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Higher-order and Weil Grassmannian as a space of subalgebras of a Weil algebra

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Abstract. Let $G^r_{k,m}$ be the subgroup of the jet group G^r_k formed by elements projectable to $G^r_m \simeq G^r_m \times \{j^r_0 \operatorname{id}_{\mathbb{R}^{k-m}}\}$. We define a partition $\mathcal{V}$ on $\operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ of G^r_m-orbits with respect to the left action defined by the jet composition. Elements of $\operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ are considered as vectors from $\mathbb{R}^k$ with the standard inner product over them. The r-th order Grassmannian $\operatorname{Gr}(r, k, m)$ is defined as the basis of the principal bundle $\hat{p}^\# : \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0 \to \mathcal{V}$ identified with the reduction of the principal bundle $\hat{p} : G^r_k \to G^r_{k,m} \backslash G^r_k$ to the structure group $G^r_m \simeq G^r_m \times \{j^r_0 \operatorname{id}_{\mathbb{R}^{k-m}}\}$. Adding the claim of the first-order orthonormality and modifying $\mathcal{V}$ to $\mathcal{V}_{Ort}$ we obtain the geometrical structure over $\operatorname{Gr}(r, k, m)$ in the form of the so-called orthonormal Grassmann bundle $\hat{p}^\#_{Ort}$. We construct an atlas on $\operatorname{Gr}(r, k, m)$ from a finite system of local sections of $\hat{p}^\#_{Ort}$ and define the r-th order Grassmannian bundle functor with standard fiber $\operatorname{Gr}(r, k, m)$ on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphism.

For the jet algebra $\mathbb{D}^r_k$, a Weil algebra $A = \mathbb{D}^r_k / I$ and the projection homomorphism $p_A : \mathbb{D}^r_k \to A$ we define the partition $\mathcal{V}_A$ on $\operatorname{reg} T^A_0 \mathbb{R}^m$ formed by orbits of the left action of the group of all $T^A_0 h; h \in \operatorname{Diff}_0 \mathbb{R}^m$, which is by [29] identified with G^r_m. We prove that the local sections of $\hat{p}^\#_{Ort}$ above satisfy some kind of T^A-respecting property and determine an atlas on $\mathcal{V}_A$. We define the Weil Grassmannian $\operatorname{Gr}(A, m) = \mathcal{V}_A$ and the bundle functor Gr^A defined on $\mathcal{M}f_m$ with $\operatorname{Gr}(A, m)$ as its standard fiber. We prove the coincidence of $\operatorname{Gr}(A, m)$ with the quotient of $\mathcal{V}$ by the map $[\widetilde{p}_{A,\mathbb{R}^m}] : \mathcal{V} \to \mathcal{V}_A$ induced by p_A.

We define the principle bundle $\hat{p}^\#_A : \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0 \to \operatorname{Gr}(A, m)$ with the structure group G^r_m. We define a partition $\mathcal{V}^A$ on $\operatorname{reg} T^A_0 \mathbb{R}^m$ coarser then $\mathcal{V}_A$ and some auxiliary partitions within $\operatorname{reg} T^A_0 \mathbb{R}^m$. It is proved that the factorization $[\hat{p}^\#_A] : \operatorname{reg} T^A_0 \mathbb{R}^m \to \mathcal{V}_A$ of $\hat{p}^\#_A$ to $\operatorname{reg} T^A_0 \mathbb{R}^m$ can be considered as the disjoint union of bundles with standard fiber identified with some sublagebra of A of width $A = m$.

1. Introduction

We give the contribution to the theory of homogeneous spaces studied from the point of view of jet spaces and the Weil theory. In the centre of our investigations there is a generalization of the classical Grassmannian (Grassmann manifold) $\operatorname{Gr}(k, m)$ considered for $k \geq m$. It is the basis of the geometrical structure called Grassmann bundle, which forms the principal bundle over $\operatorname{Gr}(k, m)$. The Grassmannian itself is usually defined as the space of m-planes containing 0 in the k-dimensional real affine space, in

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other words the space of all m-dimensional linear subspaces of $\mathbb{R}^k$. $\mathrm{Gr}(k,m)$ is a fundamental example of a homogeneous space obtained from orthogonal groups as $O(k)/O(m)\times O(k-m)$. From this point of view, $\mathrm{Gr}(k,m)$ is obtained from the right action of the orthogonal group $O(k)$ on $\mathbb{R}^k$ acting transitively on $\mathrm{Gr}(k,m)$. Since the isotropy group of the plane $\mathbb{R}^m\times\{0\}^{k-m}$ is identified with $O(m)\times O(k-m)$ (see [9], Section 10), $\mathrm{Gr}(k,m)$ coincides with the homogeneous space above. We recall some other ways of defining $\mathrm{Gr}(k,m)$, e.g. by the so-called Plücker coordinates, see [7].

In [9], Section 12 and in [11], [12] $\mathrm{Gr}(k,m)$ is defined and studied by means of the contact elements and is generalized to higher orders. By [9], 12.16 and 12.17, $\mathrm{Gr}(k,m)$ can be considered as the r-th order contact element space $(K^r_m\mathbb{R}^k)_0=\mathrm{reg}\,J^r_0(\mathbb{R}^m,\mathbb{R}^k)_0/G^r_m$, by reg indicating immersions. It is a standard fiber of the bundle functor K^r_m of contact elements defined on the category $\mathcal{M}f_k$. The space K^r_mM consists of all $j^r_0\psi\in\mathrm{reg}\,J^r_0(\mathbb{R}^m,M)$ corresponding to local parametrizations ψ of m-dimensional submanifolds factorized by the parametrizations determining locally the same submanifold.

In the present paper we essentially follow the results of Grigore and Krupka from [5], besides those from [11],[12] (see also [13]). In [5] there was properly investigated and described the geometrical structure over the higher-order Grassmanniannian manifold in the form of a principal bundle (the so-called Grassmann bundle) with the total space $\mathrm{reg}\,T^r_kM$ and the structure group G^r_m. In addition to that there were studied problems of invariants on such spaces.

We apply another approach, defining $\mathrm{Gr}(k,m)$ as the standard fiber of a natural bundle defined on the category $\mathcal{M}f_m$, in contrary to K^r_m defined on $\mathcal{M}f_k$. In this situation, reg indicates submersions. Without loss of generality, a tangent space to a submanifold M of $\mathbb{R}^k$ at any x can be replaced by a linear subspace of $T_0\mathbb{R}^k\simeq\mathbb{R}^k$ of dimension m, applying the obvious translation map and the coordinate system of M adopted to a coordinate system of $\mathbb{R}^k$. A linear operator transforming a basis $dy^1,\dots dy^m$ on $T_xM\simeq T_0M$ induced by such coordinates to the system of linearly independent vectors expressed in the natural coordinates of $\mathbb{R}^k$ can be identified with an element of the geometrical structure over $\mathrm{Gr}(k,m)$. Elements of $\mathrm{Gr}(k,m)$ themselves are obtained by the factorization of the elements of the geometrical structure to the linear subspaces of the target space $\mathbb{R}^k$. From the mechanical point of view, basis tangent vectors at points of M corresponding to the spatial coordinates are transformed to the linearly independent tangent vectors considered in the reference configuration. Every M above can be considered as a submanifold of $M\times\mathbb{R}^{k-m}$. Since the concepts of the tangent space and subspace are of local character, the system of $\mathrm{Gr}(k,M)$ can be considered as the bundle functor on $\mathcal{M}f_m$, with possible insertions of objects to k-dimensional manifolds containing them as their submanifolds.

In the next step of generalization, tangent spaces and linear subspaces are replaced by higher-order jet algebras and their subalgebras while linear maps are replaced by higher-order jets. The investigations are continued to Weil algebras, their subalgebras, Weil functors and Weil functor morphisms. Besides the higher-order and Weil generalization of Grassmannian we also investigate and discuss the geometrical structures or its modifiactions from the point of view of the Weil theory.

The studied objects are also put to context with generalized frames and frame bundles (see Section 2 below) and some elementary concepts of mechanics from [4]. We remark that in [30] we have sketched some simple ideas of mechanical applications of the discussed concepts in connection with the uniformity of elastic materials, following [4]. Nevertheless, exact proofs have not been performed.

If the orthogonality in the forefront of the definition of Grassmannian is not required we can define it by means of the homogeneous space $G^1_{k,m}\backslash G^1_k$ in the classical case and by $G^r_{k,m}\backslash G^r_k$ in the higher-order case (see Section 2). Roughly speaking, the r-th order Grassmannian $\mathrm{Gr}(r,k,m)$ will be identified with the space of orbits of the jet group G^r_m acting on the jet space $\mathrm{reg}\,J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ from the left. On the other hand, in Section 2 and 3 we present the constructions preserving orthonormality of elements of the geometrical structure in the first order.

1.1. Elementary categories on manifolds and jet bundle functors

Let us recall the elementary concepts of r-jet bundle functor from [9]. For a smooth curve $\gamma:\mathbb{R}\to M$, the r-jet $j^r_{t_0}\gamma$ expresses the equivalence class of all curves having the r-th order contact at $x=\gamma(t_0)\in M$ while for a smooth map $f:M\to N$ between manifolds the r-jet j^r_xf conceptually expresses the set of $\dim N$-valued Taylor polynomials obtained in individual coordinate systems on M centered at x. The composition of jets is correctly defined by $j^r_y g \circ j^r_x f = j^r_x(g \circ f)$ where $y = f(x)$. The jet spaces determine the bundle functor $J^r : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}$. By $\mathcal{M}f_m$ we denote the category of m-dimensional manifolds with local diffeomorphisms, by $\mathcal{M}f$ the category of smooth manifolds with smooth maps and by $\mathcal{FM}$ the category of fibered manifolds with smooth fibered maps. A bundle functor is defined as a functor $F : C \to \mathcal{M}f$ defined on a suitable admissible category (e.g. $\mathcal{M}f$, $\mathcal{M}f_m$ or $\mathcal{FM}$) satisfying the base preserving and locality conditions (for rigorous definitions see [9]). The jet functor J^r assigns the space of r-jets $J^r(M,N)$ of smooth maps $M \to N$ to any couple $(M,N) \in \mathrm{Obj}(\mathcal{M}f_m \times \mathcal{M}f)$ and the map $J^r(g,h) : J^r(M_1,N_1) \to J^r(M_2,N_2)$ defined by $j^r_x f \mapsto j^r_{f(x)} h \circ j^r_x f \circ (j^r_x g)^{-1}$ to any couple $(g,h) \in \mathrm{Morph}(\mathcal{M}f_m \times \mathcal{M}f)$. For $0 \le s \le r$ there is the subordinate natural transformation $\pi^r_s : J^r \to J^s$ mapping $j^r_x f$ onto $j^s_x f$. A bundle functor defined on the category $\mathcal{M}f_m$ is said to be a natural bundle.

1.2. *Weil functors*

Weil functors are bundle functors of the principal meaning. Their history goes back to 1953 when A. Weil in [32] defined and investigated spaces of infinitely near points on manifolds. This entailed to the contravariant definition of Weil functor. Nevertheless, we prefer the covariant approach presented below. By the classical result of Kainz and Michor ([6]) and others ([2], [15]), they are exactly those bundle functors on $\mathcal{M}f$ which preserve products. On the other hand, Weil functors generalize many significant geometrical spaces like tangent, iterated tangent, higher-order velocity bundles T^r_k, non-holonomic and semi-holonomic velocity bundles. They have been studied by many authors in e.g. [1], [8], [14], [17], [21], [28], [33].

Every Weil functor is associated to a Weil algebra. It is an algebra of the form $A = \mathbb{R} \oplus N_A$ where N_A is its nilpotent ideal. Respecting our aims, we prefer defining A as $\mathcal{E}(k)/I$ where $\mathcal{E}(k)$ is the algebra of germs of smooth functions $\mathbb{R}^k_0 \to \mathbb{R}$ factorized by an ideal I of finite codimension. Germs from $\mathcal{E}(k)$ are decomposition classes of the algebra of functions $C^\infty(\mathbb{R}^k_0, \mathbb{R})$ associated to the equivalence $\simeq$ defined by $f \simeq g$ if and only if f and g coincide on some neighbourhood of $0 \in \mathbb{R}^k$. A Weil algebra can be also defined as $A = \mathbb{D}^r_k/J$ for the so called jet algebra $\mathbb{D}^r_k$ and its ideal J. In other words, A is considered as an algebra of polynomials of k indeterminates of order at most r factorized by some of its ideals J. We put width A to $\dim(N_A/N^2_A)$ and height A to the minimal r for which $A = \mathbb{D}^r_k/J$. By $p_A : \mathbb{D}^r_k \to A$ we denote the projection homomorphism. $A = \mathbb{D}^r_k/J$ is said to be monomial if and only if J is generated by monomials.

The covariant approach to the definition of a Weil functor T^A is based on the I-factorization of germs in the following sense. For $A = \mathbb{D}^r_k/I$ two germs $\mathrm{germ}_0\, g : \mathbb{R}^k_0 \to M_x$ and $\mathrm{germ}_0\, h : \mathbb{R}^k_0 \to M_x$ are said to be I-equivalent if and only if $\mathrm{germ}_x \gamma \circ \mathrm{germ}_0\, g - \mathrm{germ}_x \gamma \circ \mathrm{germ}_0\, h \in I$ for any function $\gamma : M \to \mathbb{R}$ defined near x. Classes of such equivalence relation are denoted by $j^A g$ and the space of them by $T^A M$. For a smooth map $\varphi : M \to N$ we define the map $T^A\varphi$ by $T^A\varphi(j^A g) = j^A(\varphi \circ g)$. Clearly, $T^r_k = T^{\mathbb{D}^r_k}$. There is the bijective correspondence between Weil algebras and Weil functors determined by the assignments $A \mapsto T^A$ and $F \mapsto F\mathbb{R}$, applying the product preserving property of Weil functors.

1.3. *Automorphism of Weil algebras*

By [8] and [9], natural transformations $\widetilde{t_M} : T^B M \to T^A M$ are in bijection with homomorphisms $t : B \to A$, which holds particularly for the projections $\widetilde{p_A} : T^r_k \to T^A$. By [8], natural transformations above correspond bijectively the so-called B-admissible A-velocities defined as follows. For Weil algebras $A = \mathcal{E}(k)/I$, $B = \mathcal{E}(p)/J$ and smooth $f : \mathbb{R}^k_0 \to \mathbb{R}^p_0$ an A-velocity $j^A f$ is said to be B-admissible if and only if

$$\mathrm{germ}_0\, \varphi \in J \Rightarrow \mathrm{germ}_0(\varphi \circ f) \in I. \tag{1.3.1}$$

Thus every B-admissible A-velocity $j^A f$ is bijectively assigned a natural transformation $\widetilde{t_M} : T^B M \to T^A M$ defined as follows

$$\widetilde{t_M}(j^B\varphi) = \widetilde{t}^{j^A f}_M(j^B\varphi) = j^A(\varphi \circ f). \tag{1.3.2}$$

In particular, all automorphisms of A are determined by reparametrizations of indeterminates satisfying the conditions of A-admissibility (1.3.1).

The jet group G^r_k is defined as inv $J^r_0(\mathbb{R}^k,\mathbb{R}^k)_0$ with the multiplication defined by the composition of r-jets. The automorphism $t_{j^r_0 g}$ of $\mathbb{D}^r_k$ and the associated natural equivalence $\widetilde{t_{j^r_0 g}}$ on T^r_k determined by $j^r_0 g \in G^r_k$ are defined by the assignments

$$(1.3.3) \qquad j^r_0\eta \mapsto j^r_0\eta \circ (j^r_0 g)^{-1} \quad \text{or} \quad j^r_0\eta_x \mapsto j^r_0\eta_x \circ (j^r_0 g)^{-1}$$

for every $j^r_0\eta \in \mathbb{D}^r_k$ and $j^r_0\eta_x \in (T^r_k)_x M$. If M is a numeric space and $x = 0$ we sometimes use the notation $t_{j^r_0 g}$ for natural equivalences as well. Thus for a B-admissible A-velocity $j^r_0 g$ we have the natural equivalence $t_{j^r_0 g} : T^A \to T^B$.

For $A = \mathbb{D}^r_k/I$ and the projection homomorphism $p : \mathbb{D}^r_k \to A$ Alonso defined the subgroups G_A and $G^A \subseteq G^r_k \simeq \operatorname{Aut}(\mathbb{D}^r_k)$ ([1]) as follows

$$(1.3.4) \qquad G_A = \{j^r_0 g \in G^r_k; p \circ t_{j^r_0 g} = p\} \quad \text{and} \quad G^A = \{j^r_0 g \in G^r_k; \operatorname{Ker}(p \circ t_{j^r_0 g}) = \operatorname{Ker} p\}.$$

The first subgroup is A-stabilizing while the second one is A-respecting. In [1] it is proved that G_A is a normal subgroup of G^A and G^A/G_A is identified with the group $\operatorname{Aut} A$. Clearly, $j^r_0 g$ determines an automorphism of A if and only if $j^r_0 g \in G^A$.

1.4. Some subalgebras, factors and properties of Weil algebras

Let $B = \mathbb{D}^r_k/I$ be a Weil algebra of height r and width k with the projection homomorphism $p_B : \mathbb{D}^r_k \to B$. For $s \le r$, denote by $B_{(s)}$ its subordinate Weil algebra obtained by truncating B to the s-th order. For $s \le q \le r$ we have the projection homomorphism $\pi^q_{s,B} : B_{(q)} \to B_{(s)}$. For $q = r$ we write simply $\pi_{s,B} : B \to B_{(s)}$ and for $B = \mathbb{D}^r_k$ we have $\pi_{s,B} = \pi^r_s$.

Let $\mathbb{D}^r_{i_1\dots i_m}$ be the subalgebra of $\mathbb{D}^r_k$ of polynomials in indeterminates $\tau_{i_1}, \dots \tau_{i_m}$ only. $\mathbb{D}^r_{i_1\dots i_m}$ is obviously isomorphic to $\mathbb{D}^r_m$. There is the subalgebra $B_{i_1\dots i_m}$ of B defined by $B_{i_1\dots i_m} = p_B(\mathbb{D}^r_{i_1\dots i_m})$. There are the insertion homomorphisms $\iota_{i_1\dots i_m} : \mathbb{D}^r_m \to \mathbb{D}^r_k$ and $\iota_{i_1\dots i_m,B} : B_{i_1\dots i_m} \to B$ where the first one is obvious and the second one is defined by $p_B(\tau_j) \mapsto p_B(\iota_{i_1\dots i_m}(\tau_j)) = p_B(\tau_j)$ for $j \in \{i_1, \dots, i_m\}$. Clearly, $B_{i_1\dots i_m} = \mathbb{D}^r_m/I_{i_1\dots i_m}$ for $I_{i_1\dots i_m}$ defined by $a \in I_{i_1\dots i_m}$ if and only if $a \in I$.

Let $J_{m,k}$ denote the ideal $\langle \tau_{i_{m+1}}, \dots, \tau_{i_k}\rangle$ in $\mathbb{D}^r_k$. Then $\mathbb{D}^r_{i_1\dots i_m}$ is idetntified with $\mathbb{D}^r_k/J_{m,k}$. For monomial B there is the obvious identification of $B_{i_1\dots i_m}$ with $\mathbb{D}^r_k/I \vee J_{m,k}$. In such case there is the homomorphism $p_{i_1\dots i_m,B} : B \to B_{i_1\dots i_m}$ defined by

$$(1.4.1) \qquad \tau_i = p_B(\tau_i) \mapsto p_B(\Sigma^m_{l=1}\delta^i_{i_l}\tau_i).$$

Clearly, $\operatorname{Im} p_{i_1\dots i_m,B} = \operatorname{Im} \iota_{i_1\dots i_m,B}$. For $B = \mathbb{D}^r_k$ we write simply $p_{i_1\dots i_m}$. The notation of the induced natural transformations $(\widetilde{p}_{i_1\dots i_m})_M$ and $(\widetilde{p}_{i_1\dots i_m,B})_M$ are simplified to $p^m_{i_1\dots i_m}$ and $p^m_{i_1\dots i_m,B}$ in case of $M = \mathbb{R}^m$.

An ideal $I \subseteq \mathbb{D}^r_k$ is said to be normal if $I = J \vee \langle \tau_{j_1}, \dots, \tau_{j_{k-l}}\rangle$ for an ideal J satisfying $J \subseteq \langle \tau_{i_1}, \dots, \tau_{i_l}\rangle^2$ provided $\{j_1, \dots, j_{k-l}\} \cup \{i_1, \dots, i_l\} = \{1, \dots, k\}$. A Weil algebra $A = \mathbb{D}^r_k/I$ is said to be normal if I is normal. Let $\mu = \langle \tau_1, \dots, \tau_k\rangle$ be the maximal ideal of $\mathbb{D}^r_k$. If width $A = k$ then we can reduce the definition of normality to the condition $I \subseteq \mu^2$. Every Weil algebra is isomorphic to a normal one, which is easy to deduce from (1.3.3) (see also the text after (1.1) in [29]).

Let width $A = k$. There is the insertion $\iota^r_{1,k} : \mathbb{D}^1_k \to \mathbb{D}^r_k$, which is a linear map and a section of $\pi^r_1 : \mathbb{D}^r_k \to \mathbb{D}^1_k$. Moreover, there is the map

$$(1.4.2) \qquad i_A = p_A \circ \iota^r_{1,k} : \mathbb{D}^1_k \to A$$

and its extension $i^m_A : J^1_0(\mathbb{R}^k,\mathbb{R}^m) \to T^A_0\mathbb{R}^m$ from $A = T^A\mathbb{R}$ to $T^A\mathbb{R}^m = A^m$ defined by components. Clearly, $\iota^r_{1,k} = i_{\mathbb{D}^r_k}$. In case of normal A the maps i_A and i^m_A are global sections. We remark that i_A is not a homomorphism of Weil algebras.

1.5. The coincidence of A-covelocities and jets to the classical ones

Recall the natural bundle $T^{r*}M$ of r-th order covelocities from [9] and [29] defined by $T^{r*}_xM = J^r_x(M,\mathbb{R})_0$ on objects and by $T^{r*}_x g(j^r_x f) = j^r_x f \circ (j^r_x g)^{-1}$ on morphisms. It is proved that the system of spaces $T^{r*}M = \bigcup_{x\in M} T^{r*}_x M$ with their T^{r*}-maps determines the natural bundle $P^rM[N^r_m, \ell]$ where N^r_m denotes the nilpotent ideal of $\mathbb{D}^r_m$ and $\ell : G^r_m \times N^r_m \to N^r_m$ denotes the left action on the standard fiber defined by $\ell(j^r_0 g, j^r_0\varphi) = j^r_0(\varphi \circ g^{-1})$.

In [29] the spaces $T^{A*}M$ of A-covelocities consisting of all $T^A_x f : T^A_x M \to T^A_0\mathbb{R} \simeq N_A$ are investigated for a general Weil algebra A and its nilpotent ideal N_A. The so-called T^{A*}-maps are defined by $T^{A*}_x g(T^A_x f) = T^A_x f \circ (T^A_x g)^{-1}$. It is proved that if height $A = r$ then $T^{A*}M$ and $T^{r*}M$ coincide. For any $M \in \mathrm{Obj}(\mathcal{M}f_m)$ and $N \in \mathrm{Obj}(\mathcal{M}f)$ there is defined the space $J^A(M,N) = \{T^A_x f; f : M \to N\}$ and for a local diffeomorphism $g : M_1 \to M_2$ and a smooth map $h : N_1 \to N_2$ there is the map $J^A(g,h) : J^A(M_1,N_1) \to J^A(M_2,N_2)$ defined by $J^A(g,h)(T^A_x f) = T^A_{f(x)} h \circ T^A_x f \circ (T^A_x g)^{-1}$. It is proved that the spaces $J^A(M,N)$ and $J^r(M,N)$ coincide as well. Moreover, for $m \le k = \mathrm{width}\, A$, every $T^A_x f : T^A_x M \to T^A_0\mathbb{R}$ is determined by its $\max\{m,k\}$ values over linearly independent 1-jets of elements from $T^A_x M$.

1.6. Homogeneous spaces

For a Lie group G and its closed subgroup K, the homogeneous space G/K is defined as the space of right cosets gK for $g \in G$ with the quotient topology induced by the factor projection $\pi : G \to G/K$. Such topology coincides with the unique smooth manifold topology on G/K with respect to which $\pi : G \to G/K$ is a surjective submersion (see [9], 5.11 and 10.5 and also [26], Chapter 1 and 4). We recall the free right action r of K on G which preserves fibers of π and is transitive over them. It is obtained from the multiplication in G and determines the principal bundle structure $(G, \pi, G/K, K)$ on π.

The right cosets gK correspond bijectively to the left cosets Kg since $g_1K = g_2K$ if and only if $Kg_1^{-1} = Kg_2^{-1}$. There is the group antiisomorphism $^{-1} : G \to G$ giving the identification of G/K with $K\backslash G$ by the assignment $gK \mapsto Kg^{-1}$. This particularly yields the transformation of the principal bundle $\pi : G \to G/K$ to the principal bundle $\pi_\ell : G \to K\backslash G$, exchanging the right action r of K on G for the left action ℓ. If possible we omit the index in π_ℓ and write simply π.

1.7. Some kinds of manifolds, submanifolds, partitions, foliations and group notations

Jets, jet functors and product preserving bundle functors are also studied in context with foliations, e.g. in [16], [10], [27]. Both kinds of objects are also studied and applied in theoretical physics, e.g. in [24], [4] and in Riemannian geometry, e.g. in [22]. Investigating foliations requires to distinguish between different kinds of submanifolds. An n-dimensional submanifold N in the sense of [9], which is an equivalent of a regular submanifold N in [26] is defined in terms of coordinates on M adapted to N, i.e. by pre-images of $\mathbb{R}^n \times \{0\}^{m-n}$ with respect to local maps on M. Nevertheless, the formulation of Frobenious theorem requires the concept of the initial submanifold from [9], Def. 2.14, which is an equivalent of a submanifold in [26], Chapter 1, Def. 2.2. In the present paper, we will prefer the terminology from [9].

A (regular) foliation on M of codimension $m - n$ is usually defined in terms of a distribution $\mathcal{D}$ on M (without singularities) determining the decomposition of M to maximal integrable submanifolds, the so-called leaves. The leaves are n-dimensional initial submanifolds in M. By the classical Frobenius theorem, the existence of a foliation determined by $\mathcal{D}$ is equivalent to the involutiveness of $\mathcal{D}$, see [26], Theorem 4.1. A foliation can be equivalently defined by an $(m - n)$-codimensional foliated atlas on M, see [26], Chapter 2, par. 6. Its leaves are again initial submanifolds. In any case they connected.

If a foliation consists of submanifolds, the situation is more simple, particularly for fiber bundles. In such case we can speak about vertical foliations (see [26]), despite the non-connectivity of fibers. In our investigations, fibres mostly consist of at most two connected components. Nevertheless, we use the concept of smooth partition rather than that of foliation in order to avoid a possible misunderstanding.

Notation 1.1. *By* $\mathrm{Diff}^{r,M}_x$ *we denote the group* $\mathrm{inv}\, J^r_x(M,M)_x$ *and by* $\mathrm{Diff}^{r,M}_{x,0}$ *the group* $\mathrm{Diff}^{r,M\times\mathbb{R}^{k-m}}_{(x,0)}$*, both with the jet composition multiplication. By* $(\mathrm{Diff}^{r,M}_{k,m})_{(x,0)}$ *we denote the subgroup of* $\mathrm{Diff}^{r,M}_{(x,0)}$ *of elements projectable to* $\mathrm{Diff}^{r,M}_x$*. For* $j^r_0\alpha_x \in \mathrm{inv}\, J^r_0(\mathbb{R}^m, M)_x$ *denote by* $j^r_0\bar{\alpha}_x$ *the element* $(j^r_0\alpha_x, j^r_0\,\mathrm{id}_{\mathbb{R}^{k-m}}) \in \mathrm{inv}\, J^r_0(\mathbb{R}^k, M \times \mathbb{R}^{k-m})_{(x,0)}$*. For* $M = \mathbb{R}^m$ *and* $x = 0$ *the groups* $\mathrm{Diff}^{r,M}_x$ *and* $(\mathrm{Diff}^{r,M}_{k,m})_{(x,0)}$ *coincide with the jet groups* G^r_m *and* $G^r_{k,m}$*.*

2. Higher-order Grassmannians

We continue the motivation part started in Introduction. If we do not accent the orthogonality in our definition of $\mathrm{Gr}(k,m)$, we can define its support as the space of orbits of the space $\mathrm{reg}\, J_0^1(\mathbb{R}^k,\mathbb{R}^m)_0$ of bases of m-dimensional linear subspaces of $\mathbb{R}^k$ with respect to the linear group G_m^1 acting from the left, recalling the obvious identification of linear morphisms with 1-jets of zero preserving maps. From this point of view, the i-th basis vector of $\mathbb{R}^k$ formed by the i-th component of $j_0^1\varphi \in \mathrm{reg}\, J_0^1(\mathbb{R}^k,\mathbb{R}^m)_0$ is of the form $(a_1^i,\dots,a_k^i)$. Prolonging the definition to higher orders, we define the support of $\mathrm{Gr}(r,k,m)$ as the space of orbits on $\mathrm{reg}\, J_0^r(\mathbb{R}^k,\mathbb{R}^m)_0$ with respect to G_m^r acting from the left. The topology is introduced by means of that of the homogeneous space $G_{k,m}^r\backslash G_k^r$, from which it is transmitted to $\mathrm{Gr}(r,k,m)$ by means of the reduction of the principal bundle $G_k^r \to G_{k,m}^r\backslash G_k^r$ to the total space $\mathrm{reg}\, J_0^r(\mathbb{R}^k,\mathbb{R}^m)_0 \times \{j_0^r\,\mathrm{id}_{\mathbb{R}^{k-m}}\}$ and the structure group $G_m^r \simeq G_m^r \times \{j_0^r\,\mathrm{id}_{\mathbb{R}^{k-m}}\}$. Nevertheless, we can insist on the orthonormality when defining the geometrical structure over $\mathrm{Gr}(r,k,m)$, at least in the first order. Before introducing the corresponding inner product we recall the basic facts concerning frames and frame bundles.

Ther is an concept of the r-th order frame bundle P^r introduced in [9], which forms a natural bundle on the category $\mathcal{M}f_n$. For $N \in Ob(\mathcal{M}f_n)$ we have $P^rN = \mathrm{inv}\, J_0^r(\mathbb{R}^n,N)$ and for a local diffeomorphism $f: N_1 \to N_2$ we have the map $P^rf: P^rN_1 \to P^rN_2$ defined by $j_0^r\varphi \mapsto j_0^r(f\circ\varphi)$. The natural equivalences are of the form $\widetilde{t_{j_0^r g}}$ (see (1.3.3)). Coming back to the first order, let (a_j^i) be the matrix coordinate form of elements from $P_{x_0}^1\mathbb{R}^k$ identified with $P_0^1\mathbb{R}^k$ by the translation map. Then any frame $j_0^1\eta$ can be considered as a linear map assigning the vector $j_0^1\eta_i = (a_i^1,\dots a_i^k)$ to the i-th canonical basis vector of $\mathbb{R}^k$. There is the standard inner product defined by $j_0^1\eta \cdot j_0^1\zeta = \eta^l \cdot \zeta^l$, which yields $(j_0^1\eta_i)\cdot(j_0^1\eta_j) = a_i^l \cdot a_j^l$. Let $\dim M = m$ and $x \in M$. We transmit our deductions to $P_{(x,0)}^1(M\times\mathbb{R}^{k-m})$, identifying a coordinate system on M mapping x to 0 together with its prolongation (in the sense of Definition 2.1 below) mapping $(x,0)\in M\times\mathbb{R}^{k-m}$ to $0\in\mathbb{R}^k$ with the local isometry between M and $\mathbb{R}^m$ defined near x, in case of taking its prolonged form near $(x,0)\in M\times\mathbb{R}^{k-m}$.

Let $\mathrm{V}(k,M)_x$ be the space of the so-called pull-back m-frames $j_0^1\widetilde{\varphi} \in P_{(x,0)}^1(M\times\mathbb{R}^{k-m})$ (see Definition 2.1). Their meaning is visible from their coordinates a_j^i and the assignments

$$dy_M^i \mapsto a_j^i dx^j, \qquad i = 1,\dots,m,\ j = 1,\dots k. \tag{2.1}$$

$\mathrm{V}(k,M)_x$ can be considered as the so-called Stiefel space at x. The Grassmannian space $\mathrm{Gr}(k,M)_x$ is from $\mathrm{V}(k,M)_x$ obtained by the factorization to m-dimensional linear subspaces spanned by elements of $\mathrm{V}(k,M)_x$. For $M = \mathbb{R}^m$ and $x = 0$, the values of the assignments (2.1) over dy_M^i are of the form $(a_1^i,\dots,a_k^i)\in\mathbb{R}^k$.

Then the formula for the inner product on vectors from (2.1) reads

$$j_0^1\alpha \cdot j_0^1\beta = \alpha_l\cdot\beta_l \qquad \text{or} \qquad j_0^1\varphi^i \cdot j_0^1\varphi^j = a_l^i \cdot a_l^j, \tag{2.2}$$

the second formula determining the values of the inner product on the images $j_0^1\varphi^i = (a_1^i,\dots,a_m^i)$ and $j_0^1\varphi^j = (a_1^j,\dots,a_m^j)$ over $dy_{\mathbb{R}^m}^i$ and $dy_{\mathbb{R}^m}^j$. We remark that $0\in\mathbb{R}^m$ can be replaced by $x_0\in\mathbb{R}^m$, applying the translation map. Any coordinate system transforming x to 0 can be identified with a local isometry, which enables to transmit locally the inner product (2.2) to M. Let $f: M\to N$, $g: N\to P$ be local diffeomorphisms. Then $du_P^p \mapsto g_l^p dz_N^l$, $dz_N^l \mapsto f_i^l dy_M^i$ and (2.1) implies $du^p \mapsto g_l^p f_i^l a_j^i dx^j$. This corresponds to the composition of 1-jets. We remark that that the Riemannian geometry concepts like isometries, geodesic maps and related concepts are in details studied in e.g.. [18], [31], [19], [22] etc.

We close the motivation part by giving the intention of generalizing the concepts discussed above from linear spaces, subspaces, linear bases and maps to Weil algebras, subalgebras, algebraic bases and Weil functor morphisms.

We give the technical concept of the prolongation map mentioned above. Let $\mathrm{pr}_{1,M}^k$ and $\mathrm{pr}_{2,M}^k$ be the canonical product projections $M\times\mathbb{R}^{k-m}\to M$ and $M\times\mathbb{R}^{k-m}\to\mathbb{R}^{k-m}$. If $M=\mathbb{R}^m$ we simplify their notations to $\mathrm{pr}_{1,m}^k$ and $\mathrm{pr}_{2,m}^k$.

Definition 2.1. *An invertible element* $j_0^r\widetilde{\varphi}_x \in \operatorname{reg} J_0^r(\mathbb{R}_0^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ *is said to be a prolongation of* $j_0^r\varphi_x \in \operatorname{reg} J_0^r(\mathbb{R}^k, M)_x$ *with respect to A if*

$$T^r_{(x,0)}\operatorname{pr}^k_{1,M}(j_0^r\widetilde{\varphi}_x) = j_0^r\varphi_x \text{ and } j^A\widetilde{\varphi}_{x,1} = j^A\widetilde{\varphi}_{x,2} \text{ whenever } j^A\varphi_{x,1} = j^A\varphi_{x,2}. \tag{2.3}$$

A local section $\widetilde{\ }: \operatorname{reg} J_0^r(\mathbb{R}^k, M)_x \to \operatorname{inv} J_0^r(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ *of* $T_0^r\operatorname{pr}^k_{1,M}$ *is said to be a prolongation map at x with respect to A if each* $j_0^r\varphi_x \in \operatorname{Dom}(\widetilde{\ })$ *is assigned some of its prolongations and the map* $j_0^r\varphi_x \mapsto T^r_{(x,0)}\operatorname{pr}^k_{2,M}\circ j_0^r\widetilde{\varphi}_x$ *is constant on every* $D_{j_0^r\varphi_x} \in \mathcal{V}_{x,M}$.

We define some equivalences and partitions on fibers of $\operatorname{reg} T_k^r M$ and $\operatorname{reg} T^A M$.

Definition 2.2. *(a) We define the equivalence* $\rho_{x,M}$ *on* $\operatorname{reg} J_0^r(\mathbb{R}^k, M)_x$ *by*

$$(j_0^r\varphi_x, j_0^r\psi_x) \in \rho_{x,M} \text{ if and only if } j_0^r\psi_x = j_x^r h \circ j_0^r\varphi_x \tag{2.4}$$

for some $j_x^r h \in \operatorname{Diff}_x^{r,M}$. *Further, we define the partition* $\mathcal{V}_{x,M}$ *on* $\operatorname{reg} J_0^r(\mathbb{R}^k, M)_x$ *formed by decomposition classes of* $\rho_{x,M}$.

(b) We define the equivalence $(\rho_A)_{x,M}$ *on* $\operatorname{reg} T_x^A M$ *by*

$$(j^A\varphi_x, j^A\psi_x) \in (\rho_A)_{x,M} \text{ if and only if there are } j_0^r\varphi_{x,0} \in j^A\varphi_x \text{ and}$$

$$j_0^r\psi_{x,0} \in j^A\psi_x \text{ such that } (j_0^r\varphi_{x,0}, j_0^r\psi_{x,0}) \in \rho_{x,M}. \tag{2.5}$$

(c) We define the equivalence $\rho^A_{x,M}$ *on* $\operatorname{reg} T_x^A M$ *by* $(j^A\varphi_x, j^A\psi_x) \in \rho^A_{x,M}$ *if and only if there are*

$$j_0^r\varphi_{x,0} \in j^A\varphi_x \text{ and } j_0^r\psi_{x,0} \in j^A\psi_x \text{ such that } j_x^r h \circ j_0^r\varphi_{x,0} \circ j_0^r g = j_0^r\psi_{x,0} \tag{2.6}$$

for some $j_0^r g \in G^A$ *and* $j_x^r h \in \operatorname{Diff}_x^{r,M}$.

(d) For normal A and $M = \mathbb{R}^m$, *we define the relations of equivalence* $(\rho_{Ort})_{x,M}$, $(\rho_{A,Ort})_{x,M}$ *and* $(\rho^{A,Ort})_{x,M}$ *by adding the claim of the first-order orthonormality to the components of all elements acting in* (2.4), (2.5) *and* (2.6) *with respect to the inner product* (2.2). *For general M and x, we claim the first-order orthonormality for the components of all elements acting in* (2.4), (2.5) *and* (2.6) *with respect to the inner product introduced by local coordinates, in the first order acting as local isometries transforming* 0 *to x. All* $j_x^r h \in \operatorname{Diff}_x^{r,M}$ *above are claimed to cover x-preserving local isometries on M in the first order.*

We must check that the relations $(\rho_A)_{x,M}$ and $\rho^A_{x,M}$ are equivalences, which is a non-trivial step. Then the relations from (d) are obviously equivalences as well.

Lemma 2.3. *The definition condition* (2.5) *can be equivalently replaced by the claim of the existence of invertible* $T_x^A h : T_x^A M \to T_x^A M$ *satisfying* $j^A\psi_x = T_x^A h(j^A\varphi_x)$. *The definition condition* (2.6) *can be equivalently replaced by the claim of the existence of invertible* $T_x^A h : T_x^A M \to T_x^A M$ *satisfying* $j^A\psi_x = T_x^A h \circ \widetilde{t}_{j_0^r g}(j^A\varphi_x)$ *for some* $j_0^r g \in G^A$. *Finally,* $(\rho_A)_{x,M}$ *and* $\rho^A_{x,M}$ *are equivalences.*

Proof: It is easy to check that (2.5) and (2.6) imply the conditions formulated in the assertion. To prove the converse recall the rigidity result from Subsection 1.5. It enables to replace $T_x^A h$ by $j_x^r h \in \operatorname{Diff}_x^{r,M}$, without loss of generality. Setting $(j_0^r\varphi_{x,0}, j_0^r\psi_{x,0})$ in (2.5) to $(j_0^r\varphi_{x,0}, j_x^r h \circ j_0^r\varphi_{x,0})$ for arbitrary $j_0^r\varphi_{x,0} \in j^A\varphi_x$ verifies the claim corresponding to (b). As for (c), $T_x^A h$ can be due to the rigidity result replaced by $j_x^r h \in \operatorname{Diff}_x^{r,M}$ again. We check that $(j_0^r\varphi_{x,0}, j_0^r\psi_{x,0})$ satisfies (2.6) if we set it to $(j_0^r\varphi_{x,0}, j_x^r h \circ \widetilde{t}_{j_0^r g}(j_0^r\varphi_{x,0}))$ for any $j_0^r\varphi_{x,0} \in j^A\varphi_x$. Indeed, for any $j_0^r g_0 \in G_A$ we have $\widetilde{p}_{A,\mathbb{R}^m} \circ j_x^r h \circ j_0^r\varphi_{x,0} \circ j_0^r g_0 \circ j_0^r g^{-1} = \widetilde{p}_{A,\mathbb{R}^m} \circ j_x^r h \circ j_0^r\varphi_{x,0} \circ j_0^r g^{-1} \circ (j_0^r g \circ j_0^r g_0 \circ j_0^r g^{-1}) = \widetilde{t}_{j_0^r g \circ j_0^r g_0^{-1} \circ j_0^r g^{-1}}(j^A\psi_x)$, applying normality of G_A in G^A. Now it is obvious that $(\rho_A)_{x,M}$ and $\rho^A_{x,M}$ are equivalences on $\operatorname{reg} T_x^A M$. □

Since all relations in Definition 2.2 are equivalences we can define the partitions $(\mathcal{V}_A)_{x,M}$ and $(\mathcal{V}^A)_{x,M}$ on $\operatorname{reg} T_x^A M$ corresponding to $(\rho_A)_{x,M}$ and $\rho^A_{x,M}$. For normal A we analogously define the partitions $(\mathcal{V}_{Ort})_{x,M}$, $(\mathcal{V}_{A,Ort})_{x,M}$ and $(\mathcal{V}^A_{Ort})_{x,M}$ corresponding to the equivalences from Definition 2.2(d). For $M = \mathbb{R}^m$ and $x = 0$ we simplify the notation by omitting indices. We write simply ρ, ρ_A etc.

Proposition 2.4. *There is the unique structure of a smooth manifold on the space of left cosets $G^r_{k,m}\backslash G^r_k$ for which the projection $\hat{p} : G^r_k \to G^r_{k,m}\backslash G^r_k$ is a surjective submersion. Moreover, $\hat{p}$ determines the principal bundle with the structure group $G^r_{k,m}$ and its free left action transitive on fibers.*

For any $x\in M$, the projection $\hat{p}_{x,M} : \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)} \to (\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}\backslash \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ defined for every $j^r_0\alpha_x \in P^r_x M$ by some of the formulas

$$j^r_0\eta_x \mapsto j^r_0\bar{\alpha}_x \circ [(j^r_0\bar{\alpha}_x)^{-1}\circ j^r_0\eta_x]_\rho \quad \text{or} \quad j^r_0\bar{\alpha}_x\circ j^r_0 g \mapsto j^r_0\bar{\alpha}_x\circ[j^r_0 g]_\rho, \tag{2.7}$$

(see Notation 1.1) determines the unique smooth structure on $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}\backslash \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ for which $\hat{p}_{x,M}$ is a surjective submersion. Moreover, $\hat{p}_{x,M}$ is principal bundle with the structure group $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$.

Proof: $G^r_{k,m}$ is a closed subgroup of G^r_k and the elementary facts of the homogeneous space theory yield the homogeneous space structure on $G^r_{k,m}\backslash G^r_k$, the principal bundle structure on $\hat{p}$ and the surjective submersion of $\hat{p}$.

For a manifold M, select $j^r_0\alpha_x \in P^r_x M$. Then $j^r_0 g_x = j^r_0\bar{\alpha}_x\circ j^r_0 g$ and $j^r_0 h_x = j^r_0\bar{\alpha}_x\circ j^r_0 h \in \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ share the same left coset with respect to $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$ if and only if $j^r_0 h_x \circ (j^r_{(0,0)} g_x)^{-1} \in (\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$ and $j^r_0 h\circ j^r_{(0,0)} g^{-1} \in G^r_{k,m}$. For another $j^r_0\beta_x \in P^r_x M$ we have $(j^r_0\bar{\beta}_x)^{-1}\circ j^r_0\bar{\alpha}_x \in G^r_{k,m}$ and $j^r_0\bar{\beta}_x\circ(j^r_0\bar{\alpha}_x)^{-1} \in (\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$, which implies the independence of (2.7) and $\hat{p}_{x,M}$ on the choice of a frame. Clearly, $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)} = j^r_0\bar{\alpha}_x \circ G^r_{k,m}\circ(j^r_0\bar{\alpha}_x)^{-1} = \operatorname{conj}(j^r_0\bar{\alpha}_x)(G^r_k)$.

The smooth structure on $\hat{p}_{x,M}$ is transmitted from $G^r_{k,m}\backslash G^r_k$ by means of frames. Indeed, for any $j^r_0\alpha_x\in P^r_x M$, $j^r_0 h\in G^r_{k,m}$ and $(j^r_0\varphi_1, j^r_0\varphi_2) = (T^r_0\operatorname{pr}^k_{1,m}(j^r_0\varphi), T^r_0\operatorname{pr}^k_{2,m}(j^r_0\varphi)) \in \operatorname{inv} J^r_0(\mathbb{R}^k,\mathbb{R}^k)_0$ an element $j^r_0\bar{\alpha}_x\circ j^r_0 h\circ(j^r_0\varphi_1, j^r_0\varphi_2)$ is obtained as $(j^r_0\bar{\alpha}_x\circ j^r_0 h\circ(j^r_0\bar{\alpha}_x)^{-1})\circ j^r_0\bar{\alpha}_x\circ(j^r_0\varphi_1, j^r_0\varphi_2)$. □

Corollary 2.5. *For any $x\in M$, there is a smooth manifold structure on $\mathcal{V}_{x,M} = \operatorname{Diff}^r_{x,M}\backslash \operatorname{reg} J^r_0(\mathbb{R}^k, M)_x$. The factor projection $\hat{p}^\#_{x,M} : \operatorname{reg} J^r_0(\mathbb{R}^k, M)_x \to \mathcal{V}_{x,M}$ is obtained by the identification of $\mathcal{V}_{x,M}$ with $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}\backslash \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ and the identification of $\hat{p}^\#_{x,M}$ with $\hat{p}_{x,M}$ as follows*

$$\hat{p}^\#_{x,M} = T^r_{(x,0)}\operatorname{pr}^k_{1,m}\circ\hat{p}_{x,M}\circ\widetilde{\ } \quad \text{and} \quad \hat{p}_{x,M} = \widetilde{\ }\circ\hat{p}^\#_{x,M}\circ T^r_{(x,0)}\operatorname{pr}^k_{1,M} \tag{2.8}$$

where $\widetilde{\ }$ is an arbitrary, locally defined prolongation map with respect to $\mathbb{D}^r_k$.

Proof: Clearly, $j^r_0\psi_2 = j^r_{(x,0)} g\circ j^r_0\psi_1$ implies $j^r_{(x,0)}\operatorname{pr}^k_{1,M}\circ j^r_0\psi_2 = \underline{j^r_{(x,0)} g}\circ j^r_{(x,0)}\operatorname{pr}^k_{1,M}\circ j^r_0\psi_1$ where $j^r_0\psi_1$, $j^r_0\psi_2 \in \operatorname{reg} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ and $j^r_{(x,0)} g \in (\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$, by $\underline{j^r_{(x,0)} g} \in \operatorname{Diff}^r_{x,M}$ denoting the subordinate element from G^r_m. Conversely, $j^r_0\varphi_2 = j^r_x h\circ j^r_0\varphi_1$ for $j^r_0\varphi_1$, $j^r_0\varphi_2 \in J^r_0(\mathbb{R}^k, M)_x$ and $j^r_x h \in \operatorname{Diff}^{r,M}_x$ implies $j^r_0\widetilde{\varphi_2} = j^r_{(x,0)}\bar{h}\circ j^r_0\widetilde{\varphi_1}$. This way, $(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}$-classes are transformed to $\operatorname{Diff}^r_{x,M}$-classes and vice versa. Thus $U\subseteq \mathcal{V}_{x,M}$ will be open if and only if $\widetilde{U}\subseteq(\operatorname{Diff}^{r,M}_{k,m})_{(x,0)}\backslash \operatorname{inv} J^r_0(\mathbb{R}^k, M\times\mathbb{R}^{k-m})_{(x,0)}$ is open. Since $T^r_0\operatorname{pr}^k_{1,m}$ is an open map, the topology we have just defined coincides with the quotient topology on $\operatorname{reg} J^r_0(\mathbb{R}^k, M)_x$ obtained by $\hat{p}^\#_{x,M}$.

It remains to give a prolongation map $\widetilde{\ }$ on a neighbourhood of every $j^r_0\varphi \in J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ with respect to $\mathbb{D}^r_k$. If there is a universal element $j^r_0\theta \in J^r_0(\mathbb{R}^k,\mathbb{R}^{k-m})_0$ satisfying $(j^r_0\varphi, j^r_0\theta) \in \operatorname{inv} J^r_0(\mathbb{R}^k,\mathbb{R}^k)_0$ for every $j^r_0\varphi \in U$ then the assignment $j^r_0\varphi \mapsto (j^r_0\varphi, j^r_0\theta)$ determines the required prolongation map. Applying an arbitrary frame $j^r_0\alpha_x$ with its prolonged form $j^r_0\bar{\alpha}_x$ (see Notation 1.1) we transmit the prolongation map from an open subset of $\operatorname{reg} J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ to an open subset of $\operatorname{reg} J^r_0(\mathbb{R}^k, M)_x$. □

We remark that $\hat{p}^{\#}_{x,M}$ is a surjective submersion, which follows from left formula of (2.8) and the definition of $\hat{p}_{x,M}$. Applying the well-known universal property of surjective submersions to $\hat{p}^{\#}_{x,M}$ we observe that $\widetilde{\ }$ determines a smooth map on $\hat{p}^{\#}_{x,M}(U) \subseteq \mathcal{V}_{x,M}$ whenever U is contained in the domain of $\widetilde{\ }$.

Having the identification of $\mathcal{V}_{x,M}$ with the basis $(\mathrm{Diff}^{r,M}_{k,m})_{(x,0)} \setminus \mathrm{inv}\, J^r_0(\mathbb{R}^k, M \times \mathbb{R}^{k-m})_{(x,0)}$ of the principal bundle $\hat{p}_{x,M}$ we can consider the reduction of its structure group to the subgroup $\mathrm{Diff}^{r,M}_x \times \{j^r_0\, \mathrm{id}_{\mathbb{R}^{k-m}}\} \simeq \mathrm{Diff}^{r,M}_x$. By the formula (2.8) of Corollary 2.5 the total space of $\hat{p}_{x,M}$ can be reduced to $\widetilde{\ } \circ T^r_{(x,0)}\, \mathrm{pr}^k_{1,M}(J^r_0(\mathbb{R}^k, M \times \mathbb{R}^{k-m})_{(x,0)}) \simeq \mathrm{reg}\, J^r_0(\mathbb{R}^k, M)_x$, independently on the choice of the prolongation map. Hence we have deduced

Corollary 2.6. *$\hat{p}^{\#} : \mathrm{reg}\, J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0 \to \mathcal{V}$ from (2.8) is a principle bundle obtained by the identification of the reduction of $\hat{p}$ to the subgroup $G^r_m \simeq G^r_m \times \{j^r_0\, \mathrm{id}_{\mathbb{R}^{k-m}}\}$ of $G^r_{k,m}$ and to the subspace $\widetilde{\ } \circ (T^r_k)_0\, \mathrm{pr}^k_{1,m}(\mathrm{inv}\, J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0) \simeq \mathrm{reg}\, J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ of its total space $\mathrm{inv}\, J^r_0(\mathbb{R}^k, \mathbb{R}^k)_{(0,0)}$.*

More generally, $\hat{p}^{\#}_{x,M}$ from (2.8) is a principal bundle obtained by the identification of the reduction of $\hat{p}_{x,M}$ to the subgroup $\mathrm{Diff}^{r,M}_x \simeq \mathrm{Diff}^{r,M}_x \times \{j^r_0\, \mathrm{id}_{\mathbb{R}^{k-m}}\}$ of $(\mathrm{Diff}^{r,M}_{k,m})_{(x,0)}$ and to the subspace $\widetilde{\ } \circ (T^r_k)_{(x,0)}\, \mathrm{pr}^k_{1,M}(\mathrm{inv}\, J^r_0(\mathbb{R}^k, M \times \mathbb{R}^{k-m})_{(x,0)}) \simeq \mathrm{reg}\, J^r_0(\mathbb{R}^k, M)_x$ of its total space $\mathrm{inv}\, J^r_0(\mathbb{R}^k, M \times \mathbb{R}^{k-m})_{(x,0)}$.

Definition 2.7. *The basis $\mathcal{V}$ of the principal bundle $\hat{p}^{\#}$ is said to be the r-th order Grassmannian $\mathrm{Gr}(r, k, m)$. The basis $\mathcal{V}_{x,M}$ of the principal bundle $\hat{p}^{\#}_{x,M}$ is said to be the r-th order Grassmann space $\mathrm{Gr}(r, k, M)_x$ at $x \in M$. The space $\mathrm{Gr}(r, k, M)$ with the topology defined in Proposition 2.8 below is said to be the Grassmann manifold over M. Speaking about Grassmann manifold without specifying M means that $M = \{0\}$, $0 \in \mathbb{R}^m$ and Grassmann manifold coincides with Grassmannian $\mathrm{Gr}(r, k, m)$. The principal bundle $\hat{p}^{\#}_{x,M}$ or its total space is said to be the geometrical structure over $\mathrm{Gr}(r, k, M)_x$ at x. The system $\hat{p}^{\#}_M = \bigcup_{x \in M} \hat{p}^{\#}_{x,M}$ is said to be the geometrical structure or Grassmann bundle over the Grassmann manifold $\mathrm{Gr}(r, k, M)$. Speaking about geometrical structure without specifying M means that $M = \{0\}$, $0 \in \mathbb{R}^m$ and the geometrical structure or Grassmann bundle coincides with $\hat{p}^{\#}$.*

For $r = 1$ we write simply $\mathrm{Gr}(k, m)$ or $\mathrm{Gr}(k, M)_x$ instead $\mathrm{Gr}(1, k, m)$ and $\mathrm{Gr}(1, k, M)_x$. On the other hand we can write $\mathcal{V}_1$ or $(\mathcal{V}_1)_{x,M}$ instead $\mathcal{V}$ or $\mathcal{V}_{x,M}$. Further, any prolongation map $\widetilde{\ }$ and (2.8) imply the existence of a well-defined correspondence between local sections $s_{x,M}$ and $s^{\#}_{x,M}$ of the principal bundles $\hat{p}_{x,M}$ and $\hat{p}^{\#}_{x,M}$ as follows

$$s_{x,M} = \widetilde{s^{\#}_{x,M}} \circ T^r_{(x,0)}\, \mathrm{pr}^k_{1,M} \qquad s^{\#}_{x,M} = T^r_{(x,0)}\, \mathrm{pr}^k_{1,M} \circ s \circ \widetilde{\ }. \tag{2.9}$$

This way, $(\mathrm{Diff}^{r,M}_{k,m})_{(x,0)}$-classes are transformed to $\mathrm{Diff}^r_{x,M}$-classes and vice versa.

For any m-dimensional manifold M put $\mathrm{Gr}(r, k, M)$ to $\bigcup_{x \in M} \mathrm{Gr}(r, k, M)_x = \bigcup_{x \in M} \mathcal{V}_{x,M}$ and for any local diffeomorphism $f : M \to N$ define $\mathrm{Gr}(r, k, m)f : \mathrm{Gr}(r, k, M) \to \mathrm{Gr}(r, k, N)$ by $[j^r_0 \varphi_x]_{\rho_{x,M}} \mapsto [j^r_x f \circ j^r_0 \varphi_x]_{\rho_{f(x),N}}$. Finally, we define the projection $\pi_{\mathrm{Gr},M} : \mathrm{Gr}(r, k, M) \to M$ assigning x to $D \in \mathrm{Gr}(r, k, M)$ in case of $D \in \mathcal{V}_{x,M}$. Hence we obtain a bundle functor as follows

Proposition 2.8. *The system of $\mathrm{Gr}(r, k, M)$ with maps $\mathrm{Gr}(r, k, m)f$ and projections $\pi_{\mathrm{Gr},M}$ forms a bundle functor on $\mathcal{M}f_m$ identified with $P^r M[\mathrm{Gr}(r, k, m), \ell_{\mathrm{Gr}}]$ where ℓ_{Gr} is the trivial left action of G^r_m on $\mathrm{Gr}(r, k, m)$ defined by $\ell_{\mathrm{Gr}}(j^r_0 h, D_{j^r_0 \varphi}) = D_{j^r_0 h \circ j^r_0 \varphi} = D_{j^r_0 \varphi}$.*

There is a natural transformation $trans_M : \mathrm{reg}\, T^r_k M \to \mathrm{Gr}(r, k, M)$ defined by $trans_M(j^r_0 \varphi_x) = [j^r_0 \varphi_x]_{\rho_{x,M}} = D_{j^r_0 \varphi_x}$.

Finally, there is a bijective correspondence between natural equivalences over bundle functors $\mathrm{Gr}(r, k, \cdot)$ and $P^r[\mathrm{Gr}(r, k, m), \ell_{\mathrm{Gr}}]$ assigning $\{\mathrm{id}_{P^r M}, [\widetilde{t}]_{\mathrm{Gr}(r,k,m)}\}$ to $[\widetilde{t}]_{\mathrm{Gr}(r,k,M)}$ where $[\widetilde{t}]_{\mathrm{Gr}(r,k,M)}$ is obtained by the factorization of $\widetilde{t}_{j^r_0 g, M}$ (see (1.3.3)) acting on $T^r_k M$ to $\mathrm{Gr}(r, k, M)$.

Proof: The first assertion follows from the theory of bundle functors, namely from [9], 14.5 and 14.6. Indeed, there is a bijective correspondence ξ_M identifying the spaces $\mathrm{Gr}(r, k, M)$ with objects $P^r M[\mathrm{Gr}(r, k, m), \ell_{\mathrm{Gr}}]$ and the maps $\mathrm{Gr}(r, k, m)f$ with morphisms $\{P^r f, \mathrm{id}_{\mathrm{Gr}(r,k,m)}\}$. It is defined by the mutually inverse assignments $D_{j^r_0 \varphi_x} \mapsto \{j^r_0 \alpha_x, D_{(j^r_0 \alpha_x)^{-1} \circ j^r_0 \varphi_x}\}$ and $\{j^r_0 \alpha_x, D_{j^r_0 \varphi}\} \mapsto D_{j^r_0 \alpha_x \circ j^r_0 \varphi}$. It is easy to check that ξ_M is invertible and commutes with the maps $\mathrm{Gr}(r, k, m)f$ and morphisms $\{P^r f, \mathrm{id}_{\mathrm{Gr}(r,k,m)}\}$.

The second assertion is verified by checking the equivariance of the resriction $trans : \mathrm{reg}(T^r_k)_0\mathbb{R}^m \to \mathrm{Gr}(r,k,m)$ of $trans_{\mathbb{R}^m}$ acting between standard fibers with respect to the obvious left action of G^r_m on $\mathrm{reg}(T^r_k)_0\mathbb{R}^m$ and ℓ_{Gr} on $\mathrm{Gr}(r,k,m)$.

The last assertion follows from the first and second one. It is also a corollary of Proposition 4.8 below. □

Remark 2.9. *(a) The basis* $\mathrm{Gr}(r,k,m)$ *of* $\hat{p}^\#$ *can be identified with the space of m-wide subalgebras of* $\mathbb{D}^r_k$ *isomorphic to* $\mathbb{D}^r_m$ *while fibers of* $\hat{p}^\#$ *can be considered as spaces of algebraic bases spanning the subalgebra represented by the basis element. The standard fiber of* $\hat{p}^\#$ *can be viewed as the space of algebraic bases of* $\mathbb{D}^r_m$.

(b) $V(r,k,k-m) = G^r_m\backslash G^r_k$ *considered as the space of* $(k-m)$*-frames can be viewed as the higher-order, generalized Stiefel manifold. The group* $\bar{G}^r_m \simeq G^r_m \times \{j^r_0\,\mathrm{id}_{\mathbb{R}^{k-m}}\}$ *is again considered as a subgroup of* G^r_k*. The case of the classical Stiefel manifold* $V(k,k-m)$ *with the additional claim of orthonormality of frames corresponds to* $O(m)\backslash O(k)$.

Let $(\pi^r_1)^{-1}(\mathrm{reg}\,J^1_{Ort})_0(\mathbb{R}^k,\mathbb{R}^m)_0$ be the subspace of $\mathrm{reg}\,J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ consisting of elements with mutually orthonormal components in the first order (with respect to (2.2)). We simplify its notation to $(J^r_{Ort})_0(\mathbb{R}^k,\mathbb{R}^m)_0$. We analogously consider the subspace $(J^r_{Ort})_0(\mathbb{R}^k,M)_x = (\pi^r_1)^{-1}(\mathrm{reg}\,J^1_{Ort})_0(\mathbb{R}^k,M)_x$ of $\mathrm{reg}\,J^r_0(\mathbb{R}^k,M)_x$ consisting of elements with mutually orthonormal components in the first order in a local isometry with $\mathbb{R}^m$ determined by a selected coordinate system transforming x to 0. Let $\mathcal{I}so_x(M)$ denote the group of x-preserving local isometries on M. We define the principal bundles $\hat{p}^\#_{Ort}$ and $(\hat{p}^\#_{Ort})_{x,M}$ as follows

Definition 2.10. $\hat{p}^\#_{Ort}$ *is defined as the principal bundle obtained from* $\hat{p}^\#$ *by the reduction of its total space to* $(J^r_{Ort})_0(\mathbb{R}^k,\mathbb{R}^m)_0$ *and of its structure group to* $(\pi^r_1)^{-1}(O(m))$*. The principal bundle* $\hat{p}^\#_{Ort}$ *or its total space is said to be the geometrical structure or Grassmann bundle over* $\mathrm{Gr}(r,k,m)$.

$(\hat{p}^\#_{Ort})_{x,M}$ *is defined as the principle bundle obtained from* $\hat{p}^\#_{x,M}$ *by the reduction of its total space to* $(J^r_{Ort})_0(\mathbb{R}^k,M)_x$ *and its structure group to* $(\pi^r_1)^{-1}(\mathcal{I}so(x,M))$*. The principal bundle* $(\hat{p}^\#_{Ort})_{x,M}$ *or its total space is said to be the geometrical structure over* $\mathrm{Gr}(r,k,M)_x$*. The system* $(\hat{p}^\#_{Ort})_M = \bigcup_{x\in M}(\hat{p}^\#_{Ort})_{x,M}$ *is said to be the geometrical structure or Grassmann bundle over the Grassmann manifold* $\mathrm{Gr}(r,k,M)$*. Speaking about Grassmann bundle or geometrical structure without specifying M means that* $M = \{0\}$, $0 \in \mathbb{R}^m$ *and Grassmann bundle coincides with* $\hat{p}^\#_{Ort}$.

The claims of the first-order orthonormality can be extended from $\hat{p}^\#$ and $\hat{p}^\#_{x,M}$ to $\hat{p}$ and $\hat{p}_{x,M}$ and from the group $\mathcal{I}so_x(M)$ to $\mathcal{I}so_{(x,0)}(M\times\mathbb{R}^{k-m})$), obtaining the principal bundles $\hat{p}_{Ort}$ and $(\hat{p}_{Ort})_{x,M}$. We conclude by the identification of $\mathrm{Gr}(r,k,m)$ considered as the basis of $\hat{p}^\#_{Ort}$ with $(\pi^r_1)^{-1}(O(m))\times(\pi^r_1)^{-1}(O(k-m))\backslash(\pi^r_1)^{-1}(O(k))$ (see Corollary 3.5).

3. The construction of an atlas on the higher-order Grassmannian

3.1. A finite system of local sections on $\hat{p}^\#_{Ort}$ of the first order

We construct a local section of $\hat{p}^\#_{Ort}$ defined on a dense and open subset of $\mathcal{V} = \mathrm{Gr}(r,k,m)$. We give a finite set of local sections of this kind and an atlas on $\mathcal{V} = \mathrm{Gr}(r,k,m)$ determined by them. Despite Definition 2.7 we suppress the notation $\mathrm{Gr}(r,k,m)$ and write mostly $\mathcal{V}$ until Section 4 where the bundle functor Gr^A of Weil Grassmannian is introduced. In the first-order case corresponding to $\mathrm{Gr}(k,m)$ we write also $\mathcal{V}_1$.

We need some other notations as follows. By $\det_{i_1\dots i_m}$ we denote the value of the m-th order determinant of the matrix of the orthogonal projection of k-dimensional vectors $j^1_0\varphi^1,\dots,j^1_0\varphi^m \in \mathrm{reg}\,J^1_0(\mathbb{R}^k,\mathbb{R}^m)_0$ to the linear subspace $L(j^1_0\,\mathrm{pr}_{i_1},\dots,j^1_0\,\mathrm{pr}_{i_m})$ of $\mathbb{R}^k$ spanned by the selected canonical basis vectors. Let $V_{i_1,\dots,i_m}$ be the domain of regularity of $\det_{i_1\dots i_m}$. Since $\det_{i_1\dots i_m}$ takes the zero value on every $\mathcal{V}$-class either nowhere or everywhere we put $U_{i_1,\dots,i_m}$ to $\hat{p}^\#(V_{i_1\dots i_m})$. Clearly, any $U_{i_1\dots i_m}$ is dense in $\mathcal{V}_1$. Extending our notations to higher order cases we define

$$\mathcal{V}_{i_1\dots i_m} = \{D\in\mathcal{V};\pi^r_1(D)\in U_{i_1\dots i_m}\},\qquad \text{in particular}\qquad \overset{\circ}{\mathcal{V}} = \mathcal{V}_{1\dots m} \tag{3.1.1}$$

We essentially use the convention based on Corollary 2.5, which identifies the basis $\mathcal{V}$ of $\hat{p}^\#$ or $\hat{p}^\#_{Ort}$ with the basis of $\hat{p}$ or $\hat{p}_{Ort}$. Analogously we do with $\mathcal{V}_{x,M}$, $\hat{p}^\#_{x,M}$, $(\hat{p}^\#_{Ort})_{x,M}$ and $\hat{p}_{x,M}$.

Proposition 3.1. *For $r = 1$, there is a couple of local sections s and $s^{\#}$ of the principal bundles $\hat{p}$ and $\hat{p}^{\#}_{Ort}$ connected by the correspondence* (2.9) *defined on $U_{1\dots m}$, which is dense and open in $\mathcal{V}_1$. The range of s contains $j^1_0\,\mathrm{id}_{\mathbb{R}^k}$.*

More generally, there are two finite systems of local sections $s_{i_1\dots i_m}$ of $\hat{p}_{Ort}$ and $s^{\#}_{i_1\dots i_m}$ of $\hat{p}^{\#}_{Ort}$ connected by (2.9) *containing $s = s_{1\dots m}$ and $s^{\#}_{1\dots m} = s^{\#}$ with dense and open domains $U_{i_1\dots i_m}$ covering $\mathcal{V}_1$. Both of the systems $(U_{i_1\dots i_m}, s_{i_1\dots i_m})$ and $(U_{i_1\dots i_m}, s^{\#}_{i_1\dots i_m})$ corresponding to m-elementary subsets of $\{1, \dots, k\}$ determine a finite atlas on $\mathcal{V}_1 = \mathrm{Gr}(k, m)$.*

Proof: Consider the inner product (2.2) on $\mathrm{inv}\, J^1_0(\mathbb{R}^k, \mathbb{R}^k)_0$. Clearly, any element of $\mathcal{V}_1$ can be considered as the space of linearly independent sequences of vectors of lenght m spanning the same m-dimensional vector subspace of $\mathbb{R}^k$. Consider a system of local sections $(\sigma_\alpha)_{\alpha\in I} : U_\alpha \to G^1_k$ of $\hat{p}$ with $(U_\alpha)_{\alpha\in I}$ covering $G^1_{k,m}\backslash G^1_k$. Without loss of generality suppose that $\mathrm{Im}(\sigma_\alpha)$ is formed by k-tuples of orthonormal vectors. It is possible due to the Gramm-Schmidt orthogonalization process (GSOP), which acts smoothly until the zero vector appears as its output. Applying the compactness argument we can assume the finitness of the system $(\sigma_\alpha)_{\alpha\in I}$. Nevertheless, local sections σ_α are only auxiliary objects used for the proof of the smoothness of the map $s = s_{1\dots m}$ to be constructed.

As for the value of $s_{1\dots m}$, its first m components are defined by the assignment of the orthonormalized sequence obtained by GSOP from the orthogonal projections of vectors $j^1_0\,\mathrm{pr}_1, \dots, j^1_0\,\mathrm{pr}_m$ to the current $\mathcal{V}$-class. The claim of regularity corresponding to non-zero outputs of GSOP yields the limits for the domain of $s^{\#}_{1\dots m}$, which coincides to $U_{1\dots m}$ defined before (3.1.1). The $k - m$ additional components of $j^1_0\widetilde{\varphi}$ in $\mathrm{Im}\, s_{1\dots m}$ can be obtained as in Corollary 2.5 in case of $\hat{p}$ while in case of $\hat{p}_{Ort}$ we continue the procedure with $j^1_0\,\mathrm{pr}_{m+1}, \dots, j^1_0\,\mathrm{pr}_k$.

The smoothness of $s_{1\dots m}$ is verified on the intersection of its domain with arbitrary U_α above, if non-empty. Let $\mathcal{F} = (\vec{f^\alpha_j})$ $(j = 1, \dots, m)$ be the orthonormalized system of the first m vectors from the value of σ_α. Then the coordinate expression of $s^{\#}_{1\dots m}$ obtained by the standard computations of the orthogonal projection of a vector to the vector subspace of $\mathbb{R}^k$ determined by the vectors $\vec{f^\alpha_1}, \dots, \vec{f^\alpha_m}$ reads

$$j^1_0\varphi = \Big(\Sigma^m_{j=1}\frac{\vec{e_i}\cdot\vec{f^\alpha_j}}{\|\vec{f^\alpha_j}\|^2}\cdot\vec{f^\alpha_j}\Big)_{i=1\dots m} = \Big(\Sigma^m_{j=1}\frac{(f^\alpha_j)^i}{\|\vec{f^\alpha_j}\|^2}\cdot\vec{f^\alpha_j}\Big)_{i=1\dots m}. \tag{3.1.2}$$

For orthonormal $\mathcal{F}$ we have $j^1_0\varphi = (\Sigma^m_{j=1}((f^\alpha_j)^i\cdot\vec{f^\alpha_j})_{i=1\dots m}$. This proves the smoothness of a local section of $\hat{p}$ defined by the assignment of the sequence of orthogonal projections of the first m-canonical vectors of $\mathbb{R}^k$ to the current class $D \in \mathcal{V}$, independently on the choice of σ_α. Taking into account the smoothness of GSOP on its domain of regularity completes the verification of the smoothnes of $s = s_{1\dots m}$. Clearly, $U_{1\dots,m}$ contains $\hat{p}^{\#}(j^1_0\,\mathrm{pr}_1, \dots, j^1_0\,\mathrm{pr}_m)$ and $\mathrm{Im}(s_{1\dots m})$ contains $j^1_0\,\mathrm{id}_{\mathbb{R}^k}$, which completes the proof of the first assertion.

Local sections $s_{i_1\dots i_m}$ with domains $U_{i_1\dots i_m}$ are constructed analogously. The supports of local maps $(U_{i_1\dots i_m}, s_{i_1\dots i_m})$ are dense in $\mathcal{V}_1$ and cover it. The system determines a smooth atlas on $\mathrm{Gr}(k, m) = \mathcal{V}_1$ determined by at most $\binom{k}{m}$ local maps with the transition maps $s_{i_1\dots i_m} \circ s^{-1}_{j_1\dots j_m}$ restricted to $\mathrm{Im}\, s_{j_1\dots j_m}(U_{i_1\dots i_m} \cap U_{j_1\dots j_m})$. □

3.2. *A finite system of local sections of $\hat{p}^{\#}_{Ort}$ of higher order and an atlas on* $\mathrm{Gr}(r, k, m)$

In what follows, we need the subgroup $B^r_k = \mathrm{Ker}\, \pi^r_1 = (\pi^r_1)^{-1}(j^1_0\,\mathrm{id}_{\mathbb{R}^k})$ of G^r_k. By [9], Section 13 it is endowed with some significant properties like the globality of the exponential map and its identification with the semi-direct product $B^r_k \ltimes G^1_k$, which can be defined by (3.2.5), (3.2.6) and (3.2.7) below.

Let $\bar{p}^{\#} : (\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m}) \to B^r_m\backslash(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ be the factorization map with values in the space of B^r_m-orbits with respect to the left action defined on $(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ by the jet composition. Let $\mathcal{V}_0$ be the subspace of $\mathring{\mathcal{V}}$ defined by $D \in \mathcal{V}_0$ if and only if $j^1_0\,\mathrm{pr}^k_{1,m} \in \pi^r_1(D)$.

Lemma 3.2. *(a) The space $\mathcal{V}_0$ is identified with $B^r_m\backslash(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ equipped with the quotient topology of that on $(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ by $\bar{p}^\#$. It is the basis of the principal bundle $\bar{p}^\#$ obtained by the reduction of the corestriction of $\hat{p}^\#$ to $\mathcal{V}_0$ to $(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ on the level of the total space and to B^r_m on the level of the structure group. Analogously, $\bar{p}: B^r_k \to B^r_{k,m}\backslash B^r_k \simeq \mathcal{V}_0$ is obtained by the reduction of the corestriction of $\hat{p}$ to $\mathcal{V}_0$ to $(\pi^r_1)^{-1}(j^1_0\,\mathrm{id}_{\mathbb{R}^k}) = \ker\pi^r_1$ on the level of the total space and to $B^r_{k,m}$ on the level of the structure group. The maps $\bar{p}^\#$ and $\bar{p}$ are related by* (2.8)*.*

(b) There is a couple of global sections $\bar{s}^\#$ and $\bar{s}$ of the principal bundles $\bar{p}^\#$ and $\bar{p}$, related by (2.9)*. $\mathrm{Im}\,\bar{s}^\#$ is a submanifold of $(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ and $\mathrm{Im}\,\bar{s}$ a submanifold of B^r_k.*

Proof: (a) Clearly, $j^r_0\varphi$ and $j^r_0\psi \in (\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ share the same element of $\mathcal{V}_0$ if and only if they share the same element of $B^r_m\backslash(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$. Since $j^1_0\,\mathrm{pr}^k_{1,m} \in \pi^r_1(D)$ for any $D \in \mathcal{V}_0$ by definition of $\mathcal{V}_0$, we have the bijection between $\mathcal{V}_0$ and $B^r_m\backslash(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$.

From the topological point of view $\mathcal{U} \subseteq \mathcal{V}_0$ is open if and only if so is $(\hat{p}^\#)^{-1}(\mathcal{U})$ in $(\hat{p}^\#)^{-1}(\mathcal{V}_0)$. Any $P = (\hat{p}^\#)^{-1}(\mathcal{U})$ of this kind contains only those elements $j^r_0\varphi \in \mathrm{reg}\,J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ which satisfy $\varphi^i_j = 0$ whenever $j > m$. This follows from the definition of $\mathcal{V}_0$ and the jet composition formula. On the other hand $\mathcal{U} \subseteq \mathcal{V}_0$ is open with respect to the quotient topology on $B^r_m\backslash(\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ if and only if $(\bar{p}^\#)^{-1}(\mathcal{U})$ is open in $(\bar{p}^\#)^{-1}(\mathcal{V}_0)$. This happens if and only if $(\bar{p}^\#)^{-1}(\mathcal{U})$ is of the form $P \cap P_0$ for some P above and $P_0 = (\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$. The required identification of topologies on $\mathcal{V}_0$ follows from the fact that any element of P can be transformed to $P \cap P_0$ by the application of a suitable $j^r_0h \in G^r_m$ from the left.

The identification of topologies on $\mathcal{V}_0$ introduced by means of $\hat{p}^\#$ and $\bar{p}^\#$ we have just proved implies directly the assertion concerning the reduction of $\hat{p}^\#$ to $\bar{p}^\#$. As for $\bar{p}$ and $\hat{p}$ we proceed quite analogously, applying (2.8) and the convention on $\mathcal{V}$-classes presented before Proposition 3.1.

(b) It follows from the jet composition formula that for any $j^r_0\varphi \in (\pi^r_1)^{-1}(j^1_0\,\mathrm{pr}^k_{1,m})$ there is $j^r_0\varphi_D$ sharing the same $\mathcal{V}_0$-class such that $p^m_{1\ldots m}(j^r_0\varphi_D) = p^m_{1\ldots m}(j^r_0\,\mathrm{pr}^k_{1,m}) \simeq j^r_0\,\mathrm{id}_{\mathbb{R}^m}$ (for the notation see (1.4.1)). The jet composition formula further implies that $p^m_{1\ldots m}(j^r_0\varphi_D) \simeq j^r_0\,\mathrm{id}_{\mathbb{R}^m}$ is stabilized by $j^r_0h \in B^r_m$ acting from the left by composition if and only if $j^r_0h = j^r_0\,\mathrm{id}_{\mathbb{R}^m}$. We obtain the assignment of $j^r_0\varphi_D$ to $j^r_0\varphi$ as follows

$$j^r_0\varphi \mapsto (p^m_{1\ldots m}(j^r_0\varphi))^{-1}(j^r_0\varphi) = j^r_0\varphi_D. \tag{3.2.1}$$

On the other hand, let $j^r_0\psi_D \in D_{j^r_0\varphi}$ be obtained from $j^r_0\psi \in D_{j^r_0\varphi}$ by (3.2.1). Clearly, $p^m_{1\ldots m}(j^r_0\psi_D) \simeq j^r_0\,\mathrm{id}_{\mathbb{R}^m}$. Since $j^r_0\varphi_D$ and $j^r_0\psi_D$ share the same $\mathcal{V}_0$-class there is $j^r_0k \in B^r_m$ satisfying $j^r_0k \circ j^r_0\varphi_D = j^r_0\psi_D$. Clearly, j^r_0k coincides to $j^r_0\,\mathrm{id}_{\mathbb{R}^m}$ and consequently, $j^r_0\varphi_D = j^r_0\psi_D$. In other words, the elements $j^r_0\varphi_D$ from (3.2.1) act as invariants of $\mathcal{V}_0$-classes. In coordinates, the sets (x^i_α) indexed by all multiindices α containing at least one $q > m$ correspond bijectively to elements of $\mathcal{V}_0$. We conclude by defining $\bar{s}^\#$ in the form of the assignment $D_{j^r_0\varphi} \mapsto j^r_0\varphi_D$. The section $\bar{s}$ is obtained from $\bar{s}^\#$ by (2.9). The last assertion regarding the submanifold structure follows from the coordinate expression of (3.2.1) and the globality of $\bar{s}$. □

For $s < r$ and the jet projection π^r_s a local section $\eta^\#$ of $\hat{p}^\#$ is said to be π^r_s-projectable if and only if

$$\pi^r_s \circ \eta^\#(D_{j^r_0\varphi}) = \pi^r_s \circ \eta^\#(D_{j^r_0\psi}) \quad \text{whenever} \quad \pi^r_s(D_{j^r_0\varphi}) = \pi^r_s(D_{j^r_0\psi}). \tag{3.2.2}$$

In this case we have the projection $(\pi^r_s)_*\eta^\#$ assigning correctly $\pi^r_s \circ \eta^\#(D)$ to $\pi^r_s(D)$. Analogously we define the π^r_s-projectability of the section η of $\hat{p}$ (see (2.8) and (2.9)).

Proposition 3.3. *There is a couple of π^r_1-projectable smooth local sections $\hat{s}$ and $\hat{s}^\#$ of the principal bundles $\hat{p}_{Ort}$ and $\hat{p}^\#_{Ort}$ connected by the correspondence* (2.9) *defined on $\mathring{\mathcal{V}}$ (see* (3.1.1)*), which is a dense and open subset of $\mathcal{V}$. Moreover, $\hat{s}^\#$ and $\hat{s}$ contain $j^r_0\,\mathrm{pr}^k_{1,m}$ and $j^r_0\,\mathrm{id}_{\mathbb{R}^k}$ in their ranges.*

Further, there is a finite system of couples of π^r_1-projectable local sections $\hat{s}^\#_{i_1\ldots i_m}$ of $\hat{p}^\#_{Ort}$ and $\hat{s}_{i_1\ldots i_m}$ of $\hat{p}_{Ort}$ connected by (2.9) *containing $\hat{s}^\#$ and $\hat{s}$ with domains formed by open and dense subsets of $\mathcal{V}$. Finally, there is finite atlas on $\mathcal{V}$ determined by the local maps $(\mathring{\mathcal{V}}_{i_1\ldots i_m}, \hat{s}^\#_{i_1\ldots i_m})$.*

Proof: By (3.1.1) $\overset{\circ}{\mathcal{V}}$ is dense and open in $\mathcal{V}$. Let $s = s_{1\dots m}$ and $U_{1\dots m}$ be from Proposition 3.1. Set $s_1 = (\pi^r_1)_*\hat{s}$ to s and $s^\#_1 = (\pi^r_1)_*\hat{s}^\#$ to $s^\#$. Applying the convention on $\mathcal{V}$-classes introduced between (3.1.1) and Proposition 3.1 consider s_1 as a map defined on $G^1_{k,m}\backslash G^1_k$ with values on G^1_k. Let $\zeta_{s_1} : G^r_k \to G^r_k \simeq \mathrm{Aut}(\mathbb{D}^r_k)$ be the map defined by

$$\zeta_{s_1}(j^r_0 g) = t_{i^k_{\mathbb{D}^r_k}\circ s_1\circ\pi^r_1\circ\hat{p}(j^r_0 g)} \tag{3.2.3}$$

For any $D \in \mathcal{V}$, ζ_{s_1} is constant on $[\pi^r_1(D)]_{\rho_1}$ by definition. The symbols [ ] and $[\ ]_{\rho_1}$ indicate $\mathcal{V}$-classes and $\mathcal{V}_1$-classes (see the very beginning of Subsection 3.1).

Let $\bar{s} : B^r_{k,m}\backslash B^r_k \to B^r_k$ be the global section of $\bar{p}$ from Lemma 3.2. We assemble the local section $\hat{s}$ by means of the following formula

$$\hat{s}([j^r_0 g]) = (\bar{s}\circ\bar{p})(\zeta_{s_1}(j^r_0 g)(j^r_0 g)) \circ i^k_{\mathbb{D}^r_k} \circ s_1([j^1_0 g]_{\rho_1}) \tag{3.2.4}$$

The domain of $\hat{s}$ obviously coincides to $(\pi^r_1)^{-1}(U_{1\dots m}) = \overset{\circ}{\mathcal{V}}$ defined in (3.1.1).

Analogously to $\hat{s}$ acting as $\hat{s}_{1\dots m}$ we construct the local sections $\hat{s}_{i_1\dots i_m}$ projectable to $s_{i_1\dots i_m}$ from Proposition 3.1, by which we replace s_1 in (3.2.4). Their domains are $(\pi^r_1)^{-1}(U_{i_1\dots i_m})$, obviously dense and open in $\mathcal{V}$ and covering $\mathcal{V}$. The transition maps of the atlas are the restrictions of $s^\#_{j_1\dots j_m}\circ(s^\#_{i_1\dots i_m})^{-1}$ to $\mathrm{Im}\, s^\#_{i_1\dots i_m}(\mathcal{V}_{i_1\dots i_m}\cap\mathcal{V}_{j_1\dots j_m})$. □.

Proposition 3.4. *The local section $\hat{s}$ can be considered in the form of $\bar{s} \ltimes s_1 : \bar{p}(B^r_k)\times\hat{p}(G^1_k) \to B^r_k \ltimes G^1_k$ where $\bar{s}$ is from Lemma 3.2 and $s_1 = (\pi^r_1)_*\hat{s} : G^1_{k,m}\backslash G^1_k \to G^1_k$ is the map defined before* (3.2.3).

Proof: By [9], Section 13, G^r_k is identified with $B^r_k \ltimes G^1_k$. In what follows, we apply this identification in the form compatible to 5.16 of this book. We further observe that the map $i^k_{\mathbb{D}^r_k}$ defined in (1.4.2) is a Lie group homomorphism $G^1_k \to G^r_k$. Then the semidirect product $B^r_k \ltimes G^1_k$ can be considered with respect to the left action of G^1_k on B^r_k defined by

$$(j^1_0 h, j^r_0 g) \mapsto i^k_{\mathbb{D}^r_k}(j^1_0 h)\cdot j^r_0 g\cdot(i^k_{\mathbb{D}^r_k}(j^1_0 h))^{-1}, \tag{3.2.5}$$

which corresponds to the exact sequence of Lie group homomorphisms

$$\{j^r_0\,\mathrm{id}_{\mathbb{R}^k}\} \to B^r_k \to^i G^r_k \underset{\pi^r_1}{\overset{i^k_{\mathbb{D}^r_k}}{\leftrightarrows}} G^1_k \to \{j^r_0\,\mathrm{id}_{\mathbb{R}^k}\} \tag{3.2.6}$$

with the splitting $i^k_{\mathbb{D}^r_k}$ and the insertion i. Then the identification $G^r_k \simeq B^r_k \ltimes G^1_k$ is given by the mutually converse assignments

$$j^r_0 g \mapsto (j^r_0 g\cdot(i^k_{\mathbb{D}^r_k}(j^1_0 g))^{-1}, j^1_0 g) \qquad \text{and} \qquad (j^r_0 k, j^1_0 h) \mapsto j^r_0 k\cdot i^k_{\mathbb{D}^r_k}(j^1_0 h). \tag{3.2.7}$$

Select $j^r_0 g \in G^r_k$, $j^r_0 g \simeq (j^r_0 g\circ(i^k_{\mathbb{D}^r_k}(j^1_0 g))^{-1}, j^1_0 g)$. Then (3.2.4) can be expressed by

$$D_{j^r_0 g} \mapsto (\bar{s}\circ\bar{p}(j^r_0 g\circ(i^k_{\mathbb{D}^r_k}\circ s_1(D_{j^1_0 g}))^{-1}), s_1(D_{j^1_0 g})). \tag{3.2.8}$$

We conclude by checking the correctness of (3.2.8), which is easy. □

Let $\mathrm{V}(r,k,m) = (J^r_{Ort})_0(\mathbb{R}^k,\mathbb{R}^m)_0$ be the space of all m-frames on $\mathbb{R}^k$ orthonormal in the first order. Then we have

Corollary 3.5. $\mathrm{Gr}(r,k,m)$ *is identified with $(\pi^r_1)^{-1}(O(m))\times(\pi^r_1)^{-1}(O(k-m))\backslash(\pi^r_1)^{-1}(O(k))$. Its geometrical structure* $\mathrm{reg}\, J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0$ *is identified with the Stiefel manifold* $\mathrm{V}(r,k,m) = (\pi^r_1)^{-1}(O(k-m))\backslash(\pi^r_1)^{-1}(O(k))$. *Moreover, there is a principal bundle $\pi_{V,G} : \mathrm{V}(r,k,m) \to \mathrm{Gr}(r,k,m)$ with the structure group $(\pi^r_1)^{-1}(O(m))$ defined by $\pi_{V,G}(j^r_0\varphi) = D_{j^r_0\varphi}$.*

Proof: The first assertion is a direct corollary of Proposition 3.3 if we identify $\mathrm{Gr}(r,k,m)$ and its geometrical structure with $\hat{p}$ and put the first-order additional $(k-m)$ components of its geometrical structure to the orthonormalized form. This can be achieved by the Gramm-Schmidt orthogonalization process (see the proof of Proposition 3.1) followed by the application of the map $i^{k-m}_{\mathbb{D}^r_k}$ to the first-order additional components obtained by GSOP. The second assertion follows from the definition of $\mathrm{V}(r,k,m)$ in the form of $(J^r_{Ort})_0(\mathbb{R}^k,\mathbb{R}^m)_0$ and (2.8). The third assertion follows from the free left action of $(\pi^r_1)^{-1}(O(m))$ considered as a subgroup of $O(k)$ defined by the jet composition , which is transitive on fibers of $\pi_{V,G}$. □

Remark 3.6. In a wide class of problems on geometric objects like tensors, velocities, invariants, operators etc. the techniques of annihilation of some components of the studied objects with a possible selection of the coordinate system are often applied. Such methods can particularly lead to the decompositions of such objects to the compoments with properties of principal meaning. As an interesting example we give the paper [25] where the application of such methods was used for deducing a remarkable result of decomposing the Riemann, Ricci, Weyl, Einstein and deformation tensor together with obtaining the new criteria for Einstein spaces, spaces of constant curvature and conformal flat spaces. Coming back to our problems we remark that one of such decomposition methods has been applied to the regular r-th order velocities in order to construct the local sections in Lemma 3.2, Proposition 3.3 and Proposition 3.4. As a result we obtain that any r-th order $\mathbb{R}^m$-valued regular velocity can be decomposed to the element representing the Grassmannian $\mathrm{Gr}(r,k,m)$ and its location in the fiber of the geometrical structure represented by fiber coordinates.

Remark 3.7. *Having $\bar{s}^\#$ from Lemma 3.2 we can also define $\hat{s}^\#$ from Proposition 3.3 as follows. For any $j^r_0\varphi \in (\hat{p}^\#)^{-1}(\overset{\circ}{\mathcal{V}})$ define $\theta_1(j^r_0\varphi) = i^m_{\mathbb{D}^r_k} \circ (\pi^r_1)_*\hat{s}^\# \circ \pi^r_1 \circ \hat{p}^\#(j^r_0\varphi)$ where $(\pi^r_1)_*\hat{s}^\#$ is from the very beginning of the proof of Proposition 3.3. Put $\theta_2(j^r_0\varphi) = j^r_0\varphi - \theta_1(j^r_0\varphi) + i^m_{\mathbb{D}^r_k}(j^r_0\,\mathrm{pr}^k_{1,m})$. Finally, we define the map $\hat{s}^\#$ by $\hat{s}^\#(D_{j^r_0\varphi}) = \bar{s}^\#(D_{\theta_2(j^r_0\varphi)}) + \theta_1(j^r_0\varphi) - i^m_{\mathbb{D}^r_k}(j^r_0\,\mathrm{pr}^k_{1,m})$.*

4. Weil Grassmannian

4.1. Definition and construction

We define the support of the Weil Grassmannian as $\mathcal{V}_A$ from Definition 2.2(b). Before discussing the topology on $\mathcal{V}_A$ we introduce the notations

$$(4.1)\qquad (\mathcal{V}_A)_{i_1\dots i_m} = [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_{i_1\dots i_m}),\quad \text{particularly}\quad \overset{\circ}{\mathcal{V}}_A = [\widetilde{p}_{A,\mathbb{R}^m}](\overset{\circ}{\mathcal{V}})$$

where $[\widetilde{p}_{A,\mathbb{R}^m}] : \mathcal{V} \to \mathcal{V}_A$ is the map defined by $D_{j^r_0\varphi} \mapsto D_{j^A\varphi}$. Recalling the subordinate Weil algebra projection homomorphism $\pi_{1,A} : A \to A_{(1)}$ from Subsection 1.4 we observe easily that for normal A we have $A_{(1)} = \mathbb{D}^1_k$ and $\overset{\circ}{\mathcal{V}}_A = [\widetilde{\pi}_{1,A}]^{-1}(\overset{\circ}{\mathcal{V}}_1)$.

The smooth manifold topology on $\mathcal{V}_A$ cannot be introduced by a direct identification with a homogeneous space as for $A = \mathbb{D}^r_k$. Thus we need a finite system of maps defined on $\mathcal{V}$ with values in $\mathrm{reg}\,T^A_0\mathbb{R}^m$ satisfying the so-called T^A-respecting property. As for a possible geometrical structure over $\mathcal{V}_A$, we deduce the principal bundle $\hat{p}^\#_A : \mathrm{reg}\,J^r_0(\mathbb{R}^k,\mathbb{R}^m)_0 \to \mathcal{V}_A$ with the structure group G^r_m.

For a generic Lie group G and $g,h \in G$ denote hgh^{-1} by $\mathrm{conj}(h)(g)$. We give the so-called transformation lemma as follows

Lemma 4.1. *Let $j^r_0h \in G^r_k$ and $t_{j^r_0h} : A \to B$ and $\widetilde{t}_{j^r_0h} : T^A \to T^B$ be the corresponding Weil algebra isomorphism and natural equivalence (see (1.3.3)). Then*

$$(4.2)\qquad G_B = \mathrm{conj}(j^r_0h)(G_A) \quad \textit{and} \quad G^B = \mathrm{conj}(j^r_0h)(G^A)$$

Moreover, $\widetilde{t}_{j^r_0h}(D)$ is a $\mathcal{V}_B$-class if and only if D is a $\mathcal{V}_A$-class. Analogously, $\widetilde{t}_{j^r_0h}(D)$ is a $\mathcal{V}^B$-class if and only if D is a $\mathcal{V}^A$-class. In other words, $\widetilde{t}_{j^r_0h}(\mathcal{V}_A) = \mathcal{V}_{t_{j^r_0h}(A)}$ and $\widetilde{t}_{j^r_0h}(\mathcal{V}^A) = \mathcal{V}^{t_{j^r_0h}(A)}$.

Proof: Every $j^A\alpha \in A$ is stabilized by $j^r_0 g \in G^A$ if and only if any element of B, which is of the form $j^A\eta \circ (j^r_0 h)^{-1} = t_{j^r_0 h}(j^A\eta)$ for some $j^A\eta$ is stabilized by $\mathrm{conj}(j^r_0 h)(j^r_0 g)$. Analogously $j^A\alpha \in A$ is respected by $j^r_0 g \in G^r_k$ in the sense of $t_{j^r_0 g}(j^A\alpha) = j^A\beta$ for some $j^A\beta \in A$ if and only if any element of B, which is of the form $j^A\eta \circ (j^r_0 h)^{-1} = t_{j^r_0 h}(j^A\eta)$ for some $j^A\eta$ is respected by $\mathrm{conj}(j^r_0 h)(j^r_0 g)$.

To prove the second assertion, suppose that given $j^r_0\psi, j^r_0\varphi \in \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)$ there are $j^r_0 k \in G^r_m$ and $j^r_0 g \in G_A$ such that $j^r_0\psi \circ j^r_0 g = j^r_0 k \circ j^r_0\varphi$ (see Definition 2.2(b) and Lemma 2.3). Clearly, $j^r_0\psi \circ (j^r_0 h)^{-1} \circ \mathrm{conj}(j^r_0 h)(j^r_0 g) = j^r_0 k \circ j^r_0\varphi \circ (j^r_0 h)^{-1}$ and the first formula of (4.2) yields $\widetilde{t_{j^r_0 h}}(\mathcal{V}_A) = \mathcal{V}_B = \mathcal{V}_{t_{j^r_0 h}(A)}$. The proof of $\widetilde{t_{j^r_0 h}}(\mathcal{V}^A) = \mathcal{V}^B = \mathcal{V}^{t_{j^r_0 h}(A)}$ is almost the same with the only difference in $j^r_0 g \in G^A$ instead $j^r_0 g \in G_A$ and the application of the second formula of (4.2) instead the first one. □

Let $D \in \mathcal{V}_A$ and $\mathcal{W}_A \subseteq \mathcal{V}_A$. We define the subsets $\bar{D}$ and $\bar{\mathcal{W}}_A$ of $\mathcal{V}$ as follows

$$\bar{D} = \{D_0 \in \mathcal{V}; D_0 \subseteq D\} \qquad \text{and} \qquad \bar{\mathcal{W}}_A = \bigcup_{D \in \mathcal{W}_A} \bar{D}. \tag{4.3}$$

We remark that $\bar{D}$ is defined even in case $\operatorname{height} A < r$. Let us define the so-called T^A-respecting property as follows.

Definition 4.2. *Let $\mathcal{W}_A \subseteq \mathcal{V}_A$. A map $\sigma^\# : \mathcal{V} \to \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ is said to satisfy the T^A-respecting property on $\mathcal{W}_A$ if the following claims are satisfied*

(i) for any $\mathcal{V}$-class $D_0 \in \bar{\mathcal{W}}_A$ the set $\operatorname{Im}(\sigma^\#) \cap D_0$ is one-elementary and so is the set $\widetilde{p}_{A,\mathbb{R}^m} \circ \sigma^\#(\bar{D})$ for any $\mathcal{V}_A$-class $D \in \mathcal{W}_A$

(ii) For any $\mathcal{V}_A$-class $D \in \mathcal{W}_A$ and $D_1 \in \bar{D}$ it holds $[\widetilde{p}_{A,\mathbb{R}^m} \circ \sigma^\#(D_1)]_{\rho^A} = [D_1]_{\rho^A} = D$.

For $\sigma^\#$ of this kind, the map $u^\# = \widetilde{p}_{A,\mathbb{R}^m} \circ \sigma^\# : \mathcal{V} \to \operatorname{reg} T^A_0\mathbb{R}^m$ is said to be of the T^A-respecting property on $\mathcal{W}_A$ as well.

We remark that if there is a prolongation map with respect to A defined on the whole $\bar{\mathcal{W}}_A$ we can extend the concept of the T^A-respecting property on $\mathcal{W}_A$ from $\sigma^\#$ and $u^\#$ to σ and u by (2.8) and (2.9).

Proposition 4.3. *(a) Let A be normal Weil algebra, $\operatorname{width} A = k$ and $\hat{p}^\#_{Ort}$ be from Definition 2.10. Then $\hat{s}^\#$ and the other local sections $\hat{s}^\#_{i_1\dots i_m}$ of $\hat{p}^\#_{Ort}$ from Proposition 3.3 together with $\hat{s}_{i_1\dots i_m}$ satisfy the T^A-respecting property on $(\mathcal{V}_A)_{i_1\dots i_m} = [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_{i_1\dots i_m})$ (see (4.1)).*

(b) For any general Weil algebra A satisfying $\operatorname{width} A = k$ there is a system of local sections $\sigma^\#_{i_1\dots i_m} : \mathcal{V} \to \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ of $\hat{p}^\#$ (see (2.8) and Corollary 2.6) and maps $u^\#_{i_1\dots i_m} = \widetilde{p}_{A,\mathbb{R}^m} \circ \sigma^\#_{i_1\dots i_m} : \mathcal{V} \to \operatorname{reg} T^A_0\mathbb{R}^m$ satisfying the T^A-respecting property on $(\mathcal{V}_A)_{i_1\dots i_m}$. For normal A, the maps $\sigma^\#_{i_1\dots i_m}$ coincide to $\hat{s}^\#_{i_1\dots i_m}$ from (a).

Proof: (a) Let $\mathcal{V}_0 \subseteq \mathcal{V}$ be from Lemma 3.2(a). We construct a map $\theta^\#_1 : \mathcal{V}_0 \to \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ satisfying the T^A-respecting property on $(\mathcal{V}_A)_0 = [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_0)$ and check its coincidence with the section $\bar{s}^\#$ from Lemma 3.2(b). For any $D \in [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_0)$ and $j^A\varphi \in D$ there is $D_0 \in \bar{D}$, $j^r_0\varphi_0 \in D_0 \cap j^A\varphi$ and $T^r_0 h = j^r_0 h \in G^r_m$ such that $T^r_0 h(j^r_0\varphi_0) = j^r_0\varphi_{D_0}$ (see (3.2.1)). Clearly, $\widetilde{p}_{A,\mathbb{R}^m} \circ T^r_0 h(j^r_0\varphi_0) = T^A_0 h(j^A\varphi) = j^A\varphi_D$, $j^r_0\varphi_{D_0} \in j^A\varphi_D$. We are searching for a subset S_D of $j^A\varphi_D$ intersecting any $D_1 \in \bar{D}$ at exactly one element. Then the map $(\theta^\#_1)_{|\bar{D}}$ defined by $D_1 \mapsto D_1 \cap S_D$ satisfies the T^A-respecting property on D in case of its smoothness. Nevertheless, it suffices to put $S_D = \operatorname{Im} \bar{s}^\#_{|\bar{D}} = \operatorname{Im} \hat{s}^\#_{|\bar{D}}$ for $\bar{s}$ and $\hat{s}$ from Proposition 3.3 and its proof. Then $(\theta^\#_1)_{|\bar{D}}$ coincides to $\bar{s}^\#_{|\bar{D}}$ and consequently, $\bar{s}^\# = \theta^\#_1$.

It follows from (3.2.1) that besides the T^A-respecting property satisfied on $(\mathcal{V}_A)_0 = [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_0)$ the map $\theta^\#_1$ satisfies the T^B-respecting property on $(\mathcal{V}_B)_0 = [\widetilde{p}_{B,\mathbb{R}^m}](\mathcal{V}_0)$ for any normal Weil algebra B. This holds particularly for $B = f^{-1}(A)$ where $f : A \to C$ is an isomorphism of Weil algebras obtained by the linear reparametrization of indeterminates $\tau_1, \dots, \tau_k$ of A by linear polynomials corresponding to the components $s^i_{1\dots m}$ of the local section s from Proposition 3.1. Applying (3.2.4) we obtain the T^A-respecting property for

$\hat{s}_{1\dots m}$ from Proposition 3.3 on the set of all $\mathcal{V}^A$-classes contained in the current $\mathcal{V}_{A_{(1)}}$-class and consequently on the whole $\overset{\circ}{\mathcal{V}}{}^A = (\mathcal{V}_A)_{1\dots m}$.

In the last step consider a permutation $\omega = (i_1, \dots, i_k)$ with the reparametrization given by the assignments $\tau_{\omega(i)} \mapsto \tau_i$. Let $t_\omega : B \to A$ and $\widetilde{t_\omega} : T^B \to T^A$ be the Weil algebra isomorphism and the induced natural equivalence (see (1.3.1), (1.3.2)). Applying Lemma 4.1 we observe that $\hat{s}_{i_1\dots i_m}$ coincides to $\widetilde{t_\omega} \circ \hat{s} \circ \widetilde{t_\omega}^{-1}$ for $\hat{s}$ from (a), now considered as endowed with the T^B-respecting property on $\overset{\circ}{\mathcal{V}}_B$.

(b) Everything follows from Propostion 6(a) and the existence of an isomorphism of any Weil algebra with some normal one (see Subsection 1.4) and Lemma 4.1. □

Corollary 4.4. *For any A from Proposition 4.3(b) there is a finite system of maps $w^{\#}_{i_1\dots i_m} : \mathcal{V}_A \to \operatorname{reg} T^A_0\mathbb{R}^m$ with domains $(\mathcal{V}_A)_{i_1\dots i_m} = [\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_{i_1\dots i_m})$ determining the atlas $((\mathcal{V}_A)_{i_1\dots i_m}, w^{\#}_{i_1\dots i_m})$ on $\mathcal{V}_A$. Such atlas is compatible with the topology on $\mathcal{V}_A$ obtained as the quotient of the topology on $\mathcal{V}$ by $[\widetilde{p}_{A,\mathbb{R}^m}]$. Moreover, $(\mathcal{V}_A)_{i_1\dots i_m}$ are dense in $\mathcal{V}_A$.*

Proof: Let us define the maps $w^{\#}_{i_1\dots i_m} : \mathcal{V}_A \to T^A_0\mathbb{R}^m$ by setting them to the factorization of $u^{\#}_{i_1\dots i_m}$ from Proposition 4.3 to $[\widetilde{p}_{A,\mathbb{R}^m}](\mathcal{V}_{i_1\dots i_m})$. Their injectivity follows from the T^A-respecting property of $u^{\#}_{i_1\dots i_m}$ satisfied on $(\mathcal{V}_A)_{i_1\dots i_m}$. The transition maps over them are of the form $w^{\#}_{i_1\dots i_m} \circ (w^{\#}_{j_1\dots j_m})^{-1}$, restricted to $\operatorname{Im} w^{\#}_{j_1\dots j_m}((\mathcal{V}_A)_{i_1\dots i_m} \cap (\mathcal{V}_A)_{j_1\dots j_m})$. Then the couples $((\mathcal{V}_A)_{i_1\dots i_m}, w^{\#}_{i_1\dots i_m})$ are local maps of a smooth atlas on $\mathcal{V}_A$.

As for the second assertion, $U \subseteq \mathcal{V}_A$ is open in the quotient topology under discussion if and only if $[\widetilde{p}_{A,\mathbb{R}^m}]^{-1}(U)$ is open in $\mathcal{V}$. In such topology, $(\mathcal{V}_A)_{i_1\dots i_m}$ are obviously open. Values of the T^A-respecting local sections constructed in Proposition 4.3 can be identified with values of the restrictions of $[\widetilde{p}_{A,\mathbb{R}^m}]$ to $\mathcal{V}_{i_1\dots i_m}$, which proves our claim. □

Definition 4.5. *Let* $\operatorname{width} A = k \geq m$. *Then $\mathcal{V}_A = \operatorname{Gr}(A, m)$ with the manifold structure defined in Corollary 4.4 is said to be the Weil Grassmannian associated to A.*

Consider the projection $\hat{p}^{\#}_A = [\widetilde{p}_{A,\mathbb{R}^m}] \circ \hat{p}^{\#} : \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0 \to \mathcal{V}_A$ mapping $j^r_0\varphi$ to $D_{j^A\varphi}$ and the factorization $[\hat{p}^{\#}_A] : \operatorname{reg} T^A_0\mathbb{R}^m \to \mathcal{V}_A$ mapping $j^A\varphi$ to $D_{j^A\varphi}$. We resume our deductions to Proposition 4.6 as follows.

Proposition 4.6. *$\mathcal{V}_A = \operatorname{Gr}(A, m)$ is the basis of the principal bundle $\hat{p}^{\#}_A : \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0 \to \mathcal{V}_A$ with the structure group G^r_m identified with $\{T^A_0 h, h \in \operatorname{Diff}_0 \mathbb{R}^m\}$ and the free left action ℓ of G^r_m on $J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ defined by the jet composition, which is transitive on fibers of $\hat{p}^{\#}_A$.*

Proof: Let $j^r_0\varphi \in \operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ be arbitrary. For $D_{j^A\varphi} = \hat{p}^{\#}_A(j^r_0\varphi)$ there is a local map $((\mathcal{V}_A)_{i_1\dots i_m}, w^{\#}_{i_1\dots i_m})$ from Corollary 4.4 the support of which contains $D_{j^A\varphi}$. Let $\sigma^{\#}_{i_1\dots i_m}$ be the map from Proposition 4.3. It satisfies the T^A-respecting property on $(\mathcal{V}_A)_{i_1\dots i_m}$. Clearly, there is exactly one $j^r_0 h \in G^r_m$ such that $j^r_0\varphi \in \operatorname{Im} j^r_0 h \circ \sigma^{\#}_{i_1\dots i_m}$. Assigning $(D_{j^A\varphi}, j^r_0 h)$ to $j^r_0\varphi$ implies that $\hat{p}^{\#}_A$ is a surjective submersion. It is easy to see that the left action of G^r_m on $\operatorname{reg} J^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$ is free and is transitive on fibers. By [9], 10.3 $\hat{p}^{\#}_A$ is a principal bundle. □

Remark 4.7. *(a) The smooth manifold structure on* $\operatorname{Gr}(A, m)$ *and the smooth manifold structure on any $\mathcal{V}_A$-classes guaranteed by Definition 2.2 and Lemma 2.3 imply that $\mathcal{V}_A$ is a smooth partition of* $\operatorname{reg} T^A_0\mathbb{R}^m$. *Moreover, any $D \in (\mathcal{V}_A)_{i_1\dots i_m}$ determines its subset $D^+ = \{T^A_0 h \circ w^{\#}_{i_1\dots i_m}(D); \det(j^1_0 h) > 0\}$. Further, put $(\mathcal{V}_A)^+_{i_1\dots i_m}$ to $\{D^+; D \in (\mathcal{V}_A)_{i_1\dots i_m}\}$. Since the topologies of both spaces are identified we deduce that $(\mathcal{V}_A)^+_{i_1\dots i_m}$ is a foliation of $\bigcup_{D\in(\mathcal{V}_A)_{i_1\dots i_m}} D^+ \subseteq \operatorname{reg} T^A_0\mathbb{R}^m$.*

(b) On the other hand, $[\hat{p}^{\#}_A] : \operatorname{reg} T^A_0\mathbb{R}^m \to \operatorname{Gr}(A, m)$ defined after Definition 4.5 does not have to be a bundle with standard fiber since m-wide subalgebras over individual elements of $\mathcal{V}_A = \operatorname{Gr}(A, m)$ do not have to be isomorphic.

(c) In case of normal A we can add the claim of the first-order orthonormality on elements of the total space of the principal bundle $\hat{p}^{\#}_A$. We obtain the principal bundle $(\hat{p}^{\#}_A)_{Ort}$ determined by the reduction of the total space of $\hat{p}^{\#}_A$ to $(J^r_{Ort})_0(\mathbb{R}^k, \mathbb{R}^m)_0$ and the structure group G^r_m to $(\pi^r_1)^{-1}(O(m))$.

4.2. Weil Grassmannian bundle functor

For any m-dimensional manifold M and $x \in M$, set $\mathrm{Gr}(A, M)_x$ to $(\mathcal{V}_A)_{x,M}$ and define the atlas on $(\mathcal{V}_A)_{x,M}$ by transmitting the topology from $\mathrm{Gr}(A, m)$ to $\mathrm{Gr}(A, M)_x$, applying $T^A_0 h$ for some local diffeomorphism mapping $0 \in \mathbb{R}^k$ to x. In other words, U_x is open in $(\mathcal{V}_A)_{x,M}$ if and only if $(T^A_0 h)^{-1}(U_x)$ is open in $\mathrm{Gr}(A, m)$ for any $T^A_0 h$ above. Further, put $\mathrm{Gr}(A, M)$ to $\bigcup_{x\in M} \mathrm{Gr}(A, M)_x$ and define the map $\pi_{\mathrm{Gr},M} : \mathrm{Gr}(A, M) \to M$ by the assignment of $x \in M$ to $D \in \mathrm{Gr}(A, M)_x$. For a local diffeomorphism $f : M \to N$ we define $\mathrm{Gr}(A, f) : \mathrm{Gr}(A, M) \to \mathrm{Gr}(A, N)$ by $D_{j^A\varphi_x} \mapsto D_{j^A(f\circ\varphi_x)}$. In what follows, write $\mathrm{Gr}^A M$ instead $\mathrm{Gr}(A, M)$ and $\mathrm{Gr}^A f$ instead $\mathrm{Gr}(A, f)$. Let $\ell_{\mathrm{Gr},A}$ be the trivial left action of G^r_m on $\mathrm{Gr}^A_0 \mathbb{R}^m = \mathrm{Gr}(A, m)$ defined by $\ell_{\mathrm{Gr},A}(j^r_0 h, D_{j^A\varphi}) = D_{T^A_0 h(j^A\varphi)} = D_{j^A\varphi}$. Generalizing Proposition 2.8 we obtain the natural bundle Gr^A, which enables to speak about Weil Grassmann manifolds.

Proposition 4.8. *The system of spaces* $\mathrm{Gr}^A M$ *with* Gr^A*-maps* $\mathrm{Gr}^A f$ *forms a bundle functor on* $\mathcal{M}f_m$ *identified with* $P^r M[\mathrm{Gr}(A, m), \ell_{\mathrm{Gr},A}]$.

Proof: We modify the proof of Proposition 2.8 to our more general situation. Elements of $\mathrm{Gr}^A_x M$, which are of the form $D_{j^A\varphi_x}$ are identified with elements $\{j^r_0\alpha_x, D_{(T^A_0\alpha_x)^{-1}(j^A\varphi_x)}\}$ for any $j^r_0\alpha_x \in P^r_x M$. Conversely, any $\{j^r_0\alpha_x, j^A\varphi\}$ is assigned $T^A_0\alpha_x(j^A\varphi)$, applying the coincidence of T^A-morphisms and T^r_k-morphisms (see Subsection 1.5). Both of the assignments are well defined since $\{j^r_0\alpha_x \circ j^r_0 h, \ell_{\mathrm{Gr},A}(j^r_0 h^{-1}, j^A\varphi\})$ determines the same element of $\mathrm{Gr}(A, M)$ as $\{j^r_0\alpha_x, j^A\varphi\}$, which holds for the converse assignments as well. The assignments are compatible with the identification of $\mathrm{Gr}^A f$ with $\{P^r f, \mathrm{id}_{\mathrm{Gr}(A,m)}\}$, which is easy to check.

Proposition 4.9. *Natural equivalences* $tr_M : \mathrm{Gr}^A M \to \mathrm{Gr}^B M$ *are in a bijective correspondence with factorizations* $[\widetilde{t}\,] : \mathcal{V}_A \to \mathcal{V}_B$ *of natural equivalences* $\widetilde{t} : T^A \to T^B$ *induced by Weil algebra isomomorphisms* $t : A \to B$.

Proof: For a Weil algebra isomorphism $t : A \to B$, put $tr_{\mathbb{R}^m}$ to $[\widetilde{t}\,] = [\widetilde{t}_{j^r_0 g}]$ for some $j^r_0 g \in G^r_k$ (see Subsection 1.3). Since $\widetilde{t}$ transforms $\mathcal{V}_A$-classes to $\mathcal{V}_B$-classes by Lemma 4.1, $tr_{\mathbb{R}^m}$ is well defined and transforms $\mathrm{Gr}(A, m)$ to $\mathrm{Gr}(B, m)$. It is easy to see that $tr_{\mathbb{R}^m}$ is a G^r_m-equivariant map between standard fibers $\mathrm{Gr}(A, m)$ and $\mathrm{Gr}(B, m)$ of bundle functors Gr^A, Gr^B with respect to the left actions $\ell_{\mathrm{Gr},A}$ and $\ell_{\mathrm{Gr},B}$. This implies that tr_M is a natural equivalence (see [9], Section 14).

To prove the converse, assume a natural equivalence $tr_M : \mathrm{Gr}^A M \to \mathrm{Gr}^B M$. Since Gr^A and Gr^B are defined on $\mathcal{M}f_m$, tr is over the identity by 14.11, [9]. This yields a G^r_m-equivariant map $tr_{\mathbb{R}^m} : \mathrm{Gr}(A, m) \to \mathrm{Gr}(B, m)$, which can be viewed as a factorization of a G^r_m-equivariant map between A^m and B^m. By 42.7 in [9] it is induced by a Weil algebra isomomorphism $t : A \to B$. □

5. Some kind of partitions and foliations and a modified geometrical structure over Gr(A, m)

Consider submanifolds $(L_{T^A_0 h})_{i_1\dots i_m} \subseteq \mathrm{reg}\, T^A_0\mathbb{R}^m$ of the form $T^A_0 h(\mathrm{Im}(w^\#_{i_1\dots i_m}))$, $T^A_0 h \in G^r_m$ (see Subsection 1.5) where $w^\#_{i_1\dots i_m}$ is from Corollary 4.4. Let $[\hat{p}^\#_A]$ be the map defined after Definition 4.5. In the present subsection, let ℓ be the factorization of the left action of G^A on $(\mathrm{reg}\, T^r_k)_0\mathbb{R}^m$ induced by (1.3.3) to $\mathrm{reg}\, T^A_0\mathbb{R}^m$.

The system $\mathcal{L}_{i_1\dots i_m}$ of submanifolds $(L_{T^A_0 h})_{i_1\dots i_m}$ is in general not a partition of $\widetilde{p}_{A,\mathbb{R}^m} \circ (\hat{p}^\#)^{-1}(\mathcal{V}_{i_1\dots i_m}) = [\hat{p}^\#_A]^{-1}(\mathcal{V}_A)_{i_1\dots i_m}$. Indeed, $T^A_0 h_1(j^A\varphi) = T^A_0 h_2(j^A\varphi)$ for some $j^A\varphi \in \mathrm{Im}(u^\#_{i_1\dots i_m})$ does not imply $T^A_0 h_1(j^A\psi) = T^A_0 h_2(j^A\psi)$ for another $j^A\psi \in \mathrm{Im}(u^\#_{i_1\dots i_m})$ unless $j^A\psi$ is an element of $Orb(j^A\varphi) = \widetilde{t}_{G^A}(j^A\varphi)$, the orbit with respect to ℓ. Hence it is easy to see that any of the systems $\mathcal{L}_{i_1\dots i_m}$ cannot be interpreted as a partition extending the trivial foliation $\mathcal{L}_{i_1\dots i_m|D_{j^A\varphi}}$, $D_{j^A\varphi} \in \mathcal{V}_A$ formed by 1-elementary sets $T^A_0 h(j^A\varphi)$.

To extend the system of trivial foliations above to the system of non-trivial partions consider for any $\mathcal{V}^A$-class D the system $\mathcal{F}_D$ of submanifolds

$$\mathcal{F}_D = \{(F_{T^A_0 h})_D, h \in \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0\} = \{T^A_0 h(Orb), h \in \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0\}, \tag{5.1}$$

where Orb is an arbitrary $\mathrm{Aut}\, A$-orbit of the left action ℓ of $\mathrm{Aut}\, A = G^A/G_A$ on $\mathrm{reg}\, T^A_0\mathbb{R}^m$ mentioned above.

We remark that if D is considered as the disjoint union of its components of connectivity and the space of all $T_0^A h$ under discussion is considered as the disjoint union of its two components of connectivity determined by the value of determinant $j_0^1 h$ then $\mathcal{F}_D$ can be viewed as a foliation. Hence we have deduced

Proposition 5.1. *For every $\mathcal{V}^A$-class D there is a smooth partion $\mathcal{F}_D$ of the form* (5.1)*, which induces a foliation determined by* $\det j_0^1 h$ *and connected components of D in* (5.1)*. In other words, there is a distribution $\mathcal{F}$ assigning a smooth partition (foliation) $\mathcal{F}_D$ on D to any $\mathcal{V}^A$-class D.*

Nevertheless, any leaf of the partition $\mathcal{V}_A$ may intersect a given leaf of $\mathcal{F}_D$ in more elements. Therefore it is rather problematic to assign a couple of leaves from $\mathcal{L}_{i_1\dots i_m}$ and $\mathcal{F}_D$ to an element of $\operatorname{reg} T_0^A\mathbb{R}^m$. On the other hand, for any $w^\#_{i_1\dots i_m}$ and $\operatorname{Aut} A$-orbit Orb specified above define

$$(5.2)\qquad E_{i_1\dots i_m,Orb} = \operatorname{Im} w^\#_{i_1\dots i_m} \cap Orb \quad \text{and} \quad (E_{T_0^A h,Orb})_{i_1\dots i_m} = T_0^A h(E_{i_1\dots i_m,Orb})$$

where $h \in \operatorname{inv} J_0^r(\mathbb{R}^m,\mathbb{R}^m)_0$ is arbitrary and $[\hat{p}_A^\#]$ has been defined after Definition 4.5. Further, let us define subsets of $\mathcal{V}_A$ as follows

$$(5.3)\qquad \mathcal{V}_{i_1\dots i_m,Orb} = [\hat{p}_A^\#](E_{i_1\dots i_m,Orb}) = [\hat{p}_A^\#](\operatorname{Im} w^\#_{i_1\dots i_m} \cap Orb)$$

Hence we have the partition

$$(5.4)\qquad \mathcal{E}_{i_1\dots i_m,Orb} = \{(E_{T_0^A h,Orb})_{i_1\dots i_m}, h \in \operatorname{inv} J_0^r(\mathbb{R}^m,\mathbb{R}^m)_0\} = \{T_0^A h(E_{i_1\dots i_m,Orb}), h \in \operatorname{inv} J_0^r(\mathbb{R}^m,\mathbb{R}^m)_0\}$$

of $[\hat{p}_A^\#]^{-1}(\mathcal{V}_{i_1\dots i_m,Orb}) \subseteq \operatorname{reg} T_0^A\mathbb{R}^m$. Then we have

Proposition 5.2. *Let $\mathcal{W} \subseteq \mathcal{V}_{i_1\dots i_m,Orb}$ be a submanifold of $\mathcal{V}_A$ and $D_0 \in \mathcal{V}_{i_1\dots i_m,Orb}$. Then the corestriction $[\hat{p}_A^\#]_{\mathcal{W}}$ of $[\hat{p}_A^\#] : \operatorname{reg} T_0^A\mathbb{R}^m \to \mathcal{V}_A$ to $\mathcal{W}$ determines a bundle with standard fiber D_0. Moreover, D_0 consists of all algebraic bases of the Weil subalgebra A_{D_0} of A spanned by m components of an arbitrary element of D_0.*

Proof: Let $[\hat{p}_A^\#](j^A\varphi) \in \mathcal{W}$. Let $j^A\alpha = [\hat{p}_A^\#](j^A\varphi) \cap E_{i_1\dots i_m,Orb}$ and $j^A\beta = D_0 \cap E_{i_1\dots i_m,Orb}$. There is a well-defined element $(j^A\alpha)^{-1} \circ j^A\beta \in \operatorname{Aut} A$ with respect to the choice of r-jets contained in $j^A\alpha$, $j^A\beta$ and the induced natural equivalence $\widetilde{t}_{(j^A\beta)^{-1}\circ j^A\alpha}$ over T^A, see (1.3.3). Finally, we define a fiber bundle local map $\gamma : [p_A^\#]^{-1}(\mathcal{W}) \to \mathcal{W} \times D_0$ by the assignment of $([\hat{p}_A^\#](j^A\varphi), \widetilde{t}_{j^A\beta\circ(j^A\alpha)^{-1}}(j^A\varphi)) \in \mathcal{W} \times D_0$ to $j^A\varphi$. □

Let $D_{(1)} \in \mathcal{V}_1$ be arbitrary(for the notation see the very beginning of Subsection 3.1 and also 1.4) and Orb be any $\operatorname{Aut} A$-orbit of $\operatorname{reg} T_0^A\mathbb{R}^m$. Let $E_{i_1\dots i_m,Orb,D_{(1)}}$ be defined as $E_{i_1\dots i_m,Orb} \cap \widetilde{p}^\#_{A,\mathbb{R}^m}(\bar{D}_{(1)})$ and $\mathcal{V}_{i_1\dots i_m,Orb,D_{(1)}} \subseteq \mathcal{V}_A$ as $[\hat{p}^\#_{A,\mathbb{R}^m}](E_{i_1\dots i_m,Orb,D_{(1)}})$ or $[\widetilde{p}^\#_{A,\mathbb{R}^m}](\bar{D}_{(1)}) \cap \mathcal{V}_{i_1\dots i_m,Orb}$, recalling the notation from (4.1), (4.3) and that after Definition 4.5. In the very end we give a corollary of Proposition 5.2 as follows.

Corollary 5.3. *The projection $[\hat{p}_A^\#] : \operatorname{reg} T_0^A\mathbb{R}^m \to \mathcal{V}_A$ defined after Definition 4.5 is identified with the disjoint union od bundles $[\hat{p}_A^\#]_{\mathcal{W}}$ with standard fiber where $\mathcal{W}$ are of the form $\mathcal{V}_{i_1\dots i_m,Orb,D_{(1)}}$. The standard fiber of any $[\hat{p}_A^\#]_{\mathcal{W}}$ is identified with an arbitrary $D_0 \in \mathcal{W}$. The fibers of individual $[\hat{p}_A^\#]_{\mathcal{W}}$ consist of all algebraic bases of subalgebras A_D of A, $D \in \mathcal{W}$ where A_D is spanned by m components of any $j^A\varphi \in D$. For every $D \in \mathcal{V}_A$, it holds* $\operatorname{width} A_D = m$.

Proof: It remains to prove that every $\mathcal{V}_{i_1\dots i_m,Orb,D_{(1)}}$ is a submanifold of $\mathcal{V}_A$, the rest follows from Proposition 5.2. Let A be normal, Orb_0 be the $\operatorname{Aut} A$-orbit containing $j^A \operatorname{pr}^k_{1,m}$ and $D_{(1)}$ be the $\mathcal{V}_1$-class containing $j_0^1 \operatorname{pr}^k_{1,m}$. Recalling $j_0^r\varphi_D$ from (3.2.1) we observe that $E_{1\dots m,Orb_0,D_{(1)}} = \{j^A\varphi_D \in \operatorname{reg} T_0^A\mathbb{R}^m; \exists j_0^r\varphi_{0,D} \in j^A\varphi_D \cap (T_k^r)_0 \operatorname{pr}^k_{1,m}(G^A)\}$ is a submanifold of $\operatorname{reg} T_0^A\mathbb{R}^m$. This follows from the fact that G^A is a Lie subgroup and a submanifold of G_k^r as its closed subgroup, the fact of $(T_k^r)_0 \operatorname{pr}^k_{1,m}(G^A) = Orb_0$ and (3.2.1) itself. Since $\operatorname{Im} w^\#_{1\dots m}$ represents a submanifold of $\mathcal{V}_A$ we obtain that $\mathcal{V}_{1\dots m,Orb_0,D_{(1)}}$ is a submanifold of $\mathcal{V}_A$.

Let Orb_1 be another G^A-orbit, which admits a non-empty intersection $\pi_1^r(Orb_1) \cap D_{(1)}$ for $D_{(1)}$ from the first step. Without loss of generality, suppose that $E_{1\dots m,Orb_1,D_{(1)}}$ is non-empty. In such case, $E_{1\dots m,Orb_1,D_{(1)}} = \{j^A\psi_D \in$

$\operatorname{reg} T_0^A\mathbb{R}^m; \exists j_0^r\psi_{0,D} \in j^A\psi_D \cap \widetilde{p}_{A,\mathbb{R}^m}^{-1}(Orb_1)\}$ is a submanifold of $\operatorname{reg} T_0^A\mathbb{R}^m$ since not only G^A and $(T_k^r)_0 \operatorname{pr}_{1,m}^k(G^A)$ are submanifolds in G_k^r and $\operatorname{reg}(T_k^r)_0\mathbb{R}^m$ but so are all G^A-orbits. Applying the same argument as in the end of the previous step we obtain that $\mathcal{V}_{1\ldots,Orb_1,D_{(1)}}$ is a submanifold of $\mathcal{V}_A$.

Let $D'_{(1)}$ be a $\mathcal{V}_1$ class intersecting $\pi_1^r(\operatorname{Im} w_{1\ldots m}^\#)$ in a non-empty set and Orb_2 be an $\operatorname{Aut} A$-orbit with non-empty $E_{1\ldots m,Orb_2,D'_{(1)}}$. To prove that $E_{1\ldots m,Orb_2,D'_{(1)}}$ is submanifold of $\operatorname{reg} T_0^A\mathbb{R}^m$ we apply the same arguments as in the middle step of the proof of Proposition 4.3(a). More exactly, consider $j_0^1 g_1 \in D'_{(1)} \cap \operatorname{Im} s_1$ for s_1 from the very beginning of the proof of Proposition 3.3, applying the convention on $\mathcal{V}$-classes made between (3.1.1) and Proposition 3.1. Let $j_0^r g = i_{\mathbb{D}_k^r}^k(j_0^1 g_1)$, for $i_{\mathbb{D}_k^r}^k$ see (1.4.2). By means of the natural equivalence $\widetilde{t_{j_0^r g}}$ we transform $E_{1\ldots m,Orb_2,D'_{(1)}}$ under discussion to an object of this kind from the previous step, with the only difference in a Weil algebra B isomorphic to A instead A and an $\operatorname{Aut} B$-orbit instead the $\operatorname{Aut} A$-orbit Orb_2, applying Lemma 4.1

It remains to investigate the case of general $E_{i_1\ldots i_m,Orb,D}$, $D \in \mathcal{V}_1$. Applying the permutation $\omega = (i_1, \ldots, i_k)$ from the last step of the proof of Proposition 4.3(a) together with the natural equivalence $\widetilde{t_\omega}$ we analogously as in the proof of Proposition 4.3(a) transform a general object $E_{i_1\ldots i_m,Orb,D}$ above to an object of this kind from the previous step with the only difference in a Weil algebra B isomorphic to A instead A and consequently, in an $\operatorname{Aut} B$-orbit instead the $\operatorname{Aut} A$-orbit Orb. The fact that any Weil algebra is isomorphic to a normal Weil algebra and Lemma 4.1 prove our claim for a general Weil algebra. □

References

[1] J. R. Alonso, *Jet manifolds associated to a Weil bundle*, Arch. Math. (Brno), **36** (2000), 195-209.
[2] D. J. Eck, *Product-preserving functors on smooth manifolds*, J. Pure Appl. Algebra, **6** (42) (1986), 133-140.
[3] C. Ehresmann, *Extension du calcules jets aux jets non-holonomes*, C.R.A.S., **239** (1954), 1763-1764.
[4] M. Epstein, M. Elzanowski, *Material Inhomogenities and their Evolution*, Springer, 2007.
[5] D. R. Grigore, D. Krupka, *Invariants of velocities and higher-order Grassmann bundle*, J. Math. Phys., **24** (1998), 244-264.
[6] G. Kainz, P. W. Michor, *Natural transformations in Differential Geometry*, Czechoslovak Math. J., **37** (1987), 584-607.
[7] A. Karger, J. Novák, *Space kinematics and Lie groups*, N.Y. Gordon and Breach, 1978.
[8] I. Kolář, *Covariant Approach to natural transformations of Weil bundles*, Comment. Math. Univ. Carolin., **27** (1986), 723-729.
[9] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.
[10] I. Kolář, M. Doupovec and W. M. Mikulski, *On the jets of foliated respecting maps*, Czechoslovak Math. J., **60** (2010), 951-960.
[11] D. Krupka, *The contact ideal*, Differential Geom. Appl. **5** (3) (1995), 257-276.
[12] D. Krupka, M. Krupka, *Jets and contact elements*, Proc. Semin. Diff. Geom.- Math. Publ. Opava, Silesian Univ. in Opava bf 2-C (2000), 39-45.
[13] D. Krupka, D. Saunders, *Handbook of Global Analysis*, Elsevier, 2011.
[14] M. Kureš, *Weil algebras associated to functors of third order semiholonomic velocities*, Math. J. Okayama Univ., **56** (2014), 117-127.
[15] O. O. Luciano, *Categories of multiplicative functors and Weil's infinitely near points*, Nagoya Math. J., **109** (1988), 69-89.
[16] W. M. Mikulski, *Product preserving bundles on foliated manifolds*, Ann. Polon. Math., **84** (2004), 67-74.
[17] W. M. Mikulski, *Product preserving bundle functors on fibered manifolds*, Arch. Math. (Brno), **32** (1996), 307-316.
[18] J. Mikeš, V. Berezovski, E. Stepanova, A. Vanžurová A., Bácsó S. et al, *Differential geometry of special mappings*, Elsevier, 2011.
[19] J. Mikeš, *Geodesic mappings of special-connected and Riemannian space*, J. Math. Sci (N. Y.), **78:3** (1996), 311-333.
[20] A. Morimoto, *Prolongation of geometric structures*, Math. Inst. Nagoya Univ., ChikusaKu, Nagoya, 1969.
[21] F. J. Muriel, J. Muñoz, J. Rodriguez, *Weil bundles and jet spaces*, Czechoslovak Math. J., **50** (2000), 721-748.
[22] S. M. Minči c, L. S. Velimirović, *Integrability conditions of derivational equations of a submanifold of a generalized Riemannian space*, Appl. Math. Comput. **226** (2014), 3-9.
[23] M. A. Naymark, *Group representation theory (in Russian)*, Nauka, Moscow, 1976.
[24] I. Y. Park, *Foliation, Jet Bundle and Quantization of Einstein Gravity*, Front. Phys., **4** (2016), 25/1-14.
[25] P. Peška, M. Jukl, J. Mikeš, *Tensor decompositions and their properties*, Mathematics, **11 (17)** (2023).
[26] R. W. Sharpe, *Differential Geometry*, Springer, 1997.
[27] V. V. Shurygin, *Lie Jets and Higher-Order Partial connections*, Russian Math. Surveys **148** (2018), 122-129.
[28] V. V. Shurygin, *Manifolds over algebras and their application to the geometry of jet bundles*, Russian Math. Surveys **48** (1993), 75-104.
[29] J. Tomáš, *A general rigidity result on A-covelocities and A-jets*, Czechoslovak Math. J. **67** (2017), 297-316.
[30] J. Tomáš, *A higher-order and Weil generalization of Grassmannian*, In Proc. MSMF 10. Proc. Structural Integrity. 2023, 65-70.
[31] S. Minčić, L. Velimirović, *Riemannian subspaces of generalized Riemannian spaces*, Univ. Din Bacau Studii Cerc. Stiint., Ser.: Mat., **9** (1999), 111-128.
[32] A. Weil, *Théorie des points proches sur les variétés différentiables*, Colloque de Géometrie Diff., C.N.R.S. (1953), 111-117.
[33] S. K. Zubkova, V. V. Shurygin, *Transverse Lie jets and holomorphic objects on transverse bundles*, Russian Math. **60** (5) (2015), 70-74.