Filomat 38:21 (2024), 7609–7626
https://doi.org/10.2298/FIL2421609G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and qualitative behavior of mild solutions for fuzzy boundary value problem with nonlocal conditions

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Abstract. The focus of this study is on boundary-value issues for Caputo-type fractional differential equations of order $1 < q < 2$. To begin, we demonstrate the existence theorem of mild solutions under certain lesser constraints combining the measure of non-compactness and Darbo's fixed-point theorem. The generalised Ulam-Hyers stability requirements are then investigated. The insights provided here enlarge and enhance on several already established discoveries. subsequently an illustration is provided to demonstrate the truthfulness of what has been found.

1. Introduction

Fuzzy set theory has been attracting increasing interest in recent years as it is widely used in several fields such as mechanics, electrical engineering, signal processing, etc. As a result, in recent decades, fuzzy set theory has become a hot and current topic and has received much attention from researchers (see for instance [7, 8, 16, 17, 19, 21–23]).

Consider that Kaleva [11] explored the features of differentiable fuzzy set value relationships using the idea of H-differentiability introduced by Puri and Ralescu [12], who provided the existence and uniqueness theorems for a solution of the fuzzy differential equation.

$$x'(s) = f(s, x(s)); \quad x(0) = x_0.$$

where $f : J \times \mathcal{F}_{\mathbb{R}^n} \to \mathcal{F}_{\mathbb{R}^n}$ satisfies the Lipschitz condition.

In [13] Bhaskar Dubey and Raju K. George investigated linear-time-invariant processes with fuzzy starting points.

$$x'(s) = Ax(s) + Bc(s), \quad x(s_0) = x_0.$$

where $c(s) \in (\mathcal{F}_{\mathbb{R}})^p$ a control and A, B, are $q \times q, q \times p$ real matrices, accordingly, $s_0 \geq 0$.

In [14] Nguyen Thi Kim Son investigates the existence of fuzzy moderate solutions to non-linear fuzzy fractional evolution equations to illustrate the efficacy of mathematical conclusions.

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^q u(s) & = Au(s) + f(s, u(s)), s \in [0, a]. \\ u(0) & = \varphi_0. \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 47S40; Secondary 34A34

Keywords. Boundary value problem, Fuzzy fractional differential equation, Measure of non-compactness, Darbo's fixed point theorem, Generalized Ulam Hyers stability, Caputo fractional differential equations.

Received: 08 December 2023; Revised: 11 February 2024; Accepted: 14 February 2024

Communicated by Maria Alessandra Ragusa

Research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

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here ${}^{C}_{gH}\mathcal{D}^{q}$ is the fuzzy Caputo fractional derivative of class $q \in (0,1)$ and A is the infinitesimal generator of a strongly continuous semi-group $\{T(s)\}_{t\geq 0}$ on the space of all triangular fuzzy elements $\mathcal{T}$.

By the inspire of above works, in this article, We handle the existence of mild solutions to a set of fuzzy fractional semi-linear integrodifferential equations of order $1 < \alpha < 2$:

$$\begin{cases} D_t^q u(t) = Au(t) + f(t, u(t)) + \int_0^t p(t-s)g(s, u(s))ds, & t \in J = [0, T], \\ u(0) + m(u) = u_0 \in \mathcal{T}, \quad u'(0) + n(u) = u_1 \in \mathcal{T}, \end{cases} \tag{1}$$

where D_t^{α} is Caputo's fractional derivative of $1 < q < 2, A$ is the infinitesimal generator of a strongly continuous semigroup $E_{q,n}(At^q)$ on $\mathcal{T}$. The functions f, g are defined from $J \times \mathcal{T} \to \mathcal{T}$ and continuous. $p : J \to \mathcal{T}$ is an integrable map on J, the non-local condition $m : \mathcal{T} \to \mathcal{T}; n : \mathcal{T} \to \mathcal{T}$ are continuous fuzzy maps.

The following is how this work is organized. Section 2 gives a few definitions and initial information that shall be utilized to support our primary findings. Section 3 provides an adequate definition of mild solution to the (1). Section 4 contains evidence for our primary findings. Lastly, in Section 5, a case study is provided to demonstrate the efficacy of the findings gained.

2. Preliminaries and Lemmas

In this part we recall some basic notions that will be useful in the rest of our article.

2.1. The metric space $\mathcal{F}_{\mathbb{R}}$

Definition 1. *The fuzzy number is a fuzzy set $u : \mathbb{R} \to [0,1]$ that satisfies these conditions:*

1. *u is normal, i.e. there's a $t_0 \in \mathbb{R}$ such as $u(t_0) = 1$;*
2. *u is a fuzzy convex set;*
3. *u is upper semicontinuous;*
4. *u closure of $\{t \in \mathbb{R}, \quad u(t) > 0\}$ is compact.*

The set of all fuzzy elements on $\mathbb{R}$ is symbolized by $\mathcal{F}_{\mathbb{R}}$.

$$\mathcal{F}_{\mathbb{R}} = \{u : \mathbb{R} \to [0,1], \quad u \quad \texttt{satisfies } (1-4) \textit{ below} \}.$$

The r-cut of a $\mathcal{F}_{\mathbb{R}}$ component is given by

$$u^r = \{s \in \mathbb{R}, u(s) \geq r\} \texttt{ For all } r \in (0,1]$$

We may write using the previous items

$$u^r = [\underline{u}(r), \bar{u}(r)]. \tag{2}$$

The distance separating two elements of $\mathcal{F}_{\mathbb{R}}$ can be measured by (see [1])

$$d(\mathcal{M}, \mathcal{N}) = \sup_{r\in(0,1]} \max\{|\bar{\mathcal{M}}(r) - \bar{\mathcal{N}}(r)|, |\underline{\mathcal{M}}(r) - \underline{\mathcal{N}}(r)|\} \tag{3}$$

And the following properties are valid:

1. $d(\mathcal{M} + \epsilon, \mathcal{N} + \epsilon) = d(\mathcal{M}, \mathcal{N})$;
2. $d(\iota\mathcal{M}, \iota\mathcal{N}) = |\iota| d(\mathcal{M}, \mathcal{N})$;
3. $d(\mathcal{M} + \mathcal{N}, w + \epsilon) \leq d(\mathcal{M}, w) + d(\mathcal{N}, \epsilon)$;

The fuzzy number additions and scalar multiplication computations on $\mathbb{R}_{\mathcal{F}}$ take the structure

$$[\mathcal{M} \oplus \mathcal{N}]^r = [\mathcal{M}]^r + [\mathcal{N}]^r \text{ and } [\rho \odot \mathcal{M}]^r = \rho[\mathcal{M}]^r, \rho \in \mathbb{R}. \tag{4}$$

where

$$[\mathcal{M}]^r + [\mathcal{N}]^r = \{\mu + \nu : \mu \in [\mathcal{M}]^r, \nu \in [\mathcal{N}]^r\}. \tag{5}$$

is the Minkowski summation of $[\mathcal{M}]^r$ and $[\mathcal{N}]^r$ and

$$\rho[\mathcal{M}]^r = \{\rho\mu : \mu \in [\mathcal{M}]^r\}. \tag{6}$$

For $\mathcal{M}, y \in \mathcal{F}_{\mathbb{R}}$, the gH difference [2] of $\mathcal{M}$ and $\mathcal{N}$, expressed as $\mathcal{M} \ominus_{gH} \mathcal{N}$, is given as the number $\mathcal{H} \in \mathcal{F}_{\mathbb{R}}$ such as

$$\mathcal{M} \ominus_{gH} \mathcal{N} = z \Longleftrightarrow \begin{cases} \text{(i) } \mathcal{M} = \mathcal{N} + z \text{ or} \\ \text{(ii) } \mathcal{N} = \mathcal{M} + (-1)\mathcal{H} \end{cases}. \tag{7}$$

In regard to r-cuts, we receive

$$\left(\mathcal{M} \ominus_{gH} y\right)^{\alpha} = [\min\{\underline{\mathcal{M}}(r) - \underline{y}(r), \bar{\mathcal{M}}(r) - \bar{y}(r)\}, \max\{\underline{\mathcal{M}}(r) - \underline{y}(r), \bar{\mathcal{M}}(r) - \bar{y}(r)\}].$$

And the prerequisites for the existence of $\mathcal{H} = \mathcal{M} \ominus_{gH} \mathcal{N} \in \mathcal{F}_{\mathbb{R}}$ are

$$\text{case (i)} \quad \begin{cases} \underline{\mathcal{H}}(r) = \underline{\mathcal{M}}(r) - \underline{\mathcal{N}}(r) \text{ and } \overline{\mathcal{H}}(r) = \overline{\mathcal{M}}(r) - \overline{\mathcal{N}}(r) \\ \text{with } \underline{\mathcal{H}}(r) \text{ increasing, } \overline{\mathcal{H}}(r) \text{ decreasing, } \underline{\mathcal{H}}(r) \leq \overline{\mathcal{H}}(r) \end{cases} \tag{8}$$

$$\text{case (ii)} \quad \begin{cases} \underline{\mathcal{H}}(r) = \overline{\mathcal{M}}(r) - \overline{\mathcal{N}}(r) \text{ and } \overline{\mathcal{H}}(r) = \underline{\mathcal{M}}(r) - \underline{\mathcal{N}}(r) \\ \text{with } \underline{\mathcal{H}}(r) \text{ increasing, } \overline{\mathcal{H}}(r) \text{ decreasing, } \underline{\mathcal{H}}(r) \leq \overline{\mathcal{H}}(r) \end{cases} \tag{9}$$

for all $r \in [0, 1]$.

In general, with $x \in \mathcal{F}_{\mathbb{R}}$, There is no such thing as $y \in \mathcal{F}_{\mathbb{R}}$ such as $x \oplus y = 0$. Sadly, then, $\mathcal{F}_{\mathbb{R}}$ isn't a linear field with additions and scalar multiplication. Hence, $(\mathcal{F}_{\mathbb{R}}, \|\cdot\|)$ is not a Banach space, where $\|x\| = d(x, \tilde{0}), x \in \mathcal{F}_{\mathbb{R}}$.

Denote $\mathcal{T}$ the space of all triangular fuzzy elements in $\mathcal{F}_{\mathbb{R}}$. $(\mathcal{T}, d)$ is a subset of the metric set $(\mathcal{F}_{\mathbb{R}}, d)$. It is a complete metric space. Moreover, Bede [3] showed that if $x, y \in \mathcal{T}$, then the difference $x \ominus_{gH} y$ always exists in $\mathcal{T}$ and $x \ominus_{gH} y = (-1) \odot \left(y \ominus_{gH} x\right)$.

Let $\mathcal{T}$ be a subset of $\mathcal{F}_{\mathbb{R}}$, $J \subset \mathbb{R}$, and denote $C(J, \mathcal{T})$ by the set of all continuous mappings $f : J \to \mathcal{T}$.

2.2. Hukuhara's derivative

Let $u : J \to \mathcal{F}_{\mathbb{R}}$ a fuzzy-valued function. The r-cut of u is given by

$$u(t, r) = \left[\underline{u}(t, r), \bar{u}(t, r)\right], \forall t \in J, \forall r \in [0, 1].$$

Definition 2. *[4] Let $t_0 \in J$ and h be such that $t_0 + h \in (0, T)$, then the generalized Hukuhara derivative of a fuzzy value function $u : J \to \mathcal{F}_{\mathbb{R}}$ at t_0 is defined as*

$$\lim_{h \to 0} \left\| \frac{u(t_0 + h) -_{gH} u(t_0)}{h} -_{gH} u'_{gH}(t_0) \right\|_1 = 0. \tag{10}$$

If $u_{gH}(t_0) \in \mathcal{F}_{\mathbb{R}}$ satisfying (10) exists, we say that u is generalized Hukuhara differentiable at t_0.

Definition 3. *[4] Let $u : J \to \mathcal{F}_{\mathbb{R}}$ and $t_0 \in (0, T)$, with $\underline{u}(t, r)$ and $\bar{u}(t, r)$ both differentiable at t_0.*

We say that

1. *u is* $[(i)-gH]$*-differentiable at* t_0 *if*

$$u'_{i,gH}(t_0) = \left[\underline{u}'(t,r), \bar{u}'(t,r)\right]. \tag{11}$$

2. *u is* $[(ii)-gH]$*-differentiable at* t_0 *if*

$$u'_{ii,gH}(t_0) = \left[\bar{u}'(t,r), \underline{u}'(t,r)\right]. \tag{12}$$

Theorem 1. *[6]Let* $u : J \to \mathcal{F}_{\mathbb{R}}$ *and* $\phi : J \to \mathbb{R}$ *and* $t \in J$*. Suppose that* $\phi(t)$ *is differentiable function at t and the fuzzy-valued function u is gH-differentiable at t. So*

$$(u\phi)'_g(t) = \left(u'\phi\right)_g(t) + \left(u\phi'\right)_g(t). \tag{13}$$

Definition 4. *[5] Let* $u : J \to \mathcal{F}_{\mathbb{R}}$ *and* $u'_{gH}(t)$ *be gH-differentiable at* $t_0 \in (0,T)$*, moreover there isn't any switching point on* $(0,T)$ *and* $\underline{u}(t,r)$ *and* $\overline{u}(t,r)$ *both differentiable at* t_0*. We say that*

- u' *is* $[(i)-gH]$*-differentiable at* t_0 *if*

$$u''_{i,gH}(t_0) = \left[\underline{u}''(t,r), \overline{u}''(x,r)\right].$$

- u' *is* $[(ii)-gH]$*-differentiable at* t_0 *if*

$$u''_{ii,gH}(t_0) = \left[\overline{u}''(t,r), \underline{u}''(t,r)\right].$$

2.3. Fuzzy fractional derivative

The extended fuzzy fractional derivative and its characteristics are presented.

Definition 5. *[9] Let* $u \in L^{\mathcal{F}_{\mathbb{R}}}(J)$*. The fuzzy Riemann-Liouville integral of u is given by:*

$$I^q_{RL}u(t) = \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot u(s)ds, \quad 0 < s < t, \quad 0 < q < 1. \tag{14}$$

Definition 6. *[6] Let* $u \in L^{\mathcal{F}_{\mathbb{R}}}(J)$*. The fuzzy Riemann-Liouville derivative of u is given by:*

$$_{gH}D^q_{RL}u(s) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \odot \left(\dfrac{d}{ds}\right)^n \displaystyle\int_0^s (s-t)^{n-q-1} \odot u(t)dt, & n-1 < q < n \\ \left(\dfrac{d}{ds}\right)^{n-1} u(s), & q = n-1. \end{cases} \tag{15}$$

Definition 7. *[6] In the definition of RL fractional derivative, assume the integer degree of the derivative is an operator inside of the integral and operating on function* $u \in \mathcal{F}_{\mathbb{R}}, t \in J$*. We get the definition of Caputo gH derivative of u*

$$^{C}_{gH}D^q u(t) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \odot \displaystyle\int_0^s (t-s)^{n-q-1} \odot u^{(n)}_{gH}(s)ds, & n-1 < q < n, \\ \left(\dfrac{d}{dt}\right)^{n-1} u(t), & q = n-1 \end{cases} \tag{16}$$

Also we call u is $[(i)-gH]$-differentiable at t_0 if

$$_{gH}D^q_t u(x,t;r) = \left[D^q\underline{u}(x,t;r), D^q\bar{u}(x,t;r)\right], \quad \forall q \in (0,1) \tag{17}$$

and u is $[(ii)-gH]$-differentiable at t_0 if

$$_{gH}D^q_t u(x,t;r) = \left[D^q\bar{u}(x,t;r), \underline{u}(x,t;r)\right], \quad \forall q \in (0,1). \tag{18}$$

Definition 8. *[10] Let $u : J \to \mathcal{T} \subset \mathcal{F}_{\mathbb{R}}$ be a continuous function such as $e^{-ts} \odot u(s)$ is integrable. So the fuzzy Laplace transformation of u, expressed by $\mathbf{L}[u(t)]$, is*

$$\mathbf{L}[u(t)] := U(s) = \int_0^T e^{-st} \odot u(t)dt, t > 0. \tag{19}$$

A fuzzy mapping u is exponent bounded of rank β if there's $M > 0$ and

$$\exists t_0 > 0, d(u(t), \tilde{0}) \leq Me^{\beta t}, \forall t \geq t_0$$

Proposition 1. *If $u(t)$ is a fuzzy peacewise continuous map on J and of exponential order β, then*

$$\mathbf{L}((u \star v)(t)) = \mathbf{L}(u(t)) \odot \mathbf{L}(v(t)). \tag{20}$$

where $v(t)$ is a peace-wise continuous real function on J.

Proof.

$$\begin{aligned}\mathbf{L}(u(t)) \odot \mathbf{L}(v(t)) &= \left(\int_0^T e^{-s\tau} \odot u(\tau)d\tau\right) \odot \left(\int_0^T e^{-s\sigma} \odot v(\sigma)d\sigma\right)\\ &= \int_0^T \left(\int_0^T e^{-s(\tau+\sigma)} \odot u(\tau)d\tau\right) \odot v(\sigma)d\sigma.\end{aligned}$$

Let us to hold τ fixed in the interior integral, substituting $t = \tau + \sigma$ and $d\sigma = dt$, we obtain

$$\begin{aligned}\mathbf{L}(u(t)) \odot \mathbf{L}(v(t)) &= \int_0^T \left(\int_\sigma^T e^{-st} \odot u(\tau) \odot v(t-\tau)dt\right) d\tau\\ &= \int_0^T \int_\sigma^T e^{-st} \odot u(\tau) \odot v(t-\tau)dtd\tau\\ &= \int_0^T e^{-st} \odot \left(\int_0^t u(t-\sigma) \odot v(\sigma)d\tau\right) d\sigma\\ &= \mathbf{L}((u \star v)(t)).\end{aligned}$$

□

Definition 9. *[15]*

1. *The Gamma function is defined as*

$$\forall s > 0, \quad \Gamma(s) = \int_0^{+\infty} w^{s-1}e^{-w}dw. \tag{21}$$

2. *The $\mathbb{B}$ function is defined by*

$$\forall \mu, \nu > 0, \quad \mathbb{B}(\mu, \nu) = \int_0^1 s^{\mu-1}(1-t)^{\nu-1}ds. \tag{22}$$

Proposition 2. *[15] We have*

1) *For all $q, p \in \mathbb{R}_+^*, \mathbb{B}(q, p) = \dfrac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}$.*

2) *For all $q > 0, \Gamma(q+1) = q\Gamma(q)$.*

It is easy to show the following proposition.

Proposition 3. *For all $q > 0$, we get the following result*

$$\int_0^t E_{q,1}(As^q)\,ds = tE_{q,2}(At^q). \tag{23}$$

Proof.

$$\begin{aligned}\int_0^t E_{q,1}(As^q)\,ds &= \int_0^t \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(nq+1)} A^n ds \\ &= \sum_{n=0}^{\infty} \frac{t^{nq+1}}{(nq+1)\Gamma(nq+1)} A^n \\ &= \sum_{n=0}^{\infty} \frac{t^{nq+1}}{\Gamma(nq+2)} A^n \\ &= tE_{q,2}(At^q).\end{aligned}$$

□

Proposition 4. *For all $q \in [1,2]$ and $s > 0$,*

1. $s^{q-1}(s^q - A)^{-1} = \mathbf{L}\left(E_{q,1}(At^q)\right)(s)$,
2. $s^{q-2}(s^q - A)^{-1} = \mathbf{L}\left(tE_{q,2}(At^q)\right)(s)$,
3. $(s^q - A)^{-1} = \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_0^t (t-s)^{q-2}E_{q,1}(As^q)\,ds\right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned}\mathbf{L}\left(E_{q,1}(At^q)\right)(s) &= \mathbf{L}\left(\sum_{n=0}^{+\infty} \frac{t^{qn}A^n}{\Gamma(qn+1)}\right) \\ &= \sum_{n=0}^{+\infty} \mathbf{L}(t^{qn}) \frac{A^n}{\Gamma(qn+1)} \\ &= \sum_{n=0}^{+\infty} \frac{1}{s^{nq+1}} A^n \\ &= s^{q-1}(s^q - A)^{-1}.\end{aligned}$$

2. For $s > 0, s^{q-1}(s^q - A)^{-1} = \mathbf{L}\left(E_{q,1}(At^q)\right)(s)$, then

$$\begin{aligned}s^{q-2}(s^q - A)^{-1} &= s^{-1}s^{q-1}(s^q - A)^{-1} \\ &= \mathbf{L}(1)(s)\mathcal{L}\left(E_{q,1}(At^q)\right)(s) \\ &= \mathbf{L}\left(1 * E_{q,1}(At^q)\right)(s) \\ &= \mathbf{L}\left(\int_0^t E_{q,1}(At^q)\right)(s) \\ &= \mathbf{L}\left(tE_{q,2}(t^qA)\right)(s).\end{aligned}$$

3. From (1), we get

$$\begin{aligned}
(s^q - A)^{-1} &= s^{1-q}\mathbf{L}\left(E_{q,1}\left(At^q\right)\right)(s)\\
&= \mathbf{L}\left(\frac{t^{q-2}}{\Gamma(q-1)}\right)\mathcal{L}\left(E_{q,1}\left(At^q\right)\right)(s)\\
&= \mathbf{L}\left(\frac{t^{q-2}}{\Gamma(q-1)} * E_{q,1}\left(At^q\right)\right)(s)\\
&= \mathbf{L}\left(\int_0^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} E_{q,1}\left(A\delta^q\right)d\delta\right)(s),
\end{aligned}$$

hence the desired result. □

Lemma 1. *[10]*

(1) Let $u, v : J \to \mathcal{T}$ be continuous functions, $c_1, c_2 \in \mathbb{R}^+$. Then

$$\mathbf{L}\left[c_1 \odot u(t) + c_2 \odot v(t)\right] = c_1 \odot \mathbf{L}[u(t)] + c_2 \odot \mathbf{L}[v(t)].$$

(2) Let $u : J \to \mathcal{T}$ be a continuous function. Then

$$\mathbf{L}\left[e^{at} \odot u(t)\right] = U(s-a), s-a > 0.$$

(3) Let $u \in C^1(J, \mathcal{T})$ be exponent bounded of order β. Then

(i) if u is (i)-gH differentiable, then $\mathbf{L}\left[\mathcal{D}^i_{gH}u(t)\right] = s \odot \mathbf{L}[u(t)] \ominus u(0)$,

(ii) if u is (ii)-gH differentiable, then $\mathbf{L}\left[\mathcal{D}^{ii}_{gH}u(t)\right] = (-1) \odot u(0) \ominus (-s) \odot \mathbf{L}[u(t)]$.

Following that, the Kuratowski measure of non compactness is defined, and some of its key aspects are examined.

Definition 10. *[18] The Kuratowski measure of non compactness $\mathcal{M}(\cdot)$ constructed on the bound subset $\mathcal{V}$ of E is indeed:*

$$\mathcal{M}(\mathcal{V}) := \inf\left\{\varepsilon > 0 : \mathcal{V} = \cup_{i=1}^n \mathcal{V}_i \text{ and } \operatorname{diam}(\mathcal{V}_i) \le \varepsilon \text{ for } i = 1, 2, \ldots, n\right\}.$$

The Kuratowski measure of non compactness has the very next well-known features.

Lemma 2. *[18] Let E be a Banach space and $\mu, \nu \subset E$ be bounded. The following aspects are met:*

(1) $\mathcal{M}(\mu) \le \mathcal{M}(\nu)$ if $\mu \subset \nu$;

(2) $\mathcal{M}(\mu) = \mathcal{M}(\bar{\mu}) = \mathcal{M}(\overline{\text{conv}}\mu)$

(3) $\mathcal{M}(\mu) = 0$ iff μ is relatively compact;

(4) $\mathcal{M}(\lambda\mu) = |\lambda|\mathcal{M}(\mu)$, where $\lambda \in \mathbb{R}$;

(5) $\mathcal{M}(\mu \cup \nu) = \max\{\mathcal{M}(\mu), \mathcal{M}(\nu)\}$;

(6) $\mathcal{M}(\mu + \nu) \le \mathcal{M}(\mu) + \mathcal{M}(\nu)$, where $\mu + \nu = \{w \mid w = m + n, m \in \mu, n \in \nu\}$;

(7) $\mathcal{M}(\mu + y) = \mathcal{M}(\mu)$, $\forall y \in E$.

Lemma 3. *[20] Assume $\mathcal{V} \subset C(I, E)$ to be bound and equicontinuous subset. Therefore, the function $s \to \mathcal{M}(\mathcal{V}(s))$ is continuous on I :*

$$\mathcal{M}_C(\mathcal{V}) = \max_{s\in I} \mathcal{M}(\mathcal{V}(s)),$$

and

$$\mathcal{M}\left(\int_I u(s)\mathrm{d}s\right) \leq \int_I \mathcal{M}(\mathcal{V}(s))\mathrm{d}s,$$

where $\mathcal{V}(s) = \{u(s) : u \in \mathcal{V}\}, s \in I$.

Next consider Darbo's fixed-point theorem.

Theorem 2. *(Darbo's fixed-point theorem [18]). Suppose M be a non-empty, bounded, convex, and closed subspace of a Banach space X and $\mathcal{L} : M \longrightarrow M$ is a continuous operator fulfilling $u(\mathcal{L}Z) \leq Ku(Z)$ for any non-empty subspace Z of M and for a constant $K \in [0, 1)$. Therefore, $\mathcal{L}$ has at least one fixed point in M.*

3. Definition of a mild solution

The following theorem demonstrates the relationship between a fuzzy fractional differentiation equation (1) and a fuzzy integro-differential equation.

Theorem 3. *Let A be the infinitesimal generator of a strongly continuous semi-group $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$, the unique solution of* (1) *is provided by*

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(q-1)} \int_0^t \int_s^t (t-\delta)^{q-2} E_{\alpha,1}\left(A(\delta - s)^q\right) f(s, u(s)) d\delta ds \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_s^t \int_0^s (t-\delta)^{q-2} E_{q,1}\left(A(\delta - s)^q\right) p(s-x) g(x, u(x)) d\delta dx ds \\ &+ E_{q,1}\left(At^q\right)\left(u_0 - m(u)\right) + tE_{q,2}\left(At^q\right)\left(u_1 - n(u)\right). \end{aligned} \tag{24}$$

Proof. By the Laplace transform, we may get

$$\begin{aligned} \mathbf{L}\left(D_t^q u(t)\right)(\lambda) &= \lambda^q (\mathbf{L}u)(\lambda) - \sum_{j=0}^{m-1} \lambda^{q-j-1} \left(D^j u\right)(0) \\ &= \lambda^q (\mathbf{L}u)(\lambda) - \lambda^{q-1} u(0) - \lambda^{q-2} u'(0). \end{aligned} \tag{25}$$

It follows that

$$\begin{aligned} \lambda^q(\mathbf{L}u)(\lambda) - \lambda^{q-1}u(0) - \lambda^{q-2}u'(0) &= A(\mathbf{L}u)(\lambda) + (\mathbf{L}f)(\lambda) + (\mathbf{L}\int_0^t p(t-s)g(s,u(s))ds)(\lambda), \\ \left(\lambda^q I - A\right)(\mathbf{L}u)(\lambda) &= (\mathbf{L}f)(\lambda) + (\mathbf{L}\int_0^t q(t-s)g(s,u(s))ds)(\lambda) + \lambda^{q-1}u(0) \\ &\quad + \lambda^{q-2}u'(0), \\ (\mathbf{L}u)(\lambda) &= \left(\lambda^\alpha I - A\right)^{-1}\Big[(\mathbf{L}f)(\lambda) + (\mathbf{L}\int_0^t q(t-s)g(s,u(s))ds)(\lambda) \\ &\quad + \lambda^{q-1}u(0) + \lambda^{q-2}u'(0)\Big], \\ (\mathbf{L}u)(\lambda) &= \left(\lambda^q I - A\right)^{-1}(\mathbf{L}f)(\lambda) \\ &\quad + \left(\lambda^q I - A\right)^{-1}(\mathbf{L}\int_0^t p(t-s)g(s,u(s))ds)(\lambda) \\ &\quad + \lambda^{q-1}\left(\lambda^q I - A\right)^{-1}u(0) + \lambda^{q-2}\left(\lambda^q I - A\right)^{-1}u'(0), \end{aligned}$$

Then, by Lemma 4 we get

$$\begin{aligned}(\mathbf{L}u)(\lambda) &= \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_0^t (t-s)^{q-2}E_{q,1}(As^q)\,ds\right)(\mathbf{L}f)(\lambda)\\ &+ \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_0^t (t-s)^{q-2}E_{q,1}(As^q)\,ds\right)\mathbf{L}\left(\int_0^t p(t-s)g(s,u(s))ds\right)(\lambda)\\ &+ \mathbf{L}\left(E_{q,1}(At^q)\right)(\lambda)u(0) + \mathbf{L}\left(tE_{q,2}(At^q)\right)(\lambda)u'(0),\end{aligned}$$

Using the proposition 1, we obtain

$$\begin{aligned}(\mathbf{L}u)(\lambda) &= \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_{-\infty}^{+\infty}\int_0^{t-x}(t-x-s)^{q-2}E_{q,1}(As^q)f(x,u(x))dsdx\right)(\lambda)\\ &+ \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_{-\infty}^{+\infty}\int_0^{t-x}\int_0^{x}(t-x-s)^{q-2}E_{q,1}(As^q)p(x-s)g(s,u(s))dsdsdx\right)(\lambda)\\ &+ \mathbf{L}\left(E_{q,1}(At^q)\right)(\lambda)u(0) + \mathbf{L}\left(tE_{q,2}(At^q)\right)(\lambda)u'(0),\end{aligned}$$

Then,

$$\begin{aligned}(\mathbf{L}u)(\lambda) &= \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_0^t\int_s^t(t-\delta)^{q-2}E_{q,1}(A(\delta-s)^q)f(s,u(s))d\delta ds\right)(\lambda)\\ &+ \frac{1}{\Gamma(q-1)}\mathbf{L}\left(\int_0^t\int_s^t\int_0^s(t-\delta)^{q-2}E_{q,1}(A(\delta-s)^q)p(s-x)g(x,u(x))d\delta dxds\right)(\lambda)\\ &+ \mathbf{L}\left(E_{q,1}(At^q)\right)(\lambda)u(0) + \mathbf{L}\left(tE_{q,2}(At^q)\right)(\lambda)u'(0),\end{aligned}$$

Now (24) follows easily by applying the inverse Laplace transform. This conclude the evidence. □

Definition 11. *A function $u \in C(J,\mathcal{T})$ is called a mild solution of* (1) *if it satisfies the operator equation*

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t(t-\delta)^{q-2}E_{q,1}(A(\delta-s)^q)f(s,u(s))d\delta ds\\ &+ \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s(t-\delta)^{q-2}E_{q,1}(A(\delta-s)^q)p(s-x)g(x,u(x))d\delta dxds\\ &+ E_{q,1}(At^q)(u_0-m(u)) + tE_{q,2}(At^q)(u_1-n(u)).\end{aligned} \tag{26}$$

4. Existence Criteria

We will start with the introduction of some principals hypotheses:

(Hyp1) $E_{q,n}(At^q)$ is a compact operator for all $t \in J$ and $n \in \mathbb{N}$ ie, There's $M > 0$ such as $\forall t \in J$,

$$E_{q,n}(At^q) \le M,$$

(Hyp2) $f, g : J \times \mathcal{T} \to \mathcal{T}$ is continuous and for every $k > 0$ there's a positive functions $\mu_k, \nu_k \in L^\infty(J, R^+)$ such as

$$\sup_{d(u,\tilde{0})\le k} d(f(t,u),\tilde{0}) \le \mu_k(t), \quad \sup_{d(u,\tilde{0})\le k} d(g(t,u),\tilde{0}) \le \nu_k(t).$$

(Hyp3) There exist $q_1 \in [0,q)$, $B_\lambda := \left\{u \in \mathcal{T}, d(u,\tilde{0}) \le \lambda\right\} \subset \mathcal{T}$, $\lambda > 0$, and $\rho(.), \varrho(.) \in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that for any $u, v \in C(J, B_\lambda)$ we have

$$d(f(t,u(t)), f(t,v(t)) \le \rho(t)d(u(t),v(t)), t \in J.$$

$$d(g(t,u(t)), g(t,v(t)) \le \varrho(t)d(u(t),v(t)), t \in J.$$

(**Hyp4**) $m, n : \mathcal{T} \to \overline{D(A)}$ is continuous, and there's a constants θ, η such as

$$d\left(m(u), m(v)\right) \leq \theta d(u, v), \quad d\left(n(u), n(v)\right) \leq \eta d(u, v), \quad u, v \in \mathcal{T},$$

We shall now demonstrate the existence findings for the Eq. (1). Our initial discovery is founded on Darbo's fixed-point principle.

Theorem 4. *Assume that (**Hyp1**)-(**Hyp4**) holds, and*

$$M\left[\frac{T^q\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(\alpha+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta\right] < 1. \tag{27}$$

Then, the Eq. (1) *has at least one solution on* $C(J, \mathcal{T})$.

Proof. Consider the operator

$$\mathcal{L} : C(J, \mathcal{T}) \to C(J, \mathcal{T}), \tag{28}$$

defined by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q-1)} \int_0^t \int_s^t (t-\delta)^{q-2} E_{\alpha,1}\left(A(\delta - s)^q\right) f(s, u(s)) d\delta ds \\ &+ \frac{1}{\Gamma(q-1)} \int_0^t \int_s^t \int_0^s (t-\delta)^{q-2} E_{q,1}\left(A(\delta - s)^q\right) p(s-x) g(x, u(x)) d\delta dx ds \\ &+ E_{q,1}\left(At^q\right)\left(u_0 - m(u)\right) + t E_{q,2}\left(At^q\right)\left(u_1 - n(u)\right). \end{aligned}$$

The operator $\mathcal{L} : C(J, \mathcal{T}) \to C(J, \mathcal{T})$, in (28) is clearly defined based on the characteristics of fractional integrals and the continuity of functions. Consequently it is sufficient to demonstrate that the operator $\mathcal{L}$ possesses a fixed point u, and that fixed point corresponds to a solution of the Eq. (1).

Allow $\lambda \geq M\left[\frac{\|\mu_r\|_{L^\infty(J,\mathbb{R}^+)}T^\alpha}{\Gamma(\alpha+1)} + \frac{p\|\nu_r\|_{L^\infty(J,\mathbb{R}^+)}T^{\alpha+1}}{\Gamma(q+2)} + \bar{Z}\right]$ and consider the following set:

$$B_\lambda = \left\{u \in C(J, \mathcal{T}), d\left(u, \tilde{0}\right) \leq \lambda\right\}.$$

Obviously, B_λ is nonempty, convex, bounded, and closed.

In five phases, we will demonstrate that $\mathcal{L}$ fulfills Theorem 2.

Step 1: $\mathcal{L}(B_\lambda) \subseteq (B_\lambda)$.

For $u \in B_\lambda$, by Proposition 2 and (**Hyp2**), we get

$$
\begin{aligned}
d\left(\mathcal{L}u(t),\tilde{0}\right) &= d\Bigg(\frac{1}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{\alpha-2}E_{\alpha,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds \\
&+E_{q,1}\left(At^q\right)(u_0-m(u))+tE_{q,2}\left(At^q\right)(u_1-n(u)),\tilde{0}\Bigg) \\
&\leq \frac{M}{\Gamma(\alpha-1)}\int_0^t\int_s^t (t-\delta)^{\alpha-2}d\left(f(s,u(s)),\tilde{0}\right)d\delta ds \\
&+\frac{Mp}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{\alpha-2}d\left(g(x,u(x)),\tilde{0}\right)d\delta dxds \\
&+Md\left((u_0-m(u)),\tilde{0}\right)+TMd\left((u_1-n(u)),\tilde{0}\right) \\
&\leq \frac{M}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{\alpha-2}\mu_k(s)d\delta ds \\
&+\frac{Mp}{\Gamma(\alpha-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}\nu_k(x)d\delta dxds \\
&+Md\left((u_0-m(u)),\tilde{0}\right)+TMd\left((u_1-n(u)),\tilde{0}\right) \\
&\leq \frac{M\|\mu_r\|_{L^\infty(J,\mathbb{R}^+)}}{\Gamma(\alpha-1)}\int_0^t\int_s^t (t-\delta)^{q-2}d\delta ds \\
&+\frac{Mp\|\nu_r\|_{L^\infty(J,\mathbb{R}^+)}}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}d\delta dxds \\
&+Md\left(u_0,\tilde{0}\right)+Md\left(m(u),\tilde{0}\right)+TMd\left(u_1,\tilde{0}\right)+TMd\left(n(u),\tilde{0}\right) \\
&\leq M\left[\frac{\|\mu_r\|_{L^\infty(J,\mathbb{R}^+)}t^\alpha}{\Gamma(q+1)}+\frac{p\|\nu_r\|_{L^\infty(J,\mathbb{R}^+)}t^{q+1}}{\Gamma(q+2)}+\bar{Z}\right] \\
&\leq M\left[\frac{\|\mu_r\|_{L^\infty(J,\mathbb{R}^+)}T^q}{\Gamma(q+1)}+\frac{p\|\nu_r\|_{L^\infty(J,\mathbb{R}^+)}T^{q+1}}{\Gamma(q+2)}+\bar{Z}\right] \leq \lambda.
\end{aligned}
$$

Where $p = \max_{t\in J}\int_0^t |p(t-s)|ds$ and $\bar{Z} = d\left(u_0,\tilde{0}\right)+d\left(m(u),\tilde{0}\right)+Td\left(u_1,\tilde{0}\right)+Td\left(n(u),\tilde{0}\right)$

which means that $\mathcal{L}(B_\lambda) \subseteq (B_\lambda)$.

Step 2: P is continuous.

We assume that the sequence u_n converges to u in $C(J,\mathcal{T})$ and $t \in J$. So,

$$
\begin{aligned}
d(\mathcal{L}u_n(t),\mathcal{L}u(t)) &\leq \frac{M}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2} d\left(f(s,u_n(s)),f(s,u(s))\right) d\delta ds \\
&+ \frac{Mp}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^x (t-\delta)^{q-2} d\left(g(s,u_n(s)),g(s,u(s))\right) d\delta ds dx \\
&+ Md\left((u_0-m(u_n)),(u_0-m(u))\right) + TMd\left(u_1-n(u_n),u_1-n(u)\right) \\
&\leq \frac{M}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2} d\left(f(s,u_n(s)),f(s,u(s))\right) d\delta ds \\
&+ \frac{Mp}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^x (t-\delta)^{q-2} d\left(g(s,u_n(s)),g(s,u(s))\right) d\delta ds dx \\
&+ Md\left((u_0-m(u_n)),(u_0-m(u))\right) + TMd\left(u_1-n(u_n),u_1-n(u)\right) \\
&\leq \frac{M\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} t^{\alpha}}{\Gamma(\alpha+1)} d(u_n,u) + \frac{Mp\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} t^{\alpha+1} d(u_n,u)}{\Gamma(q+2)} \\
&+ M\theta d(u_n,u) + TM\eta d(u_n,u) \\
&\leq M\left[\frac{\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} T^{q}}{\Gamma(q+1)} + \frac{p\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} T^{q+1}}{\Gamma(q+2)} + \theta + T\eta\right] d(u_n,u).
\end{aligned}
\tag{29}
$$

Hence, we obtain

$$
d(\mathcal{L}u_n(t),\mathcal{L}u(t)) \to 0 \ \text{ as } \ n \to \infty.
$$

As result, $\mathcal{L}$ is a continuous on $C(J,\mathcal{T})$.

Step 3: $\mathcal{L}(B_\lambda)$ is bounded in $C(J,\mathcal{T})$.

According to **Step 1**, we have $\mathcal{L}(B_\lambda) \subseteq (B_\lambda)$. This implies that $\mathcal{L}(B_\lambda)$ is bounded set in $C(J,\mathcal{T})$.

Step 4: $\mathcal{L}(B_\lambda)$ is equi-continous set in $C(J,\mathcal{T})$.

For arbitrary $t_1, t_2 \in J$, with $t_1 < t_2$, let $u \in B_\lambda$. Estimate

$$
\begin{aligned}
d\left(\mathcal{L}u(t_2), \mathcal{L}u(t_1)\right) = d\Bigg(& \frac{1}{\Gamma(q-1)} \int_0^{t_2} \int_s^{t_2} (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds \\
& + \frac{1}{\Gamma(q-1)} \int_0^{t_2} \int_s^{t_2} \int_0^s (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds \\
& + E_{q,1}\left(At_2^q\right)(u_0 - m(u)) + t_2 E_{q,2}\left(At_2^q\right)(u_1 - n(u)), \\
& \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds \\
& + \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} \int_0^s (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds \\
& + E_{q,1}\left(At_1^q\right)(u_0 - m(u)) + t_1 E_{q,2}\left(At_1^q\right)(u_1 - n(u))\Bigg) \\
\le\; & d\Bigg(\frac{1}{\Gamma(q-1)} \int_0^{t_2} \int_s^{t_2} (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds, \\
& \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds\Bigg) \\
& + d\Bigg(\frac{1}{\Gamma(q-1)} \int_0^{t_2} \int_s^{t_2} \int_0^s (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds, \\
& \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} \int_0^s (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds\Bigg) \\
& + d\left(E_{q,1}\left(At_2^q\right)(u_0 - m(u)), E_{q,1}\left(At_1^q\right)(u_0 - m(u))\right) \\
& + d\left(t_2 E_{q,2}\left(At_2^q\right)(u_1 - n(u)), t_1 E_{q,2}\left(At_1^q\right)(u_1 - n(u))\right) \\
\le\; & d\Bigg(\frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds \\
& + \frac{1}{\Gamma(q-1)} \int_{t_1}^{t_2} \int_{t_1}^{t_2} (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds, \\
& \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s)) d\delta ds\Bigg) \\
& + d\Bigg(\frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} \int_0^s (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds \\
& + \frac{1}{\Gamma(q-1)} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \int_0^s (t_2-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds, \\
& \frac{1}{\Gamma(q-1)} \int_0^{t_1} \int_s^{t_1} \int_0^s (t_1-\delta)^{q-2} E_{q,1}\left(A(\delta-s)^q\right) p(s-x) g(x,u(x)) d\delta dx ds\Bigg) \\
& + (u_0 - m(u)) d\left(E_{q,1}\left(At_2^q\right), E_{q,1}\left(At_1^q\right)\right) \\
& + (u_1 - n(u)) d\left(t_2 E_{q,2}\left(At_2^q\right), t_1 E_{q,2}\left(At_1^q\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq d\Bigg(\frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds,\\
&\frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}(t_1-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds\Bigg)\\
&+d\Bigg(\frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}\int_{t_1}^{t_2}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds,\tilde{0}\Bigg)\\
&+d\Bigg(\frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}\int_0^{s}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds,\\
&\frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}\int_0^{s}(t_1-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds\Bigg)\\
&+d\Bigg(\frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_0^{s}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds,\tilde{0}\Bigg)\\
&+(u_0-m(u))d\left(E_{q,1}\left(At_2^q\right),E_{q,1}\left(At_1^q\right)\right)\\
&+(u_1-n(u))d\left(t_2E_{q,2}\left(At_2^q\right),t_1E_{q,2}\left(At_1^q\right)\right)\\
&\leq \frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}d\left((t_2-\delta)^{q-2},(t_1-\delta)^{q-2}\right)E_{q,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds\\
&+d\Bigg(\frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}\int_{t_1}^{t_2}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)f(s,u(s))d\delta ds,\tilde{0}\Bigg)\\
&+\frac{1}{\Gamma(q-1)}\int_0^{t_1}\int_s^{t_1}\int_0^{s}d\left((t_2-\delta)^{q-2},(t_1-\delta)^{q-2}\right)E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds\\
&+d\Bigg(\frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_0^{s}(t_2-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)p(s-x)g(x,u(x))d\delta dxds,\tilde{0}\Bigg)\\
&+(u_0-m(u))d\left(E_{q,1}\left(At_2^q\right),E_{q,1}\left(At_1^\alpha\right)\right)\\
&+(u_1-n(u))d\left(t_2E_{q,2}\left(At_2^q\right),t_1E_{q,2}\left(At_1^q\right)\right),
\end{aligned}
$$

Hence,

$$d\left(\mathcal{L}u(t_2),\mathcal{L}u(t_1)\right)\to 0 \text{ as } t_1\to t_2.$$

. This implies that $\mathcal{L}(B_\lambda)$ is equi-continuous.

Step 5: $\mathcal{L}$ is K-set contraction.

For $\mathcal{Z}\subset B_\lambda, t\in J$, we obtain

$$
\begin{aligned}
\mathcal{M}(\mathcal{L}(\mathcal{Z})(t)) &= \mathcal{M}(\{(\mathcal{L}u)(t), u \in \mathcal{Z}\}) \\
&= \mathcal{M}\Bigg(\Bigg\{\frac{1}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right) f(s,u(s))d\delta ds \\
&+ \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right) p(s-x)g(x,u(x))d\delta dxds \\
&+ E_{q,1}\left(At^q\right)(u_0-m(u)) + tE_{q,2}\left(At^q\right)(u_1-n(u)), u \in \mathcal{Z}\Bigg\}\Bigg) \\
&\leq \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)\mathcal{M}(\{f(s,u(s)), u \in \mathcal{Z}\})\,d\delta ds \\
&+ \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right) p(s-x)\mathcal{M}(\{g(x,u(x)), u \in \mathcal{Z}\})\,d\delta dxds \\
&+ E_{q,1}\left(At^q\right)\mathcal{M}(\{(u_0-m(u)), u \in \mathcal{Z}\}) + tE_{q,2}\left(At^q\right)\mathcal{M}(\{(u_1-n(u)), u \in \mathcal{Z}\}) \\
&\leq \frac{\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}\mathcal{M}(\mathcal{Z})}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2}E_{\alpha,1}\left(A(\delta-s)^q\right)d\delta ds \\
&+ \frac{\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}\mathcal{M}(\mathcal{Z})}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right) p(s-x)d\delta dxds \\
&+ E_{q,1}\left(At^q\right)\mathcal{M}(\{m(u), u \in \mathcal{Z}\}) + tE_{q,2}\left(At^q\right)\mathcal{M}(\{n(u), u \in \mathcal{Z}\}) \\
&\leq M\left[\frac{T^q\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(\alpha+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta\right]\mathcal{M}(\mathcal{Z})
\end{aligned}
$$

Thus, by (27), we conclude that $\mathcal{L}$ is a K-set contraction, and

$$
K := M\left[\frac{T^q\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta\right]
$$

Hence, since all conditions of Theorem 2 are meets we refer that $\mathcal{L}$ has a fixed point $u \in B_\lambda$.

Consequently, the problem (1) has at least one solution $u \in C(J, \mathcal{T})$. □

5. Generalized Ulam Hyers stability results

Existence criteria are necessary when we study the qualitative behavior of solutions to (1).

We begin by defining Ulam-Hyers and Generalised-Ulam-Hyers stability in relation to the Eq. (1), as below.

Allow $u \in C(J, \mathcal{T})$ and $\varepsilon > 0$. Given the subsequent inequality

$$
\begin{cases} D_t^q u(t) - Au(t) + f(t,u(t)) + \int_0^t p(t-s)g(s,u(s))ds < \varepsilon, t \in J, \\ u(0) + m(u) = u_0 \in \mathcal{T}, \quad u'(0) + n(u) = u_1 \in \mathcal{T}, \end{cases} \tag{30}
$$

Definition 12. *Assume that,* $\forall \varepsilon > 0$ *and* $\forall u \in C(J, \mathcal{T})$ *satisfying* (30)*, there exist a unique* $v \in C(J, \mathcal{T})$ *satisfying* (1) *and a constant* $\Omega_1 > 0$ *such that*

$$
d(u(t), v(t)) \leq \Omega_1 \varepsilon
$$

then problem (1) *is called Ulam-Hyers* (*UH*) *stable.*

Definition 13. *Assume that, $\forall \varepsilon > 0$ and $\forall u \in C(J,\mathcal{T})$ satisfying* (30)*, there exist a unique $v \in C(J,\mathcal{T})$ satifying* (1) *and $\kappa \in C(\mathbb{R},\mathbb{R}^+)$ with $\kappa(0) = 0$ such that*

$$d(u(t), v(t)) \leq \kappa(\varepsilon)$$

then problem (1) *is called generalized Ulam-Hyers* (*GUH*) *stable.*

Definition 14. *For each $\varepsilon > 0$ and for each solution v of* (1)*, the problems* (1) *is called Ulam-Hyers-Rassias stable with respect to $v \in C(J,\mathbb{R}^+)$ if*

$$D_t^q u(t) - Au(t) + f(t,u(t)) + \int_0^t p(t-s)g(s,u(s))ds < \epsilon v(t), t \in J, \tag{31}$$

and there exist a real number $\nu > 0$ and a solution $u \in C(J,\mathbb{R})$ of (1) *such that*

$$d(u(t), v(t)) \leq \nu\varepsilon_* v(t), t \in J.$$

where ε_* is a positive real number depending on ε.

Remark 1. *$u \in C(J,\mathcal{T})$ is a solution of inequality* (30) *iff there exists $\eta \in C(J,\mathcal{T})$ such that*

(1) $d\left(\eta(t), \tilde{0}\right) \leq \varepsilon, t \in J;$

(2) $D_t^q u(t) = Au(t) + f(t,u(t)) + \int_0^t p(t-s)g(s,u(s))ds + \eta(t), t \in J;$

(4) $u(0) + m(u) = u_0 \in \mathcal{T}, \quad u'(0) + n(u) = u_1 \in \mathcal{T}.$

Theorem 5. *Assume that conditions (Hyp1)-(Hyp4) and* (27) *hold. Then, the Eq.* (1) *is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Based on Definition 11, the inequality 30 is solved by

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(q-1)}\int_0^t\int_s^t (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)\left[f(s,u(s))+\eta(s)\right]d\delta ds \\ &+ \frac{1}{\Gamma(q-1)}\int_0^t\int_s^t\int_0^s (t-\delta)^{q-2}E_{q,1}\left(A(\delta-s)^q\right)\left[p(s-x)g(x,u(x))+\eta(s)\right]d\delta dx ds \\ &+ E_{q,1}\left(At^q\right)(u_0-m(u)) + tE_{q,2}\left(At^q\right)(u_1-n(u)). \end{aligned} \tag{32}$$

According to Theorem 4 and Definition 11, the unique solution $v \in C(J,\mathcal{T})$ of (1) satisfies (26). For all $\varepsilon > 0$, from ,The assumption **(Hyp3)** and (1) of 1 and using the same calculus as (29), we get

$$d(u,v) \leq M\left[\frac{T^q\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta\right]\varepsilon \tag{33}$$

Therefore, we know from (33) and the definitions 13 and 12 that the problem (1) is both Ulam-Hyers and generalized Ulam-Hyers stable. The proof is completed. □

Theorem 6. *Assume that **(Hyp3)** hold with $M < \Delta^{-1}$,* `where` *$\Delta = \frac{T^q\|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta$, and there exists a function $v \in C(J,\mathbb{R}^+)$ satisfying the condition* (31)*. Then the problems* (1) *is Ulam-Hyers-Rassias stable with respect to v.*

Proof. We have from the proof of Theorem 5,

$$d(u,v) \leq \varepsilon_* v.$$

where $\varepsilon_* = M\Delta$, this completes the proof. □

6. Example

Take the next FFDE:

$$\begin{cases} \frac{\partial^q u(t,x)}{\partial t^q} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,u(t,x)) + \int_0^t p(t-s)g(s,u(s,x))ds, \\ u(t,0) = u(t,\pi) = 0, \quad u'(t,0) = u'(t,\pi) = 0, \\ u(t,0) + \sum_{i=1}^p a_i u(t_i,x) = u_0(x), \quad u'(t,0) + \sum_{i=1}^p c_i u(t_i,x) = u_1(x). \end{cases} \tag{34}$$

Where $t \in J = [0,1], x \in (0,\pi), 1 < q < 2$, let $E = C([0,\pi],\mathcal{T})$ and take the operator $A : D(A) \subseteq E \to E$ provided by

$$D(A) = \left\{ u \in E, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in E \right\}, \quad A(u) = \frac{\partial^2 u}{\partial x^2}.$$

Clearly A is dense defined in E and is the infinitesimal generator of a compact C_0-semigroup $E_{q,n}(At^q)$ on $\overline{D(A)}$ and allow $u, v \in C([0,1],E)$. Define the operators $f : J \times \mathcal{Z} \to E, g : J \times \mathcal{Z} \to E$ and $p : J \to E$ by

$$f(s,u) = \frac{e^{-s}|u(s,x)|}{(12+e^s)(1+|u(s,x)|)}, \quad g(s,u) = \frac{e^s}{\sqrt{72}+|u(s,x)|},$$

$$m(u)(x) = \sum_{i=1}^p a_i u(s_i,x), \quad n(u)(x) = \sum_{i=1}^p c_i u(s_i,x),$$

$$p(s-t) = e^{s-t}.$$

Based on the estimate of the operator $E_{q,n}(At^q)$, we may get $M = 3$. in addition

$$\|f(s,u) - f(s,v)\| = \frac{e^{-s}}{12+e^s}\left\|\frac{u}{1+u} - \frac{v}{1+v}\right\| \le \frac{e^{-s}}{12+e^s}\|u-v\| \le \frac{1}{12}\|u-v\|.$$

Thus $\|\rho\| = \frac{1}{12}$.

$$\|g(s,u) - g(s,v)\| = e^s \left\| \frac{1}{\sqrt{72}+u} - \frac{1}{\sqrt{72}+v} \right\| \le \frac{e^t}{72}\|u-v\|. \le \frac{1}{24}$$

Therefore $\|\varrho\| = \frac{1}{24}$.

$$\max_{t\in[0,T]} \int_0^t |p(t-s)|ds = \max_{t\in[0,1]} \int_0^t e^{|t-s|}ds = \max_{t\in[0,1]} e^t - 1 \le 2.$$

And

$$\|m(u) - m(v)\| \le \sum_{i=1}^p |a_i| \, \|u-v\| \le b\|u-v\|,$$

$$\|n(u) - n(v)\| \le \sum_{i=1}^p |c_i| \, \|u-v\| \le d\|u-v\|$$

Let $q = \frac{3}{2}, \theta = \sum_{i=1}^p |a_i| \le \frac{1}{12}, \eta = \sum_{i=1}^p |c_i| \le \frac{1}{12}$. We shall check that condition

$$\begin{aligned} K := M &\left[\frac{T^q \|\rho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+1)} + \frac{pT^{q+1}\|\varrho\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}}{\Gamma(q+2)} + \theta + T\eta \right] \\ &< 3 \times \left(\frac{\frac{1}{12}}{\Gamma(5/2)} + 2\frac{\frac{1}{24}}{\Gamma(7/2)} + \frac{1}{12} + \frac{1}{12} \right) \\ &< 3 \times (0.062 + 0,025 + 0,083 + 0,083) = 0,759 \\ &< 1. \end{aligned}$$

Finally, based on Theorem 4, we gained that 34 has a unique mild solution.

In addition, according to Theorems 5 and 6, Eq. 34 is UH stable and GUH stable as well as UHR stable.

Remark 2. *The developed example demonstrates the potential of the improved existence and stability outcomes.*

7. Conclusions

The boundary-value issue for a Caputo nonlinear fractional differential equation of order $1 < q < 2$ is introduced in this study (recalling some recents results as [24–26] and reference therein). Three methodologies were used to effectively study the analytical solutions: the Kuratowski measure of non-compactness, Darbo's fixed-point theorem, and the generalized Ulam-Hyers stability principle. We set existence and stability requirements for the problem-solving solutions under assessments. At the conclusion, an example is provided to corroborate and confirm the viability of the acquired findings. We anticipate that the presented results will inspire scholars to pursue more study on the issue. The outcomes of the determined existence are critical in the qualitative analysis of the offered problem.

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