Filomat 38:21 (2024), 7399–7411
https://doi.org/10.2298/FIL2421399N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An analysis of best wavelet approximation problem of a function using Laguerre wavelets

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Abstract. In this paper, we derive optimal wavelet approximations for a function g that possesses both bounded second-order derivatives and bounded T^{th} derivatives using Laguerre wavelets.

1. Introduction

Wavelets have progressively penetrated diverse scientific and engineering realms in recent times, establishing a robust presence. Notably, their impact has been most pronounced in the field of signal analysis, where they excel in tasks such as waveform representation, data segmentation, time-frequency analysis, and the rapid deployment of algorithms. One standout application is the wavelet approximation method, which has emerged as a cutting-edge instrument and a contemporary trend. This method is particularly effective in discerning and scrutinizing abrupt changes in seismic signal processing. Its prowess lies in the ability to provide a detailed and nuanced analysis, allowing for a comprehensive understanding of seismic data and facilitating more informed decision-making in various applications.

Wavelet expansion is essential for signal analysis, offering localized representation, efficient compression, and fast computation, ensuring convergence in diverse applications. Laguerre wavelets play a vital role in wavelet expansion, excelling in functions with exponential decay. Their unique properties enhance signal analysis, offering precision in diverse applications.

Wavelet approximations of specific functions using the Haar wavelet have been meticulously explored by distinguished researchers including DeVore [4], Debnath [3], Nigam[10, 11], Mayer [7], Morlet [8, 9]; and Lal and Kumar [6]. However, a noticeable void exists in the realm of wavelet approximation concerning the utilization of Laguerre wavelet methods. Surprisingly, despite the diligent efforts of the researchers, this specific avenue remains uncharted territory awaiting exploration.

Embarking on a pioneering journey within this domain, this paper undertakes an ambitious endeavor to delve into an advanced study of wavelet approximation of a function g with $0 \leq \sup_{x\in[0,1]} |g''(x)| \leq A < \infty$ and $0 \leq \sup_{x\in[0,1]} |g^T(x)| \leq B < \infty$, where T is the positive integer using Laguerre wavelet, have been obtained. A critical point of emphasis is that an estimation of a function displays superior accuracy and

2020 *Mathematics Subject Classification*. 41A30, 42C40.
Keywords. Wavelet approximation; Laguerre wavelets; admissibility conditions; functions of bounded derivatives.
Received: 11 December 2023; Accepted: 09 April 2024
Communicated by Eberhard Malkowsky
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precision when it possesses a higher degree of bounded derivatives, as opposed to estimations with fewer bounded derivatives. This underscores the substantial significance of meticulously comparing estimated approximations within the framework of wavelet analysis. Such comparisons hold considerable importance as they offer insights into the nuanced distinctions and implications of these estimations, thereby enhancing our understanding of wavelet dynamics.

2. Definitions

2.1. Laguerre wavelets

By using dilation and translation of a map (as the mother wavelet), we can consider the family of continous wavelets

$$\psi_{\alpha,\beta}(l) = |\alpha|^{-\frac{1}{2}}\psi\left(\frac{l-\beta}{\alpha}\right) \ (\alpha, \beta \in \mathbb{R}, \alpha \neq 0),$$

where α and β are the dilation and translation parameters respectively. If $\alpha_0 > 1$, $\beta_0 > 0$, $\alpha = \alpha_0^{-\tau}$, $\beta = \nu\beta_0\alpha_0^{-\tau}$ and τ and ν are positive integers, then it reduces to the discrete wavelets

$$\psi_{\tau,\nu}(l) = |\alpha_0|^{\frac{\tau}{2}}\psi\left(\alpha_0^{\tau}l - \nu\beta_0\right)$$

which is a wavelet basis for $L^2(\mathbb{R})$ [1, 2, 5, 12]. If $\alpha_0 = 2$ and $\beta_0 = 1$, then $\{\psi_{\tau,\nu}(l)\}_{\tau,\nu\geq 0}$ is an orthonormal basis [1, 2, 5, 12]. The Laguerre wavelets $\psi_{\nu,\rho}(l) = \psi(\tau, \rho, \nu, l)$ have four argument: τ can assume any positive integer, $\rho = 1, 2, 3, ..., 2^{\tau-1}$, ν is the degree of Laguerre polynomial, and l is the normalize time. They are defined on the interval $[0, 1)$ as (see [1, 2, 5, 12])

$$\psi_{\rho,\nu}(l) = \begin{cases} \frac{2^{\frac{\tau}{2}}}{\nu!}L_\nu(2^\tau l - 2\rho + 1), & \frac{\rho-1}{2^{\tau-1}} \leq l < \frac{\rho}{2^{\tau-1}} \\ 0, & \text{otherwise} \end{cases}$$

where $\nu = 0, 1, 2, ..., T-1$ and T is a fixed positive integer. $L_\nu(l)$ are the Laguerre polynomials of degree ν, which are orthogonal with respect to the weight function $w(l) = 1$ on the interval $[0, \infty)$ and satisfy the following recursive formula

$$L_0(l) = 1,\ L_1(l) = 1 - l,\ L_{\nu+2}(l) = \frac{(2\nu + 3 - l)L_{\nu+1}(l) - (\nu + 1)L_\nu(l)}{\nu + 2},\ \ \nu = 0, 1, 2, 3, ...$$

2.2. Function approximation

A function g belonging to the space $L^2(\mathbb{R})$ and defined on the interval $[0, 1)$, is expressed as an expansion using a Laguerre wavelet series in the following form:

$$g(l) = \sum_{\rho=0}^{\infty}\sum_{\nu=0}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}, \tag{1}$$

where

$$c_{\rho,\nu} = \left\langle g, \psi_{\nu,\rho}\right\rangle \tag{2}$$

In equation (2), the symbol $\langle\cdot,\cdot\rangle$ represents the inner product.
When the infinite series in equation (1) is limited or truncated, it can be expressed as follows:

$$S_{2^{\tau-1},T}(l) = \sum_{\rho=1}^{2^{\tau-1}}\sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu}(l) = C^{'}\Psi(l),$$

where C and $\Psi(l)$ are $2^{\tau-1}T \times 1$ matrices, defined by

$$C = [c_{1,0}, c_{1,1}, ..., c_{1,T-1}, c_{2,0}, c_{2,1}, ..., c_{2,T-1}, ..., c_{2^{\tau-1},0}, c_{2^{\tau-1},1}, ..., c_{2^{\tau-1},T-1}]$$

and

$$\Psi(l) = [\psi_{1,0}(l), \psi_{1,1}(l)..., \psi_{1,T-1}(l), \psi_{2,0}(l), ..., \psi_{2,T-1}(l), ..., \psi_{2^{\tau-1},0}(l), ..., \psi_{2^{\tau-1},T-1}(l)].$$

2.3. Multiresolution Analysis

A multiresolution in the space $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces V_j, where j is an integer from the set of integers $\mathbb{Z}$, and it must adhere to the following set of conditions:

(i) $V_j \subset V_{j+1}$;
(ii) $g(x) \in V_j \Leftrightarrow g(2x) \in V_{j+1}$;
(iii) $g(x) \in V_0 \Leftrightarrow g(x+1) \in V_0$;
(iv) $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$, and $\bigcap_{-\infty}^{\infty} V_j = 0$;
(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x-\tau) : \tau \in \mathbb{Z}\}$ is a Riesz basis of V_0 (see [3]).

2.4. Projection $P_\rho(g)$

Consider $P_\rho(g)$ be the orthogonal projection of $L^2(\mathbb{R})$ onto V_ρ. Then

$$P_\rho(g) = \sum_{-\infty}^{\infty} a_{\rho,\tau}\phi_{\rho,\tau}, \quad \rho = 1, 2, 3, ...,$$

where

$$a_{\rho,\tau} = \left\langle g, \phi_{\rho,\tau} \right\rangle.$$

Thus,

$$P_\rho(g) = \sum_{-\infty}^{\infty} \left\langle g, \phi_{\rho,\tau} \right\rangle \phi_{\rho,\tau}, \quad \rho = 1, 2, 3, ... \text{ (see [13])}.$$

2.5. Wavelet approximation

The wavelet approximation under the supremum norm is defined by

$$E_\rho(g) = \|g - P_\rho(g)\|_\infty = \sup_x \|(g(x) - P_\rho(g))\| \text{ (see [14])},$$

$$\|g\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^p dl \right\}^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

The degree of wavelet approximation $E_\rho(g)$ of g by $P_\rho g$ under the norm $\|\cdot\|_p$ is given by

$$E_\rho(g) = \min_{P_\rho g} \|g - P_\rho g\|_p.$$

Remark 2.1. *If $E_\rho(g) \to 0$ as $\rho \to \infty$ then $E_\rho(g)$ is called the best wavelet approximation of g of order ρ (see [14]).*

3. Main Results

In this section, we prove the following main theorems:

Theorem 3.1. *If a continuous function g belongs to $L^2(\mathbb{R})$ on $[0,1]$ such that*

$$\sup_{l\in[0,1]} \|g''\| \le K < \infty,$$

is expanded as an infinite series of Laguerre wavelet

$$g(l) = \sum_{\rho=1}^{\infty} \sum_{\nu=0}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}(l),$$

where

$$c_{\rho,\nu} = \left\langle g, \psi_{\rho,\nu} \right\rangle, \tag{3}$$

then the Laguerre wavelet approximation $E_{2^{\tau-1},T}(g)$ of g by $(2^{\tau-1}, T)^{th}$ partial sums

$$S_{2^{\tau-1},T} = \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu}(l)$$

of its Laguerre wavelet series in $L^2(\mathbb{R})$, is given by

$$\begin{aligned} E_{2^{\tau-1},T}(g) &= \|g - S_{2^{\tau-1},T}\|_2 \\ &= O\Big(\frac{1}{2^{2\tau}T^3}\Big) \end{aligned}$$

Theorem 3.2. *If a continuous function g belongs to $L^2(\mathbb{R})$ on $[0,1]$ such that*

$$\sup_{l\in[0,1]} |g^T(l)| < \infty,$$

then Laguerre wavelet approximation of g by $(2^{\tau-1}, T)^{th}$ partial sums

$$S_{2^{\tau-1},T} = \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu}(l),$$

of its Laguerre wavelet series $L^2(\mathbb{R})$, is given by

$$\begin{aligned} E_{2^{\tau-1},T}(g) &= \|g - S_{2^{\tau-1},T}\|_2 \\ &= \left\| g - \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu} \right\|_2 \\ &= O\Big(\frac{1}{T!2^{T\tau}}\Big) \end{aligned}$$

Proof. **[Proof of the Theorem 3.1]** Laguerre wavelet series of $g \in L^2(\mathbb{R})$ $[0,1]$ is given by

$$\begin{aligned} g &= \sum_{\rho=0}^{\infty} \sum_{\nu=0}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} \\ &= \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu} + \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} + \sum_{\rho=2^{\tau-1}+1}^{\infty} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu} + \sum_{\rho=2^{\tau-1}+1}^{\infty} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} \\ &= S_{2^{\tau-1},T} + \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} + \sum_{\rho=2^{\tau-1}+1}^{\infty} \sum_{\nu=0}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu} + \sum_{\rho=2^{\tau-1}+1}^{\infty} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}, \end{aligned} \tag{4}$$

where

$$\psi_{\rho,\nu}(l) = \begin{cases} \frac{2^{\frac{\tau}{2}}}{\nu!} L_\nu(2^\tau l - 2\rho + 1), & \frac{\rho-1}{2^{\tau-1}} \le l < \frac{\rho}{2^{\tau-1}}; \\ 0, & \text{otherwise}, \end{cases}$$

where $\nu = 0, 1, 2, \ldots, T-1$ and $\rho = 1, 2, \ldots, 2^{\tau-1}$ (τ is any positive integer). Here $L_\nu(l)$ are Laguerre polynomials of degree ν with respect to weight function $w(l) = 1$.
From the Laguerre wavelet, we have

$$\frac{\rho-1}{2^{\tau-1}} \leq l \leq \frac{\rho}{2^{\tau-1}}.$$

If we take $\rho = 2^{\tau-1}+1$, then $\frac{2^{\tau-1}+1-1}{2^{\tau-1}} \leq l \leq \frac{2^{\tau-1}+1}{2^{\tau-1}} \Rightarrow 1 \leq l \leq 1 + \frac{1}{2^{\tau-1}}\ \forall\ \tau$.
Since $\psi_{\rho,\nu}$ vanishes outside the interval $[0,1]$, therefore the third and fourth term in (4) become zero.
Thus, (4) becomes

$$g = S_{2^{\tau-1},T} + \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}. \tag{5}$$

Now, (5) can be written as,

$$\begin{aligned}
\|g - S_{2^{\tau-1},T}\|_2^2 &= \left\| \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} \right\|_2^2 \\
&= \left\langle \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}\ ,\ \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu} \right\rangle \\
&= \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} |c_{\rho,\nu}|^2 \|\psi_{\rho,\nu}\|_2^2.
\end{aligned} \tag{6}$$

Now, we consider

$$\begin{aligned}
\|\psi_{\rho,\nu}\|_2^2 &= \int_0^\infty \psi_{\rho,\nu}(l)\overline{\psi_{\rho,\nu}(l)}dl \\
&= \int_0^\infty \frac{2^{\frac{\tau}{2}}}{\nu!} L_\nu(2^\tau l - 2\rho + 1) \overline{\frac{2^{\frac{\tau}{2}}}{\nu!} L_\nu(2^\tau l - 2\rho + 1)} dl \\
&= \frac{2^\tau}{(\nu!)^2} \int_{\frac{\rho-1}{2^{\tau-1}}}^{\frac{\rho}{2^{\tau-1}}} L_\nu(2^\tau l - 2\rho + 1)\overline{L_\nu(2^\tau l - 2\rho + 1)} dl \\
&= \frac{2^\tau}{(\nu!)^2} \int_{\frac{\rho-1}{2^{\tau-1}}}^{\frac{\rho}{2^{\tau-1}}} |L_\nu(2^\tau l - 2\rho + 1)|^2 dl.
\end{aligned} \tag{7}$$

Let $2^\tau l - 2\rho + 1 = u$ in (7), we have

$$\begin{aligned}
\|\psi_{\rho,\nu}\|_2^2 &= \frac{2^\tau}{(\nu!)^2} \int_{-1}^{1} |L_\nu(u)|^2 \frac{du}{2^\tau} \\
&= \frac{1}{(\nu!)^2} \int_{-1}^{1} |L_\nu(u)|^2 du.
\end{aligned}$$

Since $L_\nu(u)$ is continuous and integrable on (-1,1), we choose $\int_{-1}^{1} L_\nu(u)du = A$. Thus, we have

$$\|\psi_{\rho,\nu}\|_2^2 = \frac{A}{(\nu!)^2}. \tag{8}$$

From (6) and (8), we have

$$\|g - S_{2^{\tau-1},T}\|_2^2 = \sum_{\rho=1}^{2^{\tau-1}} \sum_{\nu=T}^{\infty} \frac{A\left|c_{\rho,\nu}\right|^2}{(\nu!)^2}. \tag{9}$$

Next,

$$
\begin{aligned}
c_{\rho,\nu} &= \int_0^1 g(x)\psi_{\rho,\nu}dx \\
&= \int_{\frac{\rho-1}{2^{\tau-1}}}^{\frac{\rho}{2^{\tau-1}}} g(x)\frac{2^{\frac{\tau}{2}}}{\nu!}L_\nu(2^\tau x-2\rho+1)dx
\end{aligned}
$$

Putting $2^\tau x - 2\rho + 1 = l$, we have

$$
\begin{aligned}
c_{\rho,\nu} &= \int_{-1}^1 g\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{2^{\frac{\tau}{2}}}{\nu!}L_\nu(l)\frac{1}{2^\tau}dl \\
&= \frac{2^{-\frac{\tau}{2}}}{(\nu!)}\int_{-1}^1 g\Big(\frac{2\rho-1+l}{2^\tau}\Big)L_\nu(l)dl.
\end{aligned}
$$

Using the recurrence relation $L_\nu(l) = L'_\nu(l) - \frac{L'_{\nu+1}(l)}{(\nu+1)}$, we have

$$
\begin{aligned}
c_{\rho,\nu} &= \frac{2^{-\frac{\tau}{2}}}{(\nu!)}\int_{-1}^1 g\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{d}{dl}\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl \\
&= \frac{2^{-\frac{\tau}{2}}}{(\nu!)}\Big[\Big\{g\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)\Big\}_{-1}^1 - \int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{1}{2^\tau}\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl\Big] \\
&= -\frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl \\
&= -\frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)L_\nu(l)dl + \frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{L_{\nu+1}(l)}{(\nu+1)}dl \\
&= I_1 + I_2 \ (say). \qquad (10)
\end{aligned}
$$

Now,

$$
I_1 = -\frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\left(\frac{2\rho-1+l}{2^\tau}\right)L_\nu(l)dl.
$$

Using the recurrence relation $L_\nu(l) = L'_\nu(l) - \frac{L'_{\nu+1}(l)}{(\nu+1)}$, we have

$$
\begin{aligned}
I_1 &= -\frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{d}{dl}\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl \\
&= -\frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\Big[\Big\{g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)\Big\}_{-1}^1 - \int_{-1}^1 g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{1}{2^\tau}\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl\Big] \\
&= \frac{2^{-\frac{5\tau}{2}}}{(\nu!)}\int_{-1}^1 g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)dl. \qquad (11)
\end{aligned}
$$

Now, I_2 gives

$$
I_2 = \frac{2^{-\frac{3\tau}{2}}}{(\nu!)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{L_{\nu+1}(l)}{(\nu+1)}dl.
$$

Using the recurrence relation $L_{\nu+1}(l) = L'_{\nu+1}(l) - \frac{L'_{\nu+2}(l)}{(\nu+2)}$, we have

$$
I_2 = \frac{2^{-\frac{3\tau}{2}}}{(\nu!)(\nu+1)}\int_{-1}^1 g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{d}{dl}\Big(L_{\nu+1}(l)-\frac{L_{\nu+2}(l)}{(\nu+2)}\Big)dl
$$

$$I_2 = \frac{2^{-\frac{3\tau}{2}}}{(\nu!)(\nu+1)}\Big[\Big\{g'\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_{\nu+1}(l)-\frac{L_{\nu+2}(l)}{(\nu+2)}\Big)\Big\}_{-1}^{1} - \int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\frac{1}{2^\tau}\Big(L_{\nu+1}(l)-\frac{L_{\nu+2}(l)}{(\nu+2)}\Big)dl\Big]$$
$$= -\frac{2^{-\frac{5\tau}{2}}}{(\nu!)(\nu+1)}\int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big(L_{\nu+1}(l)-\frac{L_{\nu+2}(l)}{(\nu+2)}\Big)dl.$$

Combining (6), (7) and (10), we have

$$c_{\rho,\nu} = \frac{2^{-\frac{5\tau}{2}}}{(\nu!)}\int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big\{\Big(L_\nu(l)-\frac{L_{\nu+1}(l)}{(\nu+1)}\Big)-\frac{1}{\nu+1}\Big(L_{\nu+1}(l)-\frac{L_{\nu+2}(l)}{\nu+2}\Big)dl\Big\}$$
$$= \frac{2^{-\frac{5\tau}{2}}}{(\nu!)}\int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big\{\frac{(\nu+1)L_\nu(l)-L_{\nu+1}(l)}{(\nu+1)}-\frac{(\nu+2)L_{\nu+1}(l)-L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big\}dl$$
$$= \frac{2^{-\frac{5\tau}{2}}}{(\nu!)}\int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big\{\frac{(\nu+1)(\nu+2)L_\nu(l)-2(\nu+2)L_{\nu+1}(l)+L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big\}dl. \tag{12}$$

Squaring both sides of (12), we have

$$|c_{\rho,\nu}|^2 = \Big|\Big(\frac{1}{(\nu!)2^{\frac{5\tau}{2}}}\Big)\int_{-1}^{1} g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big\{\frac{(\nu+1)(\nu+2)L_\nu(l)-2(\nu+2)L_{\nu+1}(l)+L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big\}\Big|^2 dl$$
$$\le \Big(\frac{1}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\int_{-1}^{1}\Big|g''\Big(\frac{2\rho-1+l}{2^\tau}\Big)\Big|^2 dl\int_{-1}^{1}\Big|\Big\{\frac{(\nu+1)(\nu+2)L_\nu(l)-2(\nu+2)L_{\nu+1}(l)+L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big\}\Big|^2 dl$$
$$\le \Big(\frac{1}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\int_{-1}^{1} B^2 dl\int_{-1}^{1}\Big|\Big\{\frac{(\nu+1)(\nu+2)L_\nu(l)-2(\nu+2)L_{\nu+1}(l)+L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big\}\Big|^2 dl$$
$$\le 2B^2\Big(\frac{1}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\int_{-1}^{1}\Big|\frac{(\nu+1)(\nu+2)L_\nu(l)-2(\nu+2)L_{\nu+1}(l)+L_{\nu+2}(l)}{(\nu+1)(\nu+2)}\Big|^2 dl$$
$$\le \Big(\frac{\sqrt{2}B}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\Big\{\int_{-1}^{1}\frac{(\nu+1)^2(\nu+2)^2L_\nu^2(l)+4(\nu+2)^2L_{\nu+1}^2(l)+L_{\nu+2}^2(l)}{(\nu+1)^2(\nu+2)^2}dl\Big\}$$

(other terms vanish due to orthogonal property of Laguerre polynomial)

$$\le \Big(\frac{\sqrt{2}B}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\Big\{\int_{-1}^{1} L_\nu^2(l)dl + \frac{4}{(\nu+1)^2}\int_{-1}^{1} L_{\nu+1}^2(l)dl + \frac{1}{(\nu+1)^2(\nu+2)^2}\int_{-1}^{1} L_{\nu+2}^2(l)dl\Big\}.$$

Since $L_\nu(l)$ is continuous and integrable on $[-1,1]$, we choose

$$\int_{-1}^{1} L_\nu^2(l)dl = C^2\ ;\ \int_{-1}^{1} L_{\nu+1}^2(l)dl = D^2\ ;\ \int_{-1}^{1} L_{\nu+2}^2(l)dl = E^2.$$

Thus, we have

$$|c_{\rho,\nu}|^2 \le \Big(\frac{\sqrt{2}B}{(\nu!)2^{\frac{5\tau}{2}}}\Big)^2\Big\{C^2+\frac{4}{(\nu!)^2(\nu+1)^2}D^2+\frac{1}{(\nu!)^2(\nu+1)^2(\nu+2)^2}E^2\Big\}$$
$$\le \frac{F}{2^{5\tau}(\nu!)^2}. \tag{13}$$

From (9) and (13), we have

$$\|g - S_{2^{\tau-1},T}\|_2^2 \le \sum_{\rho=1}^{2^{\tau-1}}\sum_{\nu=T}^{\infty}\frac{BF}{2^{5\tau}(\nu!)^4}$$
$$\|g - S_{2^{\tau-1},T}\|_2 = O\Big(\frac{1}{2^{2\tau}T^3}\Big).$$

□

Proof. **[Proof of the Theorem 3.2]** Let a function g is T times differentiable. Then by using Taylor's expansion, we have

$$g(p+h) = g_{T+1} = g(p) + \frac{h}{1!}g'(p) + \cdots + \frac{h^{T-1}}{(T-1)!}g^{(T-1)}(p) + \frac{h^T}{T!}g^T(p+\theta h)$$

$$g_{T+1} = g_T + \frac{h^T}{T!}g^T(p+\theta h), \quad \text{where} \quad 0<\theta<1, \tag{14}$$

where

$$g_T = g(p) + \frac{h}{1!}g'(p) + \cdots + \frac{h^{T-1}}{(T-1)!}g^{(T-1)}(p).$$

Now, from (14), we have

$$g_{T+1} - g_T = \frac{h^T}{T!}g^T(p+\theta h), \text{where} \quad 0<\theta<1$$

Using this and dividing the interval $[-1,1]$ in $[\frac{r}{2^{\tau-1}}, \frac{r+1}{2^{\tau-1}}]$ subintervals, we have

$$\begin{aligned}
\|g - S_{2^{\tau-1},T}\|_2^2 &= \int_{-1}^{1}\left|g(x) - \sum_{\rho=1}^{2^{\tau-1}}\sum_{\nu=T}^{\infty} c_{\rho,\nu}\psi_{\rho,\nu}\right|^2 dx \\
&= \sum_{r=-2^{\tau-1}}^{2^{\tau-1}-1}\int_{\frac{r}{2^{\tau-1}}}^{\frac{r+1}{2^{\tau-1}}}\left|g(x) - \sum_{\rho=1}^{2^{\tau-1}}\sum_{\nu=T}^{T-1} c_{\rho,\nu}\psi_{\rho,\nu}\right|^2 dx \\
&\le \sum_{r=-2^{\tau-1}}^{2^{\tau-1}-1}\int_{\frac{r}{2^{\tau-1}}}^{\frac{r+1}{2^{\tau-1}}}\left(\frac{1}{T!}\left(\frac{1}{2^\tau}\right)^T \sup_{x\in[-1,1]}|g^{(T)}(x)|\right)^2 dx \\
&= \int_{-1}^{1}\left(\frac{1}{T!}\left(\frac{1}{2^\tau}\right)^T \sup_{x\in[-1,1]}|g^{(T)}(x)|\right)^2 dx \\
&= \left(\frac{1}{T!}\right)^2\left(\frac{1}{2^{T\tau}}\right)^2\int_{-1}^{1}\left(\sup_{x\in[-1,1]}|g^{(T)}(x)|\right)^2 dx \\
&= 2\left(\frac{1}{T!}\right)^2\left(\frac{1}{2^{t\tau}}\right)^2 \sup_{x\in[-1,1]}|g^{(T)}(x)|^2 \\
&\le \left(\frac{\sqrt{2}}{T!2^{T\tau}}\right)^2 \sup_{x\in[-1,1]}|g^{(T)}(x)|^2.
\end{aligned}$$

Hence, $\|g - S_{2^{\tau-1},T}\|_2 = O\left(\frac{1}{T!2^{T\tau}}\right)$. □

Remark 3.3. *In Theorem 3.1,*

$$E_{2^{\tau-1},T} = O\left(\frac{1}{2^{2\tau}T^3}\right) = \frac{k_1}{2^{2\tau}T^3} \to 0 \text{ as } \tau\to\infty, T\to\infty.$$

Also in Theorem 3.2,

$$E_{2^{\tau-1},T} = O\left(\frac{1}{T!2^{T\tau}}\right) = \frac{k_2}{T!2^{T\tau}} \to 0 \text{ as } \tau\to\infty, T\to\infty,$$

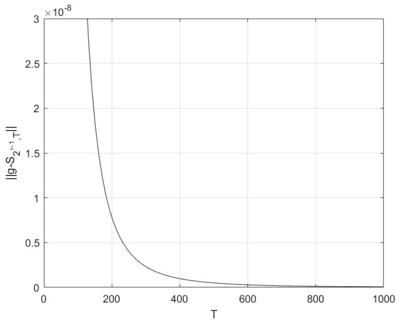
where k_1 and k_2 are positive constants. So, in Theorems 3.1 and 3.2, the Laguerre Wavelet approximation stands out as the best possible.

4. Validation

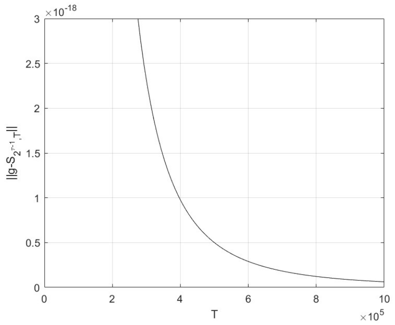
Now, we will construct the following table and draw the figures to validate our Theorem 3.1.

	$E_{2^{\tau-1},T}(g) = \|g - S_{2^{\tau-1},T}\|_2$			
T	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
1	2.5×10^{-1}	6.25×10^{-2}	1.5625×10^{-2}	3.90625×10^{-3}
2	3.125×10^{-2}	7.8125×10^{-3}	1.953125×10^{-3}	4.8828125×10^{-4}
5	2×10^{-3}	5×10^{-4}	1.25×10^{-4}	3.125×10^{-5}
10	2.5×10^{-4}	6.25×10^{-5}	1.5625×10^{-5}	3.90625×10^{-6}
100	2.5×10^{-7}	6.25×10^{-8}	1.5625×10^{-8}	3.90625×10^{-9}
1000	2.5×10^{-10}	6.25×10^{-11}	1.5625×10^{-11}	3.90625×10^{-12}
10000	2.5×10^{-13}	6.25×10^{-14}	1.5625×10^{-14}	3.90625×10^{-15}
100000	2.5×10^{-16}	6.25×10^{-17}	1.5625×10^{-17}	3.90625×10^{-18}
1000000	2.5×10^{-19}	6.25×10^{-20}	1.5625×10^{-20}	3.90625×10^{-21}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
∞	0	0	0	0

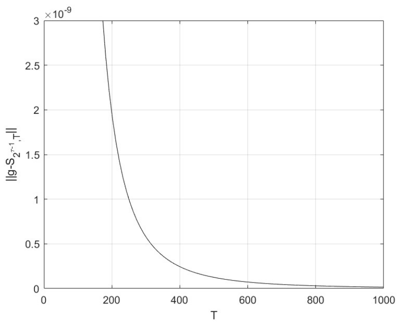
Table 1: Degree of convergence of g for different T and some values of τ in Theorem 3.1.



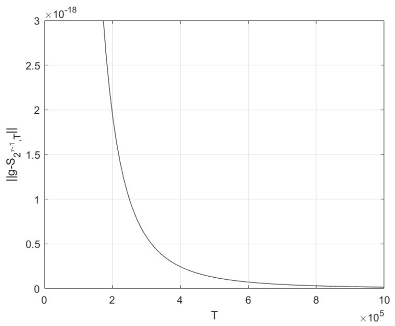
(a) For $T = 1000$ and $\tau = 2$



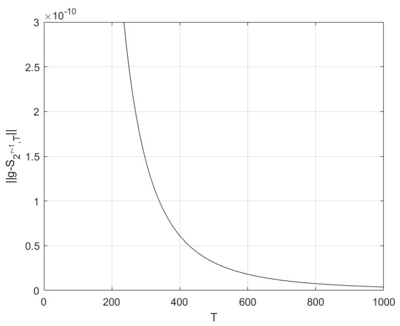
(b) For $T = 1000000$ and $\tau = 2$



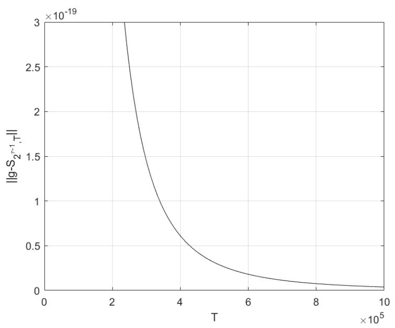
(c) For $T = 1000$ and $\tau = 3$



(d) For $T = 1000000$ and $\tau = 3$



(e) For $T = 1000$ and $\tau = 4$



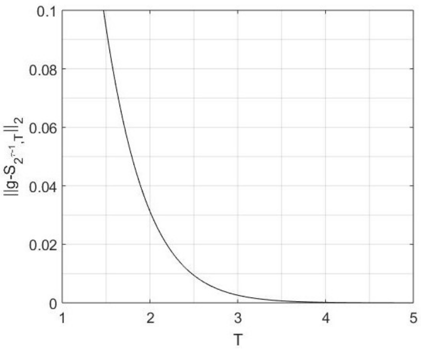
(f) For $T = 1000000$ and $\tau = 4$

Figure 1: Degree of convergence of function g for different values of T and τ in Theorem 3.1.

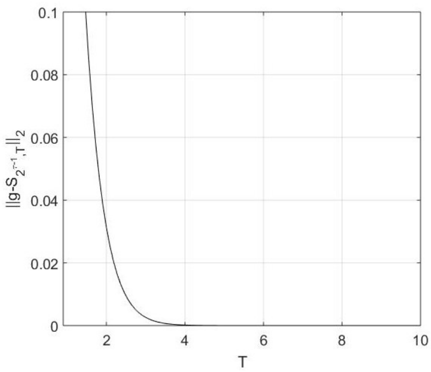
Now, we will construct the following table and draw the figures to validate our Theorem 3.2.

	$E_{2^{\tau-1},T}(g) = \|g - S_{2^{\tau-1},T}\|_2$			
T	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
1	0.5	2.5×10^{-1}	1.25×10^{-1}	6.25×10^{-2}
2	1.25×10^{-1}	3.125×10^{-2}	7.8125×10^{-3}	1.953125×10^{-3}
5	2.6041666×10^{-4}	8.13802×10^{-6}	2.54313×10^{-7}	7.94728×10^{-9}
10	2.691×10^{-10}	2.628×10^{-13}	2.566×10^{-16}	2.506×10^{-19}
100	8.452×10^{-189}	6.668×10^{-219}	5.260×10^{-249}	4.149×10^{-279}
1000	2.319×10^{-2869}	2.164×10^{-3170}	2.020×10^{-3471}	1.885×10^{-3772}
10000	$10^{-38669.75423116059}$	$10^{-41680.05418780040}$	$10^{-44690.35414444021}$	$10^{-47700.65410108003}$
100000	$10^{-10^{5.687240331902128}}$	$10^{-10^{5.713305235415010}}$	$10^{-10^{5.737893986339460}}$	$10^{-10^{5.761164860942931}}$
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
∞	0	0	0	0

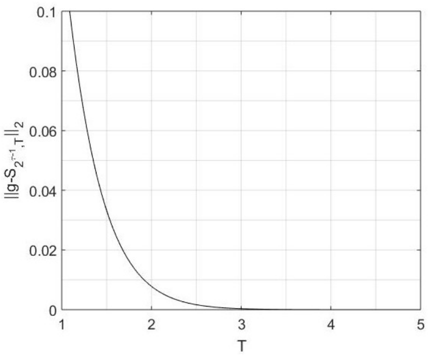
Table 2: Degree of convergence of g for different T and some values of τ in Theorem 3.2.



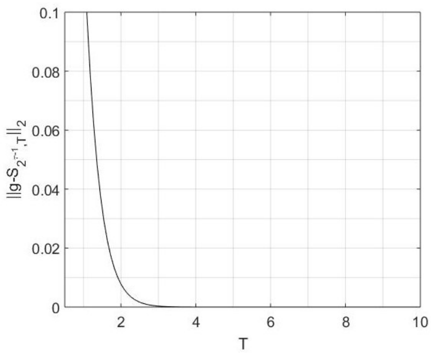
(a) For $T = 5$ and $\tau = 2$



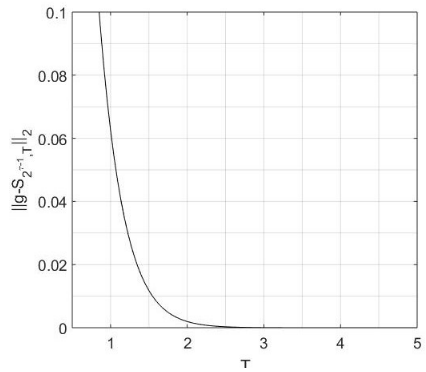
(b) For $T = 10$ and $\tau = 2$



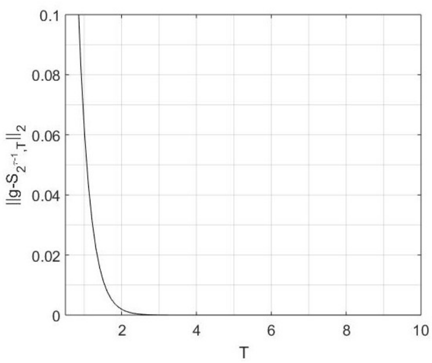
(c) For $T = 5$ and $\tau = 3$



(d) For $T = 10$ and $\tau = 3$



(e) For $T = 5$ and $\tau = 4$



(f) For $T = 10$ and $\tau = 4$

Figure 2: Degree of convergence of function g for different values of T and τ in Theorem 3.2.

5. Applications

Laguerre wavelets find valuable applications in diverse fields. They serve as effective tools for signal denoising, particularly adept at preserving essential information while filtering noise, making them pivotal in applications such as speech processing and data analysis. In quantum mechanics, they offer an efficient basis for solving Schrödinger's equation, particularly in systems with rotational symmetry.

In biomedical signal processing, Laguerre wavelets facilitate the precise analysis of vital biomedical data, including EEG and ECG signals, enabling anomaly detection and trend analysis for medical monitoring. Furthermore, in financial time series analysis, Laguerre wavelets enable trend identification and forecasting in stock markets and currency exchange, providing valuable insights into financial data. Laguerre wavelets can aid in image registration, where two or more images are aligned or matched to each other. They can help in finding correspondences between features in different images. Beyond images, Laguerre wavelets can also be applied in data compression for various types of data, such as audio signals, time series data, and more.

6. Conclusions

The wavelet approximation of a function with a higher number of bounded derivatives tends to provide a more precise estimate compared to the wavelet approximation of a function with fewer bounded derivatives. Our findings represent the most accurate results achievable.

Acknowledgement

The authors are highly thankful to the referees and editor for their insightful comments and valuable suggestions for improvement of the paper. The third author is supported by C.S.I.R, New Delhi, India through Junior Research Fellowship (JRF) vide letter no. 16/06/2019(i)EU-V (Dated-29/10/2019).

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