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On generalized forms of Hilbert's inequality

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Abstract. In this paper, we obtain the generalized form of Hilbert's inequality by using series of non-negative terms and convexity, sub-multiplicity of a function on positive real numbers and prove results for integral and discrete forms.

1. Introduction

For any two sequences (a_η) and (b_η) of non-negative real numbers, the well-known Hilbert's inequality [5] is

$$\sum_{\eta=0}^{\infty}\sum_{\theta=0}^{\infty}\frac{a_\eta b_\theta}{\theta+\eta} \leq \pi\Big(\sum_{\eta=0}^{\infty} a_\eta^2\Big)^{\frac{1}{2}}\Big(\sum_{\theta=0}^{\infty} b_\theta^2\Big)^{\frac{1}{2}}, \tag{1}$$

provided $\sum_{\eta=0}^{\infty} a_\eta^2$ and $\sum_{\eta=0}^{\infty} b_\eta^2$ are finite. The constant π in the above inequality is best possible and equality will occur if (a_η) and (b_η) both are null sequences. The extended form of above inequality is

$$\sum_{\eta=0}^{\infty}\sum_{\theta=0}^{\infty}\frac{a_\eta b_\theta}{\theta+\eta} \leq \frac{\pi}{sin\frac{\pi}{\rho}}\Big(\sum_{\eta=0}^{\infty} a_\eta^{\rho}\Big)^{\frac{1}{\rho}}\Big(\sum_{\theta=0}^{\infty} b_\theta^{\rho'}\Big)^{\frac{1}{\rho'}}, \tag{2}$$

where ρ and ρ' are two parameters such that $\rho' = \frac{\rho}{\rho-1}$, for $\rho > 1$ and $\sum_{\eta=0}^{\infty} a_\eta^{\rho}$ and $\sum_{\theta=0}^{\infty} b_\theta^{\rho'}$ are finite. The integral analogue of (1) and (2) are (see[5])

$$\int_0^\infty\int_0^\infty \frac{f(\varkappa)g(\varphi)}{\varkappa+\varphi}\,d\varkappa\,d\varphi < \pi\Big(\int_0^\infty f^2(\varkappa)d\varkappa\Big)^{\frac{1}{2}}\Big(\int_0^\infty g^2(\varphi)d\varphi\Big)^{\frac{1}{2}} \tag{3}$$

and

$$\int_0^\infty\int_0^\infty \frac{f(\varkappa)g(\varphi)}{\varkappa+\varphi}\,d\varkappa\,d\varphi < \frac{\pi}{sin(\frac{\pi}{\rho})}\Big(\int_0^\infty f^{\rho}(\varkappa)d\varkappa\Big)^{\frac{1}{\rho}}\Big(\int_0^\infty g^{\rho'}(\varphi)d\varphi\Big)^{\frac{1}{\rho'}}$$

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respectively, with the best possible constant. If we use $\varkappa = \tilde{X} + \frac{\alpha}{2}, \varphi = \tilde{Y} + \frac{\alpha}{2}, F(\tilde{X}) = f(\tilde{X} + \frac{\alpha}{2})$ and $G(\tilde{Y}) = g(\tilde{Y} + \frac{\alpha}{2}), \alpha \in \mathbf{R}$, in (3), we have

$$\int_{-\frac{\alpha}{2}}^{\infty} \int_{-\frac{\alpha}{2}}^{\infty} \frac{F(\tilde{X})G(\tilde{Y})}{\tilde{X} + \tilde{Y} + \alpha}\, d\tilde{X}\, d\tilde{Y} < \Big(\int_{-\frac{\alpha}{2}}^{\infty} F^2(\tilde{X}) d\tilde{X} \Big)^{\frac{1}{2}} \Big(\int_{-\frac{\alpha}{2}}^{\infty} G^2(\tilde{Y}) d\tilde{Y} \Big)^{\frac{1}{2}}.$$

For a non-conjugate exponent pair (ρ, ϱ), we have the following inequality (see[5])

$$\sum_{\eta=1}^{\infty} \sum_{\theta=1}^{\infty} \frac{a_\theta b_\eta}{(\theta + \eta)^\lambda} \leq K \Big(\sum_{\theta=1}^{\infty} a_\theta^\rho \Big)^{\frac{1}{\rho}} \Big(\sum_{\eta=1}^{\infty} b_\eta^\varrho \Big)^{\frac{1}{\varrho}}, \tag{4}$$

where $\rho > 1, \varrho > 1, \frac{1}{\rho} + \frac{1}{\varrho} \geq 1, 0 < \lambda = 2 - (\frac{1}{\rho} + \frac{1}{\varrho}) \leq 1$ and the constant factor $K = K(\rho, \varrho)$ is the best possible. The integral version of (4) is given by

$$\int_0^{\infty} \int_0^{\infty} \frac{f(\varkappa)g(\varphi)}{(\varkappa + \varphi)^\lambda}\, d\varkappa\, d\varphi \leq K \Big(\int_0^{\infty} f^\rho(\varkappa) d\varkappa \Big)^{\frac{1}{\rho}} \Big(\int_0^{\infty} g^\varrho(\varphi) d\varphi \Big)^{\frac{1}{\varrho}}.$$

The following inequalities, for $0 < \alpha < 1$, were given by Ingham [7] in 1936

$$\sum_{\eta=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{b_\eta b_\theta}{\theta + \eta + \alpha} \leq \pi \sum_{\theta=0}^{\infty} b_\theta^2$$

and

$$\sum_{\eta=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{b_\eta b_\theta}{\theta + \eta + \alpha} \leq \frac{\pi}{sin(\alpha\pi)} \sum_{\theta=0}^{\infty} b_\theta^2,$$

for any $\alpha \geq \frac{1}{2}$ and $0 < \alpha < \frac{1}{2}$, respectively. In 1979, Hu [6] gave a refinement of (3) as an improved Hölder's inequality

$$\int_0^{\infty} \int_0^{\infty} \frac{f(\varkappa)g(\varphi)}{\varkappa + \varphi}\, d\varkappa\, d\varphi < \pi \Big[\Big(\int_0^{\infty} f^2(\varkappa) d\varkappa \Big)^2 - \frac{1}{4} \Big(\int_0^{\infty} f^2(\varkappa) cos\sqrt{\varkappa} d\varkappa \Big)^2 \Big]^{\frac{1}{2}}.$$

The revised form of the inequality (1) has been obtained as (see[11])

$$\sum_{\eta=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{a_\eta b_\theta}{\theta + \eta + 1} \leq \pi \Big(\sum_{\eta=0}^{\infty} a_\eta^2 \Big)^{\frac{1}{2}} \Big(\sum_{\theta=0}^{\infty} b_\theta^2 \Big)^{\frac{1}{2}}. \tag{5}$$

Since for any $a_\eta, b_\theta \geq 0, \alpha \geq 1$, we have

$$\sum_{n=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{a_\eta b_\theta}{\theta + \eta + \alpha} \leq \sum_{\eta=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{a_\eta b_\theta}{\theta + \eta + 1},$$

which on using (5) yields

$$\sum_{\eta=0}^{\infty} \sum_{\theta=0}^{\infty} \frac{a_\eta b_\theta}{\theta + \eta + \alpha} \leq \pi \Big(\sum_{\eta=0}^{\infty} a_\eta^2 \Big)^{\frac{1}{2}} \Big(\sum_{\theta=0}^{\infty} b_\theta^2 \Big)^{\frac{1}{2}}. \tag{6}$$

An equivalent form of the inequality (6) is

$$\sum_{\eta=0}^{\infty} \Big(\sum_{\theta=0}^{\infty} \frac{b_\theta}{\theta + \eta + \alpha} \Big)^2 < \pi^2 \sum_{\theta=0}^{\infty} b_\theta^2,$$

where, $1 \le \alpha < 2$. An inequality similar to (3) was proved by B. G.Pachpatte [16] in 1998 as

$$\int_0^a \int_0^b \frac{f(\varkappa)g(\varphi)}{\varkappa + \varphi}\, d\varkappa\, d\varphi < \frac{\sqrt{ab}}{2}\Big[\int_0^a (a-\varkappa) f'^2(\varkappa)d\varkappa\Big]^{\frac{1}{2}}\Big[\int_0^b (b-\varphi) g'^2(\varphi)d\varphi\Big]^{\frac{1}{2}}.$$

where $a, b > 0$. More work of B. G. Pachpatte can be seen in [17].

In the literature a lot of work has been published on Hilbert's inequality as extensions, refinements and generalizations; some of which are in [1–3, 8–10, 13, 15, 18, 19]. We introduce another generalization of Hilbert's inequality. The main purpose of this paper is to describe the certain class of generalized Hilbert's inequality by introducing the series consisting of non-negative terms with the help of Jensen's inequality, Hölder's inequality involving a pair of non-conjugate exponents $\rho, \varrho \ge 0$. We prove that Pachpatte's results proved in [16] are the particular cases of our derived inequality. Furthemore, the integral and discrete forms of this inequality are give.

2. Main Results: Discrete Form

Throughout this section, we assume that $\rho > 1, \varrho > 1$ are non-conjugate exponents such that $\frac{1}{\rho} + \frac{1}{\rho'} = 1; \frac{1}{\varrho} + \frac{1}{\varrho'} = 1$ and $\{a_\theta\}, \{b_\eta\}$ are the non-negative real sequences valid for $1 \le \theta \le \kappa$ and $1 \le \eta \le \omega$, with $\kappa, \omega \in \mathbf{N}$ and $A_\theta = \sum_{\xi=1}^{\theta} a_\xi, B_\eta = \sum_{\zeta=1}^{\eta} b_\zeta$.
First, we prove the following results.

Theorem 2.1. *Let ρ, ϱ and $\{a_\theta\}, \{b_\eta\}, A_\theta, B_\eta$ be defined as above. Then, we have*

$$\begin{aligned}\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{A_\theta^{\rho}B_\eta^{\varrho}}{\theta+\eta} &\le C(\rho,\varrho,\kappa,\omega)\Big(\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)(a_\theta A_\theta^{\rho-1})^{\rho'}\Big)^{\frac{1}{\rho'}}\\ &\times \Big(\sum_{\eta=1}^{\omega}(\omega-\eta+1)(b_\eta B_\eta^{\varrho-1})^{\varrho'}\Big)^{\frac{1}{\varrho'}},\end{aligned} \tag{7}$$

provided that $\{a_\theta\}$ and $\{b_\eta\}$ are zero-sequence, where $C(\rho,\varrho,\kappa,\omega) = \frac{1}{2}\rho\varrho\kappa^{\frac{1}{\rho}}\omega^{\frac{1}{\varrho}}$ is the constant term.

Proof. From the inequality in Lemma 1 [4, 14], we have

$$\Big(\sum_{\theta=1}^{\eta} z_\theta\Big)^{\alpha} \le \alpha\sum_{\theta=1}^{\eta} z_\theta\Big(\sum_{\kappa=1}^{\eta} z_\kappa\Big)^{\alpha-1},$$

for $\alpha \ge 1$ and $z_\theta \ge 0$ $(\theta = 1, 2, ...)$, we obtain

$$A_\theta^{\rho} \le \rho\sum_{\xi=1}^{\theta} a_\xi A_\xi^{\rho-1} \qquad 1 \le \theta \le \kappa$$

and

$$B_\eta^{\varrho} \le \varrho\sum_{\zeta=1}^{\eta} b_\zeta B_\zeta^{\varrho-1} \qquad 1 \le \eta \le \omega$$

and, therefore,

$$A_\theta^{\rho}B_\eta^{\varrho} \le \rho\varrho\Big(\sum_{\xi=1}^{\theta} a_\xi A_\xi^{\rho-1}\Big)\Big(\sum_{\zeta=1}^{\eta} b_\zeta B_\zeta^{\varrho-1}\Big)$$

$$\begin{aligned} &\leq \rho\varrho\theta^{\frac{1}{\rho}}\eta^{\frac{1}{\varrho}}\Big(\sum_{\xi=1}^{\theta}(a_\xi A_\xi^{\rho-1})^{\rho'}\Big)^{\frac{1}{\rho'}}\Big(\sum_{\zeta=1}^{\eta}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big)^{\frac{1}{\varrho'}} \\ &\leq \frac{1}{2}\rho\varrho(\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}})\Big(\sum_{\xi=1}^{\theta}(a_\xi A_\xi^{\rho-1})^{\rho'}\Big)^{\frac{1}{\rho'}}\Big(\sum_{\zeta=1}^{\eta}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big)^{\frac{1}{\varrho'}}. \end{aligned} \tag{8}$$

Clearly, the second inequality in above is obtained by applying the Hölder inequality and the last inequality is the result of the inequality $(cd)^{\frac{1}{2}} \leq \frac{1}{2}(c+d)$. On dividing (8) by $\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}$ and running the summation over η and θ, we get

$$\begin{aligned} \sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{A_\theta^{\rho}B_\eta^{\varrho}}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}} &\leq \frac{1}{2}\rho\varrho\Big[\sum_{\theta=1}^{\kappa}\Big(\sum_{\xi=1}^{\theta}(a_\xi A_\xi^{\rho-1})^{\rho'}\Big)^{\frac{1}{\rho'}}\Big]\Big[\sum_{\eta=1}^{\omega}\Big(\sum_{\zeta=1}^{\eta}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big)^{\frac{1}{\varrho'}}\Big] \\ &\leq \frac{1}{2}\rho\varrho\kappa^{\frac{1}{\rho}}\omega^{\frac{1}{\varrho}}\Big[\sum_{\theta=1}^{\kappa}\Big(\sum_{\xi=1}^{\theta}(a_\xi A_\xi^{\rho-1})^{\rho'}\Big)\Big]^{\frac{1}{\rho'}}\Big[\sum_{\eta=1}^{\omega}\Big(\sum_{\zeta=1}^{\eta}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big)\Big]^{\frac{1}{\varrho'}} \\ &\leq \frac{1}{2}\rho\varrho\kappa^{\frac{1}{\rho}}\omega^{\frac{1}{\varrho}}\Big[\sum_{\xi=1}^{\kappa}(a_\xi A_\xi^{\rho-1})^{\rho'}\Big(\sum_{\theta=\xi}^{\kappa}1\Big)\Big]^{\frac{1}{\rho'}}\Big[\sum_{\zeta=1}^{\omega}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big(\sum_{\eta=\zeta}^{\omega}1\Big)\Big]^{\frac{1}{\varrho'}} \\ &= \frac{1}{2}\rho\varrho\kappa^{\frac{1}{\rho}}\omega^{\frac{1}{\varrho}}\Big[\sum_{\xi=1}^{\kappa}(a_\xi A_\xi^{\rho-1})^{\rho'}(\kappa-\xi+1)\Big]^{\frac{1}{\rho'}} \\ &\times \Big[\sum_{\zeta=1}^{\omega}(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}(\omega-\zeta+1)\Big]^{\frac{1}{\varrho'}} \\ &= C(\rho,\varrho,\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)(a_\xi A_\xi^{\rho-1})^{\rho'}\Big]^{\frac{1}{\rho'}} \\ &\times \Big[\sum_{n=1}^{\omega}(\omega-\eta+1)(b_\zeta B_\zeta^{\varrho-1})^{\varrho'}\Big]^{\frac{1}{\varrho'}}. \end{aligned}$$

In the above, the second inequality results as an application of Hölder's inequality and the third one is obtained by interchanging the order of summation [14, 15] and $C(\rho,\varrho,\kappa,\omega) = \frac{1}{2}\rho\varrho\kappa^{\frac{1}{\rho}}\omega^{\frac{1}{\varrho}}$. This completes the proof of the theorem. □

Theorem 2.2. *Let $\{a_\theta\},\{b_\eta\},A_\theta,B_\eta$ be given as in Theorem 1 and $\{\rho_\theta\},\{\varrho_\eta\}$, the positive sequences with $1 \leq \theta \leq \kappa$ and $1 \leq \eta \leq \omega$; and $P_\theta = \sum_{\xi=1}^{\theta}\rho_\xi$, $Q_\eta = \sum_{\zeta=1}^{\eta}\varrho_\zeta$. Let Φ and Υ be the non-negative, sub-multiplicative and convex functions on the set of real numbers. Then, we obtain*

$$\begin{aligned} \sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\Phi(A_\theta)\Upsilon(B_\eta)}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}} &\leq M(\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\big(\rho_\theta\Phi(a_\theta/\rho_\theta)\big)^{\rho'}\Big]^{\frac{1}{\rho'}} \\ &\times \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\big(\varrho_\eta\Upsilon(b_\eta/\varrho_\eta)\big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}, \end{aligned} \tag{9}$$

where $M(\kappa,\omega) = \frac{1}{2}\Big[\sum_{\theta=1}^{\kappa}\Big(\frac{\Phi(P_\theta)}{P_\theta}\Big)^{\rho}\Big]^{\frac{1}{\rho}}\Big[\sum_{\eta=1}^{\omega}\Big(\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big)^{\varrho}\Big]^{\frac{1}{\varrho}}$.

Proof. Using sub-multiplicity of Φ, Jensen's [12] and Hölder's inequalities, we obtain

$$\Phi(A_\theta) = \Phi\Big(\frac{P_\theta\sum_{\xi=1}^{\theta}\rho_\xi a_\xi/\rho_\xi}{\sum_{\xi=1}^{\theta}\rho_\xi}\Big)$$

$$
\begin{aligned}
&\leq \Phi(P_\theta)\Phi\Big(\frac{\sum_{\xi=1}^{\theta}\rho_\xi a_\xi/\rho_\xi}{\sum_{\xi=1}^{\theta}\rho_\xi}\Big)\\
&\leq \frac{\Phi(P_\theta)}{P_\theta}\sum_{\xi=1}^{\theta}\rho_\xi\Phi(a_\xi/\rho_\xi)\\
&\leq \theta^{\frac{1}{\rho}}\frac{\Phi(P_\theta)}{P_\theta}\Big[\sum_{\xi=1}^{\theta}\Big(\rho_\xi\Phi(a_\xi/\rho_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}. \qquad (10)
\end{aligned}
$$

Similarly,

$$
\Upsilon(B_\eta)\leq \eta^{\frac{1}{\varrho}}\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big[\sum_{\zeta=1}^{\eta}\Big(\varrho_\zeta\Upsilon(b_\zeta/\varrho_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}. \qquad (11)
$$

From (10) and (11) and using the inequality $c^{\frac{1}{2}}d^{\frac{1}{2}}\leq\frac{c+d}{2}$, we derive

$$
\begin{aligned}
\Phi(A_\theta)\Upsilon(B_\eta) \leq\ & \frac{(\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}})}{2}\frac{\Phi(P_\theta)}{P_\theta}\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big[\sum_{\xi=1}^{\theta}\Big(\rho_\xi\Phi(a_\xi/\rho_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}\\
&\times\Big[\sum_{\zeta=1}^{\eta}\Big(\varrho_\zeta\Upsilon(b_\zeta/\varrho_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}. \qquad (12)
\end{aligned}
$$

On dividing (12) by $(\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}})$ and summing over η from 1 to ω and θ from 1 to κ and then using Hölder's inequality, the following is obtained

$$
\begin{aligned}
\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\Phi(A_\theta)\Upsilon(B_\eta)}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}} \leq\ & \frac{1}{2}\Big[\sum_{\theta=1}^{\kappa}\frac{\Phi(P_\theta)}{P_\theta}\Big\{\sum_{\xi=1}^{\theta}\Big(\rho_\xi\Phi(a_\xi/\rho_\xi)\Big)^{\rho'}\Big\}^{\frac{1}{\rho'}}\Big]\\
&\times\Big[\sum_{\eta=1}^{\omega}\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big\{\sum_{\zeta=1}^{\eta}\Big(\varrho_\zeta\Upsilon(b_\zeta/\varrho_\zeta)\Big)^{\varrho'}\Big\}^{\frac{1}{\varrho'}}\Big]\\
\leq\ & \frac{1}{2}\Big[\sum_{\theta=1}^{\kappa}\Big(\frac{\Phi(P_\theta)}{P_\theta}\Big)^{p}\Big]^{\frac{1}{\rho}}\Big[\sum_{\eta=1}^{\omega}\Big(\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big)^{\varrho}\Big]^{\frac{1}{\varrho}}\\
&\times\Big[\sum_{\theta=1}^{\kappa}\sum_{\xi=1}^{\theta}\Big(\rho_\xi\Phi(a_\xi/\rho_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}\Big[\sum_{\eta=1}^{\omega}\sum_{\zeta=1}^{\eta}\Big(\varrho_\zeta\Upsilon(b_\zeta/\varrho_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}\\
\leq\ & M(\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\theta\Phi(a_\theta/\rho_\theta)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}\\
&\times\Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\eta\Upsilon(b_\eta/\varrho_\eta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}},
\end{aligned}
$$

where the last inequality is obtained by interchanging the order of the summations. This completes the proof. □

Theorem 2.3. *Let $\{a_\theta\},\{b_\eta\}$ be given as in Theorem 2.1 and Φ and Υ, the functions defined as in Theorem 2.2. If $A_\theta=\frac{1}{\theta}\sum_{\xi=1}^{\theta}a_\xi$ and $B_\eta=\frac{1}{\eta}\sum_{\zeta=1}^{\eta}b_\zeta$ with $1\leq\theta\leq\kappa$ and $1\leq\eta\leq\omega$, then*

$$
\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\theta\eta}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}}\Phi(A_\theta)\Upsilon(B_\eta) \leq C(1,1,\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\Phi(a_\theta)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}
$$

$$\times \quad \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\Upsilon(b_\eta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}, \tag{13}$$

where $C(1,1,\kappa,\omega)$ *is the constant obtained by putting* $\rho=\varrho=1$ *in* $C(\rho,\varrho,\kappa,r)$ *of Theorem* 1.

Proof. Making use of Jensen's and Hölder's inequalities, we arrive at

$$\Phi(A_\theta) \le \frac{1}{\theta}\theta^{\frac{1}{\rho}}\Big[\sum_{\xi=1}^{\theta}\Big(\Phi(a_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}$$

and

$$\Upsilon(B_\eta) \le \frac{1}{\eta}\eta^{\frac{1}{\varrho}}\Big[\sum_{\zeta=1}^{\eta}\Big(\Upsilon(b_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}.$$

The rest of the proof follows by mimicing the proofs of Theorems 2.1, 2.2. □

Theorem 2.4. *Let* $\{a_\theta\},\{b_\eta\},\{\rho_\theta\},\{\varrho_\eta\},P_\theta,Q_\eta$ *be the same as in Theorem* 2.2 *and* Φ,Υ *defined as in Theorem* 2.3. *If* $A_\theta=\frac{1}{P_\theta}\sum_{\xi=1}^{\theta}\rho_\xi a_\xi$ *and* $B_\eta=\frac{1}{Q_\eta}\sum_{\zeta=1}^{\eta}\varrho_\zeta b_\zeta$ *with* $1\le\theta\le\kappa$ *and* $1\le\eta\le\omega$*, then*

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{P_\theta Q_\eta}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}}\Phi(A_\theta)\Upsilon(B_\eta) \quad\le\quad C(1,1,\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\xi\Phi(a_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}}$$

$$\times \quad \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\zeta\Upsilon(b_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}, \tag{14}$$

where $C(1,1,\kappa,\omega)$ *is same as defined before.*

Proof. By using Jensen's and Hölder's inequalities, we obtain

$$\Phi(A_\theta) \le \frac{1}{P_\theta}\theta^{\frac{1}{\rho}}\Big[\sum_{\xi=1}^{\theta}\Big(\rho_\xi\Phi(a_\xi)\Big)^{\rho'}\Big]^{\frac{1}{\rho'}},$$

and

$$\Upsilon(B_\eta) \le \frac{1}{Q_\eta}\eta^{\frac{1}{\varrho}}\Big[\sum_{\zeta=1}^{\eta}\Big(\varrho_\zeta\Upsilon(b_\zeta)\Big)^{\varrho'}\Big]^{\frac{1}{\varrho'}}.$$

The remaining proof follows from the proofs of Theorems 2.1, 2.2. □

3. Useful Remarks

Putting $\rho=\varrho=2$, in Theorems 2.1 – 2.4, the following inequalities are obtained

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{A_\theta^2B_\eta^2}{\theta+\eta} \le 2\sqrt{\kappa\omega}\Big(\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)(a_\theta A_\theta)^2\Big)^{\frac{1}{2}}\Big(\sum_{\eta=1}^{\omega}(\omega-\eta+1)(b_\eta B_\eta)^2\Big)^{\frac{1}{2}}, \tag{15}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\Phi(A_\theta)\Upsilon(B_\eta)}{\theta+\eta} \quad\le\quad \frac{1}{2}\Big(\sum_{\theta=1}^{\kappa}\Big(\frac{\Phi(P_\theta)}{P_\theta}\Big)^2\Big)^{\frac{1}{2}}\Big(\sum_{\eta=1}^{\omega}\Big(\frac{\Upsilon(Q_\eta)}{Q_\eta}\Big)^2\Big)^{\frac{1}{2}}$$

$$\times \quad \Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\theta\Phi(a_\theta/\rho_\theta)\Big)^2\Big]^{\frac{1}{2}}$$

$$\times \quad \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\eta\Upsilon(b_\eta/\varrho_\eta)\Big)^2\Big]^{\frac{1}{2}}, \tag{16}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\theta\eta}{\theta+\eta}\Phi(A_\theta)\Upsilon(B_\eta) \leq C(1,1,\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\Phi(a_\theta)\Big)^2\Big]^{\frac{1}{2}} \times \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\Upsilon(b_\eta)\Big)^2\Big]^{\frac{1}{2}}, \tag{17}$$

with $C(1,1,\kappa,\omega)=\frac{\kappa\omega}{2}$.

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{P_\theta Q_\eta}{\theta+\eta}\Phi(A_\theta)\Upsilon(B_\eta) \leq C(1,1,\kappa,\omega)\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\xi\Phi(a_\xi)\Big)^2\Big]^{\frac{1}{2}} \times \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\zeta\Upsilon(b_\zeta)\Big)^2\Big]^{\frac{1}{2}}. \tag{18}$$

The inequality (15) is a Hilbert's type inequality and (16) – (18) are results of Pachpatte.

4. Generalized Discrete Form

Using $c^{\frac{1}{2}}d^{\frac{1}{2}} \leq \frac{c+d}{2}$, the inequalities (7), (9), (13) and (14), respectively, take the following forms.

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{A_\theta^{\rho}B_\eta^{\varrho}}{\theta+\eta} \leq \frac{1}{2}C(\rho,\varrho,\kappa,\omega)\Big[\Big(\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)(a_\theta A_\theta^{\rho-1})^{\rho'}\Big)^{\frac{2}{\rho'}} + \Big(\sum_{\eta=1}^{\omega}(\omega-\eta+1)(b_\eta B_\eta^{\varrho-1})^{\varrho'}\Big)^{\frac{2}{\varrho'}}\Big], \tag{19}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\Phi(A_\theta)\Upsilon(B_\eta)}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}} \leq \frac{1}{2}M(\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\theta\Phi(a_\theta/\rho_\theta)\Big)^{\rho'}\Big]^{\frac{2}{\rho'}} + \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\eta\Upsilon(b_\eta/\varrho_\eta)\Big)^{\varrho'}\Big]^{\frac{2}{\varrho'}}\Big\}, \tag{20}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\theta\eta}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}}\Phi(A_\theta)\Upsilon(B_\eta) \leq \frac{1}{2}C(1,1,\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\Phi(a_\theta)\Big)^{\rho'}\Big]^{\frac{2}{\rho'}} + \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\Upsilon(b_\eta)\Big)^{\varrho'}\Big]^{\frac{2}{\varrho'}}\Big\}, \tag{21}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{P_\theta Q_\eta}{\theta^{\frac{2}{\rho}}+\eta^{\frac{2}{\varrho}}}\Phi(A_\theta)\Upsilon(B_\eta) \leq \frac{1}{2}C(1,1,\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\xi\Phi(a_\xi)\Big)^{\rho'}\Big]^{\frac{2}{\rho'}}$$

$$+ \quad \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\zeta \Upsilon(b_\zeta)\Big)^{\varrho'}\Big]^{\frac{2}{\varrho'}}\Big\}. \tag{22}$$

The inequalities (19) – (22) are revised forms of our results. Moreover, for $\rho = \varrho = 2$, we obtain the following inequalities.

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{A_\theta^2 B_\eta^2}{\theta+\eta} \leq \frac{1}{2}C(2,2,\kappa,\omega)\Big[\Big(\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)(a_\theta A_\theta)^2\Big) + \Big(\sum_{\eta=1}^{\omega}(\omega-\eta+1)(b_\eta B_\eta)^2\Big)\Big], \tag{23}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\Phi(A_\theta)\Upsilon(B_\eta)}{\theta+\eta} \leq \frac{1}{2}M(\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\theta\Phi(a_\theta/\rho_\theta)\Big)^2\Big] + \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\eta\Upsilon(b_\eta/\varrho_\eta)\Big)^2\Big]\Big\}, \tag{24}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{\theta\eta}{\theta+\eta}\Phi(A_\theta)\Upsilon(B_\eta) \leq \frac{1}{2}C(1,1,\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\Phi(a_\theta)\Big)^2\Big] + \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\Upsilon(b_\eta)\Big)^2\Big]\Big\}, \tag{25}$$

$$\sum_{\theta=1}^{\kappa}\sum_{\eta=1}^{\omega}\frac{P_\theta Q_\eta}{\theta+\eta}\Phi(A_\theta)\Upsilon(B_\eta) \leq \frac{1}{2}C(1,1,\kappa,\omega)\Big\{\Big[\sum_{\theta=1}^{\kappa}(\kappa-\theta+1)\Big(\rho_\xi\Phi(a_\xi)\Big)^2\Big] + \Big[\sum_{\eta=1}^{\omega}(\omega-\eta+1)\Big(\varrho_\zeta\Upsilon(b_\zeta)\Big)^2\Big]\Big\}. \tag{26}$$

5. Main Results: Integral Form

In this section, we present integral analogues of our results proved in Theorems 2.1 – 2.4. We prove

Theorem 5.1. *Let $\rho > 1, \varrho > 1$ and $f(u) \geq 0, g(v) \geq 0$ and $0 < u < \varkappa, 0 < v < \varphi$ with $0 < \varkappa, \varphi < \infty$ and define $F(\xi) = \int_0^\xi f(u)du, G(\zeta) = \int_0^\zeta g(v)dv$, where $0 < \xi < \varkappa, 0 < \zeta < \varphi$. Then*

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{F^\rho(\xi)G^\varrho(\zeta)}{\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}}\,d\zeta\,d\xi \leq \frac{1}{2}D(\rho,\varrho,\varkappa,\varphi)\Big[\Big(\int_0^{\varkappa}(\varkappa-\xi)\Big(F^{\rho-1}(\xi)f(\xi)\Big)^{\rho'}d\xi\Big)^{\frac{2}{\rho'}} + \Big(\int_0^{\varphi}(\varphi-\zeta)\Big(G^{\varrho-1}(\zeta)g(\zeta)\Big)^{\varrho'}d\zeta\Big)^{\frac{2}{\varrho'}}\Big], \tag{27}$$

unless f or g is identically zero and $D(\rho,\varrho,\varkappa,\varphi) = \frac{1}{2}\rho\varrho\varkappa^{\frac{1}{\rho}}\varphi^{\frac{1}{\varrho}}$.

Proof. By the hypothesis, it is easily seen that

$$F^{\rho}(\xi) = \rho \int_0^{\xi} F^{\rho-1}(u) f(u) du, \qquad \xi \in (0, \varkappa), \tag{28}$$

and

$$G^{\varrho}(\zeta) = \varrho \int_0^{\zeta} G^{\varrho-1}(v) g(v) dv, \qquad \zeta \in (0, \varphi). \tag{29}$$

From (28), (29) and using Hölder's inequality and the inequality $c^{\frac{1}{2}} d^{\frac{1}{2}} \leq \frac{c+d}{2}$, we obtain

$$\begin{aligned} F^{\rho}(\xi) G^{\varrho}(\zeta) &\leq \frac{\rho\varrho}{2}(\xi^{\frac{2}{\rho}} + \zeta^{\frac{2}{\varrho}}) \Big(\int_0^{\xi} \Big(F^{\rho-1}(u) f(u) \Big)^{\rho'} du \Big)^{\frac{1}{\rho'}} \\ &\times \Big(\int_0^{\zeta} \Big(G^{\varrho-1}(v) g(v) \Big)^{\varrho'} dv \Big)^{\frac{1}{\varrho'}}. \end{aligned} \tag{30}$$

Divide (30) by $\xi^{\frac{2}{\rho}} + \zeta^{\frac{2}{\varrho}}$ and integrate over ζ from 0 to φ and then integrate over ξ from 0 to $\varkappa$, one obtains

$$\begin{aligned} \int_0^{\varkappa} \int_0^{\varphi} \frac{F^{\rho}(\xi) G^{\varrho}(\zeta)}{\xi^{\frac{2}{\rho}} + \zeta^{\frac{2}{\varrho}}} d\xi\, d\zeta &\leq \frac{\rho\varrho}{2} \Big\{ \int_0^{\varkappa} \Big(\int_0^{\xi} \Big(F^{\rho-1}(u) f(u) \Big)^{\rho'} du \Big)^{\frac{1}{\rho'}} d\xi \Big\} \\ &\times \Big\{ \int_0^{\varphi} \Big(\int_0^{\zeta} \Big(G^{\varrho-1}(v) g(v) \Big)^{\varrho'} dv \Big)^{\frac{1}{\varrho'}} d\zeta \Big\} \\ &\leq \frac{\rho\varrho}{2} \varkappa^{\frac{1}{\rho}} \varphi^{\frac{1}{\varrho}} \Big\{ \int_0^{\varkappa} \Big(\int_0^{\xi} \Big(F^{\rho-1}(u) f(u) \Big)^{\rho'} du \Big) d\xi \Big\}^{\frac{1}{\rho'}} \\ &\times \Big\{ \int_0^{\varphi} \Big(\int_0^{\zeta} \Big(G^{\varrho-1}(v) g(v) \Big)^{\varrho'} dv \Big) d\zeta \Big\}^{\frac{1}{\varrho'}} \\ &= D(\rho, \varrho, \varkappa, \varphi) \Big(\int_0^{\varkappa} (\varkappa - \xi) \Big(F^{\rho-1}(\xi) f(\xi) \Big)^{\rho'} d\xi \Big)^{\frac{1}{\rho'}} \\ &\times \Big(\int_0^{\varphi} (\varphi - \zeta) \Big(G^{\varrho-1}(\zeta) g(\zeta) \Big)^{\varrho'} d\zeta \Big)^{\frac{1}{\varrho'}} \\ &\leq \frac{1}{2} D(\rho, \varrho, \varkappa, \varphi) \Big[\Big(\int_0^{\varkappa} (\varkappa - \xi) \Big(F^{\rho-1}(\xi) f(\xi) \Big)^{\rho'} d\xi \Big)^{\frac{2}{\rho'}} \\ &+ \Big(\int_0^{\varphi} (\varphi - \zeta) \Big(G^{\varrho-1}(\zeta) g(\zeta) \Big)^{\varrho'} d\zeta \Big)^{\frac{2}{\varrho'}} \Big], \end{aligned}$$

where second inequality is achieved by applying Hölder's inequality and the last inequality is the result of the inequality $c^{\frac{1}{2}} d^{\frac{1}{2}} \leq \frac{c+d}{2}$. This completes the proof of the theorem. □

Theorem 5.2. *Let us consider f, g, F, G defined as in Theorem 5.1. Let $\rho(u)$ and $\varrho(v)$ be positive functions with $0 < u < \varkappa, 0 < v < \varphi$ and define $P(\xi) = \int_0^{\xi} \rho(u) du$ and $Q(\zeta) = \int_0^{\zeta} \varrho(v) dv$ with $0 < \xi < \varkappa, 0 < \zeta < \varphi$ and $\varkappa, \varphi \in \mathbf{R}_+$. Let Φ and Υ be the same as in Theorem 2.2. Then*

$$\begin{aligned} \int_0^{\varkappa} \int_0^{\varphi} \frac{\Phi(F(\xi)) \Upsilon(G(\zeta))}{\xi^{\frac{2}{\rho}} + \zeta^{\frac{2}{\varrho}}} d\xi\, d\zeta &\leq \frac{1}{2} L(\varkappa, \varphi) \Big\{ \Big[\int_0^{\varkappa} (\varkappa - \xi) \Big(\rho(\xi) \Phi\Big(\frac{f(\xi)}{\rho(\xi)} \Big) \Big)^{\rho'} d\xi \Big]^{\frac{2}{\rho'}} \\ &+ \Big[\int_0^{\varphi} (\varphi - \zeta) \Big(\varrho(\zeta) \Upsilon\Big(\frac{g(\zeta)}{\varrho(\zeta)} \Big) \Big)^{\varrho'} d\zeta \Big]^{\frac{2}{\varrho'}} \Big\}, \end{aligned} \tag{31}$$

where $L(\varkappa, \varphi) = \frac{1}{2} \Big(\int_0^{\varkappa} \Big(\frac{\Phi(P(\xi))}{P(\xi)} \Big)^{\rho} d\xi \Big)^{\frac{1}{\rho}} \Big(\int_0^{\varphi} \Big(\frac{\Upsilon(Q(\zeta))}{Q(\zeta)} \Big)^{\varrho} d\zeta \Big)^{\frac{1}{\varrho}}$.

Proof. From the hypothesis, we get

$$\Phi(F(\xi)) = \Phi\Big(\frac{P(\xi)\int_0^{\xi}\rho(u)\frac{f(u)}{\rho(u)}du}{\int_0^{\xi}\rho(u)du}\Big).$$

Making use of sub-multiplicity of Φ and Jensen's and Hölder's inequalities, the following is attained

$$\begin{aligned}\Phi(F(\xi)) &\leq \frac{\Phi(P(\xi))}{P(\xi)}\int_0^{\xi}\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)du\\ &\leq \xi^{\frac{1}{\rho}}\frac{\Phi(P(\xi))}{P(\xi)}\Big\{\int_0^{\xi}\Big(\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)\Big)^{\rho'}du\Big\}^{\frac{1}{\rho'}}. \qquad (32)\end{aligned}$$

Similarly, we find

$$\Upsilon(G(\zeta)) \leq \zeta^{\frac{1}{\varrho}}\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big\{\int_0^{\zeta}\Big(\varrho(v)\Upsilon\Big(\frac{g(v)}{\varrho(v)}\Big)\Big)^{\varrho'}dv\Big\}^{\frac{1}{\varrho'}}. \qquad (33)$$

From (32) and (33), we have

$$\begin{aligned}\Phi(F(\xi))\Upsilon(G(\zeta)) &\leq \xi^{\frac{1}{\rho}}\zeta^{\frac{1}{\varrho}}\frac{\Phi(P(\xi))}{P(\xi)}\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big\{\int_0^{\xi}\Big(\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)\Big)^{\rho'}du\Big\}^{\frac{1}{\rho'}}\\ &\times \Big\{\int_0^{\zeta}\Big(\varrho(v)\Upsilon\Big(\frac{g(v)}{\varrho(v)}\Big)\Big)^{\varrho'}dv\Big\}^{\frac{1}{\varrho'}}\\ &\leq \Big(\frac{\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}}{2}\Big)\Big(\frac{\Phi(P(\xi))}{P(\xi)}\Big)\Big(\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big)\Big\{\int_0^{\xi}\Big(\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)\Big)^{\rho'}du\Big\}^{\frac{1}{\rho'}}\\ &\times \Big\{\int_0^{\zeta}\Big(\varrho(v)\Upsilon\Big(\frac{g(v)}{\varrho(v)}\Big)\Big)^{\varrho'}dv\Big\}^{\frac{1}{\varrho'}}. \qquad (34)\end{aligned}$$

Divide (34) by $\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}$ and integrate over ζ from 0 to φ and then ξ from 0 to $\varkappa$ and using Hölder's inequality, we obtain

$$\begin{aligned}\int_0^{\varkappa}\int_0^{\varphi}\frac{\Phi(F(\xi))\Upsilon(G(\zeta))}{\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}}\,d\zeta\,d\xi &\leq \frac{1}{2}\Big[\int_0^{\varkappa}\Big(\frac{\Phi(P(\xi))}{P(\xi)}\Big)\Big\{\int_0^{\xi}\Big(\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)\Big)^{\rho'}du\Big\}^{\frac{1}{\rho'}}d\xi\Big]\\ &\times \Big[\int_0^{\varphi}\Big(\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big)\Big\{\int_0^{\zeta}\Big(\varrho(v)\Upsilon\Big(\frac{g(v)}{\varrho(v)}\Big)\Big)^{\varrho'}dv\Big\}^{\frac{1}{\varrho'}}d\zeta\Big]\\ &\leq \frac{1}{2}\Big[\int_0^{\varkappa}\Big(\frac{\Phi(P(\xi))}{P(\xi)}\Big)^{\rho}d\xi\Big]^{\frac{1}{\rho}}\Big[\int_0^{\varphi}\Big(\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big)^{\varrho}d\zeta\Big]^{\frac{1}{\varrho}}\\ &\times \Big[\int_0^{\varkappa}\int_0^{\xi}\Big(\rho(u)\Phi\Big(\frac{f(u)}{\rho(u)}\Big)\Big)^{\rho'}du\,d\xi\Big]^{\frac{1}{\rho'}}\\ &\times \Big[\int_0^{\varphi}\int_0^{\zeta}\Big(\varrho(v)\Upsilon\Big(\frac{g(v)}{\varrho(v)}\Big)\Big)^{\varrho'}dv\,d\zeta\Big]^{\frac{1}{\varrho'}}\\ &= L(\varkappa,\varphi)\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\rho(\xi)\Phi\Big(\frac{f(\xi)}{\rho(\xi)}\Big)\Big)^{\rho'}d\xi\Big]^{\frac{1}{\rho'}}\\ &\times \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\varrho(\zeta)\Upsilon\Big(\frac{g(\zeta)}{\varrho(\zeta)}\Big)\Big)^{\varrho'}d\zeta\Big]^{\frac{1}{\varrho'}}\\ &\leq \frac{1}{2}L(\varkappa,\varphi)\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\rho(\xi)\Phi\Big(\frac{f(\xi)}{\rho(\xi)}\Big)\Big)^{\rho'}d\xi\Big]^{\frac{2}{\rho'}}\end{aligned}$$

$$+ \quad \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\varrho(\zeta)\Upsilon\Big(\frac{g(\zeta)}{\varrho(\zeta)}\Big)\Big)^{\varrho'}d\zeta\Big]^{\frac{2}{\varrho'}}.$$

Hence, the theorem is proved. □

Theorem 5.3. *Let us consider f, g as in Theorem 5.1 and Φ, Υ as in Theorem 2.2. Let $F(\xi) = \frac{1}{\xi}\int_0^{\xi} f(u)du$ and $G(\zeta) = \frac{1}{\zeta}\int_0^{\zeta} g(v)dv$ with $0 < \xi < \varkappa, 0 < \zeta < \varphi$ for the positive real numbers $\varkappa, \varphi$. Then*

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{\xi\zeta}{\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}}\Phi(f(\xi))\Upsilon(G(\zeta))\,d\xi\,d\zeta \le \frac{1}{4}\varkappa^{\frac{1}{\rho}}\varphi^{\frac{1}{\varrho}}\Big\{\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\Phi(f(\xi))\Big)^{\rho'}d\xi\Big]^{\frac{2}{\rho'}} + \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\Upsilon(g(\zeta))\Big)^{\varrho'}d\zeta\Big]^{\frac{2}{\varrho'}}\Big\}. \tag{35}$$

Proof. The proof runs similar as the proof of Theorem 5.2. □

Theorem 5.4. *Let $f, g, \rho, \varrho, P, Q$ be the same as in Theorem 5.2 and Φ, Υ as in Theorem 2.2. Let $F(\xi) = \frac{1}{P(\xi)}\int_0^{\xi}\rho(u)f(u)du$ and $G(\zeta) = \frac{1}{Q(\zeta)}\int_0^{\zeta}\varrho(v)g(v)dv$ with $\xi \in (0, \varkappa)$, and $\zeta \in (0, < \varphi)$, for the positive real numbers $\varkappa, \varphi$. Then*

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{P(\xi)Q(\zeta)\Phi(F(\xi))\Upsilon(G(\zeta))}{\xi^{\frac{2}{\rho}}+\zeta^{\frac{2}{\varrho}}}d\xi d\zeta \le \frac{1}{4}\varkappa^{\frac{1}{\rho}}\varphi^{\frac{1}{\varrho}}\Big\{\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\rho(\xi)\Phi(f(\xi))\Big)^{\rho'}d\xi\Big]^{\frac{2}{\rho'}} + \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\varrho(\zeta)\Upsilon(g(\zeta))\Big)^{\varrho'}d\zeta\Big]^{\frac{2}{\varrho'}}\Big\}. \tag{36}$$

Proof. The proof foloows in a similar way as of Theorem 5.2. □

6. Conclusions

On considering $\rho = \varrho = 2$, inequalities (27), (31), (35) and (36) take the forms

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{F^2(\xi)G^2(\zeta)}{\xi+\zeta}d\zeta d\xi \le \frac{1}{2}D(2,2,\varkappa,\varphi)\Big[\Big(\int_0^{\varkappa}(\varkappa-\xi)\Big(F(\xi)f(\xi)\Big)^2d\xi\Big) + \Big(\int_0^{\varphi}(\varphi-\zeta)\Big(G(\zeta)g(\zeta)\Big)^2d\zeta\Big)\Big], \tag{37}$$

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{\Phi(F(\xi))\Upsilon(G(\zeta))}{\xi+\zeta} \le \frac{1}{2}L'(\varkappa,\varphi)\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\rho(\xi)\Phi\Big(\frac{f(\xi)}{\rho(\xi)}\Big)\Big)^2d\xi\Big] + \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\varrho(\zeta)\Upsilon\Big(\frac{g(\zeta)}{\varrho(\zeta)}\Big)\Big)^2d\zeta\Big], \tag{38}$$

where $L'(\varkappa,\varphi) = \frac{1}{2}\Big(\int_0^{\varkappa}\Big(\frac{\Phi(P(\xi))}{P(\xi)}\Big)^2d\xi\Big)^{\frac{1}{2}}\Big(\int_0^{\varphi}\Big(\frac{\Upsilon(Q(\zeta))}{Q(\zeta)}\Big)^2d\zeta\Big)^{\frac{1}{2}}$,

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{\xi\zeta}{\xi+\zeta}\Phi(f(\xi))\Upsilon(G(\zeta))d\xi d\zeta \le \frac{1}{4}\varkappa^{\frac{1}{2}}\varphi^{\frac{1}{2}}\Big\{\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\Phi(f(\xi))\Big)^2d\xi\Big] + \Big[\int_0^{\varphi}(\varphi-\zeta)\Big(\Upsilon(g(\zeta))\Big)^2\Big]\Big\}, \tag{39}$$

$$\int_0^{\varkappa}\int_0^{\varphi}\frac{P(\xi)Q(\zeta)\Phi(F(\xi))\Upsilon(G(\zeta))}{\xi+\zeta}d\xi d\zeta \le \frac{1}{4}\varkappa^{\frac{1}{2}}\varphi^{\frac{1}{2}}\Big\{\Big[\int_0^{\varkappa}(\varkappa-\xi)\Big(\rho(\xi)\Phi(f(\xi))\Big)^2\Big]$$

$$+ \quad \left[\int_0^{\varphi} (\varphi - \zeta)\Big(\varrho(\zeta)\Upsilon(g(\zeta))\Big)^2\right]\Bigg\} \tag{40}$$

respectively. Clearly Pachpatte's main results [16] are the special cases of our derived inequalities (37)-(40).

References

[1] L. E. Azar, *Two new form of Hilbert integral inequality*, J. Math. Inequal. Appl. **17(3)** (2014), 937–946.
[2] T. Batbold, L. E. Azar *A new form of Hilbert integral inequality*, J. Math. Inequal. **12(2)** (2018), 379–390.
[3] N. Das, S. Sahoo, *New inequalities similar to Hardy-Hilbert's inequality*, Turk. J. Math. **34(2)** (2010), 153–165.
[4] G. S. Davies, G. M. Peterson, *On an inequality of Hardy's(II)*, Quart. J. Math (Oxford) **15** (1964), 35–40.
[5] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, U. K., 1934.
[6] K. Hu, *A few important inequalities*, J. Jianxi Teach. Coll. Nat. Sci. Math. **3(1)** (1979), 1–4.
[7] A. E. Ingham, *A note on Hilbert's inequality*, J. Lond. Math. Soc. **11** (1936), 237–240.
[8] J. Jin, L. Debnath, *On a Hilbert-type linear series operator and its applications*, J. Math. Anal. Appl. **371(2)** (2010), 691–704.
[9] S. Kaushik, S. Kumar, *Certain form of Hilbert-type inequality using non-homogeneous kernel of hyperbolic functions*, Korean J. Math. **31(2)** (2023), 189–201.
[10] Q. Liu, *A Hilbert-type integral inequality under configuring free power and its applications*, J. Inequal. Appl. **2019(91)** (2019).
[11] G. Mingzhe, *On Hilbert's inequality and its applications*, J. Math. Anal. Appl. **212** (1997), 316–323.
[12] D. S. Mitrinovic, *Analytic Inequalities*, Springer, Berlin, New York, 1970.
[13] H. Mo, B. Yang, *On a new Hilbert-type integral inequality involving the upper limit functions*, J. Inequal. Appl. **2020(5)** (2020).
[14] J. Nemeth, *Generalizations of the Hardy-Littlewood inequality*, Acta. Sci. Math. (Szeged) **32** (1971), 295–299.
[15] B. G. Pachpatte, *A note on some series inequalities*, Tamkang. J. Math. **27** (1996), 77–79.
[16] B. G. Pachpatte, *On some new inequalities similar to Hilbert's inequality*, J. Math. Anal. Appl. **226** (1998), 166–179.
[17] B. G. Pachpatte, *Mathematical Inequalities*, Elsevier, Amsterdam, 2005.
[18] P. Vukovic, *Note on Hilbert-type inequalities*, Turk. J. Math. **36(2)** (2012), 253–262.
[19] M. You, Y. Guan, *On a Hilbert-type integral inequality with non-homogeneous kernel of mixed hyperbolic function*, J. Math. Inequal. **13(4)** (2019), 1197–1208.