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A study on statistically localized sequences in A-metric spaces

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Abstract. Recently, important results have been obtained by applying the basic concepts of summability theory to generalized metric spaces. The main motivation of this work is to understand the behavior of statistically localized sequences in A-metric spaces, which are a generalization of usual metric spaces. In this study we first define the concept of statistically localized sequences in A-metric spaces and explore some of their basic properties. Then, we examine the connections between statistically localized sequences and statistically Cauchy sequences. We show that a sequence is statistically Cauchy if and only if its statistical barrier is equal to zero. Furthermore, we define uniformly statistically localized sequences on A-metric spaces.

1. Introduction

The concept of metric spaces, which is foundational to various fields including mathematics and engineering, and is based on the notion of a distance function, was first introduced by Fréchet [11] in 1906. As the need to handle larger and more complex datasets grew, the study of generalizations of metric spaces gained momentum. Gähler [14] proposed the concept of 2-metric space as a generalization of the usual metric space in 1963. However, Ha et al.[15] showed in 1988 that the 2-metric function does not need to be a continuous function on its variables, thus refuting his claim. As a generalization of metric spaces, Dhage [7] introduced the concept of D-metric space and studied its topological properties in 1992. Here, the different uses of the convergence concept in D-metric space resulted in two different topologies. Although Dhage claims that these topologies are the same, it has been shown in [19] that this claim is not true. Likewise, D-metric spaces were shown to be not a generalization of the usual metric spaces. Subsequently, Mustafa and Sims [20] proposed G-metric spaces as a generalization of the usual metric spaces. For further reference, see [17]. In 2012, Sedghi et al. [25] introduced the S-metric space as a further generalization of metric spaces, and Abbas et al. [1] extended it to A-metric spaces in 2015.

The concept of statistical convergence was introduced in 1951 by Fast [9] and Steinhaus [26]. Afterward, Schoenberg [24] reintroduced it in 1959. Since then, the properties of statistical convergence have been studied by different mathematicians and applied in several areas such as summability theory, measurement theory, probability theory, number theory, optimization theory, (see, [2, 4–6, 8, 10, 12, 13, 16, 18, 21–23]). Statistical convergence is a type of convergence that includes the usual convergence based on the concept of natural density of a subset K of the set of natural numbers $\mathbb{N}$. Let's recall the definitions of natural density, and statistical convergence.

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Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. The natural density of K is the limit $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K_n|$ if it exists. A sequence (x_k) is said to be statistically convergent to x if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0.$$

A sequence (x_k) is said to be a statistically Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer $t = t(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - x_t| \geq \varepsilon\}| = 0,$$

(see [10, 12]).

In 1974, Krivonosov [28] introduced the concept of a localized sequence in metric spaces, as a generalization of a Cauchy sequence. Krivonosov [28] also obtained many results using localized and locator properties of sequences, and studied the closure operators of accounts in metric spaces. Let X be a metric space with a metric $d(\cdot,\cdot)$ and let (x_n) be a sequence of points in X. If the real number sequence $\alpha_n = d(x_n, x)$ converges for all $x \in M \subset X$ then the sequence (x_n) is said to be a localized sequence on the subset M. The maximal subset on which (x_n) is a localized sequence is said to be the locator of the sequence (x_n). If (x_n) is a localized sequence on X then (x_n) is said to be localized everywhere. If the locator of a sequence (x_n) contains all members of this sequence, except of a finite number of them, then (x_n) is said to be localized in itself [28]. It's important to remember that, every Cauchy sequence in X is localized everywhere.

In 2019, Nabiev et al. [29] presented statistically localized sequences in metric spaces, examined their basic properties, and identified necessary and sufficient conditions for localized sequences to be statistically Cauchy. Then, they [30] generalized the concept of the statistically localized sequence using the notation of ideal $\mathcal{I}$ of subset of the set $\mathbb{N}$ of positive integers. Yamancı et al. [32, 33] examined statistically localized sequences and $\mathcal{I}$-localized in 2-normed spaces. Gürdal et al. [31] examined A-statistically localized sequences in n-normed spaces and obtained some of their basic properties. Recently, Granados and Bermudez [27] introduced $\mathcal{I}_2$-localized and $\mathcal{I}_2^*$-localized double sequences in metric spaces. Also, Banerjee and Hossain [3] studied the notion of $\mathcal{I}$-localized and $\mathcal{I}^*$-localized sequences in S-metric spaces.

In this paper, we define the concepts of a statistically localized sequence and the statistical locator of the sequence (x_k) in A-metric spaces and examine their basic properties. Also, we investigate the relationships between statistically localized sequences and statistically Cauchy sequences. Furthermore, we introduce the notion of uniformly statistically localized sequences in A-metric spaces.

2. Preliminaries

In this part, we recall some fundamental definitions, notations and properties. (See [1],[10],[12],[29]).

Definition 2.1. *[29] (i) A sequence (x_k) in X is said to be statistically localized in the subset $M \subset X$ if the number sequence $d(x_k, x)$ statistically converges for all $x \in M$.*

(ii) The maximal set on which a sequence is statistically localized is said to be a statistical locator of the sequence.

(iii) A sequence (x_k) is said to be statistically localized everywhere if the statistical locator of (x_k) coincides with X.

(iv) A sequence (x_k) is said to be statistically localized in itself if the statistically locator contains (x_k) for almost all k, that is, $\delta\{k : x_k \notin loc_{st}(x_k)\} = 0$ or $\delta\{k : x_k \in loc_{st}(x_k)\} = 1$.

(v) A sequence (x_k) is said to be statistically localized if $loc_{st}(x_k)$ is not empty.

Now we recall the concept of A-metric space and its basic properties. Then, we give some fundamental definitions, notations and properties in A-metric spaces.

Definition 2.2. *[1] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is said to be an A-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$ the following conditions hold;*

(A1) $A(x_1, x_2, ..., x_{n-1}, x_n) \geq 0$,
(A2) $A(x_1, x_2, ..., x_{n-1}, x_n) = 0 \Leftrightarrow x_1 = x_2 = ... = x_n$,
(A3) $A(x_1, x_2, ..., x_{n-1}, x_n) \leq \sum_{k=1}^{n} A(\underbrace{x_k, x_k, \cdots, x_k}_{n-1}, a)$.

Also (X, A) is said to be A-metric space.

Example 2.3. *[1] Let $X = \mathbb{R}$. Define a function $A : X^n \to [0,\infty)$ by*

$$A(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|.$$

Then (X, A) is an A-metric space.

Lemma 2.4. *[1] Let (X, A) be an A-metric space. Then $A(x, x, ..., x, y) = A(y, y, ..., y, x)$ for all $x, y \in X$.*

Lemma 2.5. *[1] Let (X, A) be an A-metric space. For all $x, y \in X$ we get*

$$A(x, x, ..., x, z) \leq (n-1)A(x, x, ..., x, y) + A(y, y, ..., y, z) \text{ and}$$
$$A(x, x, ..., x, z) \leq (n-1)A(x, x, ..., x, y) + A(z, z, ..., z, y).$$

Definition 2.6. *[1] The A-metric space (X, A) is called bounded if there exists an $r > 0$ such that $A(y, y, ..., y, x) \leq r$ for every $x, y \in X$. Otherwise, X is unbounded.*

Definition 2.7. *[1] Let (X, A) be an A- metric space. For given $r > 0$ and $x \in X$ the open ball $B_A(x, r)$ and the closed ball $\overline{B}_A(x, r)$ are defined as follows:*

$$B_A(x, r) = \{y \in X : A(y, y, ..., y, x) < r\}$$

$$\overline{B}_A(x, r) = \{y \in X : A(y, y, ..., y, x) \leq r\}.$$

Definition 2.8. *[1] Let (X, A) be an A- metric space. A subset B of X is said to be an open set if for every $x \in B$, there exists an $r > 0$ such that $B_A(x, r) \subset B$. A subset $F \subset X$ is called closed, if $X \setminus F$ is open.*

Definition 2.9. *[1] Let (X, A) be an A-metric space. A sequence (x_k) in X is said to be convergent to x in X if for every $\varepsilon > 0$, there exists a positive integer K_ε such that $A(x_k, x_k, \ldots, x_k, x) < \varepsilon$ for every $k \geq K_\varepsilon$.*

Definition 2.10. *[1] Let (X, A) be an A-metric space. A sequence (x_k) in X is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists a positive integer K such that $A(x_k, x_k, \ldots, x_k, x_m) < \varepsilon$ for all $k, m \geq K$.*

Definition 2.11. *[22] Let (X, A) be an A-metric space. A sequence (x_k) in X is said to be statistically convergent to an element $x \in X$ if for every $\epsilon > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} |\{k \leq t : A(x_k, x_k, ..., x_k, x) \geq \epsilon\}| = 0$$

or equivalently

$$\lim_{t\to\infty} \frac{1}{t} |\{k \leq t : A(x_k, x_k, ..., x_k, x) < \epsilon\}| = 1$$

and is denoted by $x_k \xrightarrow{A-st} x$. In this case, we can write $st - \lim_{k\to\infty} A(x_k, x_k, \ldots, x_k, x) = 0$.

3. Results

In this section, we first present fundamental definitions and notations related to statistically localized sequences in A-metric spaces.

Definition 3.1. *Let (X, A) be an A-metric space. The sequence (x_k) in X is said to be localized in the subset $M \subset X$ if the number sequence $A(x_k, x_k, \ldots, x_k, x)$ is converges for each $x \in M$.*

Definition 3.2. *Let (X, A) be an A-metric space. The sequence (x_k) in X is said to be statistically localized in the subset $M \subset X$ if the number sequence $A(x_k, x_k, \ldots, x_k, x)$ is statistically converges for each $x \in M$.*

The maximal subset of X on which a sequence (x_k) in X is statistically localized, is said to be a statistical locator of the sequence (x_k) and it is denoted by $loc_{st}^{A}(x_k)$.

A sequence (x_k) in X is said to be statistically localized everywhere if the statistical locator of (x_k) is the whole set X.

A sequence (x_k) in X is said to be statistically localized in itself if the statistically locator contains (x_k) for almost all k, that is, $\delta(\{k : x_k \notin loc_{st}^{A}(x_k)\}) = 0$ or $\delta(\{k : x_k \in loc_{st}^{A}(x_k)\}) = 1$.

A sequence (x_k) in X is said to be statistically localized if $loc_{st}^{A}(x_k) \neq \emptyset$.

Here, we give a result that is a significant corollary of Lemma 2.5, essential for this section.

Corollary 3.3. *Let (X, A) be an A-metric space. Then for all $x, y, z \in X$,*

$$|A(x, x, \ldots, x, z) - A(z, z, \ldots, z, y)| \leq (n-1)A(x, x, \ldots, x, y).$$

Proof. For $x, y, z \in X$, by Lemma 2.5 we can write

$$A(x, x, \ldots, x, z) - A(z, z, \ldots, z, y) \leq (n-1)A(x, x, \ldots, x, y). \tag{1}$$

Similary that

$$A(y, y, \ldots, y, z) \leq (n-1)A(y, y, \ldots, y, x) + A(x, x, \ldots, x, z).$$

By Lemma 2.4, we write

$$A(z, z, \ldots, z, y) - A(x, x, \ldots, x, z) \leq (n-1)A(x, x, \ldots, x, y). \tag{2}$$

By the inequalities (1) and (2) we get

$$|A(x, x, \ldots, x, z) - A(z, z, \ldots, z, y)| \leq (n-1)A(x, x, \ldots, x, y). \tag{3}$$

□

Now, recall that the concept of a statistically Cauchy sequence was presented in [22]. Let (x_k) be a sequence in an A-metric space (X, A). The sequence (x_k) is said to be a statistically Cauchy sequence if for any $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that

$$\delta(\{k \in \mathbb{N} : A(x_k, x_k, \ldots, x_k, x_{m_\varepsilon}) > \varepsilon\}) = 0.$$

Lemma 3.4. *In A-metric spaces, every statistically Cauchy sequence is statistically localized everywhere.*

Proof. Let (x_k) be a statistically Cauchy sequence in an A-metric space (X, A). By Corollary 3.3, we can write

$$|A(x_k, x_k, \ldots, x_k, x) - A(x, x, \ldots, x, x_{m_\varepsilon})| \leq (n-1)A(x_k, x_k, \ldots, x_k, x_{m_\varepsilon})$$

we get

$$\{k \in \mathbb{N} : A(x_k, x_k, \ldots, x_k, x_{m_\varepsilon}) \geq \frac{\varepsilon}{n-1}\} \supset \{k \in \mathbb{N} : |A(x_k, x_k, \ldots, x_k, x) - A(x, x, \ldots, x, x_{m_\varepsilon})| \geq \varepsilon\}.$$

Hence the number sequence $A(x_k, x_k, \ldots, x_k, x)$ is a statistically Cauchy sequence, then $A(x_k, x_k, \ldots, x_k, x)$ is statistically convergent for all $x \in X$. So, (x_k) is statistically localized everywhere. □

We can assert from Lemma 3.4 that every statistically convergent sequence is statistically localized everywhere.

In this section of the study, the fundamental properties of statistically localized sequences in A-metric spaces will be examined. Let us define what the concept of statistical boundedness means in A-metric spaces, which is necessary for the result below.

Let (X, A) be an A-metric space. A sequence (x_k) in X is said to be statistically bounded if there exists $x \in X$ and $L > 0$ such that $\delta(k \in \mathbb{N} : A(x_k, x_k, \ldots, x_k, x) > L) = 0$.

Proposition 3.5. *In A-metric spaces, every statistically localized sequence is statistically bounded.*

Proof. Let (x_k) be a statistically localized sequence in an A-metric space (X, A). Then, $A(x_k, x_k, \ldots, x_k, x)$ is statistically converges for some $x \in X$. So, the sequence $A(x_k, x_k, \ldots, x_k, x)$ is statistically bounded. This implies that $\delta(\{k \in \mathbb{N} : A(x_k, x_k, \ldots, x_k, x) > L\}) = 0$ for some $L > 0$. Consequently, the sequence (x_k) is statistically bounded because almost all elements of (x_k) are located in the open ball $B_A(x, L)$. □

Proposition 3.6. *Let $\mathfrak{L} = loc_{st}^A(x_k)$ and $y \in X$ be a point such that for any $\varepsilon > 0$ there exists $x \in \mathfrak{L}$ satisfying*

$$\delta(\{k \in \mathbb{N} : |A(x_k, x_k, \ldots, x_k, x) - A(x_k, x_k, \ldots, x_k, y)| > \varepsilon\}) = 0. \tag{4}$$

Then $y \in \mathfrak{L}$.

Proof. Let $\varepsilon > 0$ and $x \in \mathfrak{L} = loc_{st}^A(x_k)$ be a point with the property (4). Since the sequence $A(x_k, x_k, \ldots, x_k, x)$ is statistically Cauchy sequence with the property (4), then there exists a subsequence $S = (s_k)$ of $\mathbb{N}$ with $\delta(S) = 1$ such that

$$|A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x) - A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, y)| \to 0 \text{ and}$$
$$|A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, x)| \to 0 \text{ and } k, m \to \infty.$$

Clearly, for any $\varepsilon > 0$ there exist $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $m \geq m_0$

$$|A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x) - A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, y)| < \frac{\varepsilon}{3} \tag{5}$$

$$|A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, x)| < \frac{\varepsilon}{3}. \tag{6}$$

Using (5) and (6), we can write

$$\begin{aligned} |A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, y) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, y)| &\leq |A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, y) - A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x)| \\ &\quad + |A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, x)| \\ &\quad + |A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, x) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

for all $k \geq k_0$, $m \geq m_0$, that is,

$$|A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, y) - A(x_{s_m}, x_{s_m}, \ldots, x_{s_m}, y)| \to 0 \text{ as } k, m \to \infty$$

for $S = (s_k) \subset \mathbb{N}$ with $\delta(S) = 1$. This implies that the number sequence $A(x_k, x_k, \ldots, x_k, y)$ is a Cauchy sequence. This completes the proof of the theorem. □

Lemma 3.7. *Statistically locator of any sequence (x_k) is a closed subset of the A-metric space (X, A).*

Proof. Let $y \in \overline{loc}^A_{st}(x_k)$. Then, for any $\varepsilon > 0$, the ball $B_A(y, \varepsilon)$ will contain a point $x \in loc^A_{st}(x_k)$. For each $\varepsilon > 0$, and almost all k we can write

$$|A(x_k, x_k, \dots, x_k, x) - A(x_k, x_k, \dots, x_k, y)| \leq A(x_k, x_k, \dots, x_k, y)| < \varepsilon.$$

So,

$$\delta\{k \in \mathbb{N} : |A(x_k, x_k, \dots, x_k, x) - A(x_k, x_k, \dots, x_k, y)| > \varepsilon\} = 0.$$

Therefore, the hypothesis of Proposition 3.6 is satisfied. Then $y \in loc^A_{st}(x_k)$, that is, $loc^A_{st}(x_k)$ is closed. □

Definition 3.8. *Let (X, A) be an A-metric space. A point y is called a statistical limit point of (x_k) in X if there exists a set $S = \{s_1 < s_2 < \dots\} \subset \mathbb{N}$ with $\delta(S) \neq 0$ such that $A(x_{s_k}, x_{s_k}, \dots, x_{s_k}, y) \to 0$ as $k \to \infty$.*
Similarly, a point ξ is called a statistical cluster point of (x_k) if for each $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : A(x_k, x_k, \dots, x_k, \xi) < \varepsilon\}) \neq 0.$$

Corollary 3.9. *Let (x_k) be a sequence in an A-metric space (X, A). If $y \in X$ is a statistical limit point (a statistical cluster point) of the sequence (x_k), then for each $z \in X$, the number $A(y, y, \dots, y, z)$ is a statistical limit point (a statistical cluster point) of the sequence $\{A(x_k, x_k, \dots, x_k, z)\}$.*

Proof. Let $y \in X$ be a statistical limit point of a sequence (x_k) in X. Then there exists a set $S = \{s_1 < s_2 < ...\} \subset \mathbb{N}$ such that $\delta(S) \neq 0$ and $\lim_{r\to\infty} A(x_{s_r}, x_{s_r}, \dots, x_{s_r}, y) = 0$. Then for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $A(x_{s_r}, x_{s_r}, \dots, x_{s_r}, y) < \dfrac{\varepsilon}{n-1}$ for all $r > k_0$. Let $z \in X$, by Corollary 3.3 we can write

$$|A(x_{s_r}, x_{s_r}, ..., x_{s_r}, z) - A(z, z, ..., z, y)| \leq (n-1)A(x_{s_r}, x_{s_r}, ..., x_{s_r}, y) < \varepsilon \text{ for all } r > k_0.$$

Hence $\lim_{r\to\infty} A(x_{s_r}, x_{s_r}, \dots, x_{s_r}, z) = A(z, z, \dots, z, y)$. Therefore $A(z, z, \dots, z, y)$ is a statistical limit point of the number sequence $A(x_k, x_k \dots, x_k, z)$.

Let $y \in X$ be a statistical cluster point of (x_k) in X. Then for every $\varepsilon > 0$, we get $\delta(\{k \in \mathbb{N} : A(x_k, x_k, \dots, x_k, y) < \dfrac{\varepsilon}{n-1}\}) \neq 0$. Let $z \in X$, by Corollary 3.3, we can write

$$|A(x_k, x_k, \dots, x_k, z) - A(z, z, \dots, z, y)| \leq (n-1)A(x_k, x_k, \dots, x_k, y).$$

Therefore

$$\{k \in \mathbb{N} : A(x_k, x_k, \dots, x_k, y) < \frac{\varepsilon}{n-1}\} \subset \{k \in \mathbb{N} : |A(x_k, x_k, \dots, x_k, z) - A(z, z, \dots, z, y)| < \varepsilon\}.$$

Hence

$$\delta(\{k \in \mathbb{N} : |A(x_k, x_k, \dots, x_k, z) - A(z, z, \dots, z, y)| < \varepsilon\}) \neq 0.$$

Therefore the number $A(z, z, \dots, z, y)$ is a statistical cluster point of the number sequence $\{A(x_k, x_k, \dots, x_k, z)\}$. □

Proposition 3.10. *In A-metric spaces, all statistical limit points (statistical cluster points) of the statistically localized sequence (x_k) have the same distance from each point x of the statistical locator $loc^A_{st}(x_k)$.*

Proof. Let (x_k) be a sequence in an A-metric space (X, A). If y_1 and y_2 are two statistical limit points (statistical cluster points) of the sequence (x_k), then the numbers $A(y_1, y_1, \dots, y_1, x)$ and $A(y_2, y_2, \dots, y_2, x)$ are statistical limit points of the statistically convergent sequence $A(x_k, x_k, \dots, x_k, x)$. Therefore, $A(y_1, y_1, \dots, y_1, x) = A(y_2, y_2, \dots, y_2, x)$. □

Proposition 3.11. *$loc^A_{st}(x_k)$ does not contain more than one statistical limit (cluster) point of the sequence (x_k) in an A-metric space (X, A).*

Proof. Let $\xi_1, \xi_2 \in loc^A_{st}(x_k)$ be two statistical limit or cluster points of the sequence (x_k). By the Proposition 3.10, $A(\xi_1, \xi_1, \dots, \xi_1, \xi_1) = A(\xi_1, \xi_1, \dots, \xi_1, \xi_2)$. However, $A(\xi_1, \xi_1, \dots, \xi_1, \xi_1) = 0$. This implies that, $A(\xi_1, \xi_1, \dots, \xi_1, \xi_2) = 0$ for $\xi_1 \neq \xi_2$. This is a contradiction. □

Proposition 3.12. *Let (x_k) be a sequence in an A-metric space (X, A). If the sequence (x_k) has a statistical limit point $y \in loc^A_{st}(x_k)$, then $x_k \xrightarrow{A-st} y$.*

Proof. The sequence $\{A(x_k, x_k, \ldots, x_k, y)\}$ is statistically convergent and some subsequence of this sequence converges to zero, i.e., $x_k \xrightarrow{A-st} y$. □

Definition 3.13. *Let (x_k) be a statistically localized sequence with the statistically locator $\mathfrak{L} = loc^A_{st}(x_k)$ in an A-metric space (X, A), the number*

$$\sigma_A = \inf_{x\in\mathfrak{L}}\left(st - \lim_{k\to\infty} A(x_k, x_k, \ldots, x_k, x)\right)$$

is said to be the statistical barrier of (x_k).

Theorem 3.14. *Let (X, A) be an A-metric space. Then, the sequence (x_k) in X is a statistically Cauchy sequence if and only if its statistical barrier, σ_A, is zero.*

Proof. Let (x_k) is a statistically Cauchy sequence in an A-metric space (X, A). Then, there exists a set $S = \{s_1 < s_2 < \ldots < s_k < \ldots\} \subset \mathbb{N}$ such that $\delta(S) = 1$ and $\lim_{k,m\to\infty} A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x_{s_m}) = 0$. So, for each $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that

$$A(x_{s_k}, x_{s_k}, \ldots, x_{s_k}, x_{s_{k_0}}) < \varepsilon$$

for all $k \geq k_0$. Since a statistically Cauchy sequence is statistically localized everywhere, we get $st - \lim_{k,m\to\infty} A(x_k, x_k, \ldots, x_k, x_{s_{k_0}}) \leq \varepsilon$, i.e., $\sigma_A \leq \varepsilon$. Beacuse $\varepsilon > 0$ is arbitrary, we get $\sigma_A = 0$.

Conversely, assume that $\sigma_A = 0$. Then for each $\varepsilon > 0$ there exists an $x \in \mathfrak{L} = loc^A_{st}(x_k)$ such that $\theta(x) = st - \lim_{k\to\infty} A(x_k, x_k, \ldots, x_k, x) < \varepsilon$. Therefore,

$$\delta(\{k \in \mathbb{N} : |A(x_k, x_k, \ldots, x_k, x) - \theta(x)| \geq \varepsilon - \theta(x)\} = 0.$$

Now infact, since $A(x_k, x_k, \ldots, x_k, x) = |A(x_k, x_k, \ldots, x_k, x) - \theta(x) + \theta(x)| \leq |A(x_k, x_k, \ldots, x_k, x) - \theta(x)| + \theta(x)$, therefore $\delta(\{k \in \mathbb{N} : A(x_k, x_k, \ldots, x_k, x) \geq \varepsilon\}) = 0$, i.e., $st - \lim_{k\to\infty} A(x_k, x_k, \ldots, x_k, x) = 0$. Then (x_k) is a statistically Cauchy sequence. □

Definition 3.15. *(cf. [30]) Let (x_{s_k}) be a subsequence of the sequence (x_k) in an A-metric space (X, A), if there exists an $S = \{s_1 < s_2 < \ldots < s_k < \ldots\} \subset \mathbb{N}$ such that $\delta(S) = 0$, then (x_{s_k}) is said to be a thin subsequence of (x_k). In particular, if $\delta(S) \neq 0$, (x_{s_k}) is said to be a nonthin subsequence.*

Theorem 3.16. *Let (X, A) be an A-metric space. If (x_k) in X is statistically localized in itself and (x_k) contains a nonthin Cauchy subsequence, then (x_k) is a statistically Cauchy sequence itself.*

Proof. Let (x'_k) is a nonthin Cauchy subsequence of (x_k). Not losing of generality, we can suppose that all members of (x'_k) belong to $loc^A_{st}(x_k)$. Since (x'_k) is a Cauchy sequence, by Theorem 3.14,

$$\inf_{x'_k}(\lim_{m\to\infty} A(x'_m, x'_m, \ldots, x'_m, x'_k)) = 0.$$

In other hand, since (x_k) is statistically localized in itself then,

$$st - \lim_{m\to\infty} A(x_m, x_m, \ldots, x_m, x'_k) = st - \lim_{m\to\infty} A(x'_m, x'_m, \ldots, x'_m, x'_k) = 0.$$

This implies that,

$$\sigma_A = \inf_{x\in\mathfrak{L}}(st - \lim_{m\to\infty} A(x_m, x_m, \ldots, x_m, x)) = 0,$$

i.e., (x_k) is a statistically Cauchy sequence itself. □

Let $x \in X$ and $r > 0$. Recall that a sequence (x_k) in an A-metric space (X, A) is said to be statistically bounded if there exists a subset $S = \{s_1 < s_2 < \ldots < s_k < \ldots\} \subset \mathbb{N}$ such that $\delta(S) = 1$ and $(x_{s_k}) \subset B_A(x, r)$, where $B_A(x, r)$ open the ball with center at the point x and with radius r. Evidently, (x_{s_k}) is a bounded sequence in X and it has a localized in itself subsequence. (See [28], this is valid in A-metric spaces). Consequently, the following statement is true.

Theorem 3.17. *Every statistically bounded sequence in an A-metric space (X, A) has a statistically localized in itself subsequence.*

Definition 3.18. *(cf, [30]) Let (X, A) be an A-metric spaces. An infinite subset $M \subset X$ is said to be thick relatively to a nonempty subset $Y \subset X$ if for each $\varepsilon > 0$, there exists a point $y \in Y$ such that the ball $B_A(y, \varepsilon)$ contains infinitely many points of M. In particular, if the set M is thick relatively to its subset $Y \subset M$, then M is called thick in itself.*

Proposition 3.19. *If the set M is thick relatively to some set Y, then the set M is thick in itself.*

Theorem 3.20. *The following statements are equivalent to each other in A-metric space (X, A) :*

(i) Every statistically localized in itself sequence in X is a statistically Cauchy sequence.

(ii) Every bounded subset of X is totally bounded.

(iii) Every bounded infinite set of X is thick in itself.

Proof. Let (*i*) holds, but (*ii*) does not. Then there is a subset $M \subset X$ such that M is not totally bounded. This implies that there exists $\varepsilon > 0$ and a sequence $(x_k) \subset M$ such that $A(x_k, x_k, \ldots, x_k, x_m) > \varepsilon$ for any $k \neq m$. Since (x_k) is statistically bounded by Theorem 3.17, it has a statistically localized in itself sequence (x'_k). Beacuse $A(x'_k, x'_k, \ldots, x'_k, x'_m) > \varepsilon$ for any $k \neq m$, the subsequence is not a statistically Cauchy sequence. This contradicts (*i*). So, (*i*) implies (*ii*). One can easily show that (*ii*) implies (*iii*). Now let show that (*iii*) implies (*i*). Let $(x_k) \subset X$ is statistically localized in itself. Then (x_k) is statistically bounded sequence in X. Then here is an infinite set M of points of (x_k) such that M is a bounded subset of X. By the assumption the set M is thick in itself. Then for every $\varepsilon > 0$, we can choose $x_s \in M$ such that the ball $B_A(x_s, \varepsilon)$ contains infinitely many points of X, say $x'_1, x'_2, \ldots, x'_k, \ldots$. For the sequence (x'_k) the sequence $A(x'_k, x'_k, \ldots, x'_k, x_k)_{k=1}^{\infty}$ is statistically converges and $st - \lim_{k\to\infty} A(x'_k, x'_k, \ldots, x'_k, x_S) \leq \varepsilon$ for the sequence (x'_k). So, the statistically barrier of (x_k) is equal to zero. Namely (x_k) is a Cauchy sequence. This completes the proof of the theorem. □

Using Theorem 3.16 and Theorem 3.17, we can show that (*i*) is equivalent to (*iv*) every statistically bounded sequence has a statistically convergent subsequence.

Definition 3.21. *Let (X, A) be an A-metric space. A sequence (x_k) in X is said to be uniformly statistically localized on the subset M of X if the sequence $\{A(x_k, x_k, \ldots, x_k, x)\}$ uniformly statistically converges for all $x \in M$.*

Proposition 3.22. *Let (X, A) be an A-metric space. If a sequence (x_k) in X is uniformly statistically localized on the set $M \subset X$ and $z \in Y$ is such that for every $\varepsilon > 0$, there is $y \in M$ with the property*

$$\delta(\{k \in \mathbb{N} : |A(x_k, x_k, \ldots, x_k, z) - A(x_k, x_k, \ldots, x_k, y)| \geq \varepsilon\}) = 0$$

Then, $z \in loc_{st}^{A}(x_k)$ and (x_k) is uniformly statistically localized on a set containing such points z.

Proposition 3.22 is proved in a similar way as the Proposition 3.6.

4. Conclusion

A-metric spaces are a generalization of metric spaces proposed by Abbas et al. [1]. In these spaces, studies on fixed point theorems have been conducted. However, studies related to the summability theory in this area are very limited. The concept of localized sequence one of the fundamental concepts of this study, was introduced in 1974; however, it has mostly attracted the attention of researchers in recent years. In this paper, we introduce statistically localized sequences in A-metric spaces. In A-metric spaces, we first define statistically localized sequences, statistically Cauchy sequences, and uniformly statistically localized sequences. Subsequently, we explore and prove some properties and results associated with these concepts. Therefore, the definitions and results of this study are more comprehensive than those in usual metric spaces. In this perspective, the studies done in the classical field of summability theory in generalized metric spaces are transferred and the results can be obtained in a more comprehensive way. For example, $\mathcal{I}_2$-localized double sequences may be studied in $g-$metric spaces.

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