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Hölder's seminormwise approximation of bivariable functions by the double delayed arithmetic means of Fourier series

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Abstract. The two approximation theorems pertaining to the Hölder seminormwise approximation of Lipschitz functions are proved. Some special case for Lipschitz functions are formulated as corollaries, where the Jackson orders of approximation are obtained.

1. Introduction

By L^p, $1 \leq p < \infty$ [C when $p = \infty$], we denote the space of all 2π periodic in each variable real valued bivariable functions f integrable with p-power [continuous] on $T^2 := (0, 2\pi) \times (0, 2\pi)$, and with the norm

$$\|f\|_p = \|f(\cdot,\cdot)\|_p := \begin{cases} \left(\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^p dxdy\right)^{1/p}, \ 1 \leq p < \infty, \\ \sup_{(x,y)\in T^2} |f(x,y)|, \ p = \infty. \end{cases}$$

For a seminorm P, we define the following seminormed spaces:

$$(L^p, P) = \{f \in L^p : P(f) = P(f(\cdot,\cdot)) < \infty\}$$

and

$$(C, P) = \{f \in C : P(f) = P(f(\cdot,\cdot)) < \infty\},$$

where, for any real h_1, h_2, $P[f(\cdot + h_1, \cdot + h_2)] = P[f(\cdot,\cdot)]$, and $f(\cdot + h_1, \cdot + h_2) \in (L^p, P)$ [$\in (C_{2\pi}, P)$], respectively. We say that a seminorm P is order monotonic if for all $f, g \in (L^p, P)$ [$\in (C, P)$] such that $|f(x,y)| \leq |g(x,y)|$ for every $(x,y) \in T^2$, then

$$P[f] \leq P[g]. \tag{1}$$

Let v_i, $(i = 1, 2)$ be two functions of modulus of continuity type on $[0, 2\pi]$, i.e. nondecreasing continuous functions having the following properties: $v_i(0) = 0$, $v_i(\delta_1 + \delta_2) \leq v_i(\delta_1) + v_i(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. We define the Hölder space $(H, P)^{(v_1,v_2)}$ by

$$(H, P)^{(v_1,v_2)} := \{f \in (L^p, P) : A(f; v_1, v_2) < \infty\},$$

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where

$$A(f;v_1,v_2) := \sup_{t_1\neq 0,\ t_2\neq 0} \frac{P\left[f(\cdot+t_1,\cdot+t_2)-f(\cdot,\cdot)\right]}{v_1(|t_1|)+v_2(|t_2|)}.$$

The seminorm in the space $(H,P)^{(v_1,v_2)}$ is defined by

$$P\left[f\right]^{(v_1,v_2)} := P\left[f\right] + A(f;v_1,v_2).$$

If μ_1, μ_2, v_1 and v_2 are the functions of moduli of continuity type so that $\frac{\mu_1(0)}{v_1(0)} = \frac{\mu_2(0)}{v_2(0)} = 0$ and the two-variable function $\frac{\mu_1(t_1)+\mu_2(t_2)}{v_1(t_1)+v_2(t_2)}$ has a maximum M on T^2, then it is easy to see that

$$P\left[f\right]^{(v_1,v_2)} \leq \max(1,M)\, P\left[f\right]^{(\mu_1,\mu_2)},$$

which shows that in this case, for the given spaces $(H,P)^{(\mu_1,\mu_2)}$ and $(H,P)^{(v_1,v_2)}$ we have

$$(H,P)^{(\mu_1,\mu_2)} \subseteq (H,P)^{(v_1,v_2)} \subseteq (L^p,P).$$

We write

$$\Omega_P(f;t_1,t_2) := \sup_{0\leq h_1\leq t_1; 0\leq h_2\leq t_2} P\left[f(\cdot+h_1,\cdot+h_2)-f(\cdot,\cdot)\right]$$

for the integral modulus of continuity of f, and

$$\mathrm{Lip}(w_1,w_2,P) = \{f\in(L^p,P) : \Omega_P(f;t_1,t_2) = O\left(w_1(t_1)+w_2(t_2)\right)\}$$

with the functions w_1, w_2 of moduli of continuity type. Clearly, for $w_1(t_1) = O\left(t_1^{\alpha}\right)$ and $w_2(t_2) = O\left(t_2^{\beta}\right)$, $0<\alpha\leq 1$, $0<\beta\leq 1$, the class $\mathrm{Lip}(w_1,w_2,P)$ reduces to the class $\mathrm{Lip}(\alpha,\beta,P)$, that is

$$\mathrm{Lip}(\alpha,\beta,P) = \left\{f\in(L^p,P) : \Omega_P(f;t_1,t_2) = O\left(t_1^{\alpha}\right)+O\left(t_2^{\beta}\right)\right\}.$$

Then, for $1\geq\alpha\geq\gamma\geq 0$ and $1\geq\beta\geq\delta\geq 0$, by noting $\frac{t_1^{\alpha}+t_2^{\beta}}{t_1^{\gamma}+t_2^{\delta}}$ is bounded on T^2, we have

$$\mathrm{Lip}(\alpha,\beta,P) \subseteq \mathrm{Lip}(\gamma,\delta,P) \subseteq (L^p,P).$$

Let $f\in L^1$ with the double Fourier series

$$S(f;x,y) = S(f;\cdot,\cdot) = S(f)$$

and its rectangular partial sums

$$S_{k,l}(f;x,y) = S_{k,l}(f;\cdot,\cdot) = S_{k,l}(f),$$

and their $(C,1,1)$ means

$$\sigma_{m,n}(f) = \frac{1}{(m+1)(n+1)}\sum_{k=0}^{m}\sum_{l=0}^{n} S_{k,l}(f),$$

where $k,l,m,n\in\mathbb{N}\cup\{0\}$.

To reveal our intention, we recall some other notations and notions.

We define the double delayed arithmetic mean $\sigma_{m,k;n,l}(f)$ by (see [6]):

$$\sigma_{m,k;n,l}(f) := \left(1+\frac{m}{k}\right)\left(1+\frac{n}{l}\right)\sigma_{m+k-1,n+l-1}(f) - \left(1+\frac{m}{k}\right)\frac{n}{l}\sigma_{m+k-1,n-1}(f) \qquad (2)$$

$$-\frac{m}{k}\left(1+\frac{n}{l}\right)\sigma_{m-1,n+l-1}(f)+\frac{mn}{k}\sigma_{m-1,n-1}(f),$$

when k and l are positive integers.

If k tends to ∞ with m in such a way that $\frac{m}{k}$ is bounded, and l tends to ∞ with n in such a way that $\frac{n}{l}$ is also bounded, then $\sigma_{m,k;n,l}(f)$ defines a method of summability which is at least as strong as the well-known $(C,1,1)$ summability. This means that if $\sigma_{m,n}(f)\to g$, then $\sigma_{m,k;n,l}(f)\to g$ as well. This important fact follows from (2) if we set $\sigma_{m,n}(f)=g+\xi_{m,n}$, where $\xi_{m,n}\to 0$ as $m,n\to\infty$. Introducing this mean we expect be useful in applications, particularly in approximating of 2π periodic functions in two variables.

We note that for $k=l=1$ we obtain $\sigma_{m,1;n,1}(f)=S_{m,n}(f)$, while for $m=n=0$ we get $\sigma_{0,k;0,l}(f)=\sigma_{k-1,l-1}(f)$. Moreover, for $k=m$ and $l=n$, we get

$$\sigma_{m,m;n,n}(f)=4\sigma_{2m-1,2n-1}(f)-2\sigma_{2m-1,n-1}(f)-2\sigma_{m-1,2n-1}(f)+\sigma_{m-1,n-1}(f)$$

and taking $k=2m$ and $l=2n$ we obtain

$$\sigma_{m,2m;n,2n}(f)=\frac{1}{4}\left(9\sigma_{3m-1,3n-1}(f)-3\sigma_{3m-1,n-1}(f)-3\sigma_{m-1,3n-1}(f)+\sigma_{m-1,n-1}(f)\right).$$

It is well-known (see e.g. [2, page 4])

$$\sigma_{m,n}(f;x,y)=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\Delta_{x,y}(f;t_1,t_2)F_{m,n}(t_1,t_2)\,dt_1dt_2, \tag{3}$$

where the double Fejér kernel is

$$\begin{aligned} F_{m,n}(t_1,t_2) \quad : \quad &= \frac{1}{(m+1)(n+1)}\frac{(1-\cos(m+1)t_1)(1-\cos(n+1)t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2} \\ &= \frac{4}{(m+1)(n+1)}\left(\frac{\sin\frac{(m+1)t_1}{2}\sin\frac{(n+1)t_2}{2}}{4\sin\frac{t_1}{2}\sin\frac{t_2}{2}}\right)^2 \end{aligned} \tag{4}$$

and

$$\Delta_{x,y}(f;t_1,t_2):=f(x+t_1,y+t_2)+f(x-t_1,y+t_2)+f(x+t_1,y-t_2)+f(x-t_1,y-t_2).$$

Furthermore, using (4) successively, we have

$$\begin{aligned} F_{3m-1,3n-1}(t_1,t_2) &= \frac{(1-\cos(3mt_1))(1-\cos(3nt_2))}{9mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}, \\ F_{3m-1,n-1}(t_1,t_2) &= \frac{(1-\cos(3mt_1))(1-\cos(nt_2))}{3mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}, \\ F_{m-1,3n-1}(t_1,t_2) &= \frac{(1-\cos(mt_1))(1-\cos(3nt_2))}{3mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}, \\ F_{m-1,n-1}(t_1,t_2) &= \frac{(1-\cos(mt_1))(1-\cos(nt_2))}{mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}. \end{aligned}$$

Thus, we can write

$$F_{m,2m;n,2n}(t_1,t_2) \tag{5}$$

$$
\begin{aligned}
: \quad &= \frac{1}{4}\left(9F_{3m-1,3n-1}(t_1,t_2) - 3F_{3m-1,n-1}(t_1,t_2)\right.\\
&\left.-3F_{m-1,3n-1}(t_1,t_2) + F_{m-1,n-1}(t_1,t_2)\right)\\
= \quad & \frac{1}{4mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}\left[(1-\cos(3mt_1))(1-\cos(3nt_2))\right.\\
&-(1-\cos(3mt_1))(1-\cos(nt_2)) - (1-\cos(mt_1))(1-\cos(3nt_2))\\
&\left.+ (1-\cos(mt_1))(1-\cos(nt_2))\right]\\
= \quad & \frac{(\cos(mt_1)-\cos(3mt_1))(\cos(nt_2)-\cos(3nt_2))}{4mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}\\
= \quad & \frac{\sin 2mt_1 \sin mt_1 \sin 2nt_2 \sin nt_2}{mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2} = \frac{G_{m,n}(t_1,t_2)}{mn\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2},
\end{aligned}
$$

where $G_{m,n}(t_1,t_2) = \sin 2mt_1 \sin mt_1 \sin 2nt_2 \sin nt_2$.

Therefore, using (3) and (5), we get

$$\sigma_{m,2m;n,2n}(f;x,y) = \frac{1}{mn\pi^2}\int_0^{\pi}\int_0^{\pi}\Delta_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2.$$

Throughout this paper we will put:

$$
\begin{aligned}
&\phi_{x,y}(f;t_1,t_2) = \frac{1}{4}\left(\Delta_{x,y}(f;t_1,t_2) - 4f(x,y)\right)\\
&= \frac{1}{4}(f(x+t_1,y+t_2) + f(x-t_1,y+t_2) + f(x+t_1,y-t_2) + f(x-t_1,y-t_2) - 4f(x,y)),\\
&\Phi_{x,z_1,y,z_2}(f;t_1,t_2) = \phi_{x+z_1,y+z_2}(f;t_1,t_2) - \phi_{x,y}(f;t_1,t_2)
\end{aligned}
$$

and

$$D_{m,n}(f;x,y) := \sigma_{m,2m;n,2n}(f;x,y) - f(x,y).$$

The approximation of bivariable functions by their Fourier series in the Hölder metric were considered by many authors (see e. g. [1], [3], [7]). An application of seminorm as a measure of approximation was considered in one dimension only(see [5]). The aim of this paper is to obtain the Jackson order of approximation of the deviation $D_{m,n}$ measured by seminorm $P[\cdot]$ and the Hölder norm $P[\cdot]^{(v,v)}$. We will present the simplest cases, since in the case of multiple Fourier series and more general Lipschitz functions there are only technical difficulties.

The two approximation theorems and corollaries are formulated in the second section but its proofs are done in the last forth section. Some helpful results are proved in the third section.

2. Main Results

First we present the result on seminorm approximation and its special case, for the classical Lipschitz functions.

Theorem 2.1. *Let w_1 and w_2 be two functions of moduli of continuity type. If $f \in Lip(w_1,w_2,P)$, then*

$$P\left[\sigma_{m,2m;n,2n}(f) - f\right] = O\left(w_1(h_1) + w_2(h_2)\right),$$

where $h_1 = \frac{\pi}{m}$ and $h_2 = \frac{\pi}{n}$ for $n,m \in \mathbb{N}$.

Corollary 2.2. *If $f \in Lip(\alpha,\beta,P)$, where $\alpha,\beta \in (0,1]$, then*

$$P\left[\sigma_{m,2m;n,2n}(f)-f\right] = O\left(\left(\frac{1}{m}\right)^{\alpha} + \left(\frac{1}{n}\right)^{\beta}\right),$$

for $n,m \in \mathbb{N}$.

The second more general result on approximation in the Hölder norm will be formulated in the case $w_1 = w_2 = w$ only. A corollary for the classical Lipschitz functions will be given too.

Theorem 2.3. *Let w and v be two functions of moduli of continuity type so that $\frac{w}{v}$ is non-decreasing function. If $f \in H_P^{(w,w)}$, then*

$$P\left[\sigma_{m,2m;n,2n}(f)-f\right]^{(v,v)} = O\left(\frac{w(h_1)}{v(h_1)} + \frac{w(h_2)}{v(h_2)}\right),$$

where $h_1 = \frac{\pi}{m}$ and $h_2 = \frac{\pi}{n}$ for $n,m \in \mathbb{N}$.

Corollary 2.4. *If $f \in Lip(\alpha,\alpha,P)$ and $v(t) = O\left(t^{\gamma}\right)$, where $0 < \gamma < \alpha \le 1$, then*

$$P\left[\sigma_{m,2m;n,2n}(f)-f\right]^{(v,v)} = O\left(\left(\frac{1}{m}\right)^{\alpha-\gamma} + \left(\frac{1}{n}\right)^{\alpha-\gamma}\right),$$

for $n,m \in \mathbb{N}$.

3. Auxiliary Results

Lemma 3.1. *Let $f \in (L^p, P)$ with $1 \le p < \infty$ and $K \in L^1$. If a seminorm P satisfies (1), then*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cdot+t,\cdot+s)K(s,t)\,dsdt \in (L^p,P)$$

and

$$P\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cdot+t,\cdot+s)K(s,t)\,dsdt\right] \le P\left[f\right]\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |K(s,t)|\,dsdt.$$

Proof. Using (1) we can easily show that $P(f) = P\left(|f|\right)$. Therefore applying again (1)

$$P\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cdot+t,\cdot+s)K(s,t)\,dsdt\right] \le P\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left|f(\cdot+t,\cdot+s)K(s,t)\right| dsdt\right]$$

$$\begin{aligned} &= P\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |K(s,t)|\,dsdt \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(\cdot+t,\cdot+s)K(s,t)\right|}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(s,t)|\,dsdt} dsdt\right] \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |K(s,t)|\,dsdt \\ &\quad \cdot P\left[\frac{1}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(s,t)|\,dsdt}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f(\cdot+t,\cdot+s)K(s,t)\right| dsdt\right]. \end{aligned}$$

Using the Jensen inequality and the properties of seminorms we obtain

$$P\left[\frac{1}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(s,t)|\,dsdt}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f(\cdot+t,\cdot+s)K(s,t)\right| dsdt\right]$$

$$\leq \frac{1}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(s,t)|\,dsdt}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}P\left[\left|f(\cdot+t,\cdot+s)\right|\right]|K(s,t)|\,dsdt$$
$$= \frac{1}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(s,t)|\,dsdt}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}P\left[\left|f(\cdot,\cdot)\right|\right]|K(s,t)|\,dsdt = P\left[\left|f(\cdot,\cdot)\right|\right] = P[f]$$

and our result follows. □

Lemma 3.2. *Let w_1, w_2, v_1 and v_2 be four functions of moduli of continuity type. If $f \in Lip(w_1, w_2, P)$, then for nonnegative $s_1, s_2, t_1, t_2, z_1, z_2$,*

$$P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)\right] = O\left(w_1(t_1)+w_2(t_2)\right), \tag{6}$$

$$P\left[\phi_{\cdot,\cdot}(f;t_1+s_1,t_2+s_2)-\phi_{\cdot,\cdot}(f;t_1,t_2)\right] = O\left(w_1(s_1)+w_2(s_2)\right), \tag{7}$$

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w_1(t_1)+w_2(t_2)\right), \tag{8}$$

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w_1(z_1)+w_2(z_2)\right), \tag{9}$$

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w_1(z_1)+w_2(z_2)\right) \tag{10}$$

and

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w_1(s_1)+w_2(s_2)\right). \tag{11}$$

Moreover, if $w_1 = w_2 = w$ and $v_1 = v_2 = v$ and $\frac{w(t)}{v(t)}$ is non-decreasing in t, then

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left((v(z_1)+v(z_2))\left(\frac{w(t_1)}{v(t_1)}+\frac{w(t_2)}{v(t_2)}\right)\right) \tag{12}$$

and

$$\begin{aligned} & P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] \\ = \; & O\left((v(z_1)+v(z_2))\left(\frac{w(s_1)}{v(s_1)}+\frac{w(s_2)}{v(s_2)}\right)\right). \end{aligned} \tag{13}$$

Proof. We will use the typical properties of seminorms in all parts.

Part (6). An easy calculation gives

$$\begin{aligned} & P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)\right] \\ \leq \; & \frac{1}{4}\{P[f(\cdot+t_1,\cdot+t_2)-f(\cdot,\cdot)]+P[f(\cdot-t_1,\cdot+t_2)-f(\cdot,\cdot)] \\ & +P[f(\cdot+t_1,\cdot-t_2)-f(\cdot,\cdot)]+P[f(\cdot-t_1,\cdot-t_2)-f(\cdot,\cdot)]\} \\ = \; & O\left(w_1(t_1)+w_2(t_2)\right). \end{aligned}$$

Part (7). Analogously as before

$$\begin{aligned} & P\left[\phi_{\cdot,\cdot}(f;t_1+s_1,t_2+s_2)-\phi_{\cdot,\cdot}(f;t_1,t_2)\right] \\ \leq \; & \frac{1}{4}\{P[f(\cdot+t_1+s_1,\cdot+t_2+s_2)-f(\cdot+t_1,\cdot+t_2)] \\ & +P[f(\cdot-t_1-s_1,\cdot+t_2+s_2)-f(\cdot-t_1,\cdot+t_2)] \\ & +P[f(\cdot+t_1+s_1,\cdot-t_2-s_2)-f(\cdot+t_1,\cdot-t_2)] \\ & +P[f(\cdot-t_1-s_1,\cdot-t_2-s_2)-f(\cdot-t_1,\cdot-t_2)]\} \end{aligned}$$

$$= \quad O\left(w_1(s_1) + w_2(s_2)\right).$$

Part (8). By (6) and using the properties of the seminorm P

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = P\left[\phi_{\cdot+z_1,\cdot+z_2}(f;t_1,t_2) - \phi_{\cdot,\cdot}(f;t_1,t_2)\right]$$
$$\leq \quad P\left[\phi_{\cdot+z_1,\cdot+z_2}(f;t_1,t_2)\right] + P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)\right] = 2P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)\right] = O\left(w_1(t_1) + w_2(t_2)\right).$$

Part (9). A similar reason yields

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = P\left[\phi_{\cdot+z_1,\cdot+z_2}(f;t_1,t_2) - \phi_{\cdot,\cdot}(f;t_1,t_2)\right]$$

$$\begin{aligned}
\leq \quad & \frac{1}{4}\{P\left[f(\cdot+z_1+t_1,\cdot+z_2+t_2) - f(\cdot+t_1,\cdot+t_2)\right] \\
& + P\left[f(\cdot+z_1-t_1,\cdot+z_2+t_2) - f(\cdot-t_1,\cdot+t_2)\right] \\
& + P\left[f(\cdot+z_1+t_1,\cdot+z_2-t_2) - f(\cdot+t_1,\cdot-t_2)\right] \\
& + P\left[f(\cdot+z_1-t_1,\cdot+z_2-t_2) - f(\cdot-t_1,\cdot-t_2)\right] + 4P\left[f(\cdot+z_1,\cdot+z_2) - f(\cdot,\cdot)\right]\} \\
= \quad & 2P\left[f(\cdot+z_1,\cdot+z_2) - f(\cdot,\cdot)\right] = O\left(w_1(z_1) + w_2(z_2)\right).
\end{aligned}$$

Part (10). By (9) and using the properties of the seminorm P

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2) - \Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right]$$
$$\leq \quad P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)\right] + P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w_1(z_1) + w_2(z_2)\right).$$

Part (11). By (7) and using the properties of the seminorm P

$$\begin{aligned}
& P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2) - \Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] \\
= \quad & P\left[\phi_{\cdot+z_1,\cdot+z_2}(f;t_1+s_1,t_2+s_2) - \phi_{\cdot,\cdot}(f;t_1+s_1,t_2+s_2) - \phi_{\cdot+z_1,\cdot+z_2}(f;t_1,t_2) + \phi_{\cdot,\cdot}(f;t_1,t_2)\right] \\
\leq \quad & P\left[\phi_{\cdot+z_1,\cdot+z_2}(f;t_1+s_1,t_2+s_2) - \phi_{\cdot+z_1,\cdot+z_2}(f;t_1,t_2)\right] + P\left[\phi_{\cdot,\cdot}(f;t_1+s_1,t_2+s_2) - \phi_{\cdot,\cdot}(f;t_1,t_2)\right] \\
= \quad & O\left(w_1(s_1) + w_2(s_2)\right).
\end{aligned}$$

Part (12). Using (8) and the fact $v(t)$ is non-decreasing function of t, in case of $t_1 \leq z_1$ and $t_2 \leq z_2$, we get

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(t_1) + w(t_2)\right)$$

$$= O\left(\nu(t_1)\frac{w(t_1)}{\nu(t_1)} + \nu(t_2)\frac{w(t_2)}{\nu(t_2)}\right) = O\left(\nu(z_1)\frac{w(t_1)}{\nu(t_1)} + \nu(z_2)\frac{w(t_2)}{v(t_2)}\right).$$

Since $\frac{w(t)}{v(t)}$ is non-decreasing function of t, by (9), for $t_1 \geq z_1$ and $t_2 \geq z_2$, we also obtain

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(z_1) + w(z_2)\right)$$

$$= O\left(\nu(z_1)\frac{w(z_1)}{\nu(z_1)} + \nu(z_2)\frac{w(z_2)}{\nu(z_2)}\right) = O\left(\nu(z_1)\frac{w(t_1)}{\nu(t_1)} + \nu(z_2)\frac{w(t_2)}{\nu(t_2)}\right).$$

For $t_1 \leq z_1$ and $t_2 \geq z_2$, we consider two possibilities. If $z_1 \geq t_2$, then using (6) and the monotonicity of w and v, we get

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(t_1) + w(t_2)\right) = O\left(\nu(t_1)\frac{w(t_1)}{\nu(t_1)} + \nu(z_1)\frac{w(t_2)}{\nu(z_1)}\right)$$

$$= \quad O\left(\nu(z_1)\frac{w(t_1)}{\nu(t_1)} + \nu(z_1)\frac{w(t_2)}{\nu(t_2)}\right) = O\left(\nu(z_1)\left(\frac{w(t_1)}{\nu(t_1)} + \frac{w(t_2)}{\nu(t_2)}\right)\right)$$

$$\leq \quad O\left((\nu(z_1)+\nu(z_2))\left(\frac{w(t_1)}{\nu(t_1)}+\frac{w(t_2)}{\nu(t_2)}\right)\right).$$

If $t_2 \geq z_1$, then using (9) and the monotonicity of $\frac{w}{v}$, we obtain

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(z_1)+w(z_2)\right)$$

$$= O\left(\nu(z_1)\frac{w(z_1)}{\nu(z_1)}+\nu(z_2)\frac{w(z_2)}{\nu(z_2)}\right) \leq O\left(\nu(z_1)\frac{w(t_2)}{\nu(t_2)}+\nu(z_2)\frac{w(t_2)}{\nu(t_2)}\right)$$

$$= O\left((\nu(z_1)+\nu(z_2))\,\frac{w(t_2)}{\nu(t_2)}\right) = O\left((\nu(z_1)+\nu(z_2))\left(\frac{w(t_1)}{\nu(t_1)}+\frac{w(t_2)}{\nu(t_2)}\right)\right).$$

For $t_1 \geq z_1$ and $t_2 \leq z_2$, we can use the analogous justification as above. Thus, the validity of (12) is proved.

Part (13). Similarly as in the proof of Part (12) by (10) and (11)

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(z_1)+w(z_2)\right)$$

and

$$P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+s_1,t_2+s_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right] = O\left(w(s_1)+w(s_2)\right),$$

whence our estimate follows. □

4. Proofs of the Results

In the proofs we will use Lemmas 3.1 and 3.2, the estimates

$$\begin{aligned} \left|G_{m,n}(t_1,t_2)\right| &= O(1), \ \left|G_{m,n}(t_1,t_2)\right| = O\left((mt_1nt_2)^2\right), \\ \left|G_{m,n}(t_1,t_2)\right| &= O\left((mt_1)^2\right), \ \left|G_{m,n}(t_1,t_2)\right| = O\left((nt_2)^2\right), \quad 0<t_1,t_2\leq\pi, \end{aligned} \tag{14}$$

where as above $G_{m,n}(t_1,t_2) = \sin 2mt_1 \sin mt_1 \sin 2nt_2 \sin nt_2$ and the following application of the mean value theorem in differential calculus with $\theta_i \in (0,1)\,, i=1,2,$

$$\frac{1}{\left(2\sin\frac{t_i}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_i+h_i}{2}\right)^2} = \frac{\sin^2\frac{t_i+h_i}{2}-\sin^2\frac{t_i}{2}}{4\sin^2\frac{t_i}{2}\sin^2\frac{t_i+h_i}{2}} = \frac{2h_i\sin\frac{t_i+\theta h_i}{2}\cos\frac{t_i+\theta h_i}{2}}{4\sin^2\frac{t_i}{2}\sin^2\frac{t_i+h_i}{2}}. \tag{15}$$

We will also use property (1) of seminorm P, the typical properties of seminorms and the following inequalities for the function sinus

$$2\frac{\beta}{\pi} \leq \sin\beta \text{ for } \beta \in (0,\frac{\pi}{2}) \text{ and } \sqrt{2}\frac{\beta}{6\pi} \leq \sin\beta \text{ for } \beta \in (0,\frac{3\pi}{4}). \tag{16}$$

4.1. *Proof of Theorem 1*

Using the equality

$$\int_0^{\pi}\int_0^{\pi}\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 = \frac{mn\pi^2}{4}$$

we can write

$$D_{m,n}(f;x,y) = \sigma_{m,2m;n,2n}(f;x,y)-f(x,y)$$

$$= \frac{4}{mn\pi^2}\int_0^{\pi}\int_0^{\pi}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$= \frac{4}{mn\pi^2}\left(\int_0^{h_1}\int_0^{h_2}+\int_{h_1}^{\pi}\int_0^{h_2}+\int_0^{h_1}\int_{h_2}^{\pi}+\int_{h_1}^{\pi}\int_{h_2}^{\pi}\right)\phi_{x,y}(f;t_1,t_2)\cdot$$

$$\cdot\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 = \sum_{r=1}^{4} I_r,$$

where $h_1 = \frac{\pi}{m}$ and $h_2 = \frac{\pi}{n}$. We can note that $I_2 = 0$, $I_3 = 0$ and $I_4 = 0$ when $m = 1$ or $n = 1$, therefore in estimates of these integrals we will assume $m > 1$ or $n > 1$.

It is clear that

$$P\left[D_{m,n}(f;\cdot,\cdot)\right] \le \sum_{r=1}^{4} P\left[I_r\right].$$

Then we obtain

$$\begin{aligned} P\left[I_1\right] &\le \frac{4}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)\right]\frac{\left|G_{m,n}(t_1,t_2)\right|}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &\le \frac{4}{mn\pi^2}O\left(w_1(h_1)+w_2(h_2)\right)\int_0^{h_1}\int_0^{h_2}(mn)^2\,dt_1dt_2 = O\left(w_1(h_1)+w_2(h_2)\right). \end{aligned}$$

Next, substituting $t_1 + h_1$ instead of t_1 in I_2 we get

$$I_2 = \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$\begin{aligned} &+\left(-\int_0^{h_1}-\int_{h_1}^{\pi}+\int_{\pi-h_1}^{\pi}\right)\int_0^{h_2}\frac{2\phi_{x,y}(f;t_1+h_1,t_2)G_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ =& \int_{h_1}^{\pi}\int_0^{h_2}\left[\frac{\phi_{x,y}(f;t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\frac{2G_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &-\frac{2}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}\frac{\phi_{x,y}(f;t_1+h_1,t_2)G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &+\frac{2}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_0^{h_2}\frac{\phi_{x,y}(f;t_1+h_1,t_2)G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 = \sum_{s=1}^{3} I_{2,s}. \end{aligned}$$

Lemma 3.1, Lemma 3.2, the inequalities (14) and (16) give

$$\begin{aligned} P\left[I_{2,2}\right] &\le \frac{2}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}P\left[\phi_{\cdot,\cdot}(f;t_1+h_1,t_2)\right]\frac{O\left((mt_1nt_2)^2\right)}{\left(2\frac{t_1+h_1}{\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2 \\ &= O\left(mn\right)\int_0^{h_1}\int_0^{h_2}\left(w_1(t_1+h_1)+w_2(t_2)\right)dt_1dt_2 \\ &= O\left(mn\right)\left(w_1(h_1)+w_2(h_2)\right)\int_0^{h_1}\int_0^{h_2}dt_1dt_2 = O\left(w_1(h_1)+w_2(h_2)\right) \end{aligned}$$

and

$$\begin{aligned} P\left[I_{2,3}\right] &\leq \frac{2}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_0^{h_2} P\left[\phi_{\cdot,\cdot}(f;t_1+h_1,t_2)\right]\frac{O\left((nt_2)^2\right)}{\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2 \\ &= O\left(\frac{n}{m}\right)\int_{\pi-h_1}^{\pi}\int_0^{h_2}\left(w_1(t_1+h_1)+w_2(t_2)\right)\frac{dt_1dt_2}{(t_1+h_1)^2} \\ &= O\left(\frac{n}{m}\right)\int_{\pi}^{\pi+h_1}\int_0^{h_2}\left(w_1(t_1)+w_2(t_2)\right)\frac{dt_1dt_2}{t_1^2} \end{aligned}$$

$$\begin{aligned} &= O\left(\frac{n}{m}\right)\left(w_1(\pi+h_1)+w_2(h_2)\right)\int_{\pi}^{\pi+h_1}\int_0^{h_2}\frac{dt_1dt_2}{t_1^2} \\ &= O\left(\frac{\pi}{m}\right)\left(w_1(\pi)+w_2(h_2)\right)\frac{h_1}{\pi\left(\pi+h_1\right)} \\ &= O\left(\frac{1}{m^2}\right)\left(mw_1(h_1)+w_2(h_2)\right)=O\left(w_1(h_1)+w_2(h_2)\right). \end{aligned}$$

Integral $I_{2,1}$ we write in the following form

$$I_{2,1}=\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\left[\frac{\phi_{x,y}(f;t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}\right.$$

$$\begin{aligned} &\left.+\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\frac{2G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &= \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\left[\phi_{x,y}(f;t_1,t_2)-\phi_{x,y}(f;t_1+h_1,t_2)\right]\frac{2G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_2}{2}\right)^2\left(2\sin\frac{t_1}{2}\right)^2}dt_1dt_2 \\ &+\int_{h_1}^{\pi}\int_0^{h_2}\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\frac{2\phi_{x,y}(f;t_1+h_1,t_2)G_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &= I_{2,1}^{(1)}+I_{21}^{(2)}. \end{aligned}$$

Once more, by Lemma 3.1, Lemma 3.2, the inequalities (14) and (16), we obtain

$$\begin{aligned} &P\left[I_{2,1}^{(1)}\right] \\ &\leq \int_{h_1}^{\pi}\int_0^{h_2}\frac{2P\left[\phi_{\cdot,\cdot}(f;t_1,t_2)-\phi_{\cdot,\cdot}(f;t_1+h_1,t_2)\right]\left|G_{m,n}(t_1,t_2)\right|}{mn\pi^2\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ &= O\left(\frac{1}{mn}\right)\int_{h_1}^{\pi}\int_0^{h_2}\frac{O\left(w_1(h_1)+w_2(0)\right)(nt_2)^2}{\left(2\frac{t_1}{\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2 \\ &= O\left(\frac{n}{m}\right)\left(w_1(h_1)+w_2(h_2)\right)\int_{h_1}^{\pi}\int_0^{h_2}\frac{1}{t_1^2}dt_1dt_2=O\left(\left(w_1(h_1)+w_2(h_2)\right)\right) \end{aligned}$$

and

$$P\left[I_{2,1}^{(2)}\right]$$

$$\leq \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2} P\left[\phi_{\cdot,\cdot}(f;t_1+h_1,t_2)\right]\left|\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right|\frac{O\left((nt_2)^2\right)}{\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{n}{m}\right)\int_{h_1}^{\pi}\int_0^{h_2}\left(w_1(t_1+h_1)+w_2(t_2)\right)\frac{h_1\frac{t_1+\theta h_1}{2}}{4\left(2\frac{t_1}{\pi}\right)^2\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{n}{m}\right)\int_{h_1}^{\pi}\int_0^{h_2}\left(w_1(t_1+h_1)+w_2(h_2)\right)\frac{h_1\frac{t_1+\theta h_1}{2}}{\left(2\frac{t_1}{\pi}\right)^2\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{1}{m^2}\right)\left(\int_{h_1}^{\pi}\frac{w_1(t_1+h_1)dt_1}{t_1^2\,(t_1+h_1)}+w_2(h_2)\int_{h_1}^{\pi}\frac{dt_1}{t_1^3}\right)$$

$$= O\left(\frac{1}{m^2}\right)\left(\int_{h_1}^{\pi}\frac{2w_1(h_1)dt_1}{t_1^2h_1}+\frac{w_2(h_2)}{(h_1)^2}\right)=O\left(w_1(h_1)+w_2(h_2)\right).$$

Hence

$$P\left[I_2\right]=O\left(w_1(h_1)+w_2(h_2)\right)$$

and similarly

$$P\left[I_3\right]=O\left(w_1(h_1)+w_2(h_2)\right).$$

The integral I_4 can be rewritten as a sum in the following way

$$I_4=\frac{4}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{1}{mn\pi^2}\int_0^{\pi-h_1}\int_{h_2}^{\pi}\phi_{x,y}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{\pi-h_2}\phi_{x,y}(f;t_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$
$$+\frac{1}{mn\pi^2}\int_0^{\pi-h_1}\int_0^{\pi-h_2}\phi_{x,y}(f;t_1+h_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{1}{mn\pi^2}\left(\int_{h_1}^{\pi}\int_{h_2}^{\pi}-\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}+\int_0^{h_1}\int_{h_2}^{\pi}\right)\phi_{x,y}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{1}{mn\pi^2}\left(\int_{h_1}^{\pi}\int_{h_2}^{\pi}-\int_{h_1}^{\pi}\int_{\pi-h_2}^{\pi}+\int_{h_1}^{\pi}\int_0^{h_2}\right)\phi_{x,y}(f;t_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\left[\left(\int_{h_1}^{\pi}-\int_{\pi-h_1}^{\pi}+\int_0^{h_1}\right)\left(\int_{h_2}^{\pi}-\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)\right]$$
$$\phi_{x,y}(f;t_1+h_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\phi_{x,y}(f;t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right.$$
$$\left.-\frac{\phi_{x,y}(f;t_1,t_2+h_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}+\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$-\frac{1}{mn\pi^2}\left(-\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}+\int_0^{h_1}\int_{h_2}^{\pi}\right)\phi_{x,y}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{1}{mn\pi^2}\left(-\int_{h_1}^{\pi}\int_{\pi-h_2}^{\pi}+\int_{h_1}^{\pi}\int_0^{h_2}\right)\phi_{x,y}(f;t_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\left[\int_{h_1}^{\pi}\left(-\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)+\left(-\int_{\pi-h_1}^{\pi}+\int_0^{h_1}\right)\left(\int_{h_2}^{\pi}-\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)\right]$$
$$\phi_{x,y}(f;t_1+h_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2=\sum_{s=1}^{13}I_{4,s}.$$

Elementary calculations give

$$I_{4,1}$$
$$= \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\left[\phi_{x,y}(f;t_1,t_2)-\phi_{x,y}(f;t_1+h_1,t_2)-\phi_{x,y}(f;t_1,t_2+h_2)+\phi_{x,y}(f;t_1+h_1,t_2+h_2)\right]$$
$$\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\left[\phi_{x,y}(f;t_1+h_1,t_2)-\phi_{x,y}\left(f;t_1+h_1,t_2+h_2\right)\right]$$
$$\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\left[\phi_{x,y}(f;t_1,t_2+h_2)-\phi_{x,y}\left(f;t_1+h_1,t_2+h_2\right)\right]$$
$$\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\phi_{x,y}\left(f;t_1+h_1,t_2+h_2\right)\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]$$

$$\left[\frac{1}{\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2.$$

Using Lemma 3.1, Lemma 3.2, the inequalities (14) - (16) we get

$$\begin{aligned} P\left[I_{4,1}\right] \leq\ & \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\frac{O\left(w_1(h_1)+w_2(h_2)\right)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\ & +\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\frac{O\left(w_2(h_2)\right)}{\left(2\sin\frac{t_2}{2}\right)^2}\left|\frac{2h_1\sin\frac{t_1+\theta h_1}{2}\cos\frac{t_1+\theta_1 h_1}{2}}{4\sin^2\frac{t_1}{2}\sin^2\frac{t_1+h_1}{2}}\right|dt_1dt_2 \\ & +\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\frac{O\left(w_1(h_1)\right)}{\left(2\sin\frac{t_1}{2}\right)^2}\left|\frac{2h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{4\sin^2\frac{t_2}{2}\sin^2\frac{t_2+h_2}{2}}\right|dt_1dt_2 \\ & +\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}O\left(w_1(t_1+h_1)+w_2(t_2+h_2)\right) \\ & \left|\frac{2h_1\sin\frac{t_1+\theta h_1}{2}\cos\frac{t_1+\theta_1 h_1}{2}}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_1+h_1}{2}\right)^2}\frac{2h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{\left(2\sin\frac{t_2}{2}\right)^2\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right|dt_1dt_2 \\ = & \ O\left(w_1(h_1)+w_2(h_2)\right). \end{aligned}$$

Combining the terms in pairs, we have

$$\begin{aligned} & I_{4,2}+I_{4,8} \\ = & \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2 \\ = & \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\phi_{x,y}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right. \\ & \left.+\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2 \\ = & \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\phi_{x,y}(f;t_1+h_1,t_2)-\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right. \\ & \left.+\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\left(\frac{1}{\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right)\right]G_{m,n}(t_1,t_2)dt_1dt_2 \\ = & \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\phi_{x,y}(f;t_1+h_1,t_2)-\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right. \\ & \left.+\frac{\phi_{x,y}(f;t_1+h_1,t_2+h_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\left(\frac{h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{\left(2\sin\frac{t_2}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right)\right]G_{m,n}(t_1,t_2)dt_1dt_2, \end{aligned}$$

whence similarly as above

$$P\left[I_{4,2}+I_{4,8}\right] \leq \frac{1}{mn\pi^{2}} \int_{\pi-h_{1}}^{\pi} \int_{h_{2}}^{\pi} \frac{O\left(w_{2}(h_{2})\right)}{\left(t_{1}+h_{1}\right)^{2} t_{2}^{2}} dt_{1} dt_{2}$$

$$+\frac{h_{2}}{mn\pi^{2}} \int_{\pi-h_{1}}^{\pi} \int_{h_{2}}^{\pi} \frac{O\left(w_{1}(t_{1}+h_{1})+w_{2}(t_{2}+h_{2})\right)}{\left(t_{1}+h_{1}\right)^{2}} \frac{t_{2}+h_{2}}{\left(t_{2}+h_{2}\right)^{2} t_{2}^{2}} dt_{1} dt_{2}$$

$$\leq O\left(w_{1}(h_{1})+w_{2}(h_{2})\right).$$

Analogously

$$I_{4,3}+I_{4,11}$$

$$= \frac{1}{mn\pi^{2}} \int_{0}^{h_{1}} \int_{h_{2}}^{\pi}\left[-\frac{\phi_{x,y}(t_{1}+h_{1}, t_{2})}{\left(4 \sin \frac{t_{1}+h_{1}}{2} \sin \frac{t_{2}}{2}\right)^{2}}+\frac{\phi_{x,y}(t_{1}+h_{1}, t_{2}+h_{2})}{\left(4 \sin \frac{t_{1}+h_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}}\right] G_{m,n}(t_{1}, t_{2}) dt_{1} dt_{2},$$

whence

$$P\left[I_{4,3}+I_{4,11}\right] \leq O\left(w_{1}(h_{1})+w_{2}(h_{2})\right).$$

In the same way

$$I_{4,4}+I_{4,6}$$

$$= \frac{1}{mn\pi^{2}} \int_{h_{1}}^{\pi} \int_{\pi-h_{2}}^{\pi}\left[\frac{\phi_{x,y}(f; t_{1}, t_{2}+h_{2})}{\left(4 \sin \frac{t_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}}-\frac{\phi_{x,y}(f; t_{1}+h_{1}, t_{2}+h_{2})}{\left(4 \sin \frac{t_{1}+h_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}}\right] G_{m,n}(t_{1}, t_{2}) dt_{1} dt_{2}$$

and

$$I_{4,5}+I_{4,7}$$

$$= \frac{1}{mn\pi^{2}} \int_{h_{1}}^{\pi} \int_{0}^{h_{2}}\left[-\frac{\phi_{x,y}(f; t_{1}, t_{2}+h_{2})}{\left(4 \sin \frac{t_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}}+\frac{\phi_{x,y}(f; t_{1}+h_{1}, t_{2}+h_{2})}{\left(4 \sin \frac{t_{1}+h_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}}\right] G_{m,n}(t_{1}, t_{2}) dt_{1} dt_{2},$$

whence

$$P\left[I_{4,4}+I_{4,6}\right]+P\left[I_{4,5}+I_{4,7}\right] \leq O\left(w_{1}(h_{1})+w_{2}(h_{2})\right).$$

Finally,

$$I_{4,9}+I_{4,10}+I_{4,12}+I_{4,13}$$

$$= \frac{1}{mn\pi^{2}}\left(-\int_{\pi-h_{1}}^{\pi}+\int_{0}^{h_{1}}\right)\left(-\int_{\pi-h_{2}}^{\pi}+\int_{0}^{h_{2}}\right) \phi_{x,y}(f; t_{1}+h_{1}, t_{2}+h_{2})$$

$$\frac{G_{m,n}(t_{1}, t_{2})}{\left(4 \sin \frac{t_{1}+h_{1}}{2} \sin \frac{t_{2}+h_{2}}{2}\right)^{2}} dt_{1} dt_{2},$$

whence

$$P\left[I_{4,9}+I_{4,10}+I_{4,12}+I_{4,13}\right] \leq P\left[I_{4,9}\right]+P\left[I_{4,10}\right]+P\left[I_{4,12}\right]+P\left[I_{4,13}\right]$$

$$\leq \left(\int_{\pi-h_{1}}^{\pi}+\int_{0}^{h_{1}}\right)\left(\int_{\pi-h_{2}}^{\pi}+\int_{0}^{h_{2}}\right) \frac{O\left(w_{1}(t_{1}+h_{1})+w_{2}(t_{2}+h_{2})\right)}{mn\left(t_{1}+h_{1}\right)^{2}\left(t_{2}+h_{2}\right)^{2}}\left|G_{m,n}(t_{1}, t_{2})\right| dt_{1} dt_{2}$$

$$\begin{aligned}
&\leq \frac{1}{mn}\left(\int_{\pi-h_1}^{\pi}+\int_{0}^{h_1}\right)\left(\int_{\pi-h_2}^{\pi}+\int_{0}^{h_2}\right) \\
&\quad \left[\frac{O\left(w_1(h_1)\right)}{h_1\left(t_1+h_1\right)\left(t_2+h_2\right)^2}+\frac{O\left(w_2(h_2)\right)}{\left(t_1+h_1\right)^2\left(t_2+h_2\right)h_2}\right]\left(mt_1nt_2\right)^2 dt_1dt_2 \\
&\leq O\left(w_1(h_1)+w_2(h_2)\right)
\end{aligned}$$

and therefore

$$P\left[I_4\right]=O\left(w_1(h_1)+w_2(h_2)\right).$$

Thus our theorem is proved. □

4.2. *Proof of Theorem 2*

In the notations of the above proof and by definition

$$P\left[D_{m,n}\left(f\right)\right]^{(v,v)}=P\left[D_{m,n}(f;\cdot,\cdot)\right]+\sup_{z_1\neq 0,\ z_2\neq 0}\frac{P\left[D_{m,n}(f;\cdot+z_1,\cdot+z_2)-D_{m,n}(f;\cdot,\cdot)\right]}{v(|z_1|)+v(|z_2|)},$$

where

$$D_{m,n}(f;x,y)=\sigma_{m,2m;n,2n}(f;x,y)-f(x,y)=\frac{4}{mn\pi^2}\int_0^{\pi}\int_0^{\pi}\phi_{x,y}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

and

$$\begin{aligned}
&D_{m,n}(f;x+z_1,y+z_2)-D_{m,n}(f;x,y) \\
&= \frac{4}{mn\pi^2}\int_0^{\pi}\int_0^{\pi}\left[\phi_{x+z_1,y+z_2}(f;t_1,t_2)-\phi_{x,y}(f;t_1,t_2)\right]\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\
&= \frac{4}{mn\pi^2}\int_0^{\pi}\int_0^{\pi}\Phi_{x,z_1,y,z_2}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\
&= \frac{4}{mn\pi^2}\left(\int_0^{h_1}\int_0^{h_2}+\int_{h_1}^{\pi}\int_0^{h_2}+\int_0^{h_1}\int_{h_2}^{\pi}+\int_{h_1}^{\pi}\int_{h_2}^{\pi}\right) \\
&\quad \Phi_{x,z_1,y,z_2}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2=\sum_{r=1}^{4}J_r,
\end{aligned}$$

with $h_1=\frac{\pi}{m}$ and $h_2=\frac{\pi}{n}$. We can note that $J_2=0$, $J_3=0$ and $J_4=0$ when $m=1$ or $n=1$, therefore in estimates of these integrals we will assume $m>1$ or $n>1$, respectively.

Using Lemmas 3.1 and 3.2, by the estimates (14), (15) and (16), we obtain

$$\begin{aligned}
P\left[J_1\right] &\leq \frac{4}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)\right]\frac{\left|G_{m,n}(t_1,t_2)\right|}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \\
&= O\left(\nu(|z_1|)+\nu(|z_2|)\right)mn\int_0^{h_1}\int_0^{h_2}\left(\frac{w(t_1)}{v(t_1)}+\frac{w(t_2)}{v(t_2)}\right)dt_1dt_2 \\
&= O\left(\nu(|z_1|)+\nu(|z_2|)\right)mn\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\int_0^{h_1}\int_0^{h_2}dt_1dt_2
\end{aligned}$$

$$= O\left((\nu(|z_1|)+\nu(|z_2|))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right).$$

The quantity J_2 can be estimate analogously to I_2. Substituting t_1+h_1 in the place of t_1 in J_2 we get

$$J_2 = \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 \tag{17}$$
$$-\int_0^{\pi-h_1}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\frac{2G_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$= \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$+\left(-\int_0^{h_1}-\int_{h_1}^{\pi}+\int_{\pi-h_1}^{\pi}\right)\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\frac{2S_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$= \int_{h_1}^{\pi}\int_0^{h_2}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\frac{2G_{m,n}(t_1,t_2)}{mn\pi^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$-\frac{2}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$
$$+\frac{2}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2=\sum_{s=1}^{3}J_{2,s}.$$

Applying Lemmas 3.1 and 3.2, and using the inequalities (14) and (16), we have

$$P\left[J_{2,2}\right] \tag{18}$$
$$\le \frac{2}{mn\pi^2}\int_0^{h_1}\int_0^{h_2}P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+h_1,t_2)\right]\frac{O\left((mt_1nt_2)^2\right)}{\left(2\frac{t_1+h_1}{\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2$$
$$= O\left(mn\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\int_0^{h_1}\int_0^{h_2}\left(\frac{w(t_1+h_1)}{v(t_1+h_1)}+\frac{w(t_2)}{v(t_2)}\right)dt_1dt_2$$
$$= O\left(mn\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{w(2h_1)}{v(2h_1)}+\frac{w(h_2)}{v(h_2)}\right)\int_0^{h_1}\int_0^{h_2}dt_1dt_2$$
$$= O\left(mn\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{2w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\int_0^{h_1}\int_0^{h_2}dt_1dt_2$$
$$= O\left(\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)$$

and

$$P\left(J_{2,3}\right) \tag{19}$$
$$\le \frac{2}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_0^{h_2}P\left[\Phi_{\cdot,z_1,\cdot,z_2}(t_1+h_1,t_2)\right]\frac{O\left((nt_2)^2\right)}{\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2$$

$$
\begin{aligned}
&= O\left(\frac{n}{m}\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\int_{\pi-h_1}^{\pi}\int_0^{h_2}\left(\frac{w(t_1+h_1)}{v(t_1+h_1)}+\frac{w(t_2)}{v(t_2)}\right)\frac{dt_1dt_2}{(t_1+h_1)^2}\\
&= O\left(\frac{n}{m}\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\int_{\pi}^{\pi+h_1}\int_0^{h_2}\left(\frac{w(s_1)}{v(s_1)}+\frac{w(t_2)}{v(t_2)}\right)\frac{ds_1dt_2}{s_1^2}\\
&= O\left(\frac{n}{m}\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{w(\pi+h_1)}{v(\pi+h_1)}+\frac{w(h_2)}{v(h_2)}\right)\int_{\pi}^{\pi+h_1}\int_0^{h_2}\frac{ds_1dt_2}{s_1^2}\\
&= O\left(\frac{\pi}{m}\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{2w(\pi)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\frac{h_1}{\pi\,(\pi+h_1)}\\
&= O\left(\frac{1}{m^2}\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{mw(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\\
&= O\left(\left(\nu(|z_1|)+\nu(|z_2|)\right)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).
\end{aligned}
$$

For $J_{2,1}$ we can write

$$
\begin{aligned}
J_{2,1} = {}& \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}\right.\\
&\left.+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(t_1+h_1,t_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2\\
= {}& \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\left[\Phi_{x,z_1,y,z_2}(t_1,t_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\right]\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2\\
&+\frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]\\
&\frac{G_{m,n}(t_1,t_2)}{\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2 = J_{2,1}^{(1)}+J_{2,1}^{(2)}.
\end{aligned}
\tag{20}
$$

Then,

$$
\begin{aligned}
&P\left[J_{2,1}^{(1)}\right]\\
\leq{}& \int_{h_1}^{\pi}\int_0^{h_2}P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1,t_2)-\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+h_1,t_2)\right]\frac{2\left|G_{m,n}(t_1,t_2)\right|}{mn\pi^2\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}dt_1dt_2\\
={}& O\left(\frac{1}{mn}\right)\left(\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right)\int_{h_1}^{\pi}\int_0^{h_2}\frac{(nt_2)^2}{\left(2\frac{t_1}{\pi}\right)^2\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2\\
={}& O\left(\frac{n}{m}\right)\left(\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right)\int_{h_1}^{\pi}\int_0^{h_2}\frac{1}{t_1^2}dt_1dt_2
\end{aligned}
\tag{21}
$$

$$= O\left((v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right)$$

and

$$P\left[J_{2,1}^{(2)}\right] \tag{22}$$

$$\leq \frac{2}{mn\pi^2}\int_{h_1}^{\pi}\int_0^{h_2} P\left[\Phi_{\cdot,z_1,\cdot,z_2}(f;t_1+h_1,t_2)\right]\left|\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right|\frac{O\left((nt_2)^2\right)}{\left(2\frac{t_2}{\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{n}{m}\right)(v(z_1)+v(z_2))\int_{h_1}^{\pi}\int_0^{h_2}\left(\frac{w(t_1+h_1)}{v(t_1+h_1)}+\frac{w(t_2)}{v(t_2)}\right)\frac{h_1\frac{t_1+\theta h_1}{2}}{4\left(2\frac{t_1}{\pi}\right)^2\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{n}{m}\right)(v(z_1)+v(z_2))\int_{h_1}^{\pi}\int_0^{h_2}\left(\frac{w(t_1+h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\frac{h_1\frac{t_1+\theta h_1}{2}}{\left(2\frac{t_1}{\pi}\right)^2\left(\sqrt{2}\frac{t_1+h_1}{6\pi}\right)^2}dt_1dt_2$$

$$= O\left(\frac{1}{m^2}\right)(v(z_1)+v(z_2))\left(\frac{1}{v(h_1)}\int_{h_1}^{\pi}\frac{w(t_1+h_1)dt_1}{t_1^2(t_1+h_1)}+\frac{w(h_2)}{v(h_2)}\int_{h_1}^{\pi}\frac{dt_1}{t_1^3}\right)$$

$$= O\left(\frac{1}{m^2}\right)(v(z_1)+v(z_2))\left(\frac{1}{v(h_1)}\int_{h_1}^{\pi}\frac{2w(h_1)dt_1}{t_1^2h_1}+\frac{w(h_2)}{v(h_2)}\int_{h_1}^{\pi}\frac{dt_1}{t_1^3}\right)$$

$$= O\left((v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right).$$

So, from (20), (21) and (22), we have

$$P\left[J_{2,1}\right] = O\left((v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right). \tag{23}$$

Now, taking into account (17), (18), (19) and (23), we have

$$P\left[J_2\right] = O\left((v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right). \tag{24}$$

By analogy, we conclude that

$$P\left[J_3\right] = O\left((v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\right). \tag{25}$$

Finally, let us estimate the quantity J_4 analogously to I_2 too. So

$$J_4 = \frac{4}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\Phi_{x,z_1,y,z_2}(f;t_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$= \int_{h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right.$$
$$\left.-\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]\frac{G_{m,n}(t_1,t_2)}{mn\pi^2}dt_1dt_2$$

$$-\frac{1}{mn\pi^2}\left(-\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}+\int_{0}^{h_1}\int_{h_2}^{\pi}\right)\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2$$

$$-\frac{1}{mn\pi^2}\left(-\int_{h_1}^{\pi}\int_{\pi-h_2}^{\pi}+\int_{h_1}^{\pi}\int_{0}^{h_2}\right)\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2$$
$$+\frac{1}{mn\pi^2}\left[\int_{h_1}^{\pi}\left(-\int_{\pi-h_2}^{\pi}+\int_{0}^{h_2}\right)+\left(-\int_{\pi-h_1}^{\pi}+\int_{0}^{h_1}\right)\left(\int_{h_2}^{\pi}-\int_{\pi-h_2}^{\pi}+\int_{0}^{h_2}\right)\right]$$
$$\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2=\sum_{s=1}^{13}J_{4,s}.$$

The first integral $J_{4,1}$ we will divide into a few terms

$$\begin{aligned}&J_{4,1}\\ =\;&\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\Big[\Phi_{x,z_1,y,z_2}(f;t_1,t_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)\\ &-\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)+\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)\Big]\frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2\end{aligned}$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\Big[\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)\Big]$$
$$\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$
$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\Big[\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)\Big]$$
$$\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)\left[\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\right]$$
$$\left[\frac{1}{\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2,$$

whence

$$\begin{aligned}&P\left[J_{4,1}\right]\\ \le\;&\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}O\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)\frac{1}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2}{2}\right)^2}dt_1dt_2\\ &+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi}O\left(v(z_1)+v(z_2)\right)\frac{w(h_2)}{v(h_2)}\frac{1}{\left(2\sin\frac{t_2}{2}\right)^2}\left|\frac{2h_1\sin\frac{t_1+\theta h_1}{2}\cos\frac{t_1+\theta_1h_1}{2}}{4\sin^2\frac{t_1}{2}\sin^2\frac{t_1+h_1}{2}}\right|dt_1dt_2\end{aligned}$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi} O\left(v(z_1)+v(z_2)\right)\frac{w(h_1)}{v(h_1)}\frac{1}{\left(2\sin\frac{t_1}{2}\right)^2}\left|\frac{2h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{4\sin^2\frac{t_2}{2}\sin^2\frac{t_2+h_2}{2}}\right|dt_1dt_2$$

$$+\frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{h_2}^{\pi} O\left(v(z_1)+v(z_2)\right)\left(\frac{w(t_1+h_1)}{v(t_1+h_1)}+\frac{w(t_2+h_2)}{v(t_2+h_2)}\right)$$

$$\left|\frac{2h_1\sin\frac{t_1+\theta h_1}{2}\cos\frac{t_1+\theta_1 h_1}{2}}{\left(2\sin\frac{t_1}{2}\right)^2\left(2\sin\frac{t_1+h_1}{2}\right)^2}\frac{2h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{\left(2\sin\frac{t_2}{2}\right)^2\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right|dt_1dt_2$$

$$= O(v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

By combining the terms in pairs we get

$$J_{4,2}+J_{4,8}$$

$$= \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right.$$
$$\left.+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right.$$
$$\left.+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\left(\frac{1}{\left(2\sin\frac{t_2}{2}\right)^2}-\frac{1}{\left(2\sin\frac{t_2+h_2}{2}\right)^2}\right)\right]G_{m,n}(t_1,t_2)dt_1dt_2$$

$$= \frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)-\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}\right.$$
$$\left.+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(2\sin\frac{t_1+h_1}{2}\right)^2}\left(\frac{h_2\sin\frac{t_2+\theta_2 h_2}{2}\cos\frac{t_2+\theta_2 h_2}{2}}{\left(2\sin\frac{t_2}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right)\right]G_{m,n}(t_1,t_2)dt_1dt_2,$$

whence

$$P\left[J_{4,2}+J_{4,8}\right]\leq\frac{1}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi} O\left(v(z_1)+v(z_2)\right)\frac{w(h_2)}{v(h_2)}\frac{1}{(t_1+h_1)^2\,t_2^2}dt_1dt_2$$

$$+\frac{h_2}{mn\pi^2}\int_{\pi-h_1}^{\pi}\int_{h_2}^{\pi} O\left(v(z_1)+v(z_2)\right)\left(\frac{w(t_1+h_1)}{v(t_1+h_1)}+\frac{w(t_2+h_2)}{v(t_2+h_2)}\right)\frac{1}{(t_1+h_1)^2}\frac{t_2+h_2}{(t_2+h_2)^2\,t_2^2}dt_1dt_2$$

$$\leq O\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Analogously

$$\begin{aligned} & J_{4,3}+J_{4,11} \\ = \quad & \frac{1}{mn\pi^2}\int_0^{h_1}\int_{h_2}^{\pi}\left[-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2}{2}\right)^2}+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2, \end{aligned}$$

therefore

$$P\left[J_{4,3}+J_{4,11}\right]\leq O\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Similarly

$$\begin{aligned} & J_{4,4}+J_{4,6} \\ = \quad & \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{\pi-h_2}^{\pi}\left[\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}-\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2 \end{aligned}$$

and

$$\begin{aligned} & J_{4,5}+J_{4,7} \\ = \quad & \frac{1}{mn\pi^2}\int_{h_1}^{\pi}\int_{0}^{h_2}\left[-\frac{\Phi_{x,z_1,y,z_2}(f;t_1,t_2+h_2)}{\left(4\sin\frac{t_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}+\frac{\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}\right]G_{m,n}(t_1,t_2)dt_1dt_2, \end{aligned}$$

whence

$$P\left[J_{4,4}+J_{4,6}\right]+P\left[J_{4,5}+J_{4,7}\right]\leq O\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Finally,

$$\begin{aligned} & J_{4,9}+J_{4,10}+J_{4,12}+J_{4,13} \\ = \quad & \frac{1}{mn\pi^2}\left(-\int_{\pi-h_1}^{\pi}+\int_0^{h_1}\right)\left(-\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)\Phi_{x,z_1,y,z_2}(f;t_1+h_1,t_2+h_2) \\ & \frac{G_{m,n}(t_1,t_2)}{\left(4\sin\frac{t_1+h_1}{2}\sin\frac{t_2+h_2}{2}\right)^2}dt_1dt_2, \end{aligned}$$

whence

$$\begin{aligned} & P\left[J_{4,9}+J_{4,10}+J_{4,12}+J_{4,13}\right]\leq P\left[J_{4,9}\right]+P\left[J_{4,10}\right]+P\left[J_{4,12}\right]+P\left[J_{4,13}\right] \\ \leq \quad & \frac{1}{mn}\left(\int_{\pi-h_1}^{\pi}+\int_0^{h_1}\right)\left(\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)O\left(v(z_1)+v(z_2)\right) \\ & \frac{w(t_1+h_1)/v(t_1+h_1)+w(t_2+h_2)/v\left(t_2+h_2\right)}{\left(t_1+h_1\right)^2\left(t_2+h_2\right)^2}\left|G_{m,n}(t_1,t_2)\right|dt_1dt_2 \\ \leq \quad & \frac{1}{mn}\left(\int_{\pi-h_1}^{\pi}+\int_0^{h_1}\right)\left(\int_{\pi-h_2}^{\pi}+\int_0^{h_2}\right)O\left(v(z_1)+v(z_2)\right) \\ & \left[\frac{w(h_1)/v(h_1)}{h_1\left(t_1+h_1\right)\left(t_2+h_2\right)^2}+\frac{w(h_2)/v(h_2)}{\left(t_1+h_1\right)^2\left(t_2+h_2\right)h_2}\right]\left(mt_1nt_2\right)^2dt_1dt_2 \end{aligned}$$

$$\leq \quad O\left(v(z_1)+v(z_2)\right)\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Therefore

$$P\left[J_4\right]=O(v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Thus

$$P\left[D_{m,n}(f;\cdot+z_1,\cdot+z_2)-D_{m,n}(f;\cdot,\cdot)\right]=O(v(z_1)+v(z_2))\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)$$

and

$$P\left[D_{m,n}\left(f;\cdot,\cdot\right)\right]^{(v,v)}=O\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right).$$

Using Theorem 2.1

$$P\left[D_{m,n}(f;\cdot,\cdot)\right]=O\left(w(h_1)+w(h_2)\right)\leq O\left(\frac{w(h_1)}{v(h_1)}+\frac{w(h_2)}{v(h_2)}\right)$$

our proof follows. □

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