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The maximum spectral radii of weighted uniform loose cycles and unicyclic hypergraphs

Juanxia Du[a], Peng Xiao[a,*], Weige Xi[b]

[a] *School of Mathematics and Data Science, Shaanxi University of Science and Technology, Shaanxi, Xi'an 710021, P.R. China*
[b] *College of Science, Northwest A&F University, Shaanxi, Xianyang 712100, P.R. China*

Abstract. A weighted k-uniform loose cycle of length m, denoted by $C_{m,k}$, is a cyclic list of weighted edges $e_1, e_2, \dots, e_m$ such that consecutive edges intersect in exactly one vertex, and nonconsecutive edges are disjoint, where $|e_i| = k$ for all $1 \le i \le m$. For a given positive weight set, we determine the distribution of weights of $C_{m,k}$ with the maximum spectral radius. Moreover, we characterize the unique weighted hypergraph with the maximum spectral radius in the class of all weighted uniform unicyclic hypergraphs with a given positive weight set.

1. Introduction

A hypergraph is a generalization of a graph, in which an edge can connect more than two vertices. As different edges may have different importance in representing connections among vertices, it is crucial that edges be weighted corresponding to their representative capabilities. A weighted hypergraph is a hypergraph in which each edge is assigned a weight.

Let $G = (V(G), E(G), W(G))$ be a weighted hypergraph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, edge set $E(G) = \{e_1, e_2, \dots, e_m\}$ and weight set $W(G) = \{w_G(e) \in \mathbb{R} | e \in E(G)\}$, $w_G(e)$ is the weight on the edge e of G. An unweighted hypergraph is a weighted hypergraph with each of the edges bearing weight 1. If every edge of G contains precisely k vertices, then G is called weighted k-uniform hypergraph. A weighted graph is a weighted 2-uniform hypergraph.

For $v \in V(G)$ and $e \in E(G)$, v is said to be incident to e if $v \in e$. For $v \in V(G)$, denote by $E_G(v)$ the set of all edges incident to v. The degree of vertex v of G, denoted by $d_G(v)$, is $|E_G(v)|$. If $d_G(v) = 1$, then v is called a pendent vertex. An edge $e \in E(G)$ is said to be a pendent edge if it contains exactly $k-1$ pendent vertices.

Let $P = (v_1, e_1, v_2, \dots, v_l, e_l, v_{l+1})$ be an alternating sequence of vertices and edges of G, the sequence P is called a walk of G, if v_i and v_{i+1} are incident to e_i for any $1 \le i \le l$. A walk P is referred to as a path if all vertices and all edges are distinct. A walk P is called a cycle of length l, if all vertices and all edges are distinct except $v_1 = v_{l+1}$, where $l \ge 2$. For any $u, v \in V(G)$, G is said to be connected, if u and v are connected by a path.

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* Corresponding author: Peng Xiao
Email addresses: djuanxia@qq.com (Juanxia Du), xiaopeng@sust.edu.cn (Peng Xiao), xiyanxwg@163.com (Weige Xi)

For a connected weighted k-uniform hypergraph G, let n, m be the numbers of vertices and edges of G, respectively. If $n = m(k-1)+1$, then G is called weighted k-uniform hypertree. If $n = m(k-1)$, then G is called weighted k-uniform unicyclic hypergraph.

For positive integers k and n, a real tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$ of order k and dimension n is a multidimensional array, where $a_{i_1 i_2 \dots i_k} \in \mathbb{R}, i_1, i_2, \dots, i_k \in [n], [n] = \{1, 2, \dots, n\}$.

A hypergraph G is usually represented by a tensor (also called hypermatrix). Let $\mathcal{A}$ be a real tensor with order k and dimension n, and $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. According to the product of tensors defined by Shao [14], then $\mathcal{A}x$ is a column vector in $\mathbb{C}^n$ whose ith component is

$$(\mathcal{A}x)_i = \sum_{i_2,\dots,i_k=1}^{n} a_{i i_2 \dots i_k} x_{i_2} x_{i_3} \cdots x_{i_k}, \; i \in [n]. \tag{1}$$

In 2005, Qi [11] and Lim [8] independently introduced the concepts of tensor eigenvalues and tensor spectrums.

Let $x^{[k]} = (x_1^k, x_2^k, \dots, x_n^k)^T \in \mathbb{C}^n$, and let $\mathcal{A}$ be a real tensor with order k and dimension n. If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$, such that

$$\mathcal{A}x = \lambda x^{[k-1]}, \tag{2}$$

then λ is referred to as the eigenvalue of the tensor $\mathcal{A}$, and x is known as the eigenvector of corresponding to the eigenvalue λ.

The adjacency tensor of a weighted k-uniform hypergraph is defined as follows.

Definition 1.1. *Let G be a weighted k-uniform hypergraph with n vertices. The adjacency tensor of G is defined as the n-dimensional tensor $\mathcal{A}(G)=(a_{i_1 i_2 \dots i_k})$ of order k, where*

$$a_{i_1 i_2 \dots i_k} = \begin{cases} \frac{w_G(e)}{(k-1)!}, & e = \{i_1, i_2, \dots, i_k\} \in E(G), \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

For a weighted uniform hypergraph G, the spectral radius of G is the largest modulus of the eigenvalues of its adjacency tensor $\mathcal{A}(G)$, denoted by $\rho(G)$.

During the past decade, the problem concerning hypergraphs with the extremal spectral radius of a given class of hypergraphs has been studied extensively. Li et al. [7] determined the first two hypergraphs with the largest spectral radii over all uniform hypertrees. Yuan et al. [22] determined the first eight uniform hypergraphs with the larger spectral radii among all uniform hypertrees. Among the set of uniform hypertrees with given parameters, such as independent number [25], perfect matching [23], stability number [15], degree sequence [20], the hypergraphs with the extremal spectral radii were characterized. Fan et al. [2] characterized the hypergraphs with the maximum spectral radius of several classes of uniform unicyclic hypergraphs with few edges. Yu et al. [21] determined the hypergraph with the maximum spectral radii over the unicyclic hypergraphs with a given matching number. Kang et al. [5] determined the hypergraph with the maximum spectral radius over the linear bicyclic uniform hypergraphs. Other related works can be referred to [1, 4, 6, 9, 10, 13, 17, 24].

Recently, there are few results on the problems of weighted hypergraphs in the literature. Galuppi et al. [3] obtained the properties of tensor eigenvalues and eigenvectors of weighted hypergraphs. Sun et al. [16] investigated the eigenvalues of Laplacian tensor and signless Laplacian tensor of weighted hypergraphs. For a fixed total weight sum, Wang et al. [18, 19] introduced the α-normal labelling method for comparing the spectral radii of the weighted hypergraphs, and characterized the weighted uniform hypertrees, unicyclic hypergraphs and bicyclic hypergraphs with the maximum spectral radius, respectively.

Inspired by the work [16, 18, 19], we investigate the spectral extremal problems of weighted uniform hypergraphs, and focus on the case when the weight set is given.

In what follows, we always suppose that the elements of weight set $W_m = \{w_1, w_2, \dots, w_m\}$ are positive and nondecreasingly order, that is, $w_1 \geq w_2 \geq \dots \geq w_m > 0$. Now, we are in a position to state two problems studied of this paper.

A weighted k-uniform loose cycle of length m, denoted by $C_{m,k}$, is a cyclic list of weighted edges $e_1, e_2, \ldots, e_m$ such that consecutive edges intersect in exactly one vertex, and nonconsecutive edges are disjoint, where $|e_i| = k$ for all $1 \leq i \leq m$. Let $\Gamma(C_{m,k}, W_m)$ denote the set of weighted k-uniform loose cycles $C_{m,k}$ with the weight set W_m. Our first question is: Given a positive weight set W_m and the loose cycle $C_{m,k}$, what is the optimal distribution of weights among the edges of $C_{m,k}$, so that the spectral radius of $C_{m,k}$ is maximized?

Let $\Omega(m, k, W_m)$ denote the set of weighted k-uniform unicyclic hypergraphs with m edges and weight set W_m. Our second question is: What is the maximum spectral radius among all weighted hypergraphs in $\Omega(m, k, W_m)$?

This paper is organized as follows. In Section 2, we list some definitions and operating tools which are useful to the proofs of our results. In Section 3, we characterize the unique weighted k-uniform loose cycle, which attains the maximum spectral radius in $\Gamma(C_{m,k}, W_m)$. In Section 4, we characterize the unique weighted hypergraph with the maximum spectral radius in $\Omega(m, k, W_m)$.

2. Preliminaries

In this section, we give some lemmas which will be used in the follows.

Let $G = (V, E, W)$ be a weighted uniform hypergraph on n vertices, and x be a column vector of dimension n. For a subset $U \subseteq V$, we write $x^U = \prod_{v_i \in U} x_{v_i}$, where x_{v_i} is the component of x corresponding to vertex v_i. By (1) and the definition of adjacency tensor $\mathcal{A}(G)$ of G, we have

$$(\mathcal{A}(G)x)_i = \sum_{e \in E_G(v_i)} w_G(e) x^{e \setminus \{v_i\}}, \tag{4}$$

and by (4), we get

$$x^T(\mathcal{A}(G)x) = \sum_{e \in E(G)} k w_G(e) x^e. \tag{5}$$

The tensor $\mathcal{A}$ is called symmetric if the elements of $\mathcal{A}(G)$=$(a_{i_1 i_2 \ldots i_k})$ are invariant under arbitrary permutation of their indices $i_1, i_2, \ldots, i_k$. In 2013, Qi [12] given the Rayleigh Quotient Lemma of nonnegative symmetric tensor.

Lemma 2.1. ([12]) *Let $\mathcal{A}$ be a nonnegative symmetric tensor with order k and dimension n. Then we have*

$$\rho(\mathcal{A}) = \max\{x^T(\mathcal{A}x) | x \in \mathbb{R}^n_+, ||x||^k_k = \sum_{i=1}^{n} x_i^k = 1\}.$$

Furthermore, $x \in \mathbb{R}^n_+$ is an optimal solution of above optimization problem if and only if x is an eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$.

The Perron-Frobenius theorem for adjacency tensor of weighted hypergraphs is partially described as the following.

Lemma 2.2. ([18]) *Let G be a weighted k-uniform hypergraph of order n, where $k, n \geq 3$. If all the weights of G are positive and G is connected, then $\rho(G)$ is the unique eigenvalue of the adjacency tensor $\mathcal{A}(G)$ with positive eigenvector.*

In Lemma 2.2, the unique positive eigenvector x with $\sum_{i=1}^{n} x_i^k = 1$ is called the principal eigenvector of G.

In [18], Wang et al. introduced the edge-moving operation, edge-releasing operation, weight-moving operation of weighted k-uniform hypergraphs, and studied the perturbation of the spectral radius under these operations.

Definition 2.3. *Let G be a connected weighted k-uniform hypergraph of order n, where $k, n \geq 3$, $u \in V(G)$, $e_1, \ldots, e_r \in E(G)$, and $u \notin e_i$, $i = 1, \ldots, r(r \geqslant 1)$. Suppose $v_i \in e_i$ and let $e_i' = (e_i \backslash \{v_i\}) \cup \{u\}$, where vertices $v_1, \ldots, v_r$ need not be distinct. Let $G' = (V(G'), E(G'), W(G'))$ be a weighted k-uniform hypergraph with $V(G') = V(G), E(G') = (E(G) \backslash \{e_1, \ldots, e_r\}) \cup \{e_1', \ldots, e_r'\}, W(G') = \{w_{G'}(e) | e \in E(G')\}$, where $w_{G'}(e) = w_G(e)$ if $e \neq e_i'$ and $w_{G'}(e) = w_G(e_i)$ if $e = e_i'$. Then we say that G' is obtained from G by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u.*

For a weighted k-uniform hypergraph G, if the edges e and e' of G share common k vertices, then we say e and e' are two multiple edges.

Lemma 2.4. ([18]) *Let G' and G be two weighted hypergraphs as described in definition 2.3. Suppose that G' does not contain multiple edges, and let x be the principal eigenvector of G. If $x_u \geq \max_{1 \leq i \leq r} x_{v_i}$, then $\rho(G') > \rho(G)$.*

Definition 2.5. *Let G be a connected weighted k-uniform hypergraph of order n, where $k, n \geq 3$. Let e be a non-pendent edge of G, and vertex $u \in e$. Let $\{e_1, \ldots, e_r\}$ be all the edges of G adjacent to e but not containing u. Suppose $e_i \cap e = \{v_i\}$, where $i = 1, 2, \ldots, r$. Let G' be a weighted hypergraph obtained from G by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to u. Then we say that G' is obtained from G by the edge-releasing operation on e.*

Lemma 2.6. ([18]) *Let G' and G be two weighted hypergraphs as described in definition 2.5. If G' does not have multiple edges, then $\rho(G') > \rho(G)$.*

By doing a slight modification to the Lemma 3.3 in [18], we get the following Lemma.

Lemma 2.7. *Let G be a connected weighted k-uniform hypergraph of order n. Let e_1, e_2 be two distinct edges of G, and let δ be a positive real number with $0 < \delta < w_G(e_1)$. Let $G' = (V(G'), E(G'), W(G'))$, where $V(G') = V(G)$, $E(G') = E(G)$, $W(G') = \{w_{G'}(e) | e \in E(G')\}$ with $w_{G'}(e_1) = w_G(e_1) - \delta$, $w_{G'}(e_2) = w_G(e_2) + \delta$, and $w_{G'}(e) = w_G(e)$ for $e \neq e_1, e_2$. Let x be the principal eigenvector of G. If $x^{e_1} \leq x^{e_2}$, then $\rho(G') > \rho(G)$.*

Proof. Let G' and G be two weighted k-uniform hypergraphs as described in Lemma 2.7. By Lemmas 2.1, 2.2 and (5), we get

$$\begin{aligned}
\rho(G') - \rho(G) \geq & x^T(\mathcal{A}(G')x) - x^T(\mathcal{A}(G)x) && (6)\\
= & k\Big(\sum_{e \in E(G')} w_{G'}(e)x^e - \sum_{e \in E(G)} w_G(e)x^e \Big) \\
= & k\Big(w_{G'}(e_1)x^{e_1} + w_{G'}(e_2)x^{e_2} - w_G(e_1)x^{e_1} - w_G(e_2)x^{e_2} \Big) \\
= & k\delta\Big(x^{e_2} - x^{e_1} \Big) \\
\geq & 0. && (7)
\end{aligned}$$

Where (7) follows from $0 < \delta < w_G(e_1)$ and $x^{e_1} \leq x^{e_2}$. Thus, we obtain $\rho(G') \geq \rho(G)$.

Next, we suppose $\rho(G') = \rho(G)$, it is obvious that the two equalities in (6) and (7) hold. Thus, $\rho(G') = x^T(\mathcal{A}(G')x)$. By Lemma 2.1, we have that x is also the principal eigenvector of G' corresponding to $\rho(G')$. Since e_1, e_2 are two distinct edges of G, in e_2, there exists a vertex u such that $u \notin e_1$. By using the eigenequation for vertex u of G', we have

$$\begin{aligned}
\rho(G')x_u^{k-1} = & w_{G'}(e_2)x^{e_2 \backslash \{u\}} + \sum_{e \in E_{G'}(u) \backslash \{e_2\}} w_{G'}(e)x^{e \backslash \{u\}} \\
= & (w_G(e_2) + \delta)x^{e_2 \backslash \{u\}} + \sum_{e \in E_G(u) \backslash \{e_2\}} w_G(e)x^{e \backslash \{u\}} \\
= & \delta x^{e_2 \backslash \{u\}} + \sum_{e \in E_G(u)} w_G(e)x^{e \backslash \{u\}} \\
> & \sum_{e \in E_G(u)} w_G(e)x^{e \backslash \{u\}} && (8)\\
= & \rho(G)x_u^{k-1}.
\end{aligned}$$

Where (8) follows from $0 < \delta < w_G(e_1)$ and $x^{e_2\setminus\{u\}} > 0$. Thus, we obtain $\rho(G') > \rho(G)$, which contradicts with $\rho(G') = \rho(G)$. □

3. The maximum weighted uniform loose cycle

Let $W_m = \{w_1, w_2, \ldots, w_m\}$ be a positive weight set, where $w_1 \geq w_2 \geq \ldots \geq w_m > 0$. In this section, we investigate how to assign weights $w_1, w_2, \ldots, w_m$ to the m edges of $C_{m,k}$, such that the spectral radius of $C_{m,k}$ is maximized. Let $\Gamma(C_{m,k}, W_m)$ be the set of weighted k-uniform loose cycles $C_{m,k}$ with the weight set W_m, and let G^* be the weighted hypergraph which attains the maximum spectral radius in $\Gamma(C_{m,k}, W_m)$.

Now, we investigate some properties of the principal eigenvector x of G^*.

Lemma 3.1. *Let f_1, f_2 be two distinct edges of G^*.*

(i) *If $x^{f_1} \geq x^{f_2}$, then $w_{G^*}(f_1) \geq w_{G^*}(f_2)$;*
(ii) *If $w_{G^*}(f_1) > w_{G^*}(f_2)$, then $x^{f_1} > x^{f_2}$;*
(iii) *If $x^{f_1} = x^{f_2}$, then $w_{G^*}(f_1) = w_{G^*}(f_2)$.*

Proof. (i) Assume that $w_{G^*}(f_1) < w_{G^*}(f_2)$. Put $\delta = w_{G^*}(f_2) - w_{G^*}(f_1) > 0$. Let G be the weighted hypergraph obtained from G^* by exchanging the weights of edges f_1 and f_2, i.e.,

$$w_G(f_1) = w_{G^*}(f_1) + \delta,\ w_G(f_2) = w_{G^*}(f_2) - \delta,\ w_G(e) = w_{G^*}(e) \text{ for } e \in E(G) \setminus \{f_1, f_2\}.$$

By Lemma 2.7 and $x^{f_1} \geq x^{f_2}$, we have $\rho(G) > \rho(G^*)$, a contradiction.

(ii) Assume that $x^{f_1} \leq x^{f_2}$. From Lemma 3.1(i), it can be derived that $w_{G^*}(f_1) \leq w_{G^*}(f_2)$, a contradiction.

(iii) Assume that $w_{G^*}(f_1) \neq w_{G^*}(f_2)$. If $w_{G^*}(f_1) > w_{G^*}(f_2)$, from Lemma 3.1(ii), we obtain $x^{f_1} > x^{f_2}$, a contradiction. If $w_{G^*}(f_1) < w_{G^*}(f_2)$, from Lemma 3.1(ii), we have $x^{f_1} < x^{f_2}$, a contradiction. □

Lemma 3.2. *Let $f_1 = \{v_1, v_{1,1}, \ldots, v_{1,k-2}, v_2\}$ and $f_2 = \{u_1, u_{1,1}, \ldots, u_{1,k-2}, u_2\}$ be two disjoint edges of G^*, where $d_{G^*}(v_1) = d_{G^*}(v_2) = d_{G^*}(u_1) = d_{G^*}(u_2) = 2$ and $d_{G^*}(v_{1,i}) = d_{G^*}(u_{1,i}) = 1$ for $1 \leq i \leq k-2$. Then*

$$\Big(x_{v_1} - x_{u_1}\Big)\Big(w_{G^*}(f_1)x^{f_1\setminus\{v_1\}} - w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}\Big) \geqslant 0.$$

Furthermore, $x_{v_1} = x_{u_1}$ if and only if $w_{G^*}(f_1)x^{f_1\setminus\{v_1\}} = w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}$.

Proof. Let G be the weighted k-uniform hypergraph with $V(G) = V(G^*)$, $E(G) = (E(G^*)\setminus\{f_1, f_2\}) \cup \{f_1', f_2'\}$, where $f_1' = (f_1\setminus\{v_1\}) \cup \{u_1\} = \{u_1, v_{1,1}, \ldots, v_{1,k-2}, v_2\}$, $f_2' = (f_2\setminus\{u_1\}) \cup \{v_1\} = \{v_1, u_{1,1}, \ldots, u_{1,k-2}, u_2\}$ and

$$w_G(f_1') = w_{G^*}(f_1),\ w_G(f_2') = w_{G^*}(f_2),\ w_G(e) = w_{G^*}(e) \text{ for } e \neq f_1', f_2'.$$

Obviously, $G \in \Gamma(C_{m,k}, W_m)$. By Lemmas 2.1, 2.2 and (5), we have

$$\begin{aligned}
0 \leq{}& \rho(G^*) - \rho(G) && (9)\\
\leq{}& x^T(\mathcal{A}(G^*)x) - x^T(\mathcal{A}(G)x) && (10)\\
={}& k\Big(\sum_{e\in E(G^*)} w_{G^*}(e)x^e - \sum_{e\in E(G)} w_G(e)x^e\Big)\\
={}& k\Big(w_{G^*}(f_1)x^{f_1} + w_{G^*}(f_2)x^{f_2} - w_G(f_1')x^{f_1'} - w_G(f_2')x^{f_2'}\Big)\\
={}& k\Big(w_{G^*}(f_1)x^{f_1\setminus\{v_1\}}\cdot x_{v_1} + w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}\cdot x_{u_1} - w_{G^*}(f_1)x^{f_1\setminus\{v_1\}}\cdot x_{u_1} - w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}\cdot x_{v_1}\Big)\\
={}& k\Big(w_{G^*}(f_1)x^{f_1\setminus\{v_1\}}(x_{v_1} - x_{u_1}) + w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}(x_{u_1} - x_{v_1})\Big)\\
={}& k\Big(x_{v_1} - x_{u_1}\Big)\Big(w_{G^*}(f_1)x^{f_1\setminus\{v_1\}} - w_{G^*}(f_2)x^{f_2\setminus\{u_1\}}\Big).
\end{aligned}$$

Thus,

$$\left(x_{v_1} - x_{u_1}\right)\left(w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} - w_{G^*}(f_2)x^{f_2\backslash\{u_1\}}\right) \geq 0. \tag{11}$$

Using the eigenequations for vertices u_1 and u_2 of G^*, respectively. We have

$$\rho(G^*)x_{u_1}^{k-1} = w_{G^*}(f_2)x^{f_2\backslash\{u_1\}} + \sum_{e\in E_{G^*}(u_1)\backslash\{f_2\}} w_{G^*}(e)x^{e\backslash\{u_1\}}, \tag{12}$$

and

$$\rho(G^*)x_{u_2}^{k-1} = w_{G^*}(f_2)x^{f_2\backslash\{u_1,u_2\}} \cdot x_{u_1} + \sum_{e\in E_{G^*}(u_2)\backslash\{f_2\}} w_{G^*}(e)x^{e\backslash\{u_2\}}. \tag{13}$$

If $x_{v_1} = x_{u_1}$, it is obvious that the equality in (11) holds. Then the two equalities in (9) and (10) hold. Thus, $\rho(G^*) = \rho(G)$ and $\rho(G) = x^T(\mathcal{A}(G)x)$. By Lemmas 2.1 and 2.2, x is also the principal eigenvector of G corresponding to $\rho(G)$.

Using the eigenequations for vertices u_1 and u_2 of G, respectively. We have

$$\begin{aligned}\rho(G)x_{u_1}^{k-1} &= w_G(f_1')x^{f_1'\backslash\{u_1\}} + \sum_{e\in E_G(u_1)\backslash\{f_1'\}} w_G(e)x^{e\backslash\{u_1\}}\\ &= w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} + \sum_{e\in E_{G^*}(u_1)\backslash\{f_2\}} w_{G^*}(e)x^{e\backslash\{u_1\}},\end{aligned} \tag{14}$$

and

$$\begin{aligned}\rho(G)x_{u_2}^{k-1} &= w_G(f_2')x^{f_2'\backslash\{u_2\}} + \sum_{e\in E_G(u_2)\backslash\{f_2'\}} w_G(e)x^{e\backslash\{u_2\}}\\ &= w_{G^*}(f_2)x^{f_2\backslash\{u_1,u_2\}} \cdot x_{v_1} + \sum_{e\in E_{G^*}(u_2)\backslash\{f_2\}} w_{G^*}(e)x^{e\backslash\{u_2\}}.\end{aligned} \tag{15}$$

Since $\rho(G^*) = \rho(G)$, then by (12) and (14), it can be derived that $w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} = w_{G^*}(f_2)x^{f_2\backslash\{u_1\}}$.

If $w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} = w_{G^*}(f_2)x^{f_2\backslash\{u_1\}}$, by (13) and (15), we can similarly obtain that $x_{v_1} = x_{u_1}$. This completes the proof. □

Lemma 3.3. *Let $f_1 = \{v_1, v_{1,1}, \ldots, v_{1,k-2}, w\}$ and $f_2 = \{u_1, u_{1,1}, \ldots, u_{1,k-2}, w\}$ be two edges of G^*, where $d_{G^*}(v_1) = d_{G^*}(u_1) = d_{G^*}(w) = 2$ and $d_{G^*}(v_{1,i}) = d_{G^*}(u_{1,i}) = 1$ for $1 \leq i \leq k-2$. Then*

$$\left(x_{v_1} - x_{u_1}\right)\left(w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} - w_{G^*}(f_2)x^{f_2\backslash\{u_1\}}\right) \geqslant 0.$$

Furthermore, $x_{v_1} = x_{u_1}$ if and only if $w_{G^*}(f_1)x^{f_1\backslash\{v_1\}} = w_{G^*}(f_2)x^{f_2\backslash\{u_1\}}$.

Proof. The proof is similar to the proof of Lemma 3.2. □

Next, we label all vertices and edges of G^* based on the principal eigenvector x and the weights $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ of G^*.

(*a*) We choose a vertex u such that $x_u = \max\{x_v | v \in e, w_{G^*}(e) = w_1, d(v) = 2\}$, and the vertex u labeled by u_1. If there are multiple such vertices, we choose and fix u_1 arbitrarily among them.

(*b*) We choose an edge e such that $w_{G^*}(e) = w_1$ and $u_1 \in e$. The edge e labeled by e_m. If there are multiple such edges, we choose e_m and u_m such that $x_{u_m} \geq x_{u_2}$.

(*c*) The labels of the remaining vertices and edges are shown in Figure 1.

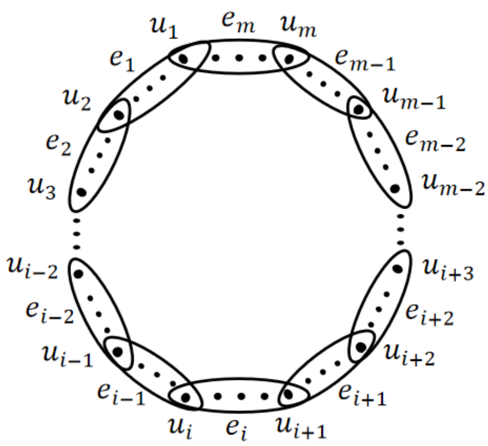

Figure 1: The loose cycle G^*.

For the weighted loose cycle G^*, (a) together with (b) imply that $x_{u_1} \geq x_{u_m}$ and $w_{G^*}(e_m) = w_1$.

In Lemma 3.4, we will determine the distribution of weights of G^*.

Lemma 3.4. *The weights of all edges of G^* satisfy:*

(i) *When $m = 2r$, $w_{G^*}(e_{2r}) \geq w_{G^*}(e_1) \geq w_{G^*}(e_{2r-1}) \geq w_{G^*}(e_2) \geq \ldots \geq w_{G^*}(e_{r+1}) \geq w_{G^*}(e_r)$;*

(ii) *When $m = 2r+1$, $w_{G^*}(e_{2r+1}) \geq w_{G^*}(e_1) \geq w_{G^*}(e_{2r}) \geq w_{G^*}(e_2) \geq \ldots \geq w_{G^*}(e_{r+2}) \geq w_{G^*}(e_r) \geq w_{G^*}(e_{r+1})$.*

Proof. Let x be the principal eigenvector of G^*. When $m = 3$, the result is obvious. Assume that $m \geq 4$.

Claim 1. $x_{u_1} = \max\limits_{1 \leq i \leq m}\{x_{u_i}\}$.

Since $x_{u_1} \geq x_{u_m}$, we may assume that there exists $2 \leq i \leq m-1$ such that $x_{u_i} > x_{u_1}$.

If $2 \leq i \leq m-2$, then we apply Lemma 3.2 to edges e_m and e_i, we have

$$\left(x_{u_1} - x_{u_i}\right)\left(w_{G^*}(e_m)x^{e_m\setminus\{u_1\}} - w_{G^*}(e_i)x^{e_i\setminus\{u_i\}}\right) \geqslant 0.$$

Since $x_{u_1} < x_{u_i}$, we have

$$w_{G^*}(e_m)x^{e_m\setminus\{u_1\}} \leq w_{G^*}(e_i)x^{e_i\setminus\{u_i\}}.$$

It follows that $x^{e_m\setminus\{u_1\}} \leq x^{e_i\setminus\{u_i\}}$, since $w_{G^*}(e_m) \geq w_{G^*}(e_i)$. Thus, $x^{e_m\setminus\{u_1\}} \cdot x_{u_1} < x^{e_i\setminus\{u_i\}} \cdot x_{u_i}$, i.e., $x^{e_m} < x^{e_i}$. By Lemma 3.1, we have $w_{G^*}(e_m) \leq w_{G^*}(e_i)$.

Noting that $x_{u_1} = \max\{x_v | v \in e, w_{G^*}(e) = w_1, d(v) = 2\}$. If $w_{G^*}(e_m) = w_{G^*}(e_i) = w_1$, then $x_{u_1} \geq x_{u_i}$, which contradicts with $x_{u_i} > x_{u_1}$. Thus, $w_{G^*}(e_m) < w_{G^*}(e_i)$, which contradicts with $w_{G^*}(e_m) = w_1 \geq w_{G^*}(e_i)$.

If $i = m-1$, the proof is similar to the case $2 \leq i \leq m-2$ in Claim 1.

Claim 2. $x_{u_m} = \max\limits_{2 \leq i \leq m}\{x_{u_i}\}$.

Assume that there exists $2 \leq i \leq m-1$ such that $x_{u_i} > x_{u_m}$.

If $2 \leq i \leq m-2$, apply Lemma 3.2 twice to the edges e_m and e_i, we have

$$\left(x_{u_m} - x_{u_i}\right)\left(w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_1} - w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_{i+1}}\right) \geqslant 0, \tag{16}$$

and

$$\left(x_{u_1} - x_{u_{i+1}}\right)\left(w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_m} - w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_i}\right) \geqslant 0. \tag{17}$$

We consider the following two possible cases.

Case 1.1. $w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \geq w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}}$.

Since $x_{u_1} \geq x_{u_{i+1}}$, we have

$$w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_1} \geq w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_{i+1}}.$$

If $w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_1} = w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_{i+1}}$, by Lemma 3.2 and (16), we have $x_{u_m} = x_{u_i}$, which is a contradiction to the assumption $x_{u_i} > x_{u_m}$. If $w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_1} > w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_{i+1}}$, by (16), we have $x_{u_m} \geq x_{u_i}$, which is also a contradiction to the assumption $x_{u_i} > x_{u_m}$.

Case 1.2. $w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} < w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}}$.

Since $x_{u_m} < x_{u_i}$, we have

$$w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_m} < w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_i}. \tag{18}$$

Combining inequalities (17) and (18), we have $x_{u_1} \leq x_{u_{i+1}}$. By Claim 1, we have $x_{u_1} = x_{u_{i+1}}$. Then the equality in (17) holds, and by Lemma 3.2, we obtain

$$w_{G^*}(e_m)x^{e_m\setminus\{u_1,u_m\}} \cdot x_{u_m} = w_{G^*}(e_i)x^{e_i\setminus\{u_i,u_{i+1}\}} \cdot x_{u_i},$$

which contradicts with (18).

If $i = m - 1$, the proof is similar to the case $2 \leq i \leq m - 2$ in Claim 2.

Claim 3. $w_{G^*}(e_1) = w_2$.

Recall that $w_{G^*}(e_m) = w_1$, we can assume, to the contrary, that there exists a positive integer $2 \leq i \leq m-1$ such that $w_{G^*}(e_1) < w_{G^*}(e_i)$.

Case 2.1. $i = 2$.

By assuming $w_{G^*}(e_1) < w_{G^*}(e_i)$ and Lemma 3.1, we have

$$x_{u_1} \cdot x^{e_1\setminus\{u_1\}} < x_{u_{i+1}} \cdot x^{e_i\setminus\{u_{i+1}\}}.$$

It follows that $x^{e_1\setminus\{u_1\}} < x^{e_i\setminus\{u_{i+1}\}}$, since $x_{u_1} \geq x_{u_{i+1}}$. Apply Lemma 3.3 to the edges e_1 and e_i, we have

$$\left(x_{u_1} - x_{u_{i+1}}\right)\left(w_{G^*}(e_1)x^{e_1\setminus\{u_1\}} - w_{G^*}(e_i)x^{e_i\setminus\{u_{i+1}\}}\right) \geqslant 0.$$

By $x^{e_1\setminus\{u_1\}} < x^{e_i\setminus\{u_{i+1}\}}$, $x_{u_1} \geq x_{u_{i+1}}$ and Lemma 3.3, we have $w_{G^*}(e_1) \geq w_{G^*}(e_i)$, which contradicts with $w_{G^*}(e_1) < w_{G^*}(e_i)$.

Case 2.2. $3 \leq i \leq m - 1$.

Apply Lemma 3.2 to the edges e_1 and e_i, we have

$$\left(x_{u_1} - x_{u_i}\right)\left(w_{G^*}(e_1)x^{e_1\setminus\{u_1\}} - w_{G^*}(e_i)x^{e_i\setminus\{u_i\}}\right) \geqslant 0.$$

By $x_{u_1} \geq x_{u_i}$ and Lemma 3.2, we have

$$w_{G^*}(e_1)x^{e_1\setminus\{u_1\}} \geq w_{G^*}(e_i)x^{e_i\setminus\{u_i\}}.$$

It follows that $x^{e_1\setminus\{u_1\}} \geq x^{e_i\setminus\{u_i\}}$, since $w_{G^*}(e_1) < w_{G^*}(e_i)$. It is easy to see that $x_{u_1} \cdot x^{e_1\setminus\{u_1\}} \geq x_{u_i} \cdot x^{e_i\setminus\{u_i\}}$, i.e., $x^{e_1} \geq x^{e_i}$. By Lemma 3.1, we have $w_{G^*}(e_1) \geq w_{G^*}(e_i)$, a contradiction.

By using the method similar to that used in Claim 2 and Claim 3, it can be proved in order that

$$x_{u_2} = \max_{2\leq i\leq m-1}\{x_{u_i}\},\ \ w_{G^*}(e_{m-1}) = w_3,\ \ x_{u_{m-1}} = \max_{3\leq i\leq m-1}\{x_{u_i}\},\ \ w_{G^*}(e_2) = w_4, \cdots$$

When m is even, by analogy, we have

$$w_{G^*}(e_m) = w_1,\ w_{G^*}(e_1) = w_2,\ w_{G^*}(e_{m-1}) = w_3,\ w_{G^*}(e_2) = w_4, \ldots, w_{G^*}(e_{\frac{m}{2}+1}) = w_{m-1},\ w_{G^*}(e_{\frac{m}{2}}) = w_m.$$

When m is odd, by analogy, we have

$$w_{G^*}(e_m) = w_1,\ w_{G^*}(e_1) = w_2,\ w_{G^*}(e_{m-1}) = w_3,\ w_{G^*}(e_2) = w_4, \ldots, w_{G^*}(e_{\frac{m-1}{2}}) = w_{m-1},\ w_{G^*}(e_{\frac{m+1}{2}}) = w_m.$$

This completes the proof. □

The conditions (i) and (ii) in Lemma 3.4 indicate that, for a given weight set $W_m = \{w_1, w_2, \ldots, w_m\}$, the distribution of weights of G^* is uniquely determined. Therefore, by Lemma 3.4, we get the following main result.

Theorem 3.5. *Let G^* be a weighted k-uniform loose cycle with m edges and a positive weight set W_m (The vertex and edge labelling as described in Figure 1), where $m, k \geq 3$. If the weights of G^* satisfy the following conditions:*

(i) *When $m = 2r$, $w_{G^*}(e_{2r}) \geq w_{G^*}(e_1) \geq w_{G^*}(e_{2r-1}) \geq w_{G^*}(e_2) \geq \ldots \geq w_{G^*}(e_{r+1}) \geq w_{G^*}(e_r)$;*

(ii) *When $m = 2r + 1$, $w_{G^*}(e_{2r+1}) \geq w_{G^*}(e_1) \geq w_{G^*}(e_{2r}) \geq w_{G^*}(e_2) \geq \ldots \geq w_{G^*}(e_{r+2}) \geq w_{G^*}(e_r) \geq w_{G^*}(e_{r+1})$.*

Then G^* is the unique weighted hypergraph in $\Gamma(C_{m,k}, W_m)$ having the maximum spectral radius.

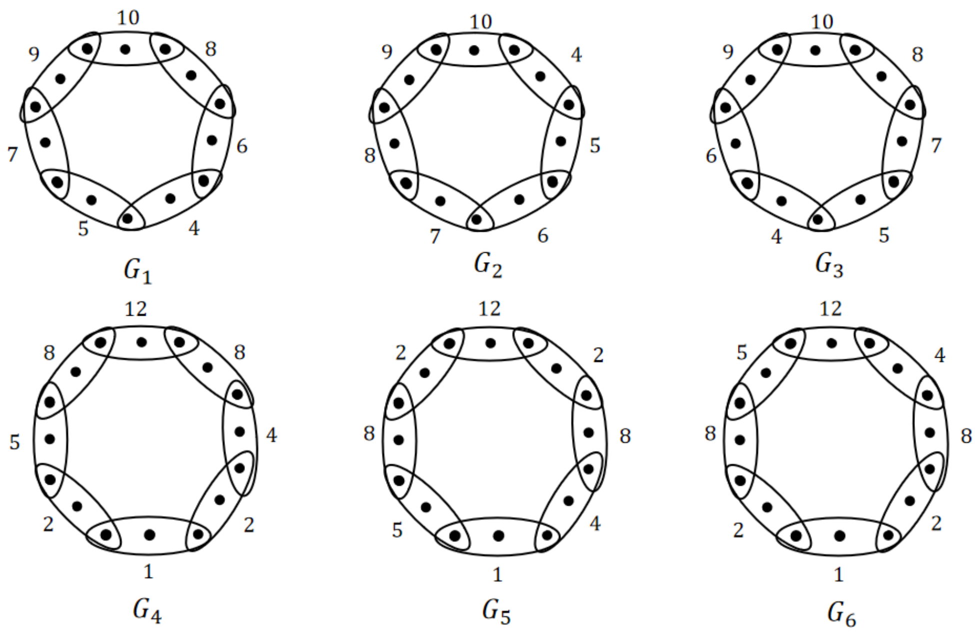

Figure 2: Six weighted 3-uniform loose cycles.

Example 3.6. *In Figure 2, six weighted 3-uniform loose cycles are displayed, where the numbers on the edges denote the weights of the edges.*

For a given positive weight set $W_7 = \{10, 9, 8, 7, 6, 5, 4\}$, three weighted loose cycles G_1, G_2 and G_3 in Figure 2 are all in $\Gamma(C_{7,3}, W_7)$. From Theorem 3.5, we know that G_1 is the weighted 3-uniform loose cycle with the maximum spectral radius in $\Gamma(C_{7,3}, W_7)$. By calculation, we have

$$\rho(G_1) = 13.0522,\ \rho(G_2) = 12.7795,\ \rho(G_3) = 13.0153.$$

This is consistent with Theorem 3.5.

Similarly, for a given positive weight set $W_8 = \{12, 8, 8, 5, 4, 2, 2, 1\}$, three weighted loose cycles G_4, G_5, G_6 in Figure 2 are all in $\Gamma(C_{8,3}, W_8)$. From Theorem 3.5, we know that G_4 is the weighted 3-uniform loose cycle with the maximum spectral radius in $\Gamma(C_{8,3}, W_8)$. We also calculate

$$\rho(G_4) = 13.8984, \rho(G_5) = 12.0532, \rho(G_6) = 12.5488.$$

This is also consistent with Theorem 3.5.

4. The weighted uniform unicyclic hypergraph with the maximum spectral radius

Let $\Omega(m, k, W_m)$ denote the set of weighted k-uniform unicyclic hypergraphs with m edges and weight set W_m. In this section, we characterize the unique weighted hypergraph with the maximum spectral radius in $\Omega(m, k, W_m)$. Let U^* be the weighted hypergraph with the maximum spectral radius in $\Omega(m, k, W_m)$. Firstly, we characterize the structure of the extremal hypergraph U^*.

Lemma 4.1. *The weighted hypergraph $U^* \in \Omega(m,k,W_m)$ is obtained from a weighted loose cycle $C_{2,k}$ by attaching $m-2$ weighted pendent edges at a non-pendent vertex of $C_{2,k}$, where $m,k \geq 3$.*

Proof. Let x be the principal eigenvector of U^*, and $C_{l,k} = v_1e_1v_2e_2\ldots v_le_lv_1$ be the unique weighted loose cycle in U^*. We partition $E(U^*)$ into $E(C_{l,k}) \cup E_1$, and denote $E_1 = \{f_1, f_2, \ldots, f_{m-l}\}$. The proof is divided into four claims.

Claim 1. The length of $C_{l,k}$ is 2.

Suppose that $l \geqslant 3$ and $x_{v_1} \geqslant x_{v_2}$. Let G_1 be the weighted hypergraph obtained from U^* by moving edge e_2 from v_2 to v_1. By Lemma 2.4, we obtain $\rho(G_1) > \rho(U^*)$, a contradiction.

Claim 2. All edges in E_1 are pendent edges.

Without loss of generality, we assume that there exists a non-pendent edge f_1. Let G_2 be the weighted hypergraph obtained from U^* by the edge-releasing operation on f_1. By Lemma 2.6, we have $\rho(G_2) > \rho(U^*)$, a contradiction.

Claim 3. All edges in E_1 share a common vertex $v^* \in V(C_{l,k})$.

By Claim 2, we know that $f_i \cap V(C_{l,k}) \neq \emptyset$ for $1 \leq i \leq m-l$. Without loss of generality, suppose that $f_1 \cap V(C_{l,k}) = \{u\}$ and $f_2 \cap V(C_{l,k}) = \{v\}$.

If $x_u \geqslant x_v$, let G_3 be the weighted hypergraph obtained from U^* by moving edge f_2 from v to u. By Lemma 2.4, we get $\rho(G_3) > \rho(U^*)$, a contradiction. If $x_u < x_v$, let G_4 be the weighted hypergraph obtained from U^* by moving edge f_1 from u to v. By Lemma 2.4, we get $\rho(G_4) > \rho(U^*)$, a contradiction.

Claim 4. The common vertex $v^* \in \{v_1, v_2\}$.

Assume that $v^* \in e_1 \setminus \{v_1, v_2\}$. If $x_{v_1} \geqslant x_{v^*}$, let G_5 be the weighted hypergraph obtained from U^* by moving edges $f_1, f_2, \ldots, f_{m-l}$ from v^* to v_1. By Lemma 2.4, we have $\rho(G_5) > \rho(U^*)$, a contradiction. If $x_{v_1} < x_{v^*}$, let G_6 be the weighted hypergraph obtained from U^* by moving edge e_2 from v_1 to v^*. By Lemma 2.4, we have $\rho(G_6) > \rho(U^*)$, a contradiction.

Combining Claims 1-4, we have our conclusion. □

An unweighted hypergraph may be regarded as a weighted hypergraph with each of the edges bearing weight 1. Therefore, the Lemma 4.1 generalizes some known results for unweighted hypergraphs.

Let U^* be the unicyclic hypergraph is shown in Figure 3. Secondly, we determine the distribution of weights of U^*.

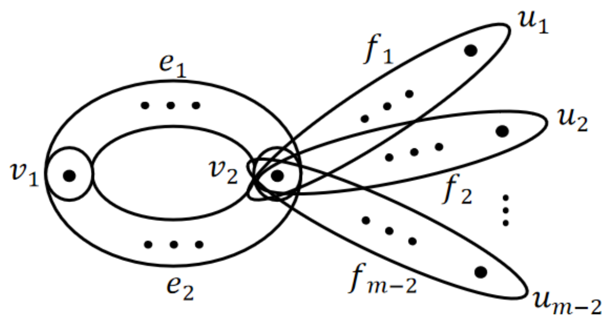

Figure 3: The unicyclic hypergraph U^*.

Lemma 4.2. *The weights of all edges of U^* satisfy*

$$min\Big\{w_{U^*}(e_1), w_{U^*}(e_2)\Big\} \geq w_{U^*}(f_i), i = 1, 2, \ldots, m-2. \tag{19}$$

Proof. Let x be the principal eigenvector of U^*. We prove $x_{v_1} > x_{u_1}$. Otherwise we can get a weighted hypergraph G_1 obtained from U^* by moving edge e_2 from v_1 to u_1. By Lemma 2.4, we have $\rho(G_1) > \rho(U^*)$, a contradiction. Thus $x_{v_1} > x_{u_1}$.

By symmetry, without loss of generality, we may assume that $w_{U^*}(f_1) \geq w_{U^*}(f_2) \geq \ldots \geq w_{U^*}(f_{m-2})$ and $w_{U^*}(e_1) \geq w_{U^*}(e_2)$. So it needs only to show $w_{U^*}(f_1) \leq w_{U^*}(e_2)$.

Suppose to the contrary that $w_{U^*}(f_1) > w_{U^*}(e_2)$. Put $\delta = w_{U^*}(f_1) - w_{U^*}(e_2) > 0$. Let G_2 be the weighted hypergraph obtained from U^* by exchanging the weights of edges f_1 and e_2, i.e.,

$$w_{G_2}(f_1) = w_{U^*}(f_1) - \delta, w_{G_2}(e_2) = w_{U^*}(e_2) + \delta, w_{G_2}(e) = w_{U^*}(e) \text{ for } e \in E(G_2) \setminus \{f_1, e_2\}. \tag{20}$$

If $x^{e_2\setminus\{v_1\}} > x^{f_1\setminus\{u_1\}}$, then by $x_{v_1} > x_{u_1}$ and Lemma 2.7, we have $\rho(G_2) > \rho(U^*)$, a contradiction.

Denote $f_1 = \{u_1, u_{1,1}, \ldots, u_{1,k-2}, v_2\}$, $e_2 = \{v_1, v_{1,1}, \ldots, v_{1,k-2}, v_2\}$. If $x^{e_2\setminus\{v_1\}} \leqslant x^{f_1\setminus\{u_1\}}$, then we construct a column vector y from x for G_2 as follows.

$$\begin{cases} y_{v_{1,j}} = x_{u_{1,j}}, & 1 \leq j \leq k-2, \\ y_{u_{1,j}} = x_{v_{1,j}}, & 1 \leq j \leq k-2, \\ y_u = x_u, & u \in V(G_2) \setminus \{v_{1,1}, \ldots, v_{1,k-2}, u_{1,1}, \ldots, u_{1,k-2}\}. \end{cases} \tag{21}$$

By (20), (21), Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\rho(G_2) - \rho(U^*) \\
\geq &\sum_{e\in E(G_2)} k w_{G_2}(e) y^e - \sum_{e\in E(U^*)} k w_{U^*}(e) x^e \\
=&k\Big(w_{G_2}(f_1) y^{f_1} + w_{G_2}(e_2) y^{e_2} - w_{U^*}(f_1) x^{f_1} - w_{U^*}(e_2) x^{e_2}\Big) \\
=&k\Big(w_{G_2}(f_1) y_{u_1} \cdot y_{v_2} \cdot \prod_{1\leq j\leq k-2} y_{u_{1,j}} + w_{G_2}(e_2) y_{v_1} \cdot y_{v_2} \cdot \prod_{1\leq j\leq k-2} y_{v_{1,j}} - w_{U^*}(f_1) x^{f_1} - w_{U^*}(e_2) x^{e_2}\Big) \\
=&k\Big(w_{G_2}(f_1) x^{e_2\setminus\{v_1\}} \cdot x_{u_1} + w_{G_2}(e_2) x^{f_1\setminus\{u_1\}} \cdot x_{v_1} - w_{U^*}(f_1) x^{f_1\setminus\{u_1\}} \cdot x_{u_1} - w_{U^*}(e_2) x^{e_2\setminus\{v_1\}} \cdot x_{v_1}\Big) \\
=&k\Big(w_{U^*}(e_2) x^{e_2\setminus\{v_1\}} \cdot x_{u_1} + w_{U^*}(f_1) x^{f_1\setminus\{u_1\}} \cdot x_{v_1} - w_{U^*}(f_1) x^{f_1\setminus\{u_1\}} \cdot x_{u_1} - w_{U^*}(e_2) x^{e_2\setminus\{v_1\}} \cdot x_{v_1}\Big) \\
=&k\Big(w_{U^*}(f_1) x^{f_1\setminus\{u_1\}} (x_{v_1} - x_{u_1}) + w_{U^*}(e_2) x^{e_2\setminus\{v_1\}} (x_{u_1} - x_{v_1})\Big) \\
=&k\Big(x_{v_1} - x_{u_1}\Big)\Big(w_{U^*}(f_1) x^{f_1\setminus\{u_1\}} - w_{U^*}(e_2) x^{e_2\setminus\{v_1\}}\Big) \\
>&0.
\end{aligned} \tag{22}$$

Where (22) follows from $x_{v_1} > x_{u_1}$, $x^{f_1\setminus\{u_1\}} \geqslant x^{e_2\setminus\{v_1\}}$ and $w_{U^*}(f_1) > w_{U^*}(e_2)$. Thus, we have $\rho(G_2) > \rho(U^*)$, a contradiction. □

Finally, the inequality (19) indicate that, for a given weight set $W_m = \{w_1, w_2, \ldots, w_m\}$, the distribution of weights of U^* is uniquely determined. Therefore, by Lemmas 4.1 and 4.2, we get the following main result.

Theorem 4.3. *Let U^* be a weighted k-uniform unicyclic hypergraph with m edges and a positive weight set W_m, where $m, k \geq 3$. If U^* satisfies the following conditions:*

(i) *The weighted hypergraph U^* is obtained from a weighted loose cycle $C_{2,k}$ by attaching $m-2$ weighted pendent edges at a non-pendent vertex of $C_{2,k}$ (The vertex and edge labelling as described in Figure 3);*

(ii) *The weights of U^* satisfy: $min\Big\{w_{U^*}(e_1), w_{U^*}(e_2)\Big\} \geq w_{U^*}(f_i), i = 1, 2, \ldots, m-2$.*

Then U^* is the unique weighted hypergraph in $\Omega(m, k, W_m)$ having the maximum spectral radius.

5. Conclusion remarks

The girth g of a hypergraph G is the minimum length of cycle in G. Let $\Theta(m, k, g, W_m)$ denote the set of weighted k-uniform unicyclic hypergraphs with m edges, girth g and weight set W_m.

For $g = m$, Theorem 3.5 determines the weighted hypergraph with the maximum spectral radius among all weighted hypergraphs in $\Theta(m, k, g, W_m)$. For $g = 2$, Theorem 4.3 determines the weighted hypergraph with the maximum spectral radius among all weighted hypergraphs in $\Theta(m, k, g, W_m)$. A natural question is to consider the case $3 \leq g \leq m-1$.

Question 5.1. *For $3 \leq g \leq m-1$, what is the maximum spectral radius among all weighted hypergraphs in $\Theta(m, k, g, W_m)$?*

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References

[1] C. X. Duan, L. G. Wang, P. Xiao, X. H. Li, *The (signless Laplacian) spectral radius (of subgraphs) of uniform hypergraphs*, Filomat **33** (2019), 4733–4745.

[2] Y. Z. Fan, Y. Y. Tan, X. X. Peng, A. H. Liu, *Maximizing spectral radii of uniform hypergraphs with few edges*, Discuss. Math. Graph Theory **36** (2016), 845–856.

[3] F. Galuppi, R. Mulas, L. Venturello, *Spectral theory of weighted hypergraphs via tensors*, Linear Multilinear Algebra **71** (2023), 317–347.

[4] H. Y. Guo, B. Zhou, *On the spectral radius of uniform hypertrees*, Linear Algebra Appl. **558** (2018), 236–249.

[5] L. Y. Kang, L. L. Liu, L. Q. Qi, X. Y. Yuan, *Spectral radii of two kinds of uniform hypergraphs*, Appl. Math. Comput. **338** (2018), 661–668.

[6] P. Keevash, J. Lenz, D. Mubayi, *Spectral extremal problems for hypergraphs*, SIAM J. Discrete Math. **28** (2014), 1838–1854.

[7] H. H. Li, J. Y. Shao, L. Q. Qi, *The extremal spectral radii of k-uniform supertrees*, J. Comb. Optim. **32** (2016), 741–764.

[8] L. H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, CAMSAP'05: Proceedings of the 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (2005), 129–132.

[9] X. X. Liu, L. G. Wang, *Distance (signless) Laplacian eigenvalues of k-uniform hypergraphs*, Taiwanese J. Math. **26** (2022), 1093–1111.

[10] C. Ouyang, L. Q. Qi, X. Y. Yuan, *The first few unicyclic and bicyclic hypergraphs with largest spectral radii*, Linear Algebra Appl. **527** (2017), 141–162.

[11] L. Q. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005), 1302–1324.

[12] L. Q. Qi, *Symmetric nonnegative tensors and copositive tensors*, Linear Algebra Appl. **439** (2013), 228–238.

[13] H. Y. Shan, Z. Y. Wang, F. F. Wang, *The smallest spectral radius of bicyclic uniform hypergraphs with a given size*, Linear Algebra Appl. **622** (2021), 166–188.

[14] J. Y. Shao, *A general product of tensors with applications*, Linear Algebra Appl. **439** (2013), 2350–2366.

[15] L. Su, H. H. Li, *The hypertree with the largest spectral radius among all hypertrees with given size and stability number*, Front. Math. **18** (2023), 697–705.

[16] R. Sun, W. H. Wang, *On the spectral radius of uniform weighted hypergraph*, Discrete Math. Algorithms Appl. **15** (2023), 2250067.

[17] R. Sun, W. H. Wang, Z. Y. Ni, *The maximal spectral radius of the uniform unicyclic hypergraphs with perfect matchings*, Filomat **37** (2023), 5949–5967.

[18] W. H. Wang, R. Sun, *The weighted hypergraph with the maximum spectral radius*, Linear Algebra Appl. **649** (2022), 47–62.

[19] W. H. Wang, L. J. Yu, *The maximum spectral radius of the weighted bicyclic hypergraphs*, Linear Multilinear Algebra (2023), 1–22.

[20] P. Xiao, L. G. Wang, Y. Lu, *The maximum spectral radii of uniform supertrees with given degree sequences*, Linear Algebra Appl. **523** (2017), 33–45.

[21] G. L. Yu, C. Yan, L. Sun, Y. R. Wu, H. L. Zhang, *Spectral radius and matching number of the unicyclic hypergraph*, Linear Algebra Appl. **610** (2021), 571–590.

[22] X. Y. Yuan, J. Y. Shao, H. Y. Shan, *Ordering of some uniform supertrees with larger spectral radii*, Linear Algebra Appl. **495** (2016), 206–222.

[23] L. Zhang, A. Chang, *Spectral radius of r-uniform supertrees with perfect matchings*, Front. Math. China **13** (2018), 1489–1499.

[24] J. B. Zhang, J. P. Li, *The maximum spectral radius of k-uniform hypergraphs with r pendent vertices*, Linear Multilinear Algebra **67** (2019), 1062–1073.

[25] L. Zhang, H. Z. Ren, *On extremal spectral radii of uniform supertrees with given independence number*, Discrete Dyn. Nat. Soc. **2022** (2022), 3930470.