Filomat 38:21 (2024), 7647–7655
https://doi.org/10.2298/FIL2421647J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Cauchy-Schwarz type inequalities and their applications

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Abstract. In this article, we present generalized improvements of certain Cauchy-Schwarz type inequalities. As applications of our results, we provide refinements of some numerical radius inequalities for Hilbert space operators. Finally, we obtain certain numerical radius inequalities of Hilbert space operators involving geometrically convex functions.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be positive, and denoted by $S \geq 0$, if $\langle Sx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and is called positive definite, denoted $S > 0$, if $\langle Sx, x\rangle > 0$ for all non zero vectors $x \in \mathcal{H}$. The *numerical range* of $S \in \mathcal{L}(\mathcal{H})$ is defined as $W(S) = \{\langle Sx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ and the *numerical radius* of S, denoted by $w(S)$, is defined by $w(S) = \sup\{|z| : z \in W(S)\}$. It is known that the set $W(S)$ is a convex subset of the complex plane and that the numerical radius $w(\cdot)$ is a norm on $\mathcal{L}(\mathcal{H})$; being equivalent to the usual operator norm $\|S\| = \sup\{\|Sx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $S \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{2}\|S\| \leq w(S) \leq \|S\|. \tag{1}$$

The inequalities in (1) are sharp. If $S^2 = 0$, then the first inequality becomes an equality, on the other hand the second inequality becomes an equality if S is a normal. In fact, for a nilpotent operator S with $S^n = 0$, Haagerup and Harpe [7] showed that $w(S) \leq \|S\|\cos(\pi/(n+1))$.

Among many techniques in obtaining numerical radius inequalities is the study of certain scalar ones. For example, the classical Young inequality which states that if $a, b \geq 0$ and $0 \leq \alpha \leq 1$, then

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b,$$

is an example of such important scalar inequalities. The numerical radius has some significant properties, such as the power inequality:

$$w(S^n) \leq w^n(S) \quad \text{for} \quad n = 1, 2, \ldots. \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 47A12; Secondary 47A30; 15A60; 47A63.
Keywords. Cauchy-Schwarz inequality; numerical radius; operator norm.
Received: 22 November 2023; Revised: 23 February 2024; Accepted: 06 March 2024
Communicated by Fuad Kittaneh
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For basic information about numerical radius one can refer [6]. The author of [11, 13] improved the inequality (1) which is stated next. If $S \in \mathcal{L}(\mathcal{H})$, then

$$w(S) \leq \frac{1}{2}\left\| |S| + |S^*| \right\| \leq \frac{1}{2}\left(\|S\| + \|S^2\|^{1/2} \right), \tag{3}$$

where $|S| = (S^*S)^{1/2}$ is the absolute value of S, and

$$\frac{1}{4}\left\| S^*S + SS^* \right\| \leq w^2(S) \leq \frac{1}{2}\left\| S^*S + SS^* \right\|. \tag{4}$$

The inequalities in (3) refines the second inequality in (1). For applications of these inequalities one can refer [11, 12].

Dragomir [4] showed the following numerical radius inequality involving the product of two operators:

$$w^r(S^*T) \leq \frac{1}{2}\left\| |T|^{2r} + |S|^{2r} \right\|, \quad r \geq 1. \tag{5}$$

The Cauchy-Schwarz inequality states that for any vectors x and y in an inner product space

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{6}$$

where $\|x\| = \langle x, x\rangle^{1/2}$. The equality holds if and only if x and y are linearly dependent. The Cauchy-Schwarz inequality is the most essential and important inequality in mathematics. Motivated by the inequality (6), Kittaneh *et al.* [14, Lemma 3] improved the Cauchy-Schwarz inequality (6). Using the improved Cauchy-Schwarz inequality, they established a result which is a refinement of (5) (for $r = 2$).

Motivated by the same inequality (6), we establish the following inequality

$$|\langle x, y\rangle| \leq \sqrt{\mu\left(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2\right) + |\langle x, y\rangle|^{2\mu}\|x\|^{2(1-\mu)}\|y\|^{2(1-\mu)}} \leq \|x\|\|y\| \text{ for } 0 \leq \mu \leq 1. \tag{7}$$

This inequality, nicely improves the Cauchy-Schwarz inequality. As a special case to the inequality (7) we get a recent result by Kittaneh *et al.* [14, Lemma 3]. Using the inequality (7), we establish a new refinement of the inequality (5) (for $r = 2$).

The Schwarz inequality for positive operators reads that if S is a positive operator in $\mathcal{L}(\mathcal{H})$, then

$$|\langle Sx, y\rangle|^2 \leq \langle Sx, x\rangle \langle Sy, y\rangle \tag{8}$$

for any vectors $x, y \in \mathcal{H}$. In 1952, Kato [9] introduced a companion inequality of (8), called the mixed Schwarz inequality, which asserts

$$|\langle Sx, y\rangle|^2 \leq \left\langle |S|^{2\alpha} x, x\right\rangle \left\langle |S^*|^{2(1-\alpha)} y, y\right\rangle, \qquad 0 \leq \alpha \leq 1, \tag{9}$$

for all operators $S \in \mathcal{L}(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$. In particular, the following inequality

$$|\langle Sx, y\rangle| \leq \sqrt{\langle |S| x, x\rangle \langle |S^*| y, y\rangle} \text{ (see[8, pp 75 – 76].} \tag{10}$$

A generalization of the mixed Cauchy-Schwarz inequality which is useful in proving our main results is presented as follows.

Lemma 1.1. [10, Theorem 1] *Let S be an operator in $\mathcal{L}(\mathcal{H})$. If f and g are non-negative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$|\langle Sx, y\rangle| \leq \|f(|S|x)\|\|g(|S^*|y)\| \text{ for all } x, y \text{ in } \mathcal{H}. \tag{11}$$

Motivated by the inequality (11), we shall establish in this article that

$$|\langle Sx,y\rangle|^2 \leq \mu\langle f^2(|S|)x,x\rangle\langle g^2(|S^*|)y,y\rangle + (1-\mu)|\langle Sx,y\rangle|\langle f^2(|S|)x,x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y,y\rangle^{\frac{1}{2}} \leq \langle f^2(|S|)x,x\rangle\langle g^2(|S^*|)y,y\rangle \tag{12}$$

for all x, y in $\mathcal{H}$, $0 \leq \mu \leq 1$ and f, g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. This inequality (12) improves the inequality (11). By employing the inequality (12) and for $f(t) = g(t) = t^{1/2}$, we obtain a considerable improvement of the second inequality in (4). As a special case to the inequality (12), we obtain an improvement of Kato's inequality (9).

The following lemma demonstrates a norm inequality involving convex functions of positive operators.

Lemma 1.2. [1, Theorem 2.3] *Let f be a non-negative, convex function on $[0,\infty)$, and let $S,T \in \mathcal{L}(\mathcal{H})$ be positive operators. Then*

$$\left\| f\left(\frac{S+T}{2}\right)\right\| \leq \left\|\frac{f(S)+f(T)}{2}\right\|.$$

In particular, if $r \geq 1$, then

$$\left\|\left(\frac{S+T}{2}\right)^r\right\| \leq \left\|\frac{S^r+T^r}{2}\right\|.$$

The following lemma is known as the operator version of the classical Jensen inequality.

Lemma 1.3. ([17, Theorem 1.2]) *Let $S \in \mathcal{L}(\mathcal{H})$ such that S is self adjoint and $\operatorname{sp}(S) \subset [m,M]$ for some scalars $m \leq M$. If $f(t)$ is a convex function on $[m,M]$, and $x \in \mathcal{H}$ be a unit vector then*

$$f(\langle Sx,x\rangle) \leq \langle f(S)x,x\rangle,$$

where $\operatorname{sp}(S)$ is spectrum of an operator S.

The following McCarthy inequality can be obtained as a special case of Lemma 1.3. For more details one can follow [10], and [17, Theorem 1.4].

Lemma 1.4. [McCarthy inequality] *Let $S \in \mathcal{L}(\mathcal{H})$ be a positive operator and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*
(i) $\langle Sx,x\rangle^r \leq \langle S^r x,x\rangle$ for $r \geq 1$;
(ii) $\langle S^r x,x\rangle \leq \langle Sx,x\rangle^r$ for $0 < r \leq 1$.

The objective of the article is to improve the inequalities (4), (5) (for $r = 2$), (6), (9), and (11).

To do this, the article is organized as follows. Section 2 contains our main results, and is of two parts. The first part presents refinements of generalized mixed Cauchy-Schwarz inequality, Kato's inequality. Using these inequalities we have established certain numerical radius inequalities for operators on Hilbert space which refine the Kittaneh's inequality (4). We also obtain certain numerical radius inequalities using generalization of Buzano's inequality. Some special cases of our results lead to the results of earlier works in the literature while in the final part, we show how geometrically convex functions can be used to calculate the numerical radius of Hilbert space operators. We emphasise that this application to numerical radius inequalities is a novel approach that we hope will be useful to field researchers.

2. Main Results

2.1. *Numerical radius inequalities via Cauchy-Schwarz type inequality*

The following lemma is a refinement of Lemma 1.1.

Lemma 2.1. *Let S be an operator in $\mathcal{L}(\mathcal{H})$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$|\langle Sx,y\rangle|^2 \leq \mu\langle f^2(|S|)x,x\rangle\langle g^2(|S^*|)y,y\rangle + (1-\mu)|\langle Sx,y\rangle|\langle f^2(|S|)x,x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y,y\rangle^{\frac{1}{2}} \leq \langle f^2(|S|)x,x\rangle\langle g^2(|S^*|)y,y\rangle \tag{13}$$

for all x, y in $\mathcal{H}$ and $0 \leq \mu \leq 1$.

Proof. Using Lemma 1.1 and for $0 \le \mu \le 1$, we have

$$
\begin{aligned}
|\langle Sx, y\rangle|^2 &= \mu|\langle Sx, y\rangle|^2 + (1-\mu)|\langle Sx, y\rangle||\langle Sx, y\rangle| \\
&\le \mu\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle + (1-\mu)|\langle Sx, y\rangle|\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y, y\rangle^{\frac{1}{2}}.
\end{aligned} \tag{14}
$$

On the other hand,

$$
\begin{aligned}
&\mu\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle + (1-\mu)|\langle Sx, y\rangle|\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y, y\rangle^{\frac{1}{2}} \\
&\le \mu\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle + (1-\mu)\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y, y\rangle^{\frac{1}{2}}\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y, y\rangle^{\frac{1}{2}} \\
&= \langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle.
\end{aligned} \tag{15}
$$

Combining (14) and (15), we get

$$
\begin{aligned}
|\langle Sx, y\rangle|^2 &\le \mu\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle + (1-\mu)|\langle Sx, y\rangle|\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)y, y\rangle^{\frac{1}{2}} \\
&\le \langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)y, y\rangle.
\end{aligned} \tag{16}
$$

□

Remark 2.2. *For $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$ in the inequality* (13)*, we find the following inequality.*

$$
\begin{aligned}
|\langle Sx, y\rangle|^2 &\le \mu\left\langle |S|^{2\alpha}x, x\right\rangle\left\langle |S^*|^{2(1-\alpha)}y, y\right\rangle + (1-\mu)|\langle Sx, y\rangle|\sqrt{\left\langle |S|^{2\alpha}x, x\right\rangle\left\langle |S^*|^{2(1-\alpha)}y, y\right\rangle} \\
&\le \left\langle |S|^{2\alpha}x, x\right\rangle\left\langle |S^*|^{2(1-\alpha)}y, y\right\rangle, \qquad 0 \le \mu \le 1.
\end{aligned} \tag{17}
$$

One can notice that the inequality (17) is a refinement of Kato's inequality (9). For $f(t) = t^{1/2}$ and $g(t) = t^{1/2}$, $\mu = \frac{1}{3}$ in the inequality (13), we find an inequality which is an improvement of the inequality (10) (see also [14, Inequality (14)]). In particular, if $S = I$ in the inequality (17), the inequality is the refinement of Cauchy-Schwarz inequality. We should remark here that the inequality (13) is the generalization of the inequality [14, inequality (14)].

As an application of Lemma 2.1, we have the following theorem.

Theorem 2.3. *Let $S \in \mathcal{L}(\mathcal{H})$ and let f, g are non-negative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$
w^2(S) \le \frac{\mu}{2}\|f^4(|S|) + g^4(|S^*|)\| + \frac{1-\mu}{2}w(S)\|f^2(|S|) + g^2(|S^*|)\| \;\; for \;\; 0 \le \mu \le 1. \tag{18}
$$

Proof. Putting $y = x$ in the first part of Lemma 2.1, and using AM-GM (arithmetic mean-geometric mean), and McCarthy inequality, we have

$$
\begin{aligned}
|\langle Sx, x\rangle|^2 &\le \mu\langle f^2(|S|)x, x\rangle\langle g^2(|S^*|)x, x\rangle + (1-\mu)|\langle Sx, x\rangle|\langle f^2(|S|)x, x\rangle^{\frac{1}{2}}\langle g^2(|S^*|)x, x\rangle^{\frac{1}{2}} \\
&\le \frac{\mu}{2}\left(\langle f^2(|S|)x, x\rangle^2 + \langle g^2(|S^*|)x, x\rangle^2\right) + \frac{(1-\mu)}{2}|\langle Sx, x\rangle|\left(\langle f^2(|S|)x, x\rangle + \langle g^2(|S^*|)x, x\rangle\right) \\
&\le \frac{\mu}{2}\left\langle\left(f^4(|S|) + g^4(|S^*|)\right)x, x\right\rangle + \frac{(1-\mu)}{2}|\langle Sx, x\rangle|\left\langle\left(f^2(|S|) + g^2(|S^*|)\right)x, x\right\rangle.
\end{aligned}
$$

Taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we have the desired inequality. □

Remark 2.4. *For $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $0 \le \alpha \le 1$ in the inequality* (18)*, we find the following inequality.*

$$
w^2(S) \le \frac{\mu}{2}\||S|^{4\alpha} + |S^*|^{4(1-\alpha)}\| + \frac{1-\mu}{2}w(S)\||S|^{2\alpha} + |S^*|^{2(1-\alpha)}\|, \;\; for \;\; 0 \le \mu \le 1.
$$

For $\alpha = \frac{1}{2}$ in Remark 2.4, we have the following corollary which demonstrates that our inequality in Corollary 2.5 is much stronger than the inequality (4).

Corollary 2.5. *Let $S \in \mathcal{L}(\mathcal{H})$. Then*

$$w^2(S) \le \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{2}w(S)\||S| + |S^*|\| \le \frac{1}{2}\||S|^2 + |S^*|^2\|, \ \ for \ \ 0 \le \mu \le 1. \tag{19}$$

Proof. Putting $\alpha = \frac{1}{2}$ in Remark 2.4, we have

$$\begin{aligned} w^2(S) &\le \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{2}w(S)\||S| + |S^*|\| \\ &\le \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{4}\||S| + |S^*|\|^2 \\ &= \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{4}\left\|\left(\frac{2|S| + 2|S^*|}{2}\right)^2\right\| \\ &\le \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{8}\left\|(2|S|)^2 + (2|S^*|)^2\right\| \\ &= \frac{\mu}{2}\||S|^2 + |S^*|^2\| + \frac{1-\mu}{2}\left\||S|^2 + |S^*|^2\right\| \\ &= \frac{1}{2}\||S|^2 + |S^*|^2\|, \end{aligned}$$

where the second inequality follows from the inequality (3), third inequality follows from Lemma 1.2. □

Remark 2.6. *For $\mu = \frac{1}{2}$ in Corollary 2.5, we have the following inequalities.*

$$w^2(S) \le \frac{1}{4}\||S|^2 + |S^*|^2\| + \frac{1}{4}w(S)\||S| + |S^*|\| \le \frac{1}{2}\||S|^2 + |S^*|^2\|, \ \ for \ \ S \in \mathcal{L}(\mathcal{H}). \tag{20}$$

The following example shows that inequality (20) is an improvement of the second part of the inequality (4).

Example 2.7. *Let $S = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then, $\frac{1}{2}\||S|^2 + |S^*|^2\| = 1$, whereas $\frac{1}{4}\||S|^2 + |S^*|^2\| + \frac{1}{4}w(S)\||S| + |S^*|\| \approx 0.8536$.*

Remark 2.8. *We mention here that Corollary 2.5 is a refinement of the Kittaneh inequality* (4). *Putting $\mu = \frac{1}{3}$ in Corollary 2.5, we get an inequality due to Kittaneh et al. [14, Corollary 2].*

Buzano [3] obtained the following extension of the celebrated Cauchy-Schwarz inequality

$$|\langle x, e\rangle\langle e, y\rangle| \le \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right), \tag{21}$$

where x, y, e are vectors in $\mathcal{H}$ with $\|e\| = 1$.

Moslehian *et al.* [15, Corollary 2.5] obtained the following generalization of Buzano's inequality

$$|\beta\langle x, e\rangle\langle e, y\rangle - \langle x, y\rangle| \le \max\{1, |\beta - 1|\}\|x\|\|y\|, \tag{22}$$

where x, y, e are vectors in $\mathcal{H}$ with $\|e\| = 1$ and $\beta \in \mathbb{C}$. Inequality (22) was also recently established by Bottazzi *et al.* [2, Proposition 3.1] using a rank one operator. If $\beta \in \mathbb{C} \setminus \{0\}$, then inequality (22) is equivalent to

$$\left|\langle x, e\rangle\langle e, y\rangle - \frac{1}{\beta}\langle x, y\rangle\right| \le \frac{1}{|\beta|}\max\{1, |\beta - 1|\}\|x\|\|y\|.$$

Using the continuity property of modulus for complex numbers, i.e. $|a-b| \geq ||a|-|b||$, we have

$$\left| |\langle x,e\rangle\langle e,y\rangle| - \frac{1}{|\beta|}|\langle x,y\rangle| \right| \leq \frac{1}{|\beta|}\max\{1,|\beta-1|\}\|x\|\|y\|,$$

which implies that

$$|\langle x,e\rangle\langle e,y\rangle| \leq \frac{1}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big). \tag{23}$$

For $\beta = 2$ in inequality (23), we have the Buzano inequality (21). Using inequality (23) and $0 \leq \mu \leq 1$, we have

$$\begin{aligned}
|\langle x,e\rangle\langle e,y\rangle|^2 &\leq |\langle x,e\rangle\langle e,y\rangle|\frac{1}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big)\\
&= |\langle x,e\rangle\langle e,y\rangle|\frac{\mu}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big)\\
&\quad + |\langle x,e\rangle\langle e,y\rangle|\frac{1-\mu}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big)\\
&\leq \frac{\mu}{|\beta|^2}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big)^2\\
&\quad + |\langle x,e\rangle\langle e,y\rangle|\frac{1-\mu}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big)\\
&= \frac{\mu}{|\beta|^2}\Big(|\langle x,y\rangle|^2 + \max\{1,|\beta-1|\}\|x\|^2\|y\|^2\Big)\\
&\quad + \frac{2\mu}{|\beta|^2}|\langle x,y\rangle|\max\{1,|\beta-1|\}\|x\|\|y\|\\
&\quad + |\langle x,e\rangle\langle e,y\rangle|\frac{1-\mu}{|\beta|}\Big(|\langle x,y\rangle| + \max\{1,|\beta-1|\}\|x\|\|y\|\Big).
\end{aligned} \tag{24}$$

As an application of the inequality (24) we have the following result.

Theorem 2.9. *Let $S \in \mathcal{L}(\mathcal{H})$ and $\mu \in [0,1]$, $\beta \in \mathbb{C} \setminus \{0\}$. Then the following inequality holds.*

$$\begin{aligned}
w^4(S) &\leq \frac{\mu}{|\beta|^2}w^2(S^2) + \frac{\mu}{2|\beta|^2}\max\{1,|\beta-1|\}\||S|^4+|S^*|^4\| + \frac{\mu}{|\beta|^2}\max\{1,|\beta-1|\}w(S^2)\||S|^2+|S^*|^2\|\\
&\quad + \frac{1-\mu}{|\beta|}w^2(S)\left[w(S^2) + \frac{\max\{1,|\beta-1|\}}{2}\||S|^2+|S^*|^2\|\right].
\end{aligned} \tag{25}$$

Proof. Let $x,y,e \in \mathcal{H}, 0 \leq \mu \leq 1$ and replacing e by x, x by Sx, y by S^*x in the inequality (24), we have

$$\begin{aligned}
|\langle Sx,x\rangle|^4 &\leq \frac{\mu}{|\beta|^2}\Big(|\langle S^2x,x\rangle|^2 + \max\{1,|\beta-1|\}\|Sx\|^2\|S^*x\|^2\Big)\\
&\quad + \frac{2\mu}{|\beta|^2}|\langle S^2x,x\rangle|\max\{1,|\beta-1|\}\|Sx\|\|S^*x\|\\
&\quad + \frac{1-\mu}{|\beta|}|\langle Sx,x\rangle|^2\Big(|\langle S^2x,x\rangle| + \max\{1,|\beta-1|\}\|Sx\|\|S^*x\|\Big).
\end{aligned}$$

So,

$$
\begin{aligned}
|\langle Sx,x\rangle|^4 &\le \frac{\mu}{|\beta|^2}|\langle S^2x,x\rangle|^2+\max\{1,|\beta-1|\}\frac{\mu}{|\beta|^2}\langle|S|^2x,x\rangle\langle|S^*|^2x,x\rangle\\
&\quad+\max\{1,|\beta-1|\}\frac{2\mu}{|\beta|^2}|\langle S^2x,x\rangle|\sqrt{\langle|S|^2x,x\rangle\langle|S^*|^2x,x\rangle}\\
&\quad+\frac{1-\mu}{|\beta|}|\langle Sx,x\rangle|^2\left(|\langle S^2x,x\rangle|+\max\{1,|\beta-1|\}\sqrt{\langle|S|^2x,x\rangle\langle|S^*|^2x,x\rangle}\right)\\
&\le \frac{\mu}{|\beta|^2}|\langle S^2x,x\rangle|^2+\max\{1,|\beta-1|\}\frac{\mu}{2|\beta|^2}\left(\langle|S|^2x,x\rangle^2+\langle|S^*|^2x,x\rangle^2\right)\\
&\quad+\max\{1,|\beta-1|\}\frac{\mu}{|\beta|^2}|\langle S^2x,x\rangle|\left(\langle|S|^2x,x\rangle+\langle|S^*|^2x,x\rangle\right)\\
&\quad+\frac{1-\mu}{|\beta|}|\langle Sx,x\rangle|^2\left(|\langle S^2x,x\rangle|+\max\{1,|\beta-1|\}\frac{\langle|S|^2x,x\rangle+\langle|S^*|^2x,x\rangle}{2}\right)\\
&\le \frac{\mu}{|\beta|^2}|\langle S^2x,x\rangle|^2+\max\{1,|\beta-1|\}\frac{\mu}{2|\beta|^2}\left\langle(|S|^4+|S^*|^4)x,x\right\rangle\\
&\quad+\max\{1,|\beta-1|\}\frac{\mu}{|\beta|^2}|\langle S^2x,x\rangle|\left\langle(|S|^2+|S^*|^2)x,x\right\rangle\\
&\quad+\frac{1-\mu}{|\beta|}|\langle Sx,x\rangle|^2\left(|\langle S^2x,x\rangle|+\frac{\max\{1,|\beta-1|\}}{2}\left\langle(|S|^2+|S^*|^2)x,x\right\rangle\right).
\end{aligned}
$$

Taking supremum over all x with $\|x\|=1$, we get our desired result. □

Putting $\mu=\frac{1}{2}$ and $\beta=2$ in Theorem 2.9, we get the first part of the following corollary.

Corollary 2.10. *Let $S\in\mathcal{L}(\mathcal{H})$. Then*

$$
\begin{aligned}
w^4(S) &\le \frac{1}{8}w^2(S^2)+\frac{1}{16}\||S|^4+|S^*|^4\|+\frac{1}{8}w(S^2)\||S|^2+|S^*|^2\|+\frac{1}{4}w^2(S)\left[w(S^2)+\frac{1}{2}\||S|^2+|S^*|^2\|\right]\\
&\le \frac{1}{2}\||S|^4+|S^*|^4\|. \qquad (26)
\end{aligned}
$$

Proof. Our aim is to prove the second inequality. Using power inequality, one may write

$$
\begin{aligned}
&\frac{1}{8}w^2(S^2)+\frac{1}{16}\||S|^4+|S^*|^4\|+\frac{1}{8}w(S^2)\||S|^2+|S^*|^2\|+\frac{1}{4}w^2(S)\left[w(S^2)+\frac{1}{2}\||S|^2+|S^*|^2\|\right]\\
&\le \frac{1}{8}w^2(S^2)+\frac{1}{16}\||S|^4+|S^*|^4\|+\frac{1}{8}w^2(S)\||S|^2+|S^*|^2\|+\frac{1}{4}w^2(S)\left[w^2(S)+\frac{1}{2}\||S|^2+|S^*|^2\|\right]\\
&\le \frac{1}{16}\||S|^4+|S^*|^4\|+\frac{1}{16}\||S|^4+|S^*|^4\|+\frac{1}{16}\||S|^2+|S^*|^2\|^2+\frac{1}{8}\||S|^2+|S^*|^2\|^2\\
&= \frac{1}{8}\||S|^4+|S^*|^4\|+\frac{3}{16}\||S|^2+|S^*|^2\|^2\\
&= \frac{1}{8}\||S|^4+|S^*|^4\|+\frac{3}{16}\left\|\left(\frac{2|S|^2+2|S^*|^2}{2}\right)^2\right\|\\
&\le \frac{1}{8}\||S|^4+|S^*|^4\|+\frac{3}{32}\left\|\left(2|S|^2\right)^2+\left(2|S^*|^2\right)^2\right\|\\
&= \frac{1}{8}\||S|^4+|S^*|^4\|+\frac{3}{8}\left\||S|^4+|S^*|^4\right\|\\
&= \frac{1}{2}\||S|^4+|S^*|^4\|,
\end{aligned}
$$

where the first inequality follows from the power inequality (2), the second inequality follows from $w^2(S^2) \leq \frac{1}{2}\| |S|^4 + |S^*|^4 \|$ and $w^2(S) \leq \frac{1}{2}\| |S|^2 + |S^*|^2 \|$, and the last inequality follows from Lemma 1.2. □

Remark 2.11. *Putting $\mu = \frac{1}{3}$ and $\beta = 2$ in Theorem 2.9, we get a recent result by Kittaneh et al. [14, Theorem 3].*

2.2. Numerical radius inequalities via geometric convexity

In this section, we present some applications of geometrically convex function to numerical radius inequalities. If I is a sub interval of $(0, \infty)$ and $f : I \to (0, \infty)$, then f is called geometrically convex, if

$$f(a^{1-\mu}b^{\mu}) \leq f^{1-\mu}(a)f^{\mu}(b), \qquad \mu \in [0, 1], \tag{27}$$

for more on this one can follow [16].

Theorem 2.12. *Let $S \in \mathcal{L}(\mathcal{H})$ and h be an increasing geometrically convex function. If in addition h is convex, then*

$$h(w^2(S)) \leq \mu\|\alpha h(|S|^2) + (1-\alpha)h(|S^*|^2)\| + \frac{1-\mu}{2}h(w(S))\|h(|S|^{2\alpha}) + h(|S^*|^{2(1-\alpha)})\|, \tag{28}$$

for $0 \leq \mu, \alpha \leq 1$.

Proof. By using the monotony of h, we have for any unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
&h(|\langle Sx, x\rangle|^2) \\
&= \mu h(|\langle Sx, x\rangle|^2) + (1-\mu)h(|\langle Sx, x\rangle|^2) \\
&\leq \mu h(\langle |S|^{2\alpha}x, x\rangle\langle |S^*|^{2(1-\alpha)}x, x\rangle) + (1-\mu)h(|\langle Sx, x\rangle|)h(\sqrt{\langle |S|^{2\alpha}x, x\rangle\langle |S^*|^{2(1-\alpha)}x, x\rangle}) \\
&\leq \mu h(\langle |S|^2x, x\rangle^{\alpha}\langle |S^*|^2x, x\rangle^{1-\alpha}) + (1-\mu)h(|\langle Sx, x\rangle|)h(\sqrt{\langle |S|^{2\alpha}x, x\rangle\langle |S^*|^{2(1-\alpha)}x, x\rangle}) \\
&\leq \mu h^{\alpha}(\langle |S|^2x, x\rangle)h^{1-\alpha}(\langle |S^*|^2x, x\rangle) + (1-\mu)h(|\langle Sx, x\rangle|)(\sqrt{h(\langle |S|^{2\alpha}x, x\rangle)h(\langle |S^*|^{2(1-\alpha)}x, x\rangle)}) \\
&\leq \mu(\alpha h(\langle |S|^2x, x\rangle) + (1-\alpha)h(\langle |S^*|^2x, x\rangle) + (1-\mu)h(|\langle Sx, x\rangle|)(\sqrt{\langle h(|S|^{2\alpha})x, x\rangle\langle h(|S^*|^{2(1-\alpha)})x, x\rangle}) \\
&\leq \mu\langle(\alpha h(|S|^2) + (1-\alpha)h(|S^*|^2))x, x\rangle + \frac{1-\mu}{2}h(|\langle Sx, x\rangle|)\langle(h(|S|^{2\alpha}) + h(|S^*|^{2(1-\alpha)}))x, x\rangle,
\end{aligned}$$

where the first inequality follows from the inequality (9), the second inequality follows from McCarthy inequality, third inequality follows from the inequality (27), fourth inequality follows from Young's inequality, and fifth inequality follows from Lemma 1.3 and AM-GM. Now, by taking supremum over $x \in \mathcal{H}$ with $\|x\|$, we have

$$h(w^2(S)) \leq \mu\|\alpha h(|S|^2) + (1-\alpha)h(|S^*|^2)\| + \frac{1-\mu}{2}h(w(S))\|h(|S|^{2\alpha}) + h(|S^*|^{2(1-\alpha)})\|,$$

as required. □

Remark 2.13. *Let $S \in \mathcal{L}(\mathcal{H})$ and setting $h(t) = t^r, r \geq 1$, we have*

$$w^{2r}(S) \leq \mu\|\alpha|S|^{2r} + (1-\alpha)|S^*|^{2r}\| + \frac{1-\mu}{2}w^r(S)\||S|^{2r\alpha} + |S^*|^{2r(1-\alpha)}\|, \ \ for \ \ 0 \leq \alpha, \mu \leq 1.$$

Remark 2.14. *Putting $\mu = \frac{1}{3}$, $\alpha = \frac{1}{2}$, $r = 1$ in Remark 2.13, we get a recent result by Kittaneh et al. [14, Theorem 2]. For $\mu = 0$, we have a result by El-Hadad et al. [5, Theorem 1] and for $\mu = 1$ we have a result by El-Hadad et al. [5, Theorem 2].*

Finally, we should mention that our analysis can be used to obtain a number of numerical radius inequalities involving generalised Cauchy-Schwarz type inequality and geometrically convexity for Hilbert space operators. Further research on this topic could lead to the discovery of an interesting area for future research. We leave the details to the interested reader.

Data availability: The author declare that data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
Conflict of interest: The author declare that there is no conflict of interest.

Acknowledgements

The authors thank the referee for the valuable suggestions and comments on an earlier version. Incorporating appropriate responses to these in the article has led to a better presentation of the results.

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