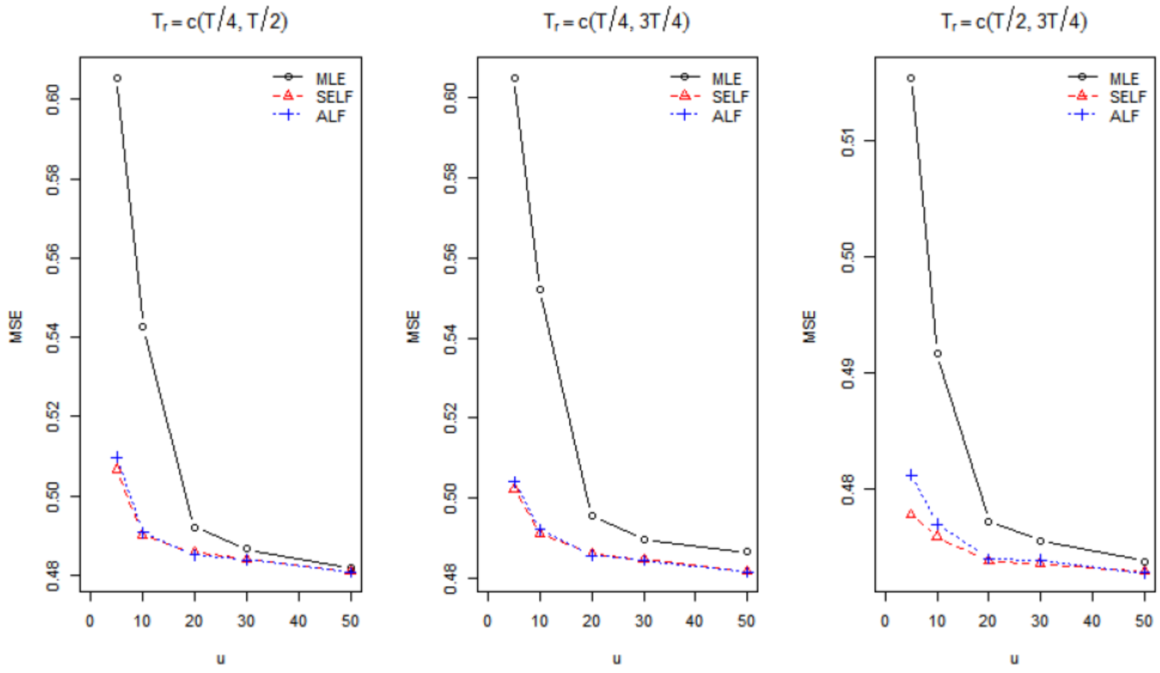

Figure .13: MSE under MLE and Bayes estimator for γ_{11}^{1} with varying u and T_r .

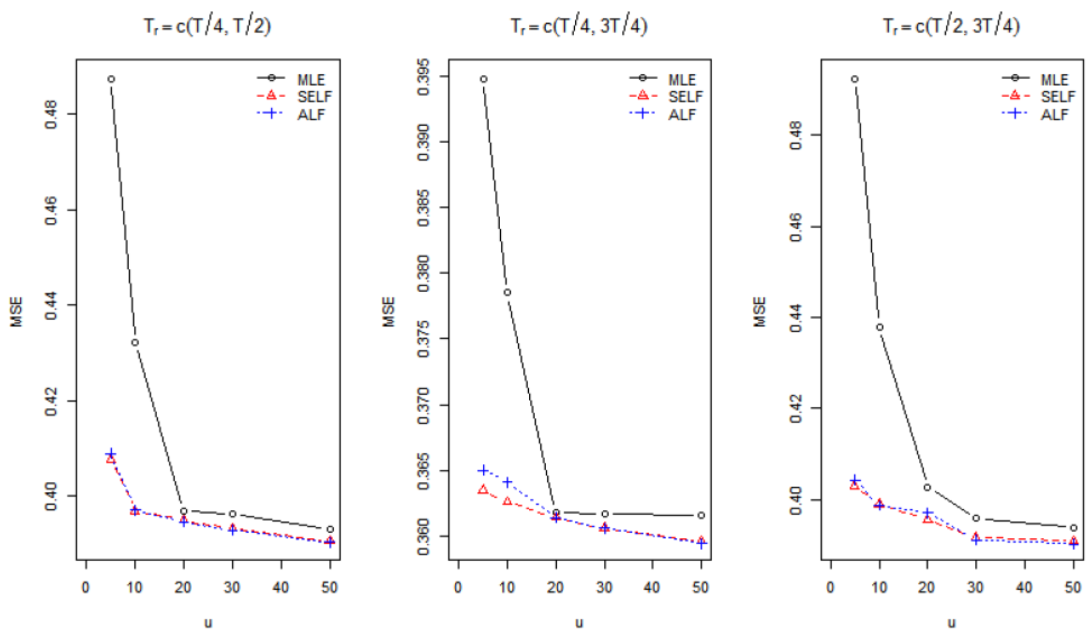

Figure .14: MSE under MLE and Bayes estimator for γ_{21}^{1} with varying u and T_r .

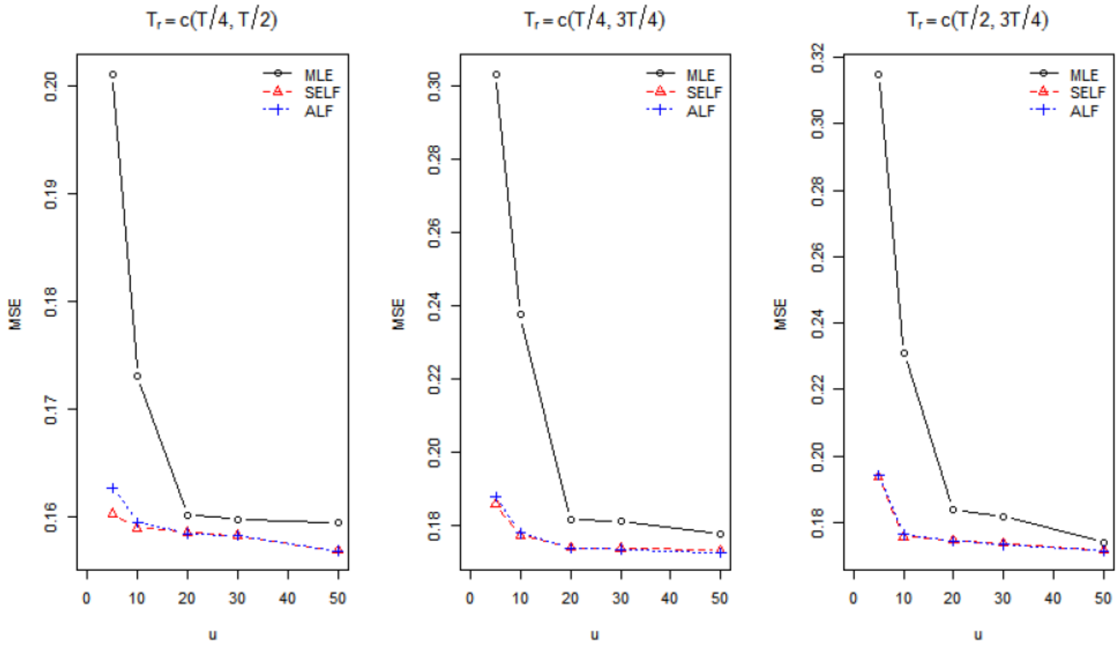

Figure .15: MSE under MLE and Bayes estimator for γ_{11}^2 with varying u and T_r .

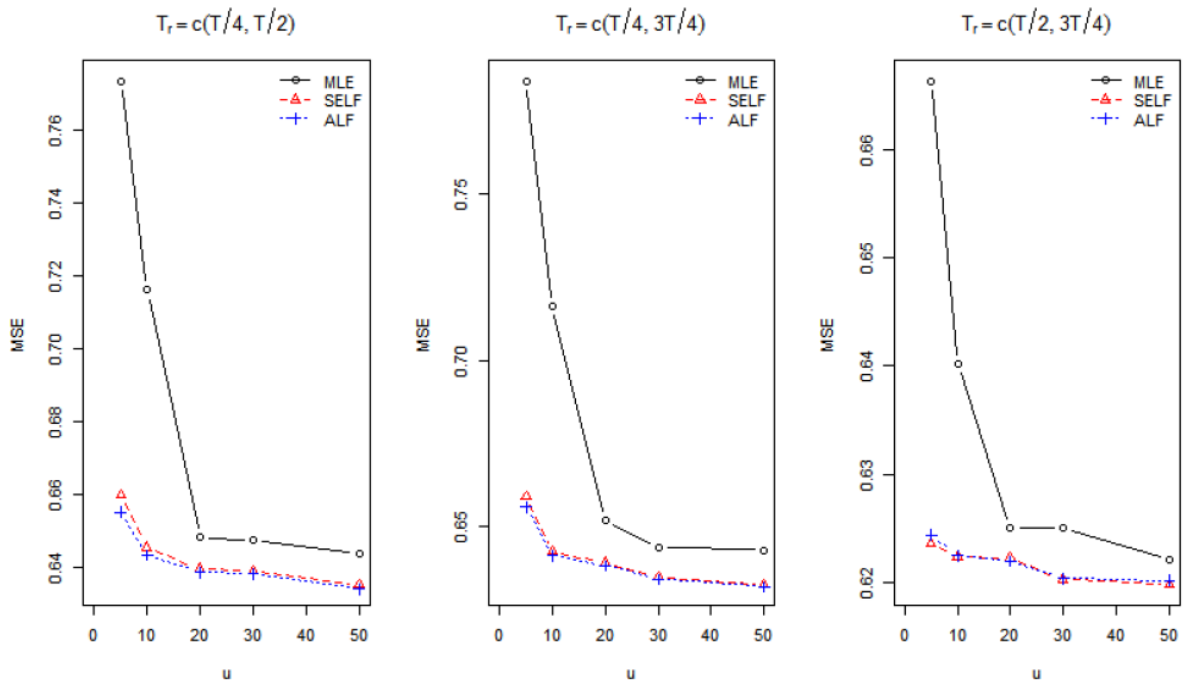

Figure .16: MSE under MLE and Bayes estimator for γ_{21}^2 with varying u and T_r .

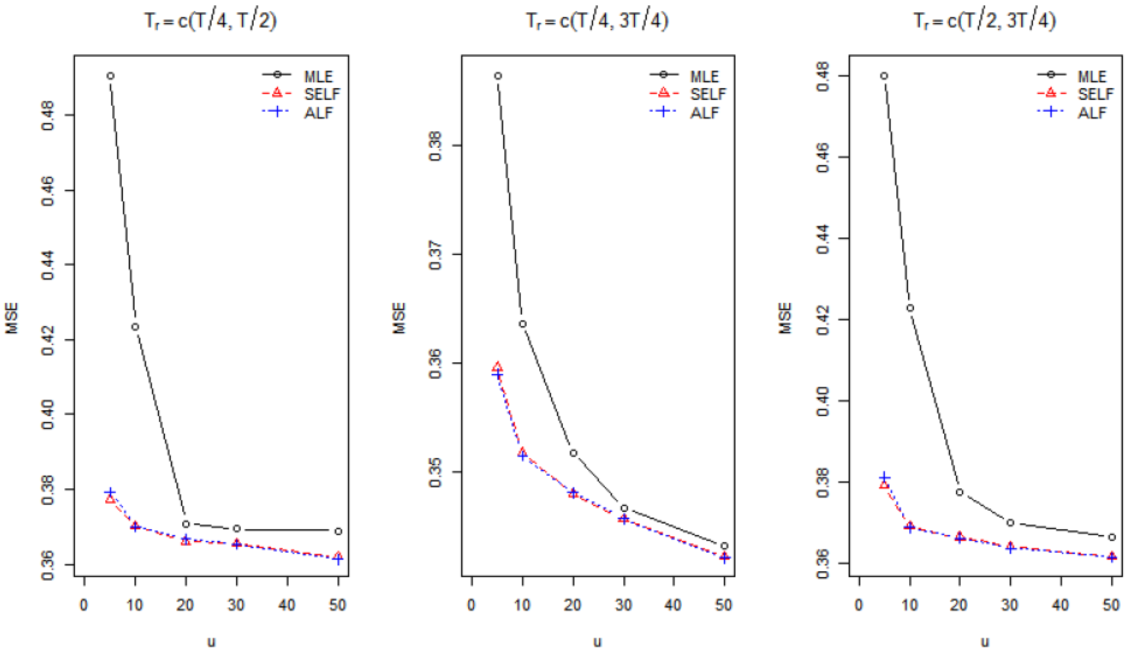

Figure .17: MSE under MLE and Bayes estimator for γ_{11}^{3} with varying u and T_r .

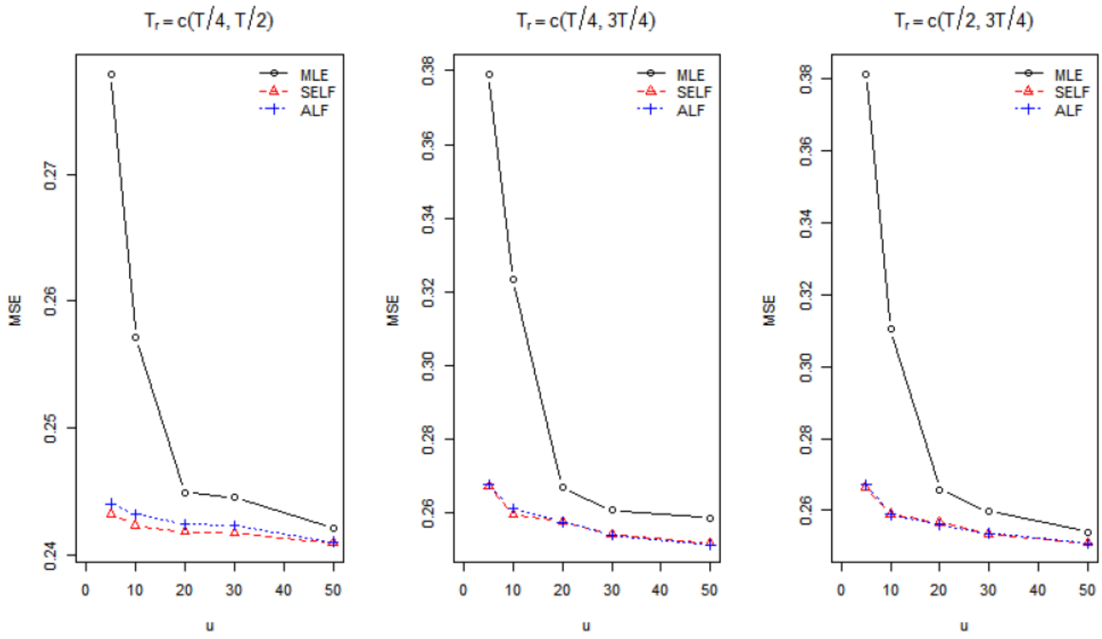

Figure .18: MSE under MLE and Bayes estimator for γ_{21}^{3} with varying u and T_r .

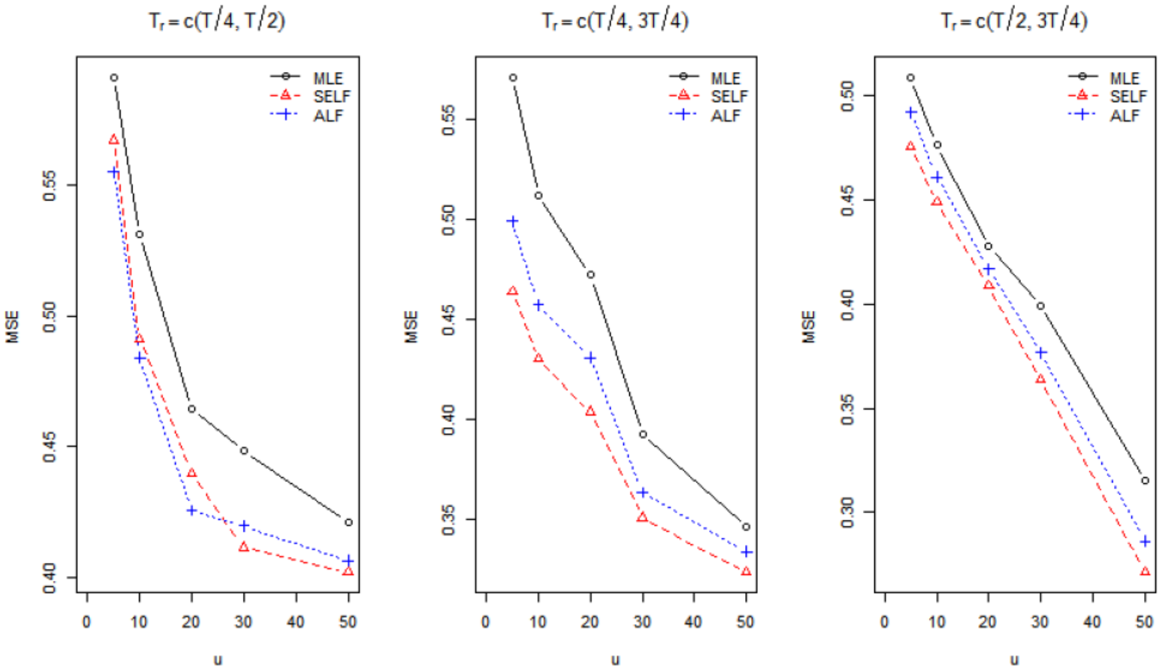

Figure .19: MSE under MLE and Bayes estimator for δ with varying u and T_r .