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Differential identities involving Engel conditions in prime rings

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Abstract. The aim of this article is to examine the commutativity criterion of a prime ring, where generalized derivation adheres to differential identities incorporating Engel conditions. Moreover, an example is presented to illustrate that the assumption of primeness cannot be entirely ignored.

1. Introduction

Over the past years, there has been growing interest in exploring the relationship between the commutativity of a ring $\mathcal{R}$ and the additive mappings on $\mathcal{R}$. The first result that appears in this direction was given by Posner [17] demonstrated that if a prime ring $\mathcal{R}$ has a non-zero centralizing derivation on $\mathcal{R}$, then $\mathcal{R}$ must be commutative. Later on, Herstein [9] proved that if $\mathcal{R}$ is a prime ring with characteristics different from 2 and $\mathfrak{D}$ is a non-zero derivation on $\mathcal{R}$ such that $[\mathfrak{D}(x),\ \mathfrak{D}(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Huang [6] extended this result by establishing the commutativity of $\mathcal{R}$ satisfying the identity $[\mathfrak{D}(x),\ \mathfrak{D}(y)]_m = [x, y]^n$ for all $x, y \in \mathcal{I}$, where $\mathfrak{D}$ is a derivation of $\mathcal{R}$ and m, n are fixed positive integers. Continuing this area of research, similar results were obtained for the differential identities involving the anti-commutator. For example, Ashraf and Rehman [1] investigated the action of a non-zero derivation on a prime ring $\mathcal{R}$ and established the commutativity of the ring $\mathcal{R}$. More precisely, they proved that if $\mathcal{R}$ is a 2-torsion free prime ring, $\mathcal{I}$ is a non zero ideal of $\mathcal{R}$ and $\mathfrak{D}$ a non-zero derivation on $\mathcal{R}$ such that $\mathfrak{D}(x) \circ \mathfrak{D}(y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. In 2007, Huang [7] studied the problem and proved that if $\mathcal{R}$ is a prime ring with $char(\mathcal{R}) \neq 2$, U a square closed Lie ideal of $\mathcal{R}$ and $\mathcal{F}$ a generalized derivation associated with a derivation $\mathfrak{D}$ of $\mathcal{R}$ such that $\mathfrak{D}(x) \circ \mathcal{F}(y) = x \circ y$ for all $x, y \in U$, then $\mathfrak{D} = 0$ or $\mathcal{R}$ is commutative. Rehman *et al.* [18] obtained the commutativity of the ring and proved that, if $\mathcal{R}$ is a prime ring and $m, n \geq 1$ fixed positive integers, $\mathcal{F}$ a generalized derivation with an associated derivation $\mathfrak{D}$ of $\mathcal{R}$, such that $\left(\mathcal{F}(x) \circ \mathfrak{D}(y)\right)^m = (x \circ y)^n$ for all x, y in some appropriate subset of $\mathcal{R}$. Rehman and his colleagues [19] continued the studies on the commutativity of prime rings with different conditions and proved that the commutativity of a prime ring $\mathcal{R}$ of characteristics different from 2 by considering the identity $[\mathcal{F}(x),\ \mathfrak{D}(y)]_m = [x, y]$ for all $x, y \in \mathcal{I}$, where $\mathcal{F}$ is a generalized derivation with an associated nonzero derivation $\mathfrak{D}$ and $m \geq 1$ a fixed positive integer. Recently Ashraf *et al.* [2]demonstrated the commutativity of a prime ring $\mathcal{R}$ and proved that, if $\mathcal{R}$ is a prime ring with characteristic different from 2 , $\mathcal{I}$ a non zero ideal of $\mathcal{R}$, $\mathcal{F}$ a generalized derivation with associated non zero derivation $\mathfrak{D}$ such that $\left(\mathcal{F}(x) \circ \mathcal{F}(y)\right)^k = \mathcal{F}(x \circ_k y)\ \forall x,\ y \in \mathcal{I}$, where k is a fixed integer, then $\mathcal{R}$ is commutative.

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Keeping in view that every derivation is a generalized derivation, we aim to answer the fundamental question of whether it is possible to consider the differential identities of the form $[\mathcal{F}(x), \mathcal{F}(y)]_m = [x, y]^n$ for all $x, y \in \mathcal{I}$ and $\mathcal{F}(x) \circ_m \mathcal{F}(y) = (x \circ y)^n$ for all $x, y \in \mathcal{I}$. The answer to this question is affirmative and lies in the Kharchenko theory of differential identities.

2. Preliminaries

We consider $\mathcal{R}$ as an associative ring with its center $\mathcal{Z}(\mathcal{R})$ and $Q = Q_{mr}(\mathcal{R})$ as the maximal right ring of quotients of $\mathcal{R}$, while $\mathfrak{U}$ represents the Utumi quotient ring of $\mathcal{R}$. The extended centroid of $\mathcal{R}$ is the center of $\mathfrak{U}$, denoted by $\mathfrak{C}$. We use the notations $[x, y]$ and $x \circ y, x, y \in \mathcal{R}$ to refer the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. For $m \geq 1$, we define

$$[x, y]_0 = x, [x, y]_1 = [x, y], \quad [x, y]_m = [[x, y]_{m-1}, y],$$
$$x \circ_0 y = x,\ x \circ_1 y = x \circ y, \quad x \circ_m y = (x \circ_{m-1} y) \circ y.$$

A ring $\mathcal{R}$ is said to be prime if $x\mathcal{R}y = \{0\}$ implies $x = 0$ or $y = 0$. A Lie Ideal of $\mathcal{R}$ is an additive subgroup U of $\mathcal{R}$ such that $ur - ru \in U$ for all $u \in U$. If a Lie Ideal $\mathfrak{U}$ satisfies $u^2 \in U$ for all $u \in U$, then U is referred to as Square closed Lie Ideal of $\mathcal{R}$. We define a derivation on $\mathcal{R}$ as an additive mapping $\mathfrak{D} : \mathcal{R} \to \mathcal{R}$ such that $\mathfrak{D}(xy) = \mathfrak{D}(x)y + x\mathfrak{D}(y), x, y \in \mathcal{R}$. A generalized derivation associated with a derivation $\mathfrak{D} : \mathcal{R} \to \mathcal{R}$ is an additive mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + x\mathfrak{D}(y)$ holds for all $x, y \in \mathcal{R}$. For a non-empty subset $\mathfrak{S}$ of $\mathcal{R}$, a mapping $\mathfrak{f} : \mathfrak{S} \to \mathcal{R}$ is said to be centralizing on $\mathfrak{S}$, if $[\mathfrak{f}(x), x] \in \mathcal{Z}(\mathcal{R}), \forall x \in \mathfrak{S}$. In particular, if $[\mathfrak{f}(x), x] = 0$ for all $x \in \mathfrak{S}$, then $\mathfrak{f}$ is said to be commuting on $\mathfrak{S}$.

The main focus of our paper is directed towards examining the annihilator condition of the identities $[\mathcal{F}(x), \mathcal{F}(y)]_m - [x, y]^n = 0$ for all $x, y \in \mathcal{I}$ and $\left(\mathcal{F}(x) \circ \mathcal{F}(y)\right)^m - (x \circ y)^n = 0$ for all $x, y \in \mathcal{I}$, where m, n are fixed positive integers and obtain the commutativity of $\mathcal{R}$ under this restriction. Also , here we mention some well known results which will be used for developing the proofs of the main results.

Fact 2.1. *[12] Let $\mathcal{R}$ be a prime ring, $\mathfrak{D}$ a nonzero derivation on $\mathcal{R}$ and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\mathcal{I}$ satisfies the differential identity*

$$f(s_1, \ldots, s_n, \mathfrak{D}(s_1), \ldots, \mathfrak{D}(s_n)) = 0 \quad \textit{for all } s_1, \ldots, s_n \in \mathcal{I}$$

then either

(*i*) *$\mathcal{I}$ satisfies the generalized polynomial identity*

$$f(s_1, \ldots, s_n, y_1, \ldots, y_n) = 0$$

or

(*ii*) *$\mathfrak{D}$ is Q-inner derivation i,e., for some $q \in Q$, $\mathfrak{D}(x) = [q, x]$ and $\mathcal{I}$ satisfies the generalized polynomial identity*

$$f(s_1, \ldots, s_n, [q, s_1], \ldots, [q, s_n]) = 0.$$

Fact 2.2. *Let $\mathcal{Y} = \{y_1, y_2, \ldots,\}$ be the countable set of non-commutating indeterminates $y_1, y_2 \ldots,$. Let $\mathfrak{C}\{\mathcal{Y}\}$ be the free algebra over $\mathfrak{C}$ in the set $\mathcal{Y}$. We denote $\mathfrak{T} = \mathfrak{U}_{*\mathfrak{C}}\mathfrak{C}\{\mathcal{Y}\}$, the free product of $\mathfrak{C}$-algebra over $\mathfrak{U}$ and $\mathfrak{C}\{\mathcal{Y}\}$. The elements of $\mathfrak{T}$ are called generalized polynomials with coefficients in $\mathfrak{U}$*

Fact 2.3. *[4] If $\mathcal{I}$ is a two sided ideal of $\mathcal{R}$, then $\mathcal{R}, \mathcal{I}, Q$ and $\mathfrak{U}$ satisfy the same generalized polynomial identities with coefficients in $\mathfrak{U}$.*

Fact 2.4. *[3] Every derivation of $\mathcal{R}$ can be uniquely extended a derivation of $\mathfrak{U}$*

Fact 2.5. *[14] Let $\mathcal{I}$ be a two sided ideal of $\mathcal{R}$, then $\mathcal{R}, \mathcal{I}$ and $\mathfrak{U}$ satisfy the same differential identities.*

3. Main results

Following are the main results of this paper.

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from* 2 *and* $\mathfrak{b}$, *a nonzero element of* $\mathcal{R}$. *Suppose that* $\mathcal{F}$ *is a generalized derivation of* $\mathcal{R}$ *associated with a nonzero derivation* $\mathfrak{D}$ *of* $\mathcal{R}$ *such that* $\mathfrak{b}\Big([\mathcal{F}(x),\mathcal{F}(y)]_m - [x,y]^n\Big) = 0$ *for all* $x, y \in \mathcal{I}$. *Then* $\mathcal{R}$ *is commutative.*

Proof. Any generalized derivation $\mathcal{F}$ that operates on a dense right ideal of $\mathcal{R}$ can be extended in a unique way to a generalized derivation of $\mathfrak{U}$. Therefore, we can assume that $\mathcal{F}$ operates on the entirety of $\mathfrak{U}$ in the form of $\mathcal{F}(x) = ax + \mathfrak{D}(x)$, where $a \in \mathfrak{U}$ and $\mathfrak{D}$ is a derivation of $\mathfrak{U}$. Additionally, based on Fact 2.3, it can be established that $\mathcal{I}$, $\mathcal{R}$, and $\mathfrak{U}$ satisfy the same generalized polynomial identities. Hence, according to our assumption, we can express this as follows:

$$\mathfrak{b}\Big(\Big[ax + \mathfrak{D}(x),\ ay + \mathfrak{D}(y)\Big]_m - [x,y]^n\Big) = 0.$$

In view of Kharchenko's theorem we divide our proof into two cases:

Case 1: If $\mathfrak{D}$ is Q-inner derivation, then applying Kharchenko's theorem [12], we have

$$\mathfrak{b}\Big(\Big[ax + s,\ ay + t\Big]_m - [x,y]^n\Big) = 0 \ \text{ for all } x, y, s, t \in \mathcal{R}.$$

In particular take $x = y = 0$, our identity reduces to

$$\mathfrak{b}\Big[s,t\Big]_m = 0 \ \text{ for all } \ s, t \in \mathcal{R}. \tag{3.1}$$

Assuming that $\mathcal{R}$ is not commutative, we take any noncentral element h from $\mathcal{R}$ and substitute $[h,s]$ for t in (3.1). This yields $\mathfrak{b}\Big[h,s\Big]_{m+1} = 0$ for all $s \in \mathcal{R}$. From [11], it follows that $\mathcal{R}$ is commutative, which contradicts our assumption.

Case 2: In this second case, we assume that $\mathfrak{D}$ is the Q-inner derivation induced by some $p \in Q$, i.e., $\mathfrak{D}(x) = [p,x]$ for all $x \in \mathcal{R}$. Since we have taken $\mathfrak{D}$ as a nonzero derivation, it is clear that $p \notin \mathfrak{C}$. We define $\psi(x,y) = \mathfrak{b}\Big(\Big[ax + [p,\ x],\ ay + [p,\ y]\Big]_m - [x,\ y]^n\Big) = 0$. It can be observed that $\psi(x,y)$ is a nontrivial generalized polynomial identity (*GPI*) for $\mathcal{R}$. According to Chuang [4, Theorem 2], $\psi(x,y)$ is also satisfied by Q. We denote by $\mathcal{F}$ either the algebraic closure of $\mathfrak{C}$ or $\mathfrak{C}$, depending on whether $\mathfrak{C}$ is infinite or finite, respectively. By using standard arguments, we conclude that $\psi(x,y)$ is also a generalized polynomial identity for $Q \otimes_{\mathfrak{C}} \mathcal{F}$. Since $Q \otimes_{\mathfrak{C}} \mathcal{F}$ is a centrally closed prime $\mathcal{F}$-algebra (for instance, see [5]), we can replace $\mathcal{R}$ by $Q \otimes_{\mathfrak{C}} \mathcal{F}$ and $\mathfrak{C}$ by $\mathcal{F}$. Thus, we can assume that $\mathcal{R}$ is centrally closed and $\mathfrak{C}$ is either finite or algebraically closed. Using Martindale's theorem [16], we can conclude that $\mathcal{R}$ is a primitive ring with a nonzero socle $\mathfrak{H}$, where $\mathfrak{C}$ is the associated division ring. According to Jacobson's theorem [10, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of a vector space $\mathcal{V}$ over $\mathfrak{C}$, and $\mathfrak{H}$ consists of the linear transformations in $\mathcal{R}$ of finite rank. If $\mathcal{V}$ is finite dimensional over $\mathfrak{C}$, then density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathfrak{M}_K\mathfrak{C}$, where $K = dim_{\mathfrak{C}}\mathcal{V}$. Suppose that $dim_{\mathfrak{C}}\mathcal{V} \geq 2$, otherwise we are done. Now we will show that for any $v \in \mathcal{V}$, v and pv are linearly $\mathfrak{C}$-dependent. Suppose that v and pv are linearly independent for some $v \in \mathcal{V}$. By density of $\mathcal{R}$ on $\mathcal{V}$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{aligned} xv &= pv, \quad & xpv &= (ap + p^2)v, \\ yv &= 0, & ypv &= -v. \end{aligned}$$

Therefore, we have

$$0 = \mathfrak{b}\Big(\Big[ax + [p,\ x],\ ay + [p,\ y]\Big]_m - [x,\ y]^n\Big)v = \mathfrak{b}v. \tag{3.2}$$

In fact, for any $u \in \mathcal{V}$, $\mathfrak{C}$-independence of u and v implies that $\mathfrak{b}u = 0$. Since $\mathfrak{b} \neq 0$, there exists $w \in \mathcal{V}$ such that $\mathfrak{b}w \neq 0$ and so w and v are linearly $\mathfrak{C}$-independent. Also $\mathfrak{b}(w+v) = \mathfrak{b}w \neq 0$ and $\mathfrak{b}(w-v) = \mathfrak{b}w \neq 0$. By the above arguments it follows that w and $\mathfrak{b}w$ are linearly $\mathfrak{C}$-dependent, as are $\{w+v, p(w+v)\}$ and $\{w-v, p(w-v)\}$. Hence there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in \mathfrak{C}$ such that

$$pw = \alpha_w w, \ \ p(w+v) = \alpha_{w+v}(w+v), \ \ p(w-v) = \alpha_{w-v}(w-v),$$

that is,

$$\alpha_w w + pv = \alpha_{w+v} w + \alpha_{w+v} v, \quad \text{and} \tag{3.3}$$

$$\alpha_w w - pv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{3.4}$$

On comparing (3.3) and (3.4), we get

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0, \quad \text{and} \tag{3.5}$$

$$2pv = (\alpha_{w+v} - \alpha_{w-v})w - (\alpha_{w+v} + \alpha_{w-v})v. \tag{3.6}$$

By (3.5) and since w and v are linearly $\mathfrak{C}$-independent and $char(\mathcal{R}) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Hence by (3.6), it follows that $2pv = 2\alpha_w v$. This leads to a contradiction with the fact that v and pv are linearly $\mathfrak{C}$-independent.

In view of this, we may assume that for any $v \in \mathcal{V}$, there exists $\alpha_v \in \mathfrak{C}$ such that $pv = \alpha_v v$ and the standard argument shows that there exists some $\alpha \in \mathfrak{C}$ such that $pv = \alpha v$ for all $v \in \mathcal{V}$. Now let $r \in \mathcal{R}, v \in \mathcal{V}$. As $pv = \alpha v$, we have

$$[p, r]v = (pr)v - (rp)v = p(rv) - r(pv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[p, r]v = 0$ for all $v \in \mathcal{V}$ i.e., $[p, r]\mathcal{V} = 0$. The faithfulness of $\mathcal{V}$ implies that $[p, r] = 0$ for all $r \in \mathcal{R}$. Thus $p \in \mathcal{Z}(\mathcal{R})$, a contradiction. This completes the proof of our theorem. □

The following result is the immediate consequence of the above Theorem.

Corollary 3.1. *[6, Theorem 2.2] Let $\mathcal{R}$ be a prime ring of characteristic different from 2 and $\mathfrak{b} \neq 0 \in \mathcal{R}$. Suppose $\mathfrak{D}$ is a derivation of $\mathcal{R}$ and $n \geq 1, m \geq 1$ fixed positive integers such that $\mathfrak{b}\Big(\big[\mathfrak{D}(x), \ \mathfrak{D}(y)\big]_m - [x, y]^n\Big) = 0$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Considering Theorem 3.1, it is reasonable to inquire whether substituting the commutator with the anti-commutator would result in the commutativity of $\mathcal{R}$. i.e., $\mathcal{R}$ satisfies $\mathfrak{b}\Big(\big(\mathcal{F}(x) \circ_p \mathcal{F}(y)\big)^m - (x \circ y)^n\Big) = 0$, where $m \geq 1, n \geq 1, p \geq 1$ are fixed positive integers. The subsequent outcome demonstrates that Theorem 3.1 still holds if $\mathcal{R}$ fulfills the differential identity $\mathfrak{b}\Big(\big(\mathcal{F}(x) \circ \mathcal{F}(y)\big)^m - (x \circ y)^n\Big) = 0$ for all $m \geq 1, n \geq 1$.

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring and $\mathfrak{b}$ an element of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ with associated non zero derivation $\mathfrak{D}$ and $m, \ n$ fixed positive integers such that $\mathfrak{b}\Big(\big(\mathcal{F}(x) \circ \mathcal{F}(y)\big)^m - (x \circ y)^n\Big) = 0$ for all $x, y \in \mathcal{I}$, then either $\mathfrak{b} = 0$ or $\mathcal{R}$ is commutative.*

Proof. Since $\mathcal{R}$ is a prime ring and $\mathcal{F}$ is a generalized derivation of $\mathcal{R}$. By the given hypothesis and using Lee [15, Theorem 3], we can write

$$\mathfrak{b}\bigg(\Big(\big(\mathfrak{z}x + \mathfrak{D}(x)\big) \circ \big(\mathfrak{z}y + \mathfrak{D}(y)\big)\Big)^m - (x \circ y)^n\bigg) = 0, \text{ for all } x, \ y \in \mathcal{I}.$$

This can be rewritten as

$$\mathfrak{b}\Big(\big(\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ \mathfrak{D}(y) + \mathfrak{D}(x) \circ \mathfrak{z}y + \mathfrak{D}(x) \circ \mathfrak{D}(y)\big)^m - (x \circ y)^n\Big) = 0, \tag{3.7}$$

for all $x, y \in \mathcal{I}$. Firstly, we assume that $\mathfrak{D}$ is an outer derivation on $\mathcal{Q}$. By kharchenko's Theorem [12], $\mathcal{I}$ satisfies the generalized polynomial identity

$$\mathfrak{b}\Big(\big(\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ z + w \circ \mathfrak{z}y + w \circ z\big)^m - (x \circ y)^n\Big) = 0, \text{ for all } x, y, z, \mathfrak{z}, w \in \mathcal{I}.$$

Taking $x = y = 0$ yields $\mathfrak{b}(w \circ z)^m = 0$ for all $w, z \in \mathcal{I}$, which is a generalized polynomial identity (GPI) for $\mathcal{I}$. By Chuang [4, Theorem 2], this GPI is satisfied by $\mathcal{Q}$ and hence by $\mathcal{R}$.

Let $s = (wz + zw)^m$. Since $\mathfrak{b}s = 0$, we have $\mathfrak{b}(tsu\mathfrak{b} + su\mathfrak{b}t)^m = 0$ for all $t, u \in \mathcal{Q}$. Thus, $\mathfrak{b}(tsu\mathfrak{b})^m = 0$, which implies $(su\mathfrak{b}t)^{m+1} = 0$ by Levitzki's lemma [9, Lemma 1.1]. Therefore, $su\mathfrak{b} = 0$ for all $u \in \mathcal{R}$, and since $\mathcal{R}$ is prime, we conclude that either $\mathfrak{b} = 0$ or $s = 0$. If $s = (wz + zw)^m = 0$ for all $w, z \in \mathcal{R}$, then this is a polynomial identity for $\mathcal{R}$. By invoking Lemma 1 in [13], there exists a field $\mathcal{E}$ such that $\mathcal{R} \subseteq \mathfrak{M}_k(\mathcal{E})$, the ring of $k \times k$ matrices over $\mathcal{E}$, and $\mathcal{R}$ and $\mathfrak{M}_k(\mathcal{E})$ satisfy the same polynomial identity. If $k \geq 2$, choosing $x = e_{12}$ and $y = e_{21}$ leads to the contradiction $0 = (wz + zw)^m = e_{11} + e_{22}$, so we must have $\mathcal{R}$ commutative. Thus, we have shown that either $\mathcal{R}$ is commutative or $\mathfrak{b} = 0$.

Our second assumption is that $\mathfrak{D}$ is a $\mathcal{Q}$-inner derivation induced by an element $q \in \mathcal{Q}$ such that $\mathfrak{D}(x) = [\mathfrak{r}, x]$ for all $x \in \mathcal{R}$. Using this assumption and equation (3.7), we obtain the following relation for all $x, y \in \mathcal{I}$:

$$\mathfrak{b}\left((\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ [\mathfrak{r}, y] + [\mathfrak{r}, x] \circ \mathfrak{z}y + [\mathfrak{r}, x] \circ [\mathfrak{r}, y])^m - (x \circ y)^n\right) = 0.$$

This relation also holds for $\mathcal{Q}$, as proven by Chuang [4, Theorem 2], so we can write:

$$\mathfrak{b}\left((\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ [\mathfrak{r}, y] + [\mathfrak{r}, x] \circ \mathfrak{z}y + [\mathfrak{r}, x] \circ [\mathfrak{r}, y])^m - (x \circ y)^n\right) = 0,$$

for all $x, y \in \mathcal{Q}$. If the center $\mathfrak{C}$ of $\mathcal{Q}$ is infinite, we can extend $\mathcal{Q}$ to $\mathcal{Q} \otimes_C \mathcal{F}$, where $\mathcal{F}$ is the algebraic closure of $\mathfrak{C}$. Then, the same relation holds for all $x, y \in \mathcal{Q} \otimes_C \mathcal{F}$. Since both $\mathcal{Q}$ and $\mathcal{Q} \otimes_C \mathcal{F}$ are prime and centrally closed (see [16, Theorems 2.5 and 3.5]), we can replace $\mathcal{R}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_C \mathcal{F}$ depending on whether $\mathcal{F}$ is finite or infinite. Therefore, we can assume that $\mathcal{R}$ is centrally closed over $\mathfrak{C}$, which is either finite or algebraically closed and

$$\mathfrak{b}\Big(\big(\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ [\mathfrak{r}, y] + [\mathfrak{r}, x] \circ \mathfrak{z}y + [\mathfrak{r}, x] \circ [\mathfrak{r}, y]\big)^m - (x \circ y)^n\Big) = 0, \tag{3.8}$$

for all $x, y \in \mathcal{R}$. According to Martindale's [16, Theorem 3], the ring $\mathcal{R}$ is primitive and has a non-zero socle $\mathfrak{H}$, where $\mathfrak{C}$ is the associated division ring. Using Jacobson's theorem [10, p.75], it follows that $\mathcal{R}$ is isomorphic to a dense ring of linear transformations on some vector space $\mathcal{V}$ over $\mathfrak{C}$, with $\mathfrak{C}$ as its associated division ring. Since $\mathcal{R}$ is dense on $\mathcal{V}$, it must be isomorphic to the ring of $k \times k$ matrices over $\mathfrak{C}$, where $K = \dim_{\mathfrak{C}} \mathcal{V}$. However, if K is less than 2, then we are finished with the proof.
Now our aim is to demonstrate that for all $v \in \mathcal{V}$, both v and $\mathfrak{r}v$ are linearly dependent over $\mathfrak{C}$ If $\mathfrak{r}v = 0$, then v and $\mathfrak{r}v$ are $\mathfrak{C}$-dependent. Suppose on the contrary that v and $\mathfrak{r}v$ are linearly $\mathfrak{C}$-dependent for some $v \in \mathfrak{C}$. By the density of $\mathcal{R}$ on $\mathcal{V}$, there exist $x, y \in \mathcal{R}$ such that

$$xv = 0, \quad x\mathfrak{r}v = -v,$$
$$yv = 0, \quad y\mathfrak{r}v = -v.$$

Thus from the relation (3.8), we find that

$$0 = \mathfrak{b}\Big(\big(\mathfrak{z}x \circ \mathfrak{z}y + \mathfrak{z}x \circ [\mathfrak{r}, y] + [\mathfrak{r}, x] \circ \mathfrak{z}y + [\mathfrak{r}, x] \circ [\mathfrak{r}, y]\big)^m - (x \circ y)^n\Big)v = 2^m \mathfrak{b}v.$$

From now on wards, we apply the same logic as in the proof of the Theorem 3.1 after the relation (3.2) to get the desired result. This completes the proof of the theorem. □

We have the following immediate consequence of the above theorem.

Corollary 3.2. *[1, Theorem 4.4] Let $\mathcal{R}$ be a prime ring of characteristics different from* 2 *and Suppose that $\mathfrak{D}$ is a non zero derivation of $\mathcal{R}$ and $\mathcal{I}$ a non zero ideal of $\mathcal{R}$ such that $\mathfrak{D}(x) \circ \mathfrak{D}(y) - x \circ y = 0$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Next we provide an example which shows the existence of primeness is essential in our Theorems.

Example 3.1. *Let $\mathfrak{G}$ be the set of integers and Let* $\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathfrak{G} \right\}$.

Define a map $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ by $\mathcal{F}(xe_{12} + ye_{13} + ze_{23}) = xe_{13}$. Then it is clear that $\mathcal{F}$ is a generalized derivation with an associated derivation $\mathfrak{D} : \mathcal{R} \to \mathcal{R}$ defined by $\mathfrak{D}(xe_{12} + ye_{13} + ze_{23}) = ye_{13}$ for all $x, y, z \in \mathfrak{G}$. One can very easily see that $\mathcal{F}$ satisfies the assumptions of our Theorems 3.1 and 3.2. However $\mathcal{R}$ is not commutative. Hence the primness of $\mathcal{R}$ can not be ignored.

In retrospect, we have the following open problem:

Problem 1. *Let $\mathcal{R}$ be a prime ring and $\mathfrak{b}$ an element from $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ is a generalized derivation associated with a non zero derivation $\mathfrak{D}$ and m, n, p, fixed positive integers such that* $\mathfrak{b}\Big(\big(\mathcal{F}(x) \circ_p \mathcal{F}(y)\big)^m - (x \circ y)^n\Big) = 0$ *for all $x, y \in \mathcal{I}$, then either $\mathfrak{b} = 0$ or $\mathcal{R}$ is commutative.*

Conflict of interests:

The authors declare that they have no conflict of interest.

References

[1] Ashraf, M., Rehman, N., *On commutativity of rings with derivations,* Results Math., **42** (1-2) (2002), 3-8.
[2] Ashraf, M., Pary, S.A., Raza, M.A., *On generalized derivations in semiprime rings involving anticommutator* Beitr Algebra Geom., **60** (2019), 587-598.
[3] Beidar, K. I., Martindale, III, W. S., Mikhalev, A. V., *Rings with Generalized Identities,* Pure and Applied Mathematics, vol **196**, Marcel Dekker, New York (1996).
[4] Chuang, C. L., *GPIs having coefficients in Utumi quotient rings,* Proc. Amer. Math. Soc., **103** (3) (1988), 723-728.
[5] Erickson, J. S., Martindale, W. S., Osborn, J. M., *Prime nonassociative algebras,* Pacific J. Math., **60** (1975), 49-63.
[6] Huang, S., *Derivations with engel conditions in prime and semiprime rings rings*, Czech. Math. J., **61** (4) (2011), 1135-1140.
[7] Huang, S., *Generalized derivation of prime rings,* Int. J. Math. Math. Sci., 2007(2011), Article ID 85612.
[8] Huang, S., *On generalized derivations of prime and semiprime rings,* Taiwanese J. Math., **6** (2012), 771-776.
[9] Herstein, I.N., *Topics in Ring Theory*. Univ. of Chicago, USA (1969).
[10] Jacobson, N., *Structure of rings,* Amer. Math. Soc. Colloq., Pub **37** Rhod Island (1964).
[11] Liu, C. K., *Derivations with engel and annihilator conditions on multilinear polynomials,* Commun. Algebra, **33** (2005), 719-725.
[12] Kharchenko, V. K., *Differential identities of prime rings*, Algebra i Logika, **17 (2)** (1978), 155-168.
[13] Lanski, C., *An Engel condition with derivation,* Proc. Amer. Math. Soc., **118** (1993), 731-734.
[14] Lee, T. K., *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin. **20** (1992), 27-38.
[15] Lee, T. K., Generalized derivations of left faithful rings, Comm. Algebra **27** (1998), 4057-4073.
[16] Martindale, W. S.,*Prime rings satisfying a generalized polynomial identity,* J. Algebra, **12** (1969), 576-584.
[17] Posner, E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.
[18] Rehman, N., Raza, M.A., and Bano. T., *On commutativity of rings with generalized derivations*, J.Egyptian Math. Soc., **2** 4 (2016), 151-155.
[19] Raza, M. A., Rehman, N., *On prime and semiprime rings with generalized derivations and noncommutative banach algebras*, Proc. Math. Sci., **126** (3) (2016), 389-398.