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Impulsive discrete Dirac equation with spectral parameter

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Abstract. In this study, an impulsive problem which consists a Dirac equation with eigenparameter dependent boundary conditions is studied. Scattering solutions ensuring existence of Jost solution, resolvent operator and scattering function of this impulsive problem are delivered. Furthermore, discrete spectrum and asymptotic behavior of Jost function of the problem are investigated. An example illustrating the main results is given.

1. Introduction

In the present work, we study the scattering analysis of an impulsive discrete Dirac problem on the aspect of scattering solutions, scattering function, resolvent operator, Jost function and eigenvalues. While these aspects are well-known in many cases for both Schrödinger and Sturm-Liouville equations [1, 2, 5, 6, 19, 21, 22], there are numerous open questions about scattering analysis of impulsive Dirac equations or the Dirac-Maxwell systems. The readers can find little kinds of studies about scattering analysis of impulsive Dirac equations [9, 10, 20]. But, none of them does not consist spectral parameter in boundary conditions. This property gives a new perspective to the problem. In that way, the results can be used more in applicable science such as in physics, applied mathematics, medicine and engineering.
The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons that makes valuable and accessible mathematical and physical results. The basic and comprehensive results about Dirac equation were given in [23]. In [16], eigenfunction for one-dimensional Dirac operators describing the motion of a particle in quantum mechanics is examined. Inverse nodal, spectral, eigenvalue and scattering problems for Dirac system have been studied by various authors [3, 7, 13–15, 26–28, 30, 31]. Some of these undertaking studies consist spectral parameter both in the equation and boundary conditions. Problems with spectral parameter in equations and conditions form an important part of spectral and scattering analysis of Sturm-Liouville and Dirac equations [7, 8, 11, 12, 29]. On the other hand, this kind of problems generated by Dirac equations sometimes have discontinuities inside on interval at one or more than one points. These points are called impulsive points which create extra conditions named by impulsive conditions. Note that impulsive conditions also are known as jump conditions, transmission conditions and point interactions in literature [1, 2, 5, 11]. Despite having both impulsive conditions and

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boundary conditions with spectral parameter leads additional difficulties to the problems, it increases the value and applications of them in physics, robotics, population dynamics, ecology, biology, optimal control, electronics and etc. Spectral and scattering problems for Dirac equations with impulsive (jump) conditions have been studied in [9, 17, 18, 20, 25]. But scattering properties of impulsive discrete Dirac equation consisting of eigenparameter in boundary conditions have not been studied yet. The works [9, 20], are also about the scattering properties of impulsive discrete Dirac equations, but their boundary conditions do not consist eigenparameter. This paper is concerned with the investigation of a discrete Dirac equation with impulsive at a single point and with boundary conditions depending on spectral parameter. In Section 2, required preliminary information about the problem and some notations are given. Section 3 and Section 4 consist the main results. Jost solution, scattering function and its properties are obtained in section 3. The resolvent operator and discrete spectrum of the problem is discussed in section 4. Finally, an example about the main results is examined in section 5.

2. Preliminaries

In this section, we firstly introduce our problem and present some notations that we will use. Throughout the work, we will use the following notations for the following given sets

$$\mathbb{N} := \{1, 2, 3, ...\},$$

$$\mathbb{N}^0 := \{0, 1, 2, ...\},$$

$$\mathbb{N}^*_{m_0} := \mathbb{N} \setminus \{m_0\},$$

$$\mathbb{N}^{m_0} := \{m_0 + 1, m_0 + 2, ...\},$$

$$\mathbb{N}_{m_0} := \{1, 2, ..., m_0 - 2, m_0 - 1\},$$

$$\mathbb{N}(m_0) := \mathbb{N} \setminus \{m_0 - 1, m_0, m_0 + 1\},$$

here $m_0 \geq 3$ is an integer number.

Now, let us introduce the Hilbert space $l_2\left(\mathbb{N}, \mathbb{C}^2\right)$ consisting of all vector sequences $y = \{y_n\}, \left(y_n \in \mathbb{C}^2, n \in \mathbb{N}\right)$, such that $\sum_{n=1}^{\infty} \| y_n \|^2_{\mathbb{C}^2} < \infty$ with the inner product $< y, z >= \sum_{n=1}^{\infty} (y_n, z_n)_{\mathbb{C}^2}$, where $\mathbb{C}^2$ is 2-dimensional Euclidean space $\| \, . \, \|_{\mathbb{C}^2}$ and $(., .)_{\mathbb{C}^2}$ denote the norm and inner product in $\mathbb{C}^2$, respectively. Furthermore, we denote by L the operator generated in $l_2\left(\mathbb{N}, \mathbb{C}^2\right)$ by the difference system

$$\begin{cases} a_{n+1} y^{(2)}_{n+1} + b_n y^{(2)}_n + p_n y^{(1)}_n = \lambda y^{(1)}_n \\ a_{n-1} y^{(1)}_{n-1} + b_n y^{(1)}_n + q_n y^{(2)}_n = \lambda y^{(2)}_n, \quad n \in \mathbb{N}(m_0) \end{cases} \tag{1}$$

with the boundary condition

$$(\gamma_0 + \gamma_1 \lambda) y^{(2)}_1 + (\upsilon_0 + \upsilon_1 \lambda) y^{(1)}_0 = 0, \quad \gamma_0 \upsilon_1 - \gamma_1 \upsilon_0 \neq 0, \quad \gamma_1 \neq \frac{\upsilon_0}{a_0} \tag{2}$$

and the impulsive conditions

$$\begin{pmatrix} y^{(1)}_{m_0+1} \\ y^{(2)}_{m_0+2} \end{pmatrix} = B \begin{pmatrix} y^{(2)}_{m_0-1} \\ y^{(1)}_{m_0-2} \end{pmatrix}, \tag{3}$$

where γ_j, υ_j are real numbers for $j = 0, 1$, $B = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$ is a real matrix, $\det B > 0$, $\lambda = 2 \sin \frac{z}{2}$ is a spectral parameter and $\{a_n\}_{n \in \mathbb{N}^0}$, $\{b_n\}_{n \in \mathbb{N}}$, $\{p_n\}_{n \in \mathbb{N}}$, $\{q_n\}_{n \in \mathbb{N}}$ are real sequences that satisfy the following condition

$$\sum_{n \in \mathbb{N}^*_{m_0}} (| 1 - a_n | + | 1 + b_n | + | p_n | + | q_n |) < \infty \tag{4}$$

with $a_n \neq 0$, $n \in \mathbb{N}^0$ and $b_n \neq 0$, $n \in \mathbb{N}$.
If $a_n \equiv 1$, $n \in \mathbb{N}^*_{m_0} \cup \{0\}$ and $b_n \equiv 1$, $n \in \mathbb{N}^*_{m_0}$, then the system (1) becomes the following form

$$\begin{cases} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)}, \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \quad n \in \mathbb{N}\,(m_0)\,, \end{cases} \tag{5}$$

where Δ is a forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. System (5) is the discrete analog of the well-known canonical Dirac system [23]

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Therefore, the system (5) is called a canonical discrete Dirac system.

3. Jost solution and scattering function of L

In this section, we analyze spectral properties of the discrete Dirac operator L on the aspect of scattering solutions, Jost solution, Jost function and scattering function. Assume that

$$P(z) = \{P_n(z)\}_{n \in \mathbb{N}_{m_0}} = \left\{ \begin{pmatrix} P_n^{(1)}(z) \\ P_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{N}_{m_0}}$$

and

$$Q(z) = \{Q_n(z)\}_{n \in \mathbb{N}_{m_0}} = \left\{ \begin{pmatrix} Q_n^{(1)}(z) \\ Q_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{N}_{m_0}}$$

are the fundamental solutions of (1) for $z \in \mathbb{C}$ and $n \in \mathbb{N}_{m_0}$ fulfilling the initial conditions

$$P_0^{(1)}(z) = 0, \qquad P_1^{(2)}(z) = -1$$

and

$$Q_0^{(1)}(z) = \frac{1}{a_0}, \qquad Q_1^{(2)}(z) = 0.$$

It is easy to see

$$\deg\left[P_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-1, \quad \deg\left[P_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-2 \tag{6}$$

and

$$\deg\left[Q_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-2, \quad \deg\left[Q_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right)\right] = 2n-3 \tag{7}$$

for all $n \in \mathbb{N}_{m_0}$. On the other hand, under the condition (4) for $\lambda = 2\sin\frac{z}{2}$, $n \in \mathbb{N}_{m_0}$ and $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \mathtt{Im}z \geq 0\}$, equation (1) has the following bounded solutions [4]

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \left(I_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \begin{pmatrix} e^{\frac{iz}{2}} \\ -i \end{pmatrix} e^{inz}$$

and

$$f_0^{(1)}(z) = \alpha_0^{11} \left[e^{\frac{iz}{2}} \left(1 + \sum_{m=1}^{\infty} A_{0m}^{11} e^{imz} \right) - i \sum_{m=1}^{\infty} A_{0m}^{12} e^{imz} \right],$$

where I_2 is 2x2 identity matrix,

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}.$$

Here, A_{nm} is expressed in terms of (p_n) and (q_n), $n \in \mathbb{N}$. Also A_{nm}^{ij} for $i, j = 1, 2$ satisfy

$$| A_{nm}^{ij} | \leq C \sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} \left(| 1 - a_k | + | 1 + b_k | + | p_k | + | q_k |\right),$$

where $\left[\frac{m}{2}\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is a constant. The asymptotic equality of the solution $f(z) = \{f_n(z)\}_{n \in \mathbb{N}^{m_0}}$ can be given as

$$f_n(z) = [I_2 + o(1)] \begin{pmatrix} e^{\frac{iz}{2}} \\ -i \end{pmatrix} e^{inz}, \quad z \in \overline{\mathbb{C}}_+, \quad n \to \infty. \tag{8}$$

It is known that the function $f(z) = \{f_n(z)\}_{n \in \mathbb{N}^{m_0}}$ is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $f_n(z + 4\pi) = f_n(z)$ for all z in $\mathbb{C}_+$ and continuous up to the real axis [4]. Let us define the following semi-strips

$$D_0 := \{z \in \mathbb{C} : 0 \leq \mathrm{Re} z \leq 4\pi, \mathrm{Im} z > 0\}, \quad D := D_0 \cup [0, 4\pi].$$

Definition 3.1. *The Wronskian of two solutions* $\{Y_n(z)\}_{n \in \mathbb{N}^*_{m_0}} = \left\{ \begin{pmatrix} y_n^{(1)}(z) \\ y_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{N}^*_{m_0}}$ *and* $\{U_n(z)\}_{n \in \mathbb{N}^*_{m_0}} = \left\{ \begin{pmatrix} u_n^{(1)}(z) \\ u_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{N}^*_{m_0}}$ *of the equation* (1) *is defined by*

$$W[Y_n(z), U_n(z)] = a_n \left[y_n^{(1)}(z) u_{n+1}^{(2)}(z) - y_{n+1}^{(2)}(z) u_n^{(1)}(z) \right].$$

From the definition of Wronskian, we obtain that

$$W[P(z), Q(z)] = 1, \quad z \in \mathbb{C} \quad \text{and} \quad W\left[f(z), \overline{f(z)}\right] = 2i \cos\frac{z}{2}, \quad z \in \mathbb{R}.$$

Now, we consider the following solution of (1)-(3) using $f(z)$, $P(z)$ and $Q(z)$

$$E_n(z) = \begin{cases} k(z) P_n(z) + l(z) Q_n(z), & n \in \mathbb{N}_{m_0} \\ f_n(z), & n \in \mathbb{N}^{m_0} \end{cases} \tag{9}$$

for $z \in D$. By the help of the impulsive condition (3), we get

$$k(z) = \frac{a_{m_0-2}}{\det B} \left(b(z) f_{m_0+2}^{(2)}(z) - c(z) f_{m_0+1}^{(1)}(z) \right) \tag{10}$$

and

$$l(z) = -\frac{a_{m_0-2}}{\det B} \left(d(z) f_{m_0+2}^{(2)}(z) - g(z) f_{m_0+1}^{(1)}(z) \right) \tag{11}$$

for $z \in D$, where

$$\begin{cases} b(z) = \delta_{11} Q_{m_0-1}^{(2)}(z) + \delta_{12} Q_{m_0-2}^{(1)}(z) \\ c(z) = \delta_{21} Q_{m_0-1}^{(2)}(z) + \delta_{22} Q_{m_0-2}^{(1)}(z) \\ d(z) = \delta_{11} P_{m_0-1}^{(2)}(z) + \delta_{12} P_{m_0-2}^{(1)}(z) \\ g(z) = \delta_{21} P_{m_0-1}^{(2)}(z) + \delta_{22} P_{m_0-2}^{(1)}(z). \end{cases} \tag{12}$$

The vector sequence $E(z) = \{E_n(z)\}_{n\in\mathbb{N}^*_{m_0}} = \left\{\begin{pmatrix} E_n^{(1)}(z) \\ E_n^{(2)}(z) \end{pmatrix}\right\}_{n\in\mathbb{N}^*_{m_0}}$ is called the Jost solution of (1)-(3). Using (2) and (9), we find the Jost function of L by

$$\psi(z) = -(\gamma_0 + \gamma_1\lambda)\, k(z) + (v_0 + v_1\lambda)\, \frac{l(z)}{a_0}. \tag{13}$$

We see directly from (10) and (11) that the Jost function is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$. On the other hand, for $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$, (1) admits another solution

$$F_n(z) = \begin{cases} \varphi_n(z), & n \in \mathbb{N}_{m_0} \\ m(z) f_n(z) + \eta(z)\overline{f_n(z)}, & n \in \mathbb{N}^{m_0}, \end{cases} \tag{14}$$

where $\varphi_n = \{\varphi_n(z)\}_{n\in\mathbb{N}^*_{m_0}}$ is the solution of (1) satisfying the boundary condition (2), defined by

$$\varphi_n(z) = (v_0 + v_1\lambda)\, P_n(z) + a_0\,(\gamma_0 + \gamma_1\lambda)\, Q_n(z).$$

By means of (3) and (12), $m(z)$ and $\eta(z)$ can be gotten as

$$\begin{aligned} m(z) = \frac{a_{m_0+1}}{2i\cos\frac{z}{2}} &\left\{\left[(v_0 + v_1\lambda)\, d(z) + a_0(\gamma_0 + \gamma_1\lambda)\, b(z)\right] \overline{f^{(2)}_{m_0+2}(z)}\right. \\ &\left. - \left[(v_0 + v_1\lambda)\, g(z) + a_0(\gamma_0 + \gamma_1\lambda)\, c(z)\right] \overline{f^{(1)}_{m_0+1}(z)}\right\}, \end{aligned} \tag{15}$$

$$\begin{aligned} \eta(z) = -\frac{a_{m_0+1}}{2i\cos\frac{z}{2}} &\left\{\left[(v_0 + v_1\lambda)\, d(z) + a_0(\gamma_0 + \gamma_1\lambda)\, b(z)\right] f^{(2)}_{m_0+2}(z)\right. \\ &\left. - \left[(v_0 + v_1\lambda)\, g(z) + a_0(\gamma_0 + \gamma_1\lambda)\, c(z)\right] f^{(1)}_{m_0+1}(z)\right\}. \end{aligned} \tag{16}$$

From (12), (13), (15) and (16), since $b(z) = \overline{b(z)}$, $c(z) = \overline{c(z)}$, $d(z) = \overline{d(z)}$ and $g(z) = \overline{g(z)}$, the following equalities are provided

$$\overline{m(z)} = \eta(z) = \frac{a_0 a_{m_0+1}}{2i a_{m_0-2}} \frac{\det B}{\cos\dfrac{z}{2}} \psi(z). \tag{17}$$

Lemma 3.2. *For $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$, the following equation holds*

$$W[E_n(z), F_n(z)] = \begin{cases} -\dfrac{a_{m_0-2}}{a_{m_0+1}} \dfrac{2i\cos\dfrac{z}{2}}{\det B}\eta(z), & n \in \mathbb{N}_{m_0} \\ 2i\cos\left(\dfrac{z}{2}\right)\eta(z), & n \in \mathbb{N}^{m_0}. \end{cases}$$

Proof. Using Definition 3.1, it is easy to write

$$W[E_n(z), F_n(z)] = a_0 k(z)\,(\gamma_0 + \gamma_1\lambda) - l(z)\,(v_0 + v_1\lambda)$$

for $n \in \mathbb{N}_{m_0}$. In accordance with (13) and (17), we find

$$W[E_n(z), F_n(z)] = -\frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i\cos\dfrac{z}{2}}{\det B}\eta(z), \quad n \in \mathbb{N}_{m_0}.$$

Similarly, we obtain

$$W[E_n(z), F_n(z)] = 2i\cos\left(\frac{z}{2}\right)\eta(z), \quad n \in \mathbb{N}^{m_0}.$$

□

Theorem 3.3. *For all $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$, $\eta(z) \neq 0$, where $\eta(z)$ is defined in* (16).

Proof. Assume that there exists a z_0 in $[0, 4\pi] \setminus \{\pi, 3\pi\}$, such that $\eta(z_0) = 0$. By the help of (17), we get $m(z_0) = \eta(z_0) = 0$. Then the solution $F_n(z)$ is equal to zero identically and this gives a contradiction, i.e., for all $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$, $\eta(z) \neq 0$. □

Definition 3.4. *The scattering function $S(z)$ of L is defined by*

$$S(z) := \frac{\overline{\psi(z)}}{\psi(z)}, \quad z \in [0, 4\pi] \setminus \{\pi, 3\pi\}.$$

From (17) and Definition 3.4, $S(z)$ has following representation

$$S(z) = \frac{\overline{\psi(z)}}{\psi(z)} = \frac{\overline{\eta(z)}}{\eta(z)} \quad z \in [0, 4\pi] \setminus \{\pi, 3\pi\}. \tag{18}$$

Theorem 3.5. *The function $S(z)$ satisfies*

$$S^{-1}(z) = \overline{S(z)}, \quad |S(z)| = 1$$

for all $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$.

Proof. For $z \in [0, 4\pi] \setminus \{\pi, 3\pi\}$, by using (13) and (17), we write

$$S^{-1}(z) = \frac{\psi(z)}{\overline{\psi(z)}} = \overline{S(z)}.$$

It is evident from that $|S(z)| = 1$. □

4. Resolvent operator and eigenvalues of L

In this part, we define an unbounded solution of (1)-(3) in order to obtain the resolvent operator of our problem. Then, we express the set of eigenvalues of L by using the singularities of the kernel of the resolvent operator.

Let us define the unbounded solution $G(z) = \{G_n(z)\}$ of L as follows

$$G_n(z) = \begin{cases} \varphi_n(z), & n \in \mathbb{N}_{m_0} \\ r(z) f_n(z) + p(z) \widehat{f_n}(z), & n \in \mathbb{N}^{m_0} \end{cases} \tag{19}$$

for $z \in D \setminus \{\pi, 3\pi\}$, where $\widehat{f_n}(z) = \left\{\widehat{f_n}(z)\right\}_{n \in \mathbb{N}^{m_0}} = \left\{\begin{pmatrix} \widehat{f_n}^{(1)}(z) \\ \widehat{f_n}^{(2)}(z) \end{pmatrix}\right\}_{n \in \mathbb{N}^{m_0}}$ that satisfies the following asymptotic equality

$$\lim_{n \to \infty} e^{i\left(n + \frac{1}{2}\right) z} \widehat{f_n}^{(1)}(z) = \lim_{n \to \infty} e^{inz} \widehat{f_n}^{(2)}(z) = 1, \quad z \in \overline{\mathbb{C}}_+.$$

Note that it is clear that

$$W\left[f_n(z), \widehat{f_n}(z)\right] = 2i \cos \frac{z}{2}, \quad z \in D \setminus \{\pi, 3\pi\}.$$

By the help of (3) and (12), the coefficients $r(z)$ and $p(z)$ are obtained as follows

$$r(z) = \frac{a_{m_0+1}}{2i\cos\frac{z}{2}} \left\{ \left[(v_0 + v_1\lambda) d(z) + a_0 (\gamma_0 + \gamma_1\lambda) b(z) \right] \widehat{f}^{(2)}_{m_0+2}(z) \right. \tag{20}$$

$$\left. - \left[(v_0 + v_1\lambda) g(z) + a_0 (\gamma_0 + \gamma_1\lambda) c(z) \right] \widehat{f}^{(1)}_{m_0+1}(z) \right\},$$

$$p(z) = -\frac{a_{m_0+1}}{2i\cos\frac{z}{2}} \left\{ \left[(v_0 + v_1\lambda) d(z) + a_0 (\gamma_0 + \gamma_1\lambda) b(z) \right] f^{(2)}_{m_0+2}(z) \right. \tag{21}$$

$$\left. - \left[(v_0 + v_1\lambda) g(z) + a_0 (\gamma_0 + \gamma_1\lambda) c(z) \right] f^{(1)}_{m_0+1}(z) \right\}$$

for $z \in D \backslash \{\pi, 3\pi\}$. From (16) and (21), it is obvious that $\eta(z) = p(z)$.

Similar to Lemma 3.2, for $z \in D$, the Wronskian of the solutions $E_n(z)$ and $G_n(z)$ is found as

$$W[E_n(z), G_n(z)] = \begin{cases} -\dfrac{a_{m_0-2}}{a_{m_0+1}} \dfrac{2i\cos\dfrac{z}{2}}{\det B} \eta(z), & n \in \mathbb{N}_{m_0} \\ 2i\cos\left(\dfrac{z}{2}\right) \eta(z), & n \in \mathbb{N}^{m_0}. \end{cases}$$

Theorem 4.1. *For all $z \in D \backslash \{\pi, 3\pi\}$ and $\eta(z) \neq 0$, the resolvent operator of L is defined by*

$$R_\lambda(L) h_n := U \left\{ \sum_{k=1}^{n} V \begin{pmatrix} \frac{a_{k-1}}{a_k} G^{(1)}_{k-1} \\ G^{(2)}_k \end{pmatrix} \begin{pmatrix} E^{(1)}_n \\ E^{(2)}_n \end{pmatrix} + \sum_{k=n+1}^{\infty} V \begin{pmatrix} \frac{a_{k-1}}{a_k} E^{(1)}_{k-1} \\ E^{(2)}_k \end{pmatrix} \begin{pmatrix} G^{(1)}_n \\ G^{(2)}_n \end{pmatrix} \right\},$$

where $U := -\dfrac{1}{W[E_n(z), G_n(z)]}$ *and* $V := \left(h^{(1)}_{k-1}, h^{(2)}_k \right)$.

Proof. To obtain the resolvent operator of L, it is necessary to solve the following system of equations

$$\begin{cases} a_{n+1} y^{(2)}_{n+1} + b_n y^{(2)}_n + p_n y^{(1)}_n - \lambda y^{(1)}_n = h^{(1)}_n \\ a_{n-1} y^{(1)}_{n-1} + b_n y^{(1)}_n + q_n y^{(2)}_n - \lambda y^{(2)}_n = h^{(2)}_n, \quad n \in \mathbb{N}(m_0). \end{cases} \tag{22}$$

Since $E_n(z)$ and $G_n(z)$ are the fundamental solutions of (1)-(3), the general solution of (22) can be written as

$$H^{(i)}_n = u_n E^{(i)}_n(z) + t_n G^{(i)}_n(z), \quad i = 1, 2, \tag{23}$$

where u_n, t_n are coefficients and are different from zero. Using the method of variation of parameters, we find u_n and t_n by

$$u_n = -\sum_{k=0}^{n-1} \frac{a_k h^{(1)}_k(z) G^{(1)}_k(z) + a_{k+1} h^{(2)}_{k+1}(z) G^{(2)}_{k+1}(z)}{a_{k+1} W[E_n(z), G_n(z)]} \tag{24}$$

and

$$t_n = -\sum_{k=n}^{\infty} \frac{a_k h^{(1)}_k(z) E^{(1)}_k(z) + a_{k+1} h^{(2)}_{k+1}(z) E^{(2)}_{k+1}(z)}{a_{k+1} W[E_n(z), G_n(z)]}, \tag{25}$$

respectively.
By using (23), (24) and (25), we get the resolvent operator of L. It completes the proof of Theorem 4.1. □

Using Theorem 4.1 and the definition of eigenvalues [24], we obtain the set of eigenvalues of (1)-(3) as follows

$$\sigma_d(L) = \left\{ \lambda = 2\sin\frac{z}{2} : z \in D_0, \eta(z) = 0 \right\}.$$

Theorem 4.2. *$\eta(z)$ satisfies the following asymptotic equation for $z \in D$*

$$\eta(z) = e^{5iz}\left[A + o(1)\right], \quad |\, z \,| \to \infty,$$

where $A = a_{m_0+1}\left[\delta_{11} \upsilon_1 (-1)^{m_0-1} K_2(m_0-1)\, \alpha^{22}_{m_0+2}\right]$, $\delta_{11} \neq 0$.

Proof. It is known from [9] that

$$\lim_{|z|\to\infty} P_n^{(1)}(z) e^{i\left(n-\frac{1}{2}\right)z} = i\,(-1)^n K_1(n), \quad z \in \overline{\mathbb{C}}_+,$$

$$\lim_{|z|\to\infty} P_n^{(2)}(z) e^{i(n-1)z} = i\,(-1)^n K_2(n), \quad z \in \overline{\mathbb{C}}_+, \tag{26}$$

where

$$K_1(n) := -\left(b_1 \prod_{k=2}^{n} a_k b_k\right)^{-1}, \quad K_2(n) := -\left(\prod_{k=2}^{n} a_k b_{k-1}\right)^{-1}.$$

By the help of (6), (7) and (8), we get

$$\lim_{|z|\to\infty} d(z) f^{(2)}_{m_0+2}(z) = \delta_{11} (-1)^{m_0-1} K_2(m_0-1)\, \alpha^{22}_{m_0+2} e^{4iz} \tag{27}$$

and

$$\lim_{|z|\to\infty} e(z) f^{(1)}_{m_0+1}(z) = \delta_{21} (-1)^{m_0-1} K_2(m_0-1)\, \alpha^{12}_{m_0+1} e^{3iz}. \tag{28}$$

From (27) and (28), $n(z)$ can be written as

$$\eta(z) e^{-5iz} = A + o(1),$$

where

$$A = a_{m_0+1}\left[\delta_{11} \upsilon_1 (-1)^{m_0-1} K_2(m_0-1)\, \alpha^{22}_{m_0+2}\right].$$

□

5. An example

In this part, we present a special impulsive discrete Dirac system as an example. We get the Jost solution, Jost function and scattering function of this example. Then, we determine a region for the eigenvalues of this problem.

Let us consider the following discrete Dirac system

$$\begin{cases} y^{(2)}_{n+1} - y^{(2)}_n = \lambda y^{(1)}_n \\ y^{(1)}_{n-1} - y^{(1)}_n = \lambda y^{(2)}_n, \quad n \in \mathbb{N}(2) \end{cases} \tag{29}$$

with the boundary condition

$$(\gamma_0 + \gamma_1 \lambda)\, y^{(2)}_1 + (\upsilon_0 + \upsilon_1 \lambda)\, y^{(1)}_0 = 0, \quad \gamma_0 \upsilon_1 - \gamma_1 \upsilon_0 \neq 0, \quad \gamma_1 \neq \frac{\upsilon_0}{a_0} \tag{30}$$

and the impulsive conditions

$$\begin{pmatrix} y^{(1)}_3 \\ y^{(2)}_4 \end{pmatrix} = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} y^{(2)}_1 \\ y^{(1)}_0 \end{pmatrix}, \tag{31}$$

where γ_j, υ_j are real numbers for $j = 0, 1$, δ_1 and δ_2 are real numbers such that $\delta_1\delta_2 \neq 0$. By using (9) and (13), we obtain the Jost solution and Jost function of (29)-(31)

$$E_n(z) = \begin{cases} k(z)P_n(z) + l(z)Q_n(z), & n = 0, 1 \\ f_n(z), & n = 3, 4, ..., \end{cases} \tag{32}$$

$$\psi(z) = \delta_2 (\gamma_0 + \gamma_1\lambda) e^{\frac{7iz}{2}} - i\delta_1 (\upsilon_0 + \upsilon_1\lambda) e^{4iz}, \tag{33}$$

respectively. In accordance with Definition 3.4 and (33), we find the scattering function of (29)-(31)

$$S(z) = e^{-8iz} \left[\frac{\delta_2 (\gamma_0 + \gamma_1\lambda) e^{\frac{iz}{2}} + i\delta_1 (\upsilon_0 + \upsilon_1\lambda)}{\delta_2 (\gamma_0 + \gamma_1\lambda) e^{-\frac{iz}{2}} - i\delta_1 (\upsilon_0 + \upsilon_1\lambda)} \right].$$

By the help of the definition of eigenvalues [24], we write

$$\sigma_d (L) = \left\{ \lambda = 2 \sin \frac{z}{2} : z \in D_0, \psi(z) = 0 \right\}.$$

For the simplicity on calculations, if we choose $\gamma_0 = \upsilon_1 = 1$ and $\gamma_1 = \upsilon_0 = 0$ in (30), we easily find that

$$e^{iz} = \frac{\delta_2}{\delta_1} - 1.$$

Let $\delta_2 = M\delta_1$, $M \in \mathbb{R}$. From last equation, we write

$$z_k = -i \ln | M - 1 | + Arg (M - 1) + 2k\pi, \quad k \in \mathbb{Z}.$$

There appear two special cases:
Case1. If $0 < M < 1$, then

$$z_k = -i \ln | M - 1 | + (2k + 1) \pi.$$

In this case, the problem (29)-(31) has eigenvalues if and only if $k = 0, 1$.
Case2. If $1 < M < 2$, then

$$z_k = -i \ln | M - 1 | + 2k\pi.$$

Likewise the other case, for $k = 1$, there are eigenvalues.

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