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On A-Berezin number in functional Hilbert space

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Abstract. A-Berezin radius distance and A-Berezin norm distance are presented in this study. Furthermore, by employing the notions of A-Berezin radius distance and A-Berezin norm distance, we find A-Berezin radius inequalities of the product and commutator of functional Hilbert space operators. Moreover, we generalize the A-Berezin radius distance. Finally, we prove the theorem pertaining to the A-Berezin radius distance. To recapitulate, the A-Berezin number of operator V on $\mathcal{L}(\mathcal{H}(\Theta))$ is defined by the following special type of quadratic form: $\mathrm{ber}_A(V) = \sup_{\eta\in\Theta}\left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|$, $\eta \in \Theta$, where $\hat{k}_\eta$ is the normalized reproducing kernel on $\mathcal{H}$ and a semi-inner product on $\mathcal{H}$, denoted as $\langle V\hat{k}_\eta, \hat{k}_\eta\rangle_A := \langle AV\hat{k}_\eta, \hat{k}_\eta\rangle_{\mathcal{H}}$, is induced by any positive operator A.

1. Introduction

We present the A-Berezin norm and radius distances in this publication. Using the notion of A-Berezin radius distance and A-Berezin norm distance, we also find A-Berezin radius inequalities of the product and commutator of functional Hilbert space operators. We also extend the concept of the A-Berezin radius distance. $\mathcal{H}$ establishes a non-complex Hilbert space along this work, with associated norm $\|.\|$ and an inner product $\langle ., .\rangle$. The algebra of all bounded linear operators operating on $\mathcal{H}$ is defined as $\mathcal{L}(\mathcal{H})$. Let the identity operator on $\mathcal{H}$ be represented by the symbol I. $\mathcal{N}(V)$, $\mathcal{R}(V)$ and $\overline{\mathcal{R}(V)}$ stand for null space, the range and closure of range of V, respectively, for the operator $V \in \mathcal{L}(\mathcal{H})$. V^* defines the adjoint of V. $V \in \mathcal{L}(\mathcal{H})$ is said to be positive if $\langle Vx, x\rangle \geq 0$ for every $x \in \mathcal{H}$, shown by $V \geq 0$. The absolute value of V, represented by $|V|$ for $V \in \mathcal{L}(\mathcal{H})$, is $|V| = (V^*V)^{1/2}$. Recall that the functional Hilbert space (shortly FHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ of complex-valued functions on some Θ such that the evaluation functionals $\varphi_\eta(f) = f(\eta)$, $\eta \in \Theta$, are continuous on $\mathcal{H}$ and for every $\eta \in \Theta$ there exist a function $f_\eta \in \mathcal{H}$ such that $f_\eta(\eta) \neq 0$ or, equivalently, there is no $\eta_0 \in \Omega$ such that $f(\eta_0) = 0$ for all $f \in \mathcal{H}$. Then by the Riesz representation theorem for each $\eta \in \Theta$ there exists a unique function $k_\eta \in \mathcal{H}$ which is called the reproducing kernel of the space $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\eta\rangle$ for all $f \in \mathcal{H}$. The function $\widehat{k_\eta} := \frac{k_\eta}{\|k_\eta\|}$, $\eta \in \Theta$, is called the normalized reproducing kernel of $\mathcal{H}$. The prototypical FHSs are the Hardy space $H^2(\mathbb{D})$, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc and the Fock space $\mathcal{F}(\mathbb{C})$. A detailed presentation of

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the theory of reproducing kernels and FHSs is given, for instance in Aronzajn [3]. Note that for a bounded linear operator V on $\mathcal{H}$ (i.e., for $V \in \mathcal{L}(\mathcal{H})$ its Berezin symbol $\widetilde{V}$ is defined on Θ by (see Berezin [8])

$$\widetilde{V}(\lambda) := \langle V\widehat{k}_\eta(z), \widehat{k}_\eta(z) \rangle, \eta \in \Theta.$$

In other words, Berezin symbol $\widetilde{V}$ is the function on Θ defined by restriction of the quadratic form $\langle Vx, x\rangle$ with $x \in \mathcal{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathcal{H}$. The Berezin set, Berezin number and Berezin norm of operators are defined, respectively, by (see [11, 27, 28])

$$\mathrm{Ber}(V) = \mathrm{Range}\left(\widetilde{V}\right) = \left\{\widetilde{V}(\eta) : \eta \in \Theta\right\},$$
$$\mathrm{ber}(V) = \sup_{\eta\in\Theta} \left|\widetilde{V}(\eta)\right|,$$

and

$$\|V\|_{\mathrm{Ber}} = \sup_{\eta\in\Theta} \left\|V\widehat{k}_\eta\right\|_{\mathcal{H}}.$$

It is obvious that $\mathrm{ber}(V) \leq w(V) \leq \|V\|$ and $\mathrm{Ber}(V) \subset W(V)$, where $w(V)$ denotes the numerical radius and $W(V)$ is the numerical range of operator V.

Let $\mathcal{L}(\mathcal{H})^+$ represent the positive operator cone, meaning that

$$\mathcal{L}(\mathcal{H})^+ = \{V \in \mathcal{L}(\mathcal{H}) : \langle Vx, x\rangle \geq 0, \forall x \in \mathcal{H}\}.$$

A positive semi-definite sesquilinear form

$$\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \ \langle x, y\rangle_A = \langle Ax, y\rangle, \forall x, y \in \mathcal{H},$$

is indicated by an operator $V \in \mathcal{L}(\mathcal{H})^+$. As expected, this semi-inner product generates a semi-norm $\|.\|_A$, which is represented by $\|x\|_A = \sqrt{\langle x, x\rangle_A} = \left\|A^{\frac{1}{2}}x\right\|, \forall x \in \mathcal{H}$. It's obvious that $\|x\|_A = 0$ iff $x \in \mathcal{N}(A)$. Therefore, if and only if A is injective operator, $\|x\|_A$ is a norm on $\mathcal{H}$, and iff $\mathcal{R}(A)$ is closed in $\mathcal{H}$, the semi-normed space $(\mathcal{L}(\mathcal{H}), \|.\|_A)$ is complete. The inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ is known to be induced by the semi-inner product $\langle ., . \rangle_A$. If $\mathcal{R}(A)$ is not closed in $\mathcal{H}$, then the quotient space $\mathcal{H}/\mathcal{N}(A)$ is not complete. Nonetheless, the completion $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}\left(A^{1/2}\right)$ with the inner product $\left\langle A^{1/2}x, A^{1/2}y\right\rangle_{\mathbf{R}(A)} = \left\langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\right\rangle, \forall x, y \in \mathcal{H}$, as shown by a classic construction by de Branges and Rownvak [10]. The Hilbert space $\left(\mathcal{R}\left(A^{1/2}\right), \langle ., . \rangle_{\mathcal{R}(A^{1/2})}\right)$ for the sequel shall be abbreviated as $\mathcal{R}\left(A^{1/2}\right)$ (see to [2]). Given $V \in \mathcal{L}(\mathcal{H})$,

$$\|V\|_A = \sup_{\substack{x\in\overline{\mathcal{R}(A)} \\ x\neq 0}} \frac{\|Vx\|_A}{\|x\|_A} = \sup_{\substack{x\in\overline{\mathcal{R}(A)} \\ \|x\|=1}} \|Vx\|_A < \infty$$

if there is $c > 0$ such that for every $x \in \overline{\mathcal{R}(A)}$, $\|Vx\|_A \leq c\|V\|_A$. Here after, we define

$$\mathcal{L}^A(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \|V\|_A < \infty\}$$

and assume that $A \neq 0$ is a positive operator in $\mathcal{L}(\mathcal{H})$. Observe that $\|V\|_A = 0$ iff $V^*AV = 0$, and $\mathcal{L}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{L}(\mathcal{H})$. Furthermore, we obtain

$$\|V\|_A = \left\{\left|\langle Vx, y\rangle_A\right| : x, y \in \overline{\mathcal{R}(A)} \text{ and } \|x\| = \|y\| = 1\right\}$$

for $V \in \mathcal{L}^A(\mathcal{H})$. If $\langle Vx, y\rangle_A = \langle x, Yy\rangle_A$ holds for every $x, y \in \mathcal{H}$, then an operator $Y \in \mathcal{L}(\mathcal{H})$ for $V \in \mathcal{L}(\mathcal{H})$ is termed an A-adjoint of an operator V. On the other hand, a solution to the operator equation $AX = V^*A$ can be understood as the presence of an A-adjoint of V. The equation $AX = V^*A$ has a bounded linear solution iff $R(V^*A) \subseteq R(A)$, according to Douglas' theorem in [12]. If all operators allowing A-adjoint are in $\mathcal{L}_A(\mathcal{H})$, then we get $\mathcal{L}_A(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : R(V^*A) \subseteq R(A)\}$. The unique solution to equation $AX = V^*A$ is defined as $V^{\sharp_A}$ if $V \in \mathcal{L}_A(\mathcal{H})$. Keep in mind that $V^{\sharp_A} = A^{\dagger}V^*A, R(V^{\sharp_A}) \subseteq \overline{R(A)}$ and $N(V^{\sharp_A}) \subseteq N(V^*A)$, where $A^{\dagger}$ is A's Moore–Penrose inverse. $V^{\sharp_A} \in \mathcal{L}_A(\mathcal{H})$, $(V^{\sharp_A})^{\sharp_A} = P_A V P_A$, and $((V^{\sharp_A})^{\sharp_A})^{\sharp_A} = V^{\sharp_A}$, where P_A is the orthogonal projection on $\overline{R(A)}$, may all be verified for $V^{\sharp_A}$. Also if $Y \in \mathcal{L}_A(\mathcal{H})$, then $VY \in \mathcal{L}_A(\mathcal{H})$, and $(VY)^{\sharp_A} = Y^{\sharp_A}V^{\sharp_A}$. Moreover,

$$\|V\|_A = \left\|V^{\sharp_A}\right\|_A = \left\|V^{\sharp_A}V\right\|_A^{1/2} = \left\|VV^{\sharp_A}\right\|_A^{1/2}. \tag{1}$$

Recall that the set of all operators admitting $A^{1/2}$-adjoint is denoted by $\mathcal{L}_{A^{1/2}}(\mathcal{H})$. Douglas' theorem may be used to confirm that

$$\mathcal{L}_{A^{1/2}}(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \exists c > 0, \|Vx\|_A \leq c\|x\|_A, \forall x \in \mathcal{H}\}.$$

Any operator in $\mathcal{L}_{A^{1/2}}(\mathcal{H})$ is defined the A-bounded operator. Furthermore, it was showed in [1] that if $V \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$, then

$$\|V\|_A = \sup_{x \in \overline{\mathcal{N}(A)}} \frac{\|Vx\|_A}{\|x\|_A} = \sup_{x \in \mathcal{H}, \|x\|=1} \|Vx\|_A.$$

In addition, , then $V(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and $\|Vx\|_A \leq \|V\|_A \|x\|_A$, $\forall x \in \mathcal{H}$ if V is A-bounded. Keep in mind that there are two algebras of $\mathcal{L}(\mathcal{H})$: $\mathcal{L}_A(\mathcal{H})$ and $\mathcal{L}_{A^{1/2}}(\mathcal{H})$. In $\mathcal{L}_A(\mathcal{H})$, these two algebras are likewise neither dense nor closed (see, [1]). Additionally, the subsequent inclusions $\mathcal{L}_A(\mathcal{H}) \subseteq \mathcal{L}_{A^{1/2}}(\mathcal{H}) \subseteq \mathcal{L}^A(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$.

Specifically, if AV is selfadjoint, then an operator $V \in \mathcal{L}(\mathcal{H})$ is A-selfadjoint; this guarantees that $\|V\|_A = \sup\{|\langle Vx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}$, as stated in [13]. Provided that AV is positive, an operator $V \in \mathcal{L}(\mathcal{H})$ is A-positive. It is obvious that an operator that is A-positive is always an A-selfadjoint operator. Furthermore, it should be mentioned that both $V^{\sharp_A}V$ and $VV^{\sharp_A}$ are A-positive. The authors of [29] examined the A-numerical radius of operator using these ideas. See [9, 14, 19, 30, 31, 36] for further research on the A-numerical radius of operators.

Now, we can give the following definitions, which given by Gürdal and Başaran [20].

Definition 1.1. $\mathrm{Ber}_A(V) = \left\{\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A : \eta \in \Theta\right\}$ *defines the A-Berezin set of* $\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A$ *for* $V \in \mathcal{L}(\mathcal{H})$.

It should be noted that even though $\mathcal{H}$ is finite dimensional, $\mathrm{Ber}_A(V)$ is a nonempty subset of $\mathbb{C}$ and is generally not closed.

Definition 1.2. *(a) The A-Berezin number of V is the supremum modulus of* $\mathrm{Ber}_A(V)$, *represented as* $\mathrm{ber}_A(V)$, *or* $\mathrm{ber}_A(V) = \sup_{\eta \in \Theta}\left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|$.

(b) For operators $V \in \mathcal{L}(\mathcal{H}(\Theta))$, $\|V\|_{A-\mathrm{Ber}} = \sup_{\eta \in \Theta}\left\|AV\widehat{k}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}$ *defines the A-Berezin norm.*

We can determine the Berezin number if $A = I$. Hence, this idea generalizes the Berezin number of functional Hilbert space operators, which have garnered interest from several writers lately (see, for example, [4–6, 15–18, 21, 23, 25, 26, 32–34]).

We can consult [20] for further information and proof on A-Berezin radius operators. $V = \Re_A(V) + i\Im(V)$ can be used to represent any operator $V \in \mathcal{L}(\mathcal{H})$. Here,

$$\Re_A(V) = \frac{V + V^{\sharp_A}}{2} \text{ and } \Im_A(V) = \frac{V - V^{\sharp_A}}{2i}.$$

A-selfadjoint operators are also $\Im_A(V)$ and $\Re_A(V)$. We also obtain $\|\Re_A(V)\|_{A-\mathrm{Ber}} \leq \mathrm{ber}_A(V)$ and $\|\Im_A(V)\|_{A-\mathrm{Ber}} \leq \mathrm{ber}_A(V)$. Moreover,

$$\max\{\|\Re_A(V)\|_{A-\mathrm{Ber}}, \|\Im_A(V)\|_{A-\mathrm{Ber}}\} \leq \mathrm{ber}_A(V).$$

Huban [24] discovered the inequality mentioned above. For $V \in \mathcal{L}_A(\mathcal{H})$, the following inequality

$$\frac{1}{2}\|V\|_{A-\text{Ber}} \le \text{ber}_A(V) \le \|V\|_{A-\text{Ber}} \tag{2}$$

was demonstrated by the same author. Also,

$$\|VR\|_{A-\text{Ber}} \le \|VR\|_A \le \|V\|_A \|R\|_A . \tag{3}$$

The A-Crawford number of $V \in \mathcal{L}_A(\mathcal{H})$ is denoted by

$$c_A(V) = \inf\{|\langle Vx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}$$

(see, [36]). The number $\widetilde{c}_A(V) = \inf_{\eta\in\Theta}\left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|$ is also shown. That recognizes that $c_A(V) \le \widetilde{c}_A(V) \le \text{ber}_A(V)$. Recently, refinements of A-Berezin radius inequalities are examined by [7, 20, 22, 24].

In this work, we introduce A-Berezin radius distance and A-Berezin norm distance. Also, we discover A-Berezin radius inequalities of the product and commutator of FHS operators using the concept of A-Berezin radius distance and A-Berezin norm distance. Furthermore, we generalize the A-Berezin radius distance. Finally, we prove the theorem related to the A-Berezin radius distance.

2. Preliminaries

We need the following lemmas in work. Let $V \in \mathcal{L}(\mathcal{H})$. An operator $Y \in \mathcal{L}(\mathcal{H})$ is called (A,Θ)-adjoint of V if for every $\tau, \mu \in \Theta$, the identity $\left\langle V\widehat{k}_\tau, \widehat{k}_\mu\right\rangle_A = \left\langle \widehat{k}_\tau, Y\widehat{k}_\mu\right\rangle_A$ holds. We denote the set of all operators in $\mathcal{L}(\mathcal{H})$ admitting (A,Θ)-adjoints by $\mathcal{L}_{A,\Theta}(\mathcal{H})$ (see, [20]). We denote $V^{\sharp_A}$ by (A,Θ)-adjoint operator of V.

Lemma 2.1 ([24]). *Let $V \in \mathcal{L}_{A,\Theta}(\mathcal{H})$ be an (A,Θ)-selfadjoint operator. Then*

$$\text{ber}_A(V) = \|V\|_{A-\text{Ber}} . \tag{4}$$

Lemma 2.2 ([22]). *Let $V, Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\text{ber}_A\left(VY^{\sharp_A} \mp YV\right) \le 2\|Y\|_{A-\text{Ber}}\, \text{ber}_A(V) . \tag{5}$$

Lemma 2.3 ([19]). *If $z, t \in \mathcal{H}$ with $t \ne 0$, then*

$$\inf_{\mu\in\mathbb{C}}\left\|z - \mu t\right\|_A^2 = \frac{\|z\|_A^2\|t\|_A^2 - |\langle z, t\rangle_A|^2}{\|t\|_A^2}. \tag{6}$$

Lemma 2.4 ([19]). *Let $z, t, \gamma \in \mathcal{H}$ with $\mu, \zeta \in \mathbb{C}$. Then*

$$\left|\langle z, \gamma\rangle_A \langle t, \gamma\rangle_A\right| \le |\langle z, t\rangle_A| + \inf_{\mu\in\mathbb{C}}\left\|z - \mu\gamma\right\|_A \inf_{\zeta\in\mathbb{C}}\left\|t - \zeta\gamma\right\|_A . \tag{7}$$

3. Inequalities of A-Berezin norm distance and A-Berezin radius distance

The A-Berezin norm distance and A-Berezin radius distance are introduced in this section. Furthermore, we enhance and expand upon a few inequalities concerning the FHS's A-Berezin radius and A-Berezin norm distance.

For $V \in \mathcal{L}_A(\mathcal{H})$, its A-seminorm distance of V from scalar operator is defined by $D_A(V)$, denoted as

$$D_A(V) = \inf_{\mu\in\mathbb{C}}\left\|V - \mu I\right\|_A .$$

Also, let $d_A(V)$ define the A-numerical radius of V from scalar operators, i.e.,

$$d_A(V) = \inf_{\mu\in\mathbb{C}} w_A(V - \mu I) .$$

By using compactness, we can determine that there exists μ_0 such that $d_A(V) = w_A(V - \mu_0 I)$.

Definition 3.1. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS. For $V \in \mathcal{L}_A(\mathcal{H})$, the A-Berezin norm of distance denoted by $\widetilde{D}_A(V)$, is defined by A-Berezin norm distance of V from the scalar operators, i.e.,*

$$\widetilde{D}_A(V) = \inf_{\lambda\in\mathbb{C}} \|V - \lambda I\|_{A-\mathrm{Ber}}.$$

Definition 3.2. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS. For $V \in \mathcal{L}_A(\mathcal{H})$, the A-Berezin radius of distance denoted by $\widetilde{d}_A(V)$, is defined by A-Berezin radius distance of V from the scalar operators, i.e.,*

$$\widetilde{d}_A(V) = \inf_{\lambda\in\mathbb{C}} \mathrm{ber}_A(V - \lambda I).$$

Again, applying compactness we can see that there exists λ_0 such that $\widetilde{d}_A(V) = \mathrm{ber}_A(V - \lambda_0 I)$.

It is clear that $\widetilde{D}_A(V) \le D_A(V)$ and $\widetilde{d}_A(V) \le d_A(V)$.
Let's now demonstrate the first theorem.

Theorem 3.3. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and let $V \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\sqrt{\widetilde{D}_A^2(V) + \widetilde{c}_A^2(V)} \le \|V\|_{A-\mathrm{Ber}} \le \sqrt{\widetilde{D}_A^2(V) + \mathrm{ber}_A^2(V)}. \tag{8}$$

Proof. From (6), we can write that

$$\inf_{\lambda\in\mathbb{C}} \|Vx - \lambda x\|_A^2 = \frac{\|Vx\|_A^2 \|\lambda x\|_A^2 - |\langle Vx, \lambda x\rangle_A|^2}{\|\lambda x\|_A^2}, \tag{9}$$

where $x \in \mathcal{H}$. Now, replacing x by $\widehat{k}_\eta$ in (9), we reach

$$\begin{aligned}\inf_{\lambda\in\mathbb{C}} \left\|V\widehat{k}_\eta - \lambda\widehat{k}_\eta\right\|_A^2 &= \frac{\left\|V\widehat{k}_\eta\right\|_A^2 \left\|\lambda\widehat{k}_\eta\right\|_A^2 - \left|\left\langle V\widehat{k}_\eta, \lambda\widehat{k}_\eta\right\rangle_A\right|^2}{\left\|\lambda\widehat{k}_\eta\right\|_A^2} \\ &= \left\|V\widehat{k}_\eta\right\|_A^2 - \left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|^2 \\ &\le \|V\|_{A-\mathrm{Ber}}^2 - \widetilde{c}_A^2(V).\end{aligned}$$

By taking the supremum over $\eta \in \Theta$, we obtain

$$\widetilde{D}_A^2(V) + \widetilde{c}_A^2(V) \le \|V\|_{A-\mathrm{Ber}}^2, \tag{10}$$

which has the first inequality at the theorem. Next, we prove the second inequality. From Lemma 2.3, we get

$$\|z\|_A^2 \|t\|_A^2 - |\langle z, t\rangle_A|^2 = \|t\|_A^2 \inf_{\lambda\in\mathbb{C}} \|z - \lambda t\|_A^2. \tag{11}$$

Replacing z by $V\widehat{k}_\eta$ and t by $\widehat{k}_\eta$ in (11), we reach

$$\left\|V\widehat{k}_\eta\right\|_A^2 \left\|\widehat{k}_\eta\right\|_A^2 - \left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|^2 = \inf_{\lambda\in\mathbb{C}} \left\|V\widehat{k}_\eta - \lambda\widehat{k}_\eta\right\|_A^2.$$

That is

$$\left\|V\widehat{k}_\eta\right\|_A^2 \left\|\widehat{k}_\eta\right\|_A^2 = \inf_{\lambda\in\mathbb{C}} \left\|V\widehat{k}_\eta - \lambda\widehat{k}_\eta\right\|_A^2 + \left|\left\langle V\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|^2.$$

Taking the supremum over $\eta \in \Theta$ in the above inequality, we have

$$\|V\|_{A-\mathrm{Ber}}^{2} \leq \inf_{\lambda\in\mathbb{C}} \|V-\lambda I\|_{A-\mathrm{Ber}}^{2} + \mathrm{ber}_A^2(V) = \widetilde{D}_A^2(V) + \mathrm{ber}_A^2(V). \tag{12}$$

By combining (10) and (12), we get

$$\widetilde{D}_A^2(V) + \widetilde{c}_A^2(V) \leq \|V\|_{A-\mathrm{Ber}}^{2} \leq \widetilde{D}_A^2(V) + \mathrm{ber}_A^2(V).$$

Consequently, we have

$$\sqrt{\widetilde{D}_A^2(V) + \widetilde{c}_A^2(V)} \leq \|V\|_{A-\mathrm{Ber}} \leq \sqrt{\widetilde{D}_A^2(V) + \mathrm{ber}_A^2(V)}.$$

We completes the proof. □

In [35], Yamancı and Karlı show that if $V \in \mathcal{L}_A(\mathcal{H})$, then

$$\mathrm{ber}^2(V) + \mathrm{ber}\left(V^2\right) \leq \inf_{\lambda\in\mathbb{C}} \|V-\lambda I\|^2. \tag{13}$$

The inequality (13) is generalized by the following theorem.

Theorem 3.4. *If $V \in \mathcal{L}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A^{2r}(V) \leq 2^{r-1}\left(\mathrm{ber}_A^r\left(V^2\right) + \widetilde{D}_A^{2r}(V)\right),$$

for any $r \geq 1$.

Proof. Let $\eta \in \Theta$ be an arbitrary. Replacing z by $V\widehat{k_\eta}$, t by $V^{\sharp_A}\widehat{k_\eta}$ and γ by $\widehat{k_\eta}$ in (7), we have

$$\left|\left\langle V\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A \left\langle V^{\sharp_A}\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right| \leq \left|\left\langle V\widehat{k_\eta}, V^{\sharp_A}\widehat{k_\eta}\right\rangle_A\right| + \inf_{\lambda\in\mathbb{C}}\left\|V\widehat{k_\eta} - \lambda\widehat{k_\eta}\right\|_A \inf_{\xi\in\mathbb{C}}\left\|V^{\sharp_A}\widehat{k_\eta} - \xi\widehat{k_\eta}\right\|_A.$$

Hence,

$$\left|\left\langle V\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right|^2 \leq \left|\left\langle V^2\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right| + \inf_{\lambda\in\mathbb{C}}\left\|V\widehat{k_\eta} - \lambda\widehat{k_\eta}\right\|_A \inf_{\xi\in\mathbb{C}}\left\|V^{\sharp_A}\widehat{k_\eta} - \xi\widehat{k_\eta}\right\|_A.$$

From the elementary inequality $\left(\frac{x+y}{2}\right)^r \leq \frac{x^r+y^r}{2}$, $x, y > 0$ and $r \geq 1$, we get

$$\left|\left\langle V\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right|^{2r} \leq 2^{r-1}\left(\left|\left\langle V^2\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right|^r + \inf_{\lambda\in\mathbb{C}}\left\|V\widehat{k_\eta} - \lambda\widehat{k_\eta}\right\|_A^r \inf_{\xi\in\mathbb{C}}\left\|V^{\sharp_A}\widehat{k_\eta} - \xi\widehat{k_\eta}\right\|_A^r\right).$$

Taking the supremum in the inequality above over $\eta \in \Theta$, we have

$$\mathrm{ber}_A^{2r}(V) \leq 2^{r-1}\left(\mathrm{ber}_A^r\left(V^2\right) + \inf_{\lambda\in\mathbb{C}}\|V-\lambda I\|_{A-\mathrm{Ber}}^r \inf_{\xi\in\mathbb{C}}\left\|V^{\sharp_A} - \xi I\right\|_{A-\mathrm{Ber}}^r\right).$$

Finally, by taking the infimum $\lambda, \xi \in \mathbb{C}$, we reach

$$\mathrm{ber}_A^{2r}(V) \leq 2^{r-1}\left(\mathrm{ber}_A^r\left(V^2\right) + \widetilde{D}_A^r(V)\,\widetilde{D}_A^r\left(V^{\sharp_A}\right)\right).$$

Moreover, for every $V \in \mathcal{L}_A(\mathcal{H})$ and for every $\lambda \in \mathbb{C}$ one can see that

$$\begin{aligned}\|V-\lambda I\|_{A-\mathrm{Ber}} &= \left\|(V-\lambda I)^{\sharp_A}\right\|_{A-\mathrm{Ber}} \\ &= \left\|V^{\sharp_A} - \overline{\lambda}P\right\|_{A-\mathrm{Ber}} = \left\|(V-\lambda P)^{\sharp_A}\right\|_{A-\mathrm{Ber}} \\ &= \|V-\lambda P\|_{A-\mathrm{Ber}}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\widetilde{D}_A\left(V^{\sharp_A}\right) &= \inf_{\lambda\in\mathbb{C}}\left\|V^{\sharp_A}-\lambda I\right\|_{A-\mathrm{Ber}} = \inf_{\lambda\in\mathbb{C}}\left\|V^{\sharp_A}-\lambda P\right\|_{A-\mathrm{Ber}}\\ &= \inf_{\lambda\in\mathbb{C}}\left\|\left(V-\overline{\lambda}I\right)^{\sharp_A}\right\|_{A-\mathrm{Ber}}\\ &= \inf_{\lambda\in\mathbb{C}}\left\|V-\overline{\lambda}I\right\|_{A-\mathrm{Ber}}\\ &= \widetilde{D}_A\left(V\right).\end{aligned}$$

Thus,

$$\mathrm{ber}_A^{2r}\left(V\right) \le 2^{r-1}\left(\mathrm{ber}_A^{r}\left(V^2\right)+\widetilde{D}_A^{2r}\left(V\right)\right).$$

The evidence is now complete. □

Specifically, taking into account that $r = 1$ in Theorem 3.4, we obtain the subsequent corollary.

Corollary 3.5. *If $V \in \mathcal{L}_A(\mathcal{H})$, then*

$$\mathrm{ber}_A\left(V\right) \le \sqrt{\mathrm{ber}_A\left(V^2\right)+\widetilde{D}_A^{2}\left(V\right)}.$$

Now, applying compactness argument can see that there exists $\lambda_0 \in \mathbb{C}$ such that $\widetilde{D}_A(V,R) = \inf_{\lambda_0\in\mathbb{C}}\|V-\lambda_0 R\|_{A-\mathrm{Ber}}$. Utilizing this generalizing distance $\widetilde{D}_A(V,R)$, and proceeding similarly as in Theorem 3.3, we get the subsequent consequence.

Corollary 3.6. *If $V, Y \in \mathcal{L}_A(\mathcal{H})$, then*

$$\frac{\sqrt{\widetilde{m}_A^2\left(Y\right)\widetilde{D}_A^2\left(V,Y\right)+\widetilde{c}_A^2\left(Y^{\sharp_A}V\right)}}{\|Y\|_{A-\mathrm{Ber}}} \le \|V\|_{A-\mathrm{Ber}} \le \frac{\sqrt{\|Y\|_{A-\mathrm{Ber}}^2\widetilde{D}_A^2\left(V,Y\right)+\mathrm{ber}_A^2\left(Y^{\sharp_A}V\right)}}{\widetilde{m}_A\left(Y\right)},$$

where $\widetilde{m}_A(Y) = \inf_{\eta\in\Theta}\left\|Y\widehat{k}_\eta\right\|_A$.

We shall now demonstrate the subsequent theorem.

Theorem 3.7. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be an FHS and let $V, Y \in \mathcal{L}_{A,\Theta}(\mathcal{H})$. Then*

$$\mathrm{ber}_A\left(VY\right) \le \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A\left(Y\right)+\frac{1}{2}\min\left\{\mathrm{ber}_A\left(VY+YV^{\sharp_A}\right),\mathrm{ber}_A\left(VY-YV^{\sharp_A}\right)\right\}. \tag{14}$$

Proof. Let $\theta \in \mathbb{R}$. It is clear that $\Re_A\left(e^{i\theta}VY\right)$ is an (A,Θ)-selfadjoint operator. Hence, we have

$$\begin{aligned}
\left\|\Re_A\left(e^{i\theta}VY\right)\right\|_{A-\mathrm{Ber}} &= \mathrm{ber}_A\left(\Re_A\left(e^{i\theta}VY\right)\right) \text{ (by (4))}\\
&= \mathrm{ber}_A\left(\frac{1}{2}\left(e^{i\theta}VY + e^{-i\theta}Y^{\sharp_A}V^{\sharp_A}\right)\right)\\
&= \mathrm{ber}_A\left(\frac{1}{2}\left(e^{i\theta}VY + e^{-i\theta}VY^{\sharp_A} + e^{-i\theta}Y^{\sharp_A}V^{\sharp_A} - e^{-i\theta}VY^{\sharp_A}\right)\right)\\
&= \mathrm{ber}_A\left(V\Re_A\left(e^{i\theta}Y\right) + \frac{1}{2}e^{-i\theta}\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\right)\\
&= \mathrm{ber}_A\left(V\Re_A\left(e^{i\theta}Y\right)\right) + \mathrm{ber}_A\left(\frac{1}{2}e^{-i\theta}\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\right)\\
&\leq \left\|V\Re_A\left(e^{i\theta}Y\right)\right\|_{A-\mathrm{Ber}} + \frac{1}{2}\mathrm{ber}_A\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\\
&\leq \|V\|_{A-\mathrm{Ber}}\left\|\Re_A\left(e^{i\theta}Y\right)\right\|_{A-\mathrm{Ber}} + \frac{1}{2}\mathrm{ber}_A\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\\
&\leq \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) + \frac{1}{2}\mathrm{ber}_A\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right).
\end{aligned}$$

Therefore, by taking the supremum over all $\theta \in \mathbb{R}$, we have

$$\mathrm{ber}_A(VY) \leq \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) + \frac{1}{2}\mathrm{ber}_A\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right). \tag{15}$$

On the other hand, for $\eta \in \Theta$ we observe that

$$\begin{aligned}
\left|\left\langle\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right| &= \left|\left\langle Y^{\sharp_A}V^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A - \left\langle VY^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|\\
&= \left|\left\langle Y^{\sharp_A}V^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A - \left\langle P_{\overline{\mathcal{R}(A)}}VP_{\overline{\mathcal{R}(A)}}Y^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\left|\left\langle\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right| &= \left|\left\langle Y^{\sharp_A}V^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A - \left\langle\left(V^{\sharp_A}\right)^{\sharp_A}Y^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|\\
&= \left|\left\langle\left(VY - YV^{\sharp_A}\right)^{\sharp_A}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|\\
&= \left|\left\langle\left(VY - YV^{\sharp_A}\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle_A\right|.
\end{aligned}$$

It follows that $\mathrm{ber}_A\left(Y^{\sharp_A}V^{\sharp_A} - VY^{\sharp_A}\right) = \mathrm{ber}_A\left(VY - YV^{\sharp_A}\right)$. So, the following inequality have been by (15):

$$\mathrm{ber}_A(VY) \leq \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) + \frac{1}{2}\mathrm{ber}_A\left(VY - YV^{\sharp_A}\right). \tag{16}$$

Also, by replacing V by iV in (15), we obtain

$$\mathrm{ber}_A(VY) \leq \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) + \frac{1}{2}\mathrm{ber}_A\left(VY + YV^{\sharp_A}\right). \tag{17}$$

Thus, the proof is completed by combining (16) together with (17). □

We are now prepared to demonstrate the subsequent theorem.

Theorem 3.8. *If $V, Y \in \mathcal{L}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A(VY) \leq \min\left\{\left(\|V\|_{A-\mathrm{Ber}} + \widetilde{D}_A(V)\right)\mathrm{ber}_A(Y), \left(\|Y\|_{A-\mathrm{Ber}} + \widetilde{D}_A(Y)\right)\mathrm{ber}_A(V)\right\}. \tag{18}$$

Proof. Let $\eta \in \Theta$ be an arbitrary. There exists $\lambda_0 \in \mathbb{C}$ such that $\widetilde{D}_A(V) = \inf_{\lambda_0 \in \mathbb{C}} \|V - \lambda_0 I\|_{A-\text{Ber}}$. If $\lambda_0 = 0$, then by the inequalities in (2), we have

$$\text{ber}_A(VY) \leq \|VY\|_{A-\text{Ber}} \leq \|V\|_{A-\text{Ber}} \|Y\|_{A-\text{Ber}} \leq 2\|V\|_{A-\text{Ber}} \|Y\|_{A-\text{Ber}} = \left(\|V\|_{A-\text{Ber}} + \widetilde{D}_A(V)\right) \text{ber}_A(Y).$$

Next, we choose $\lambda_0 \neq 0$ and $\xi = \frac{\lambda_0}{|\lambda_0|}$. Then, from the inequality (14), we have

$$\begin{aligned}
\text{ber}_A(VY) &\leq \text{ber}_A(\xi VY) \leq \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \frac{1}{2}\text{ber}_A\left(\xi VY - \overline{\xi} Y V^{\sharp_A}\right) \\
&= \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \frac{1}{2}\text{ber}_A\left(\overline{\xi} Y^{\sharp_A} V^{\sharp_A} - \xi \left(V^{\sharp_A}\right)^{\sharp_A} Y^{\sharp_A}\right) \\
&= \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \frac{1}{2}\text{ber}_A\left(\xi \left(V^{\sharp_A}\right)^{\sharp_A} \left(Y^{\sharp_A}\right)^{\sharp_A} - \overline{\xi}\left(Y^{\sharp_A}\right)^{\sharp_A} V^{\sharp_A}\right) \\
&= \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \frac{1}{2}\text{ber}_A\left(\xi \left(\left(V^{\sharp_A}\right)^{\sharp_A} - \lambda_0 I\right)\left(Y^{\sharp_A}\right)^{\sharp_A} - \overline{\xi}\left(Y^{\sharp_A}\right)^{\sharp_A}\left(\left(V^{\sharp_A}\right)^{\sharp_A} - \lambda_0 I\right)^{\sharp_A}\right) \\
&\leq \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \left\|\left(V^{\sharp_A}\right)^{\sharp_A} - \lambda_0 I\right\|_{A-\text{Ber}} \text{ber}_A\left(\left(Y^{\sharp_A}\right)^{\sharp_A}\right) \text{ (by (5))} \\
&\leq \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \left\|\left(V^{\sharp_A}\right)^{\sharp_A} - \lambda_0 I\right\|_{A-\text{Ber}} \text{ber}_A(Y).
\end{aligned}$$

Next, by using the $\left\|Y^{\sharp_A}\right\|_{A-\text{Ber}} = \|Y\|_{A-\text{Ber}}$, for all $Y \in \mathcal{L}_A(\mathcal{H})$ we can see that

$$\left\|\left(V^{\sharp_A}\right)^{\sharp_A} - \lambda_0 I\right\|_{A-\text{Ber}} = \left\|V^{\sharp_A} - \lambda_0 P\right\|_{A-\text{Ber}} = \left\|(V - \lambda_0 I)^{\sharp_A}\right\|_{A-\text{Ber}} = \|V - \lambda_0 I\|_{A-\text{Ber}}.$$

Hence,

$$\text{ber}_A(VY) \leq \|V\|_{A-\text{Ber}} \text{ber}_A(Y) + \|V - \lambda_0 I\|_{A-\text{Ber}} \text{ber}_A(Y) = \left(\|V\|_{A-\text{Ber}} + \widetilde{D}_A(V)\right) \text{ber}_A(Y) \tag{19}$$

Replacing V by $Y^{\sharp_A}$ and Y by $V^{\sharp_A}$ in the above inequality and since $\widetilde{D}_A\left(Y^{\sharp_A}\right) = \widetilde{D}_A(Y)$, we have

$$\text{ber}_A(VY) \leq \left(\|Y\|_{A-\text{Ber}} + \widetilde{D}_A(Y)\right) \text{ber}_A(V). \tag{20}$$

Combining the inequalities in (19) and (20), we reach the inequality

$$\text{ber}_A(VY) \leq \min\left\{\left(\|V\|_{A-\text{Ber}} + \widetilde{D}_A(V)\right) \text{ber}_A(Y), \left(\|Y\|_{A-\text{Ber}} + \widetilde{D}_A(Y)\right) \text{ber}_A(V)\right\}.$$

□

Corollary 3.9. *If $V, Y \in \mathcal{L}_A(\mathcal{H})$, then we have*

$$\widetilde{D}_A(V) \leq \|V\|_{A-\text{Ber}} \text{ and } \widetilde{D}_A(Y) \leq \|Y\|_{A-\text{Ber}},$$

$$\left(\|V\|_{A-\text{Ber}} + \widetilde{D}_A(V)\right) \text{ber}_A(Y) \leq 2\|V\|_{A-\text{Ber}} \text{ber}_A(Y),$$

and

$$\left(\|Y\|_{A-\text{Ber}} + \widetilde{D}_A(Y)\right) \text{ber}_A(V) \leq 2\|Y\|_{A-\text{Ber}} \text{ber}_A(V).$$

Now, we obtain the following inequalities, which is A-Berezin distance $\widetilde{d}_A(V)$.

Theorem 3.10. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and let $V \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\|V\|_{A-\mathrm{Ber}} \le \mathrm{ber}_A(V) + \widetilde{d_A}(V) \le 2\mathrm{ber}_A(V). \tag{21}$$

Proof. There exists $\lambda_0 \in \mathbb{C}$ such that $\widetilde{d_A}(V) = \inf_{\lambda_0 \in \mathbb{C}} \mathrm{ber}_A(V - \lambda_0 I)$. If $\lambda_0 = 0$, then $\|V\|_{A-\mathrm{Ber}} \le 2\mathrm{ber}_A(V) = \mathrm{ber}_A(V) + \mathrm{ber}_A(V - \lambda_0 I) = \mathrm{ber}_A(V) + \widetilde{d_A}(V)$.

Next, we choose $\lambda_0 \neq 0$ and $\xi = \frac{\lambda_0}{|\lambda_0|}$. Hence,

$$\begin{aligned}\|V\|_{A-\mathrm{Ber}} &= \|\xi V\|_{A-\mathrm{Ber}} = \|\Re_A(\xi V) + i\Im_A(\xi V)\|_{A-\mathrm{Ber}} \\ &\le \|\Re_A(\xi V)\|_{A-\mathrm{Ber}} + \|\Im_A(\xi V)\|_{A-\mathrm{Ber}} \\ &= \|\Re_A(\xi V)\|_{A-\mathrm{Ber}} + \|\Im_A(\xi(V - \lambda_0 I))\|_{A-\mathrm{Ber}} \\ &\le \mathrm{ber}_A(V) + \mathrm{ber}_A(V - \lambda_0 I).\end{aligned}$$

Therefore, $\|V\|_{A-\mathrm{Ber}} \le \mathrm{ber}_A(V) + \widetilde{d_A}(V)$. The second inequality follows from the fact that $\widetilde{d_A}(V) \le \mathrm{ber}_A(V)$. □

Corollary 3.11. *Let $V, Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\|VY\|_{A-\mathrm{Ber}} \le \left(\mathrm{ber}_A(V) + \widetilde{d_A}(V)\right)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right) \le 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y).$$

Proof. There exists $\lambda_0 \in \mathbb{C}$ such that $\widetilde{d_A}(V) = \inf_{\lambda_0 \in \mathbb{C}} \mathrm{ber}_A(V - \lambda_0 I)$. If $\lambda_0 = 0$, then $\|V\|_{A-\mathrm{Ber}} \le 2\mathrm{ber}_A(V) = \mathrm{ber}_A(V) + \mathrm{ber}_A(V - \lambda I) = \mathrm{ber}_A(V) + \widetilde{d_A}(V)$.

Next, we choose $\lambda_0 \neq 0$ and $\xi = \frac{\lambda_0}{|\lambda_0|}$. Hence,

$$\begin{aligned}\|VY\|_{A-\mathrm{Ber}} &\le \|V\|_{A-\mathrm{Ber}}\|Y\|_{A-\mathrm{Ber}} \le \left(\mathrm{ber}_A(V) + \widetilde{d_A}(V)\right)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right) \text{ (by (21))} \\ &\le 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y) \text{ (by } \mathrm{ber}_A(V) \ge \widetilde{d_A}(V)\text{)}.\end{aligned}$$

□

Assuming V to be A-positive, we then obtain the following inequalities.

Theorem 3.12. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and $V, Y \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. If V is A-positive, then*

$$\mathrm{ber}_A(VY) \le \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) \text{ and } \mathrm{ber}_A(YV) \le \|Y\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(V).$$

Proof. For all $\beta \in [0,1]$, we get

$$\begin{aligned}\mathrm{ber}_A(VY) &= \mathrm{ber}_A\left(\left(V - \beta\|V\|_{A-\mathrm{Ber}} I\right)Y + \beta\|V\|_{A-\mathrm{Ber}} Y\right) \\ &\le \mathrm{ber}_A\left(\left(V - \beta\|V\|_{A-\mathrm{Ber}} I\right)Y\right) + \beta\|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) \\ &\le \left\|\left(V - \beta\|V\|_{A-\mathrm{Ber}} I\right)Y\right\|_{A-\mathrm{Ber}} + \beta\|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) \\ &\le \left\|V - \beta\|V\|_{A-\mathrm{Ber}} I\right\|_{A-\mathrm{Ber}}\|Y\|_{A-\mathrm{Ber}} + \beta\|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y).\end{aligned}$$

Since V is A-positive, we can see that $\left\|V - \beta\|V\|_{A-\mathrm{Ber}} I\right\|_{A-\mathrm{Ber}} = (1-\beta)\|V\|_{A-\mathrm{Ber}}$ for all $\beta \in [0,1]$. Hence

$$\mathrm{ber}_A(VY) \le \|V\|_{A-\mathrm{Ber}}\left(1 - \beta I\|Y\|_{A-\mathrm{Ber}} + \beta\mathrm{ber}_A(Y)\right) \tag{22}$$

Therefore, by considering $\beta = 1$ in (22), we have

$$\mathrm{ber}_A(VY) \le \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y).$$

Similarly,

$$\mathrm{ber}_A(YV) \le \|Y\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(V).$$

This completes the proof. □

The following Berezin radius inequalities for the product of FHS operators are obtained by taking $A = I$ in Theorem 3.12.

Corollary 3.13. *If $V, Y \in \mathcal{L}(\mathcal{H})$, $V \geq 0$, then we have*

$$\mathrm{ber}(VY) \leq \|V\|_{\mathrm{Ber}} \, \mathrm{ber}(Y) \text{ and } \mathrm{ber}(YV) \leq \|Y\|_{\mathrm{Ber}} \, \mathrm{ber}(V).$$

We shall now demonstrate the next theorem.

Theorem 3.14. *If $V, Y \in \mathcal{L}_A(\mathcal{H})$, then we have*

$$\mathrm{ber}_A(VY \mp YV) \leq 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y). \tag{23}$$

Proof. (2) and (3) imply that

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &\leq \mathrm{ber}_A(VY) + \mathrm{ber}_A(YV)\\ &\leq \|V\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(Y) + \|Y\|_{A-\mathrm{Ber}}\,\mathrm{ber}_A(V) \text{ (by Theorem 3.12)}\\ &\leq 2\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y) + 2\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y)\\ &= 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y).\end{aligned}$$

This completes the evidence. □

We derive the following theorem from Theorem 3.14,

Theorem 3.15. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and $V, Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\mathrm{ber}_A(VY - YV) \leq 4\widetilde{d_A}(V)\,\widetilde{d_A}(Y) \leq 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y).$$

Proof. Let $\lambda_0, \xi_0 \in \mathbb{C}$ such that $\widetilde{d_A}(V) = \inf_{\lambda_0 \in \mathbb{C}} \mathrm{ber}_A(V - \lambda_0 I)$ and $\widetilde{d_A}(Y) = \inf_{\xi_0 \in \mathbb{C}} \mathrm{ber}_A(Y - \xi_0 I)$. Then, we get

$$\begin{aligned}\mathrm{ber}_A(VY - YV) &= \mathrm{ber}_A((V - \lambda_0 I)(Y - \xi_0 I) - (Y - \xi_0 I)(V - \lambda_0 I))\\ &\leq 4\mathrm{ber}_A(V - \lambda_0 I)\,\mathrm{ber}_A(Y - \xi_0 I) \text{ (by (23))}\\ &\leq 4\widetilde{d_A}(V)\,\widetilde{d_A}(Y).\end{aligned}$$

Thus,

$$\mathrm{ber}_A(VY - YV) \leq 4\widetilde{d_A}(V)\,\widetilde{d_A}(Y).$$

The second desired inequality follows from the fact that $\widetilde{d_A}(V) \leq \mathrm{ber}_A(V)$ and $\widetilde{d_A}(Y) \leq \mathrm{ber}_A(Y)$. □

We need the following theorem to prove the next corollary.

Theorem 3.16. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and let $V_1, V_2, Y_1, Y_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\mathrm{ber}_A(V_1 Y_1 \pm Y_2 V_2) \leq \sqrt{\left\|V_1^{\sharp_A} V_1 + V_2 V_2^{\sharp_A}\right\|_{A-\mathrm{Ber}}} \sqrt{\left\|Y_1 Y_1^{\sharp_A} + Y_2^{\sharp_A} Y_2\right\|_{A-\mathrm{Ber}}}.$$

Proof. Let $\eta \in \Theta$ be an arbitrary. An application of Cauchy-Schwarz inequality obtains

$$\begin{aligned}\left|\left\langle (V_1 Y_1 \pm Y_2 V_2)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_A\right| &\leq \left|\left\langle V_1 Y_1 \widehat{k}_\eta, \widehat{k}_\eta \right\rangle_A + \left\langle Y_2 V_2 \widehat{k}_\eta, \widehat{k}_\eta \right\rangle_A\right|\\ &= \left|\left\langle Y_1 \widehat{k}_\eta, V_1^{\sharp_A} \widehat{k}_\eta \right\rangle_A + \left\langle V_2 \widehat{k}_\eta, Y_2^{\sharp_A} \widehat{k}_\eta \right\rangle_A\right|\\ &\leq \left(\left\|Y_1 \widehat{k}_\eta\right\|_A \left\|V_1^{\sharp_A} \widehat{k}_\eta\right\|_A + \left\|V_2 \widehat{k}_\eta\right\|_A \left\|Y_2^{\sharp_A} \widehat{k}_\eta\right\|_A\right)\\ &\leq \left(\left\|V_1^{\sharp_A} \widehat{k}_\eta\right\|_A^2 + \left\|V_2 \widehat{k}_\eta\right\|_A^2\right)\left(\left\|Y_1 \widehat{k}_\eta\right\|_A^2 + \left\|Y_2^{\sharp_A} \widehat{k}_\eta\right\|_A^2\right)\\ &= \sqrt{\left\langle \left(V_2^{\sharp_A} V_2 + V_1 V_1^{\sharp_A}\right)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_A} \sqrt{\left\langle \left(Y_1^{\sharp_A} Y_1 + Y_2 Y_2^{\sharp_A}\right)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_A}\\ &\leq \sqrt{\left\|V_1^{\sharp_A} V_1 + V_2 V_2^{\sharp_A}\right\|_{A-\mathrm{Ber}}} \sqrt{\left\|Y_1 Y_1^{\sharp_A} + Y_2^{\sharp_A} Y_2\right\|_{A-\mathrm{Ber}}}.\end{aligned}$$

Hence,

$$\left|\left\langle (V_1Y_1 \pm Y_2V_2)\widehat{k_\eta}, \widehat{k_\eta}\right\rangle_A\right| \leq \sqrt{\left\|V_1^{\sharp_A}V_1 + V_2V_2^{\sharp_A}\right\|_{A-\text{Ber}}}\sqrt{\left\|Y_1Y_1^{\sharp_A} + Y_2^{\sharp_A}Y_2\right\|_{A-\text{Ber}}}.$$

By taking the supremum over $\eta \in \Theta$ in the above inequality, we get

$$\text{ber}_A(V_1Y_1 \pm Y_2V_2) \leq \sqrt{\left\|V_1^{\sharp_A}V_1 + V_2V_2^{\sharp_A}\right\|_{A-\text{Ber}}}\sqrt{\left\|Y_1Y_1^{\sharp_A} + Y_2^{\sharp_A}Y_2\right\|_{A-\text{Ber}}}.$$

The proof is now complete. □

Corollary 3.17. *If $V, Y \in \mathcal{L}_A(\mathcal{H})$, then we have*

$$\text{ber}_A(VY \mp YV) \leq 2\sqrt{2}\,\|V\|_{A-\text{Ber}}\,\text{ber}_A(Y). \tag{24}$$

Proof. By putting $V_1 = V_2 = V$ and $Y_1 = Y_2 = Y$ in Theorem 3.16 and then using the inequality in [22, Corollary 1] we have

$$\begin{aligned}\text{ber}_A(VY \pm YV) &\leq \sqrt{\left\|VV^{\sharp_A} + V^{\sharp_A}V\right\|_{A-\text{Ber}}}\sqrt{\left\|YY^{\sharp_A} + Y^{\sharp_A}Y\right\|_{A-\text{Ber}}}\\ &\leq 2\sqrt{\left\|VV^{\sharp_A} + V^{\sharp_A}V\right\|_{A-\text{Ber}}}\text{ber}_A(Y)\\ &\leq 2\sqrt{2}\,\|V\|_{A-\text{Ber}}\,\text{ber}_A(Y) \text{ (by (1))}.\end{aligned}$$

The proof is now complete. □

Corollary 3.17 may be generalized to provide the following conclusion.

Corollary 3.18. *Let $V, Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\text{ber}_A(VY \mp YV) \leq 2\sqrt{2}\min\{\|V\|_{A-\text{Ber}}\,\text{ber}_A(Y), \|Y\|_{A-\text{Ber}}\,\text{ber}_A(V)\}. \tag{25}$$

Proof. By replacing V by Y and Y by V respectively in (24), we have the desired result. □

It is clear that (25) provides an upper bound for the A-Berezin radius of the commutator $VY - YV$.
We can now demonstrate the following theorem.

Theorem 3.19. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and $V, Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\text{ber}_A(VY - YV) \leq 2\sqrt{2}\min\left\{\widetilde{D}_A(V)\,\widetilde{d}_A(Y), \widetilde{D}_A(Y)\,\widetilde{d}_A(V)\right\} \leq 2\sqrt{2}\,\|V\|_{A-\text{Ber}}\,\text{ber}_A(Y).$$

Proof. Let $\lambda_0, \xi_0 \in \mathbb{C}$ such that $\widetilde{D}_A(V) = \inf_{\lambda_0 \in \mathbb{C}} \|V - \lambda_0 I\|_{A-\text{Ber}}$ and $\widetilde{d}_A(Y) = \inf_{\xi_0 \in \mathbb{C}} \text{ber}_A(Y - \xi_0 I)$. Then, we get

$$\begin{aligned}\text{ber}_A(VY - YV) &= \text{ber}_A((V - \lambda_0 I)(Y - \xi_0 I) - (Y - \xi_0 I)(V - \lambda_0 I))\\ &\leq 2\sqrt{2}\,\|V - \lambda_0 I\|_{A-\text{Ber}}\,\text{ber}_A(Y - \xi_0 I)\\ &= 2\sqrt{2}\widetilde{D}_A(V)\,\widetilde{d}_A(Y).\end{aligned}$$

Thus, $\text{ber}_A(VY - YV) \leq 2\sqrt{2}\widetilde{D}_A(V)\,\widetilde{d}_A(Y)$.
Replacing V by Y and Y by V in the above inequality, we get

$$\text{ber}_A(YV - VY) \leq 2\sqrt{2}\widetilde{D}_A(Y)\,\widetilde{d}_A(V).$$

The first inequality is obtained by combining the two above inequality. Since $\widetilde{D}_A(V) \leq \|V\|_{A-\text{Ber}}$ and $\widetilde{d}_A(Y) \leq \text{ber}_A(Y)$, the second inequality is inferred. □

Next, we generalize the A-Berezin distance $\widetilde{d_A}(V,Y)$ as following from: For $V,Y \in \mathcal{L}_A(\mathcal{H})$

$$\widetilde{d_A}(V,Y) = \mathrm{ber}_A(V - \xi_0 Y).$$

Utilizing this generalized A-Berezin distance $\widetilde{d_A}(V,Y)$, we get the following inequalities.

Theorem 3.20. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and let $V,Y,W \in \mathcal{L}_A(\mathcal{H})$ be such that W commutes with both V and W. Then*

$$\mathrm{ber}_A(VY - YV) \leq 4\widetilde{d_A}(V,W)\widetilde{d_A}(Y,W) \leq 4\mathrm{ber}_A(V)\mathrm{ber}_A(Y).$$

Proof. Let $\lambda_0, \xi_0 \in \mathbb{C}$ such that $\widetilde{d_A}(V,W) = \inf_{\lambda_0\in\mathbb{C}} \mathrm{ber}_A(V - \lambda_0 W)$ and $\widetilde{d_A}(Y,W) = \inf_{\xi_0\in\mathbb{C}} \mathrm{ber}_A(Y - \xi_0 W)$. Then, we get

$$\begin{aligned}\mathrm{ber}_A(VY - YV) &= \mathrm{ber}_A((V - \lambda_0 W)(Y - \xi_0 W) - (Y - \xi_0 W)(V - \lambda_0 W)) \\ &\leq 4\mathrm{ber}_A(V - \lambda_0 W)\mathrm{ber}_A(Y - \xi_0 W) \\ &= 4\widetilde{d_A}(V,W)\widetilde{d_A}(Y,W).\end{aligned}$$

Thus, $\mathrm{ber}_A(VY - YV) \leq 4\widetilde{d_A}(V,W)\widetilde{d_A}(Y,W)$.

The second desired inequality follows from fact that $\widetilde{d_A}(V,W) \leq \mathrm{ber}_A(V)$ and $\widetilde{d_A}(Y,W) \leq \mathrm{ber}_A(Y)$. □

Theorem 3.21. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and let $V,Y,W \in \mathcal{L}_A(\mathcal{H})$ be such that W commutes with both V and W. Then*

$$\mathrm{ber}_A(VY - YV) \leq 2\sqrt{2}\min\left\{\widetilde{D_A}(V,W)\widetilde{d_A}(Y,W), \widetilde{D_A}(Y,W)\widetilde{d_A}(V,W)\right\}.$$

Proof. Let $\lambda_0, \xi_0 \in \mathbb{C}$ such that $\widetilde{D_A}(V,W) = \inf_{\lambda_0\in\mathbb{C}} \|V - \lambda_0 W\|_{A-\mathrm{Ber}}$ and $\widetilde{d_A}(Y,W) = \inf_{\xi_0\in\mathbb{C}} \mathrm{ber}_A(Y - \xi_0 W)$. Then, we get

$$\begin{aligned}\mathrm{ber}_A(VY - YV) &= \mathrm{ber}_A((V - \lambda_0 W)(Y - \xi_0 W) - (Y - \xi_0 W)(V - \lambda_0 W)) \\ &\leq 2\sqrt{2}\|V - \lambda_0 W\|_{A-\mathrm{Ber}}\mathrm{ber}_A(Y - \xi_0 W) \ \text{(by (24))} \\ &\leq 2\sqrt{2}\widetilde{D_A}(V,W)\widetilde{d_A}(Y,W).\end{aligned}$$

Thus, $\mathrm{ber}_A(VY - YV) \leq 2\sqrt{2}\widetilde{D_A}(V,W)\widetilde{d_A}(Y,W)$.

In the inequality above, if we replace V by Y and Y by V, we obtain

$$\mathrm{ber}_A(YV - VY) \leq 2\sqrt{2}\widetilde{D_A}(Y,W)\widetilde{d_A}(V,W).$$

Combining the above two inequalities we obtain the first inequality. Since $\widetilde{D_A}(V,W) \leq \|V\|_{A-\mathrm{Ber}}$ and $\widetilde{d_A}(Y,W) \leq \mathrm{ber}_A(Y)$, the second inequality is inferred. □

Finally, we will prove the theorem related to the A-Berezin distance.

Theorem 3.22. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ be a FHS and $V,Y \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &\leq 2\min\left\{\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right), \mathrm{ber}_A(Y)\left(\mathrm{ber}_A(V) + \widetilde{d_A}(V)\right)\right\} \\ &\leq 4\mathrm{ber}_A(V)\mathrm{ber}_A(Y).\end{aligned}$$

Proof. Let $\lambda_0, \xi_0 \in \mathbb{C}$ such that $\widetilde{d_A}(V) = \inf_{\lambda_0 \in \mathbb{C}} \mathrm{ber}_A(V - \lambda_0 I)$. If $\lambda_0 = 0$, then we have

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &\le 2\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right)\\ &= 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y).\end{aligned}$$

As in the Theorem 3.8 proof, we may take $\lambda_0 \neq 0$ and $\xi = \frac{\lambda_0}{|\lambda_0|}$ for granted. Then, we have

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &= \mathrm{ber}_A(V(\xi Y) + (\xi Y)V)\\ &\le \mathrm{ber}_A(V\mathfrak{R}_A(\xi Y) + iV\mathfrak{I}_A(\xi Y) + \mathfrak{R}_A(\xi Y)V + +i\mathfrak{I}_A(\xi Y)V)\\ &\le \mathrm{ber}_A(V\mathfrak{R}_A(\xi Y) + \mathfrak{R}_A(\xi Y)V) + \mathrm{ber}_A(V\mathfrak{I}_A(\xi Y) + \mathfrak{I}_A(\xi Y)V).\end{aligned}$$

It is simple to verify that

$$\mathfrak{R}_A^{\sharp_A}(\xi Y) = \left(\mathfrak{R}_A^{\sharp_A}\right)^{\sharp_A}(\xi Y) \text{ and } \mathfrak{I}_A^{\sharp_A}(\xi Y) = \left(\mathfrak{I}_A^{\sharp_A}\right)^{\sharp_A}(\xi Y).$$

Hence, from (5),

$$\begin{aligned}\mathrm{ber}_A(V\mathfrak{R}_A(\xi Y) + V\mathfrak{R}_A(\xi Y)) &= \mathrm{ber}_A\left(\mathfrak{R}_A^{\sharp_A}(\xi Y)V^{\sharp_A} + V^{\sharp_A}\mathfrak{R}_A^{\sharp_A}(\xi Y)\right)\\ &= \mathrm{ber}_A\left(V^{\sharp_A}\left(\mathfrak{R}_A^{\sharp_A}\right)^{\sharp_A}(\xi Y) + \mathfrak{R}_A^{\sharp_A}(\xi Y)V^{\sharp_A}\right)\\ &\le 2\left\|\mathfrak{R}_A^{\sharp_A}(\xi Y)\right\|_{A-\mathrm{Ber}}\mathrm{ber}_A\left(V^{\sharp_A}\right)\\ &\le 2\left\|\mathfrak{R}_A(\xi Y)\right\|_{A-\mathrm{Ber}}\mathrm{ber}_A(V).\end{aligned}$$

Similarly,

$$\mathrm{ber}_A(V\mathfrak{I}_A(\xi Y) + \mathfrak{I}_A(\xi Y)V) \le 2\left\|\mathfrak{I}_A(\xi Y)\right\|_{A-\mathrm{Ber}}\mathrm{ber}_A(V).$$

Therefore,

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &\le 2\mathrm{ber}_A(V)\left(\|\mathfrak{R}_A(\xi Y)\|_{A-\mathrm{Ber}} + \|\mathfrak{I}_A(\xi Y)\|_{A-\mathrm{Ber}}\right)\\ &= 2\mathrm{ber}_A(V)\left(\|\mathfrak{R}_A(\xi Y)\|_{A-\mathrm{Ber}} + \|\mathfrak{I}_A(\xi(Y - \lambda_0 I))\|_{A-\mathrm{Ber}}\right).\end{aligned}$$

Since $\|\mathfrak{R}_A(\xi Y)\|_{A-\mathrm{Ber}} \le \mathrm{ber}_A(\xi Y)$ and $\|\mathfrak{I}_A(\xi(Y - \lambda_0 I))\|_{A-\mathrm{Ber}} \le \mathrm{ber}_A(\xi(Y - \lambda_0 I))$, we have

$$\mathrm{ber}_A(VY + YV) \le 2\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \mathrm{ber}_A(\xi(Y - \lambda_0 I))\right) \le 2\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right).$$

Now, replacing V by Y and Y by V in the above inequality, we obtain

$$\mathrm{ber}_A(VY + YV) \le 2\mathrm{ber}_A(Y)\left(\mathrm{ber}_A(V) + \widetilde{d_A}(V)\right).$$

Combining the above inequalities we reach the first theorem. For second inequality, since $\widetilde{d_A}(V) \le 2\mathrm{ber}_A(V)$ and $\widetilde{d_A}(Y) \le 2\mathrm{ber}_A(Y)$, we have

$$\begin{aligned}\mathrm{ber}_A(VY + YV) &\le 2\min\left\{\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right), \mathrm{ber}_A(Y)\left(\mathrm{ber}_A(V) + \widetilde{d_A}(V)\right)\right\}\\ &\le 2\mathrm{ber}_A(V)\left(\mathrm{ber}_A(Y) + \widetilde{d_A}(Y)\right)\\ &\le 4\mathrm{ber}_A(V)\,\mathrm{ber}_A(Y).\end{aligned}$$

□

References

[1] M. L. Arias, G. Corach, M. C. Gonzalez, *Partial isometries in semi-Hilbertian spaces*, Linear Algebra Appl. **428**(7) (2008), 1460–1475.
[2] M. L. Arias, G. Corach, M. C. Gonzalez, *Lifting properties in operator ranges*, Acta Sci. Math. (Szeged) **75**(3-4) (2009), 635–657.
[3] N. Aronzjan, *Theory of reproducing kernel*, Trans. Amer. Math. Soc. **68** (1950), 337–404.
[4] M. Bakherad, *Some Berezin number inequalities for operator matrices*, Czechoslovak Math. J. **68**(2018), 997–1009.
[5] H. Başaran, M. Gürdal, *Berezin number inequalities via inequality*, Honam Math. J. **43**(3) (2021,) 523–537.
[6] H. Başaran, M. Gürdal, A. N. Güncan, *Some operator inequalities associated with Kantorovich and Hölder-McCarthy inequalities and their applications*, Turkish J. Math. **43**(1) (2019), 523–532.
[7] H. Başaran, M. Gürdal, *Some upper bounds of A-Berezin number inequalities*, International online Conference on Mathematical Advances and Applications (ICOMAA-2022), Conference Proceeding Science and Technology, **5**(1) (2022), 21–29.
[8] F. A. Berezin, *Covariant and contravariant symbols for operators*, Math. USSR-Izvestiya **6** (1972), 1117–1151.
[9] P. Bhunia, K. Feki, K. Paul, *Numerical radius parallelism and orthogonality of semi-Hilbertian space operators and its applications*, Bull. Iranian Math Soc. **47**(1) (2021), 435–457.
[10] L. de Branges, J. Rovnyak, *Square Summable Power Series*, Holt, Rinehert and Winston, New York, (1966).
[11] I. Chalendar, E. Fricain, M. Gürdal, M. Karaev, *Compactness and Berezin symbols*, Acta Sci. Math. (Sezeged) **78**(1-2) (2012), 315–329.
[12] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17**(2) (1966), 413–416.
[13] K. Feki, *Spectral radius of semi-Hilbertian space operators and its applications*, Ann Funct Anal. **11**(1) (2020), 929–946.
[14] K. Feki, *Generalized numerical radius inequalities of operators in Hilbert spaces*, Adv. Oper. Theor. **6**(1) (2020), 1–19.
[15] M. Garayev, F. Bouzeffour, M. Gürdal, C. M. Yangöz, *Refinements of Kantorovich type, Schwarz and Berezin number inequalities*, Extracta Math. **35** (2020), 1–20.
[16] M. T. Garayev, M. Gürdal, A. Okudan, *Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators*, Math. Inequal. Appl. **19** (2016), 883–891.
[17] M. T. Garayev, M. Gürdal, S. Saltan, *Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems*, Positivity **21** (2017), 1615–1623.
[18] M. T. Garayev, H. Guedri, M. Gürdal, G. M. Alsahli, *On some problems for operators on the reproducing kernel Hilbert space*, Linear Multilinear Algebra **69**(11) (2021), 2059–2077.
[19] M. Guesba, P. Bhunia, K. Paul, *A-numerical radius inequalities and A-translatable radii of semi-Hilbert space operators*, Filomat **37**(11) (2023), 3443–3456.
[20] M. Gürdal, H. Başaran, *A-Berezin number of operators*, Proc. Inst. Math. Mech. **48**(1) (2022), 77–87.
[21] V. Gürdal, H. Başaran, M. B. Huban, *Further Berezin radius inequalities*, Palestine J. Math. **12**(1) (2023), 757–767.
[22] M. Gürdal, H. Başaran, *On inequalities for A-Berezin radius of operators*, Afr. Mat., 35, 44, 2024.
[23] V. Gürdal, H. Başaran, *On Berezin radius inequalities via Cauchy-Schwarz type inequalities*, Malaya J. Mat. **11**(2) (2023), 127–141.
[24] M. B. Huban, *Upper and lower bounds of the A-Berezin number of operators*, Turkish J. Math. **46**(1) (2022), 189–206.
[25] M. B. Huban, H. Başaran, M. Gürdal, *New upper bounds related to the Berezin number inequalities*, J. Inequal. Spec. Funct. **12**(3) (2021), 1–12.
[26] M. B. Huban, H. Başaran, M. Gürdal, *Some new inequalities via Berezin numbers*, Turk. J. Math. Comput. Sci. **14**(1) (2022), 129–137.
[27] M. T. Karaev, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory **7** (2013), 983–1018.
[28] M. T. Karaev, R. Tapdigoglu, *On some problems for reproducing kernel Hilbert space operators via the Berezin transform*, Mediterr. J. Math. **19** (2022), 1–16.
[29] H. Qiao, G. Hai, E. Bai, *A-numerical radius and A-norm inequalities for semi-Hilbertian space operators*, Linear Multilinear Algebra, **70**(21) (2022), 6891–6907.
[30] Q. Xu, Z. Ye, A. Zamani, *Some upper bounds for the A-numerical radius of* 2×2 *block matrices*, Adv. Oper. Theor. **6**(1) (2021), 1–13.
[31] A. Saddi, *A-normal operators in semi-Hilbertian spaces*, Australian J. Math. Anal. Appl. **9**(1) (2012), 1–12.
[32] S. Saltan, R. Tapdigoglu, I. Calisir, *Some new relations between the Berezin number and the Berezin norm of operators*, Rocky Mt. J. Math. **52**(5) (2022), 1767–1774.
[33] T. Tapdigoglu, M. Gürdal, N. Altwaijry, N. Sarı, *Davis-Wielandt-Berezin radius inequalities via Dragomir inequalities*, Oper. Matrices **15**(4) (2021), 1445–1460.
[34] U. Yamancı, M. Gürdal, *On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space*, New York J. Math. **23** (2017), 1531-1537.
[35] U. Yamancı, İ. M. Karlı, *Further refinements of the Berezin number inequalities on operators*, Linear Multilinear Algebra **70**(20) (2022), 5237–5246.
[36] A. Zamani, *A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Algebra Appl. **578**(1) (2019), 159–183.