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Characterization of generalized essential spectra of 2×2 operator matrices via weakly demicompact operators

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Abstract. In this paper, we focus on the relationship between upper generalized semi-Fredholm and weakly demicompact operators acting on Banach spaces and we use this relation to extend some known results to generalized Fredholm theory. Moreover, in non-reflexive Banach spaces satisfying certain topological properties, we develop under some conditions on its entries, generalized Fredholm results for a 2×2 block operator matrix. The obtained results are used to characterize the generalized essential spectra, in particular the generalized Gustafson and Wolf essential spectra.

1. Introduction

Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$). For $T \in \mathcal{C}(X,Y)$, we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, $Y/\mathcal{R}(T)$ and $\mathcal{N}(T)$ the domain, the range, the co-kernel and the kernel of T respectively. When $X = Y$ we denote by $\rho(T)$ and $\sigma(T)$ the resolvent set and the spectrum of T respectively. $\mathcal{K}(X,Y)$ (resp. $\mathcal{W}(X,Y)$) are the subspaces of compact (resp. weakly compact) operators of $\mathcal{L}(X,Y)$. We recall that $\mathcal{W}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X,X)$ containing $\mathcal{K}(X,X)$. Further information can be found in [17, 20]. The dual (resp. the second dual or bidual) is denoted by X^* (resp. X^{**}), whenever T^* is the conjugate of an operator T and T^{**} is the second conjugate. For a subspace M of X^*, ${}^\circ M := \{x \in X : \forall\, \varphi \in M, \varphi(x) = 0\}$ and for a subspace M of X, $M^\circ := \{\varphi \in X^* : \forall\, x \in M, \varphi(x) = 0\}$. Now, a bounded linear operator T acting between X and Y is called tauberian and we denote $T \in \mathcal{T}(X,Y)$, if $T^{**-1}(Y) \subseteq X$, also T is co-tauberian and we denote $T \in \mathcal{T}^d(X,Y)$, when its conjugate T^* is tauberian. For more information on these classes, the reader can see the book [19].
In [6], the sets of upper generalized semi-Fredholm operators and lower generalized semi-Fredholm operators are respectively defined by:

$$\Phi_{g+}(X,Y) := \{T \in \mathcal{L}(X,Y) : \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\},$$

$$\Phi_{g-}(X,Y) := \{T \in \mathcal{L}(X,Y) : Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

The set $\Phi_g(X,Y) := \Phi_{g+}(X,Y) \cap \Phi_{g-}(X,Y)$ is formed by all generalized Fredholm operators and $\Phi_{g\pm}(X,Y) := \Phi_{g+}(X,Y) \cup \Phi_{g-}(X,Y)$. It is well known that the set of upper (resp. lower) generalized semi-Fredholm

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operators is strictly contained in the set of tauberian (resp. co-tauberian) operators (see [6]). Further, an operator $T \in \mathcal{L}(X,Y)$ is called g-Riesz (resp. upper g-Riesz) if $(\lambda - T) \in \Phi_g(X,Y)$ (resp. $(\lambda - T) \in \Phi_{g+}(X,Y)$) for all $\lambda \in \mathbb{C}\backslash\{0\}$ (see [7]). As an extension of weakly compact operators, we say that $F \in \mathcal{L}(X,Y)$ is a generalized Fredholm perturbation, if $(T+F) \in \Phi_g(X,Y)$ whenever $T \in \Phi_g(X,Y)$. Analogously, F is said to be upper (resp. lower) generalized semi-Fredholm perturbation, if $(T+F) \in \Phi_{g+}(X,Y)$ (resp. $(T+F) \in \Phi_{g-}(X,Y)$) whenever $T \in \Phi_{g+}(X,Y)$ (resp. $T \in \Phi_{g-}(X,Y)$). The sets of generalized Fredholm, upper generalized and lower generalized semi-Fredholm perturbations from X into Y are respectively denoted by $\mathcal{F}_g(X,Y)$, $\mathcal{F}_{g+}(X,Y)$ and $\mathcal{F}_{g-}(X,Y)$. When $X = Y$, the sets $\mathcal{C}(X,Y)$, $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, $\Phi_g(X,Y)$, $\Phi_{g+}(X,Y)$, $\Phi_{g-}(X,Y)$, $\mathcal{T}(X,Y)$, $\mathcal{T}^d(X,Y)$, $\mathcal{F}_g(X,Y)$, $\mathcal{F}_{g+}(X,Y)$, $\mathcal{F}_{g-}(X,Y)$ are replaced by $\mathcal{C}(X)$, $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi_g(X)$, $\Phi_{g+}(X)$, $\Phi_{g-}(X)$, $\mathcal{T}(X)$, $\mathcal{T}^d(X)$, $\mathcal{F}_g(X)$, $\mathcal{F}_{g+}(X)$, $\mathcal{F}_{g-}(X)$ respectively. For more details on these classes of operators we refer to [7].

As subsets of the classical essential spectra, A. Azzouz, M. Beghdadi and B. Krichen introduced in [6, 7] the so-called generalized essential spectra for a bounded linear operator T acting on a Banach space X as follow:

$$\begin{aligned} \sigma_{e_1,g}(T) &:= \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi_{g+}(X)\} \text{ and} \\ \sigma_{e_4,g}(T) &:= \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi_g(X)\}. \end{aligned}$$

Here, $\sigma_{e_1,g}(.)$ refers to the generalized Gustafson essential spectrum and $\sigma_{e_4,g}(.)$ to the generalized Wolf essential spectrum.

As an extension of demicompact linear operators, B. Krichen and D. O'Regan introduced in [26] the class of weakly relative demicompact linear operator with respect to a given linear operator. This concept asserts that if $T : \mathcal{D}(T) \subset X \longrightarrow X$ is a linear operator, then T is said to be weakly demicompact, if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n$ weakly converges in X, then there exists a weakly convergent subsequence of $(x_n)_n$. Obviously, weakly compact operators are weakly demicompacts. We refer to [26] for further examples. Moreover, if $T \in \mathcal{L}(X)$ is weakly demicompact equivalently, then $I - T$ is tauberian [19], where I is the identity operator. However, T^2 is weakly demicompact, if and only if T and $-T$ are weakly demicompact. The set of weakly demicompact operators acting on X is denoted by $\mathcal{WDC}(X)$.

The theory of block operator matrices has appeared in a variety of fields and applications, among which we mention systems theory, such as the Hamiltonians (see [15]), the estimation of PDEs as large matrices divided due to patterns of variance, and the saddle point problems of nonlinear analysis (see [11]), problems of evolution, such as linearity of second-order Cauchy problems and as a linear parameter describing the paired systems of partial differential equations. As a result, many important works have been published on the spectrum theory of operator matrices, among them we cite [14, 30, 33]. For example, in [33] author developed the essential spectra of 2×2 block operator matrices and presented a wide panorama of methods to investigate the essential spectra of block operator matrices.

Our main purpose in this work is to use the concept of weak demicompactness to describe the generalized essential spectra of an operator matrix in non-reflexive Banach spaces satisfying certain topological conditions.

The paper is organized in the following way. In Section 2, we recall some definitions and preliminaries results and give some positive answers to questions raised in [6]. Therefore, we present an example of a generalized Fredholm operator given by a tridiagonal infinite matrix acting on the generalized Hahn space. In section 3, we study weakly demicompact operators in relation to generalized Fredholm theory. In Section 4, we develop some results concerning the weak demicompactness of operator matrices under some assumptions. Among these results, we extend some theorems proved in [10]. Further, we characterize the generalized Gustafson and the generalized Wolf essential spectrum of a 2×2 block operator matrix in a non-reflexive Banach space satisfying some properties. An example of a weakly demicompact operator matrices in the James' non-reflexive space is presented.

2. Preliminary results

We start this section by recalling the following definition due to A. Azzouz, M. Beghdadi and B. Krichen in [6].

Definition 2.1. Let X be a Banach space. We say that X has the property (H_1) (resp. (H_2)) if every reflexive subspace admits a closed complementary subspace (resp. if every closed subspace with reflexive quotient space admits a closed complementary subspace).
We say that X has the property (H), if it satisfies both properties (H_1) and (H_2).

As an example of non-reflexive Banach space has the property (H_1), we cite the James' quasi-reflexive space J. Indeed, let J be the James Banach space and F_1 be a reflexive subspace of J, then by [[4], Theorem 9] F_1 is contained in a reflexive complemented subspace F of J, that is

$$J = F \oplus Z. \tag{1}$$

where Z is a closed subspace of J. Since F is a reflexive Banach space and F_1 is closed in F, then there exists F_2 a closed subspace of F such that $F = F_1 \oplus F_2$ (see [27]). By applying this result in Eq. (1) we obtain:

$$J = F_1 \oplus F_2 \oplus Z.$$

Hence F_1 is a complemented in J and consequently the last satisfies the property (H_1).
For another example of spaces having the properties (H_1) and (H_2), we refer the reader to [6]. Note that if X is reflexive and satisfies property (H_1), then it has property (H_2).

Now under the above properties, let us recall a characterization of a generalized Fredholm operator due to K. W. Yang [34].

Theorem 2.2. [34] Let X and Y be two Banach spaces satisfying the properties (H_1) and (H_2) respectively, and let $T \in \mathcal{L}(X,Y)$. Then the following assertions are equivalent.
(*i*) T is a generalized Fredholm operator.
(*ii*) There exist weakly compact operators $W_1 \in \mathcal{W}(X)$, $W_2 \in \mathcal{W}(Y)$ and an operator $T_0 \in \mathcal{L}(Y,X)$ such that $T_0T = I + W_1$ and $TT_0 = I + W_2$ and $\mathcal{R}(T)$ is closed in Y.

Next, for X and Y are two Banach spaces, we will try to find out the answer to the following relations:

(*i*) If X achieves the property (H_1), this entails that X^* satisfies the property (H_2)?

(*ii*) If X achieves the property (H_2), this implies that X^* satisfies the property (H_1)?

(*iii*) If X and Y satisfy the property (H_1), this implies that $X \times Y$ has the property (H_1)?

(*iv*) If X and Y satisfy the property (H_2), this implies that $X \times Y$ has the property (H_2)?

The answer to the above four questions is positive. Before giving answers to these questions, let us present the following result.

Remark 2.3. Let X be a Banach space. For any closed subspace F of X^*, the map A defined by:

$$\begin{aligned} A : ({}^\circ F)^* &\rightarrow X^*/F \\ \varphi &\rightarrow \widehat{\varphi} := \varphi + F. \end{aligned}$$

is an isomorphism. Indeed, firstly, let us check that $\widehat{\varphi}$ is well defined. To do so, if $\varphi, \psi \in ({}^\circ F)^*$ such that $\varphi = \psi$, then for all $x \in {}^\circ F$, $(\varphi - \psi)(x) = 0$ implies that $\varphi - \psi \in ({}^\circ F)^\circ = F$. Then, $\widehat{\varphi} = \widehat{\psi}$ and thus A is well defined. Secondly, if $\widehat{\varphi} = \tilde{0}$, then for all $x \in {}^\circ F$, $\varphi(x) = 0$. It follows that $\varphi = \tilde{0}$. Hence, A is injective. Now its remains to show that A is surjective. For this purpose, let $\psi \in X^*/F$ and let π be the projection from X^* to X^*/F. We have $(\psi \circ \pi)(F) = \psi(\pi)(F) = \psi(0) = 0$, thus $\psi \circ \pi \in ({}^\circ F)^*$. Moreover, $\widehat{\psi \circ \pi}(\varphi + F) = (\psi \circ \pi)(\varphi) = \psi(\pi)(\varphi) = \psi(\varphi + F)$. Hence, $A(\psi \circ \pi) = \psi$. Consequently, A is surjective.

Next, we state the following useful lemma.

Lemma 2.4. Let X be a Banach space. The following assertions hold.
(i) If X satisfies the property (H_1), then X^* has the property (H_2).
(ii) If X satisfies the property (H_2), then X^* has the property (H_1).

Proof. (i) Let F be a closed subspace of X^* such that X^*/F is reflexive. We have ${}^\circ F := \{x \in X \text{ such that } \forall \varphi \in F : \varphi(x) = 0\}$. Since, $({}^\circ F)^* \cong X^*/F$ (see Remark 2.3) and X^*/F is reflexive, then by using [[17], Corollary 24] we deduce that ${}^\circ F$ is also reflexive. Taking into account that X has the property (H_1), then there exists a closed subspace X_0 of X such that $X = X_0 \oplus {}^\circ F$. Thus,

$$X^* = X_0^\circ \oplus F,$$

here, $X_0^\circ := \{\varphi \in X^* \text{ such that } \forall x \in X_0, \varphi(x) = 0\}$. Hence, X^* has the property (H_2).
(ii) Let F be a closed reflexive subspace of X^*. Since $F = ({}^\circ F)^\circ \cong (X/{}^\circ F)^*$ and F is reflexive, then by [[17], Corollary 24] we infer that $X/{}^\circ F$ is also reflexive. This result combined with the fact that X satisfies the property (H_2), enable us to deduce that there exists a closed subspace X_1 of X such that $X = {}^\circ F \oplus X_1$ and so $X^* = F \oplus X_1^\circ$. Thus, X^* has the property (H_1). □

As a consequence of Lemma 2.4, we can easily obtain the following result.

Remark 2.5. Let X be a Banach space. If X has the property (H), then X^* has the property (H).

As examples of spaces that are achieved the two above results, we have the $L_p(0,1)$-spaces with $1 < p < \infty$.
Below, we present some results from [6, 7]. Before that, we would like to point out that from Lemma 2.4 (ii), [[6], Theorem 3.9] can be rewritten as follow:

Theorem 2.6. Let X be a non-reflexive Banach space having the property (H) and let T and S be two bounded linear operators on X. The following statements hold.
(i) If $ST \in \Phi_{g+}(X)$, then $T \in \Phi_{g+}(X)$.
(ii) If $ST \in \Phi_{g-}(X)$, then $S \in \Phi_{g-}(X)$.
(iii) If $ST \in \Phi_g(X)$, then $S \in \Phi_{g-}(X)$ and $T \in \Phi_{g+}(X)$.

Theorem 2.7. [6] Let X, Y and Z be three non-reflexive Banach spaces and let $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$. The following statements hold.
(i) Assume that X, Y and Z satisfy the properties (H_1), (H) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $S \in \Phi_g(Y, Z)$, then $ST \in \Phi_g(X, Z)$.
(ii) Assume that X and Y satisfy the properties (H_1) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $W \in \mathcal{W}(X, Y)$, then $(T + W) \in \Phi_g(X, Y)$.
For $X = Y = Z$, we have the following assertions:
(iii) Assume that X has the property (H_1). If $T \in \Phi_{g+}(X)$ and $S \in \Phi_{g+}(X)$, then $ST \in \Phi_{g+}(X)$.
(iv) Assume that X has the property (H_2). If $T \in \Phi_{g-}(X)$ and $S \in \Phi_{g-}(X)$, then $ST \in \Phi_{g-}(X)$.
(v) Assume that X has the property (H_1). If $T \in \Phi_{g+}(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_{g+}(X)$.

Now, let us recall the De Blasi measure of weak noncompactness. Before that, we present some standard notations useful for the sequel. We denote by $\mathcal{M}_X$ (resp. $\mathcal{K}^w(X)$) the set of bounded sets of a Banach space X (resp. the set of all weakly compact subsets of X), by $conv(A)$ the convex hull of a subset $A \subset X$ and by $B_r = B(0, r)$ the open ball centered at 0 and with radius r.
In [16], the De Blasi measure of weak noncompactness of a non empty bounded subset A of X, denoted by $\omega : \mathcal{M}_X \longrightarrow [0, +\infty[$, was introduced as follow:

$$\omega(A) = \inf\{r > 0, \text{ there exists } N \in \mathcal{K}^w(X) : A \subset N + \overline{B}_r\}. \tag{2}$$

We point out that the definition given by Eq. (2) can be also expressed as axiomatic statements [9] which is as follows.

Definition 2.8. Let X be a Banach space and let A, B be two bounded subsets of X. A function $\mu : \mathcal{M}_X \longrightarrow [0, +\infty[$ is a measure of weak noncompactness in X, if the following conditions are satisfied,
(i) $\mu(A) = 0$ if, and only if, A is relatively weakly compact set,
(ii) if $A \subset B$, then $\mu(A) \leq \mu(B)$,
(iii) $\mu(\overline{conv(A)}) = \mu(A)$,
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$, for $\lambda \in \mathbb{C}$.
From [9], the measure of weak noncompactness guarantees the Cantor intersection condition, and the following inequality for all $A \in \mathcal{M}_X$:

$$\mu(A) \leq \mu(B_r)\omega(A).$$

Recently, measures of weak noncompactness are widely used in the theory of bounded linear operators. More precisely, for $T \in \mathcal{L}(X)$ we have the following definition:

$$\overline{\omega}(T) = \inf\{k \,:\, \omega(T(A)) \leq k \cdot \omega(A), \text{ for all } A \in \mathcal{M}_X\}.$$

In the following proposition, we recall some known properties of $\overline{\omega}(\cdot)$.

Proposition 2.9. [16] Let X be a Banach space, $T, S \in \mathcal{L}(X)$ and let $B \in \mathcal{M}_X$. Then, we have the following properties:
(i) $\overline{\omega}(T) = 0$ if, and only if, T is weakly compact.
(ii) $\omega(T(B)) \leq \overline{\omega}(T)\omega(B)$.
(iii) $\overline{\omega}(TS) \leq \overline{\omega}(T)\overline{\omega}(S)$.
(iv) $\overline{\omega}(T + S) \leq \overline{\omega}(T) + \overline{\omega}(S)$.
(v) $\overline{\omega}(\lambda T) = |\lambda|\overline{\omega}(T)$, for $\lambda \in \mathbb{C}$.

Next, let us recall a generalized Fredholm result from [6] related with the measure $\overline{\omega}(\cdot)$ that we will need in the sequel.

Theorem 2.10. Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\overline{\omega}(T^n) < 1$ for some $n \in \mathbb{N}\backslash\{0\}$. Then, $(I - T) \in \Phi_g(X)$.

In the following, we give an example of generalized Fredholm operator acting on the generalized Hahn space inferred from [29]. Before that, let us recall the following notations and results.

For Λ denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$, the operator $\Delta : \Lambda \to \Lambda$ of the so-called forward differences is defined by $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.
In 1972, G. Goes [18] introduced the generalized Hahn space by

$$h_d = \left\{x \in \Lambda : \quad \sum_{k=1}^{\infty} d_k|\Delta x_k| < \infty\right\} \cap c_0,$$

with its norm is the following

$$\|x\|_{h_d} = \sum_{k=1}^{\infty} d_k|\Delta x_k| < \infty, \text{ for all } x \in h_d,$$

where $d = (d_k)_{k=1}^{\infty}$ is a given sequence of positive real numbers d_k for all $k \in \mathbb{N}$.

Recently, E. Malkowsky, V. Rakočević and O. Tuğ in [29] showed that $(h_d, \|x\|_{h_d})$ is a Banach space. Notice that, if $d_k = k$ for all k, then h_d reduces to the original Hahn space h (see [21]).

Beside, let us recall the definition of the Hausdorff measure of noncompacntess of non empty bounded subset in Banach space.

Let X be a Banach space and let $A \in \mathcal{M}_X$, then the Hausdorff measure of noncompacntess of A, $\chi(A)$, is defined by

$$\chi(A) = \inf\Big\{\varepsilon > 0 : \ A \subset \bigcup_{i=1}^{n} B(x_i, r_i), \quad x_i \in X, r_i \leq \varepsilon, i = 1, \dots, n\Big\}.$$

Moreover, for $T \in \mathcal{L}(X)$ we have

$$\|T\|_\chi = \inf\Big\{c \geq 0 : \chi(T(A)) \leq c \cdot \chi(A), \text{ for all } A \in \mathcal{M}_X\Big\}.$$

For several useful properties of this measure $\chi(\cdot)$, we refer to [8, 28].

In order to state our example, we need to fix some notations that we are using. Throughout this note, let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix. Let us also write for $m \in \mathbb{N}$, $A^{<m>} = (a_{nk}^{<m>})_{n,k=1}^{\infty}$ for the matrix with the rows $A_n^{<m>} = 0$ for $1 \leq n \leq m$ and $A_n^{<m>} = A_n$ for $n \geq m+1$ and denote by $T^{<m>}$ the operator represented by the matrix $A^{<m>}$.

Example 2.11. Let $\alpha = (\alpha_n)_{n=1}^{\infty}$, $\beta = (\beta_n)_{n=1}^{\infty}$ and $\gamma = (\gamma_n)_{n=1}^{\infty}$ be given sequences of complex numbers. For $d_k = k$, $\alpha_k = 1 - 1/k$ and $\beta_k = \gamma_k = 1/k$ for all k, the operator $T \in \mathcal{L}(h_d, h_d)$ represented by the matrix

$$\begin{aligned} A(\gamma, \alpha, \beta) &= \begin{pmatrix} \alpha_1 & \beta_1 & 0 & & & & \cdots & 0 & \cdots \\ \gamma_1 & \alpha_2 & \beta_2 & \ddots & & & \cdots & 0 & \cdots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 & & \cdots & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & & \\ 0 & & & \gamma_{n-1} & \alpha_n & \beta_n & 0 & \cdots & 0 & \cdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \\ &= A(\gamma, 0, 0) + A(0, \alpha, 0) + A(0, 0, \beta) \end{aligned}$$

is generalized Fredholm.

Indeed, writing $c_m^{<l>}(\alpha - e)$, $c_m^{<l>}(\gamma)$ and $c_m^{<l>}(\beta)$ for the previous expressions that defined in the following equation for the matrices $A(0, \alpha - e, 0)$, $A(\gamma, 0, 0)$ and $A(0, 0, \beta)$ respectively

$$\begin{aligned} c_m &= \frac{1}{d_m}\Big[\sum_{n=1}^{m-2} d_n|\Delta(\alpha_n + \beta_n + \gamma_{n-1}| + d_{m-1}|\Delta(\alpha_{m-1} + \gamma_{m-2}) + \beta_{m-1}|\Big] \\ &+ \frac{1}{d_m}\Big[d_m|\alpha_m + \Delta_{\gamma_{m-1}}| + d_{m+1}|\gamma_m|\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} c_m^{<l>}(\alpha - e) &= \frac{1}{m}\Big(\frac{1}{l+1}\sum_{n=l+1}^{m-1} n\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + \frac{m}{m}\Big) \\ &\leq \frac{2}{l}\sum_{n=l+1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big). \end{aligned}$$

Combining Lemma 4.5 and Theorem 4.8 in [29] together with the fact that $\overline{\omega}(\cdot) \leq \|\cdot\|_\chi$ then we obtain that

$$\overline{\omega}(T_{A(0,\alpha,0)} - I) \leq \limsup_{l \to \infty}\Big(\sup_m c_m^{<l>}(\alpha - e)\Big) < 1.$$

By using Theorem 2.10 we deduce that $T_{A(0,\alpha,0)} \in \Phi_g(h_d, h_d)$.
Moreover,

$$\begin{aligned} c_m^{<l>}(\gamma) &= \frac{1}{d_m}\Big(d_l|\gamma_l| + \sum_{n=l+1}^{m} d_n|\Delta\gamma_{n-1}| + d_{m+1}|\gamma_m|\Big) \\ &\leq \frac{1}{m} + \sum_{n=1}^{\infty}\Big(\frac{1}{n-1} - \frac{1}{n}\Big) + \frac{m+1}{m^2} \\ &\leq \frac{3}{l} + 2\sum_{n=l}^{\infty}\frac{1}{n^2}. \end{aligned}$$

Then

$$\overline{\omega}(T_{A(\gamma,0,0)}) \leq \limsup_{l\to\infty}\Big(c_m^{<l>}(\gamma)\Big) = 0,$$

and so $T_{A(\gamma,0,0)}$ is weakly compact.
Similarly, we have

$$\begin{aligned} c_m^{<l>}(\beta) &= \frac{1}{d_m}\Big(\sum_{n=l+1}^{m} d_n|\Delta\beta_n| + d_m|\beta_m|\Big) \\ &= \frac{1}{m}\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + \frac{m}{m}\Big) \\ &\leq \frac{1}{l} + \sum_{n=l+1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big). \end{aligned}$$

Then,

$$\overline{\omega}(T_{A(0,0,\beta)}) \leq \limsup_{l\to\infty}\Big(c_m^{<l>}(\beta)\Big) = 0,$$

and hence $T_{A(0,0,\beta)}$ is weakly compact.
Furthermore, the operator $T_{A(\gamma,\alpha,\beta)} - I$ can be written in the following form:

$$T_{A(\gamma,\alpha,\beta)} - I = T_{A(\gamma,0,0)} + T_{A(0,\alpha,0)} - I + T_{A(0,0,\beta)}.$$

Accordingly,

$$\begin{aligned} \overline{\omega}(T_{A(\gamma,\alpha,\beta)} - I) &= \overline{\omega}(T_{A(\gamma,0,0)} + T_{A(0,\alpha,0)} - I + T_{A(0,0,\beta)}) \\ &\leq \overline{\omega}(T_{A(\gamma,0,0)}) + \overline{\omega}(T_{A(0,\alpha,0)} - I) + \overline{\omega}(T_{A(0,0,\beta)}). \end{aligned}$$

Since $T_{A(\gamma,0,0)}$ and $T_{A(0,0,\beta)}$ are weakly compact and $\overline{\omega}(T_{A(0,\alpha,0)} - I) < 1$, then we obtain that $\overline{\omega}(T_{A(\gamma,\alpha,\beta)} - I) < 1$. Hence, by applying Theorem 2.10 we conclude that $T_{A(\gamma,\alpha,\beta)} \in \Phi_g(h_d, h_d)$.

3. Weakly demicompact and generalized semi-Fredholm operators

In this section, we give some generalized Fredholm results related with weakly demicompact operators. Before that, we recall the following definition.

Definition 3.1. [24] Let T be a closed linear operator on a Banach space X. For $x \in \mathcal{D}(T)$, the graph norm of x is defined by

$$\|x\|_T = \|x\| + \|Tx\|.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|x\|_T$ is a Banach space. In this new space, denoted by X_T, the operator T satisfies $\|Tx\| \leq \|x\|_T$ and, consequently, is a bounded operator (acting from X_T into X).

Let J be a linear operator on X. If $\mathcal{D}(T) \subset \mathcal{D}(J)$, then J will be called T-defined. If J is an T-defined operator, we will denote by $\widehat{J}$ the restriction of J to $\mathcal{D}(T)$. Moreover, if $\widehat{J} \in \mathcal{L}(X_T, X)$, we say that J is T-bounded. One checks easily that if J is closed or closable (see [[24], Remark 1.5, p. 191]), then J is T-bounded. Furthermore, we have the obvious relations

$$\begin{cases} \mathcal{N}(\widehat{T}) = \mathcal{N}(T), \quad \mathcal{N}(\widehat{T} + \widehat{J}) = \mathcal{N}(T + J), \\ \mathcal{R}(\widehat{T}) = \mathcal{R}(T), \quad \mathcal{R}(\widehat{T} + \widehat{J}) = \mathcal{R}(T + J). \end{cases} \tag{3}$$

Remark 3.2. From Eq. (3), we can see that $T \in \Phi_g(X)$ (resp. $\Phi_{g+}(X), \Phi_{g-}(X)$) if and only if, $\widehat{T} \in \Phi_g(X_T, X)$ (resp. $\Phi_{g+}(X_T, X), \Phi_{g-}(X_T, X)$).

The following result consists to a generalization of [[12], Theorem 3.1] and [[13], Theorem 2.8], which shows the connection between upper generalized semi-Fredholm operators and the class of weakly demicompact operators.

Theorem 3.3. Let X be a Banach space having the property (H_1) and $T \in \mathcal{C}(X)$. Then T is weakly demicompact if and only if $I - T$ is upper generalized semi-Fredholm.

Proof. Let $\Pi^1 = \{x \in \mathcal{D}(T) \colon (I - T)x = 0 \text{ and } \|x\| \leq 1\}$. Let $(x_n)_n$ be a sequence in Π^1, then $x_n - Tx_n \rightharpoonup 0$ and $\|x_n\| \leq 1$. Since $T \in \mathcal{WDC}(X)$, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that

$$x_{\varphi(n)} \rightharpoonup y,\ y \in X.$$

According to the closedness of T we deduce that $y \in \mathcal{D}(T)$ and $y = Ty$. Then $y \in \mathcal{N}(I - T)$. Since $x_{\varphi(n)} \rightharpoonup y$, then $\|y\| \leq \liminf \|x_{\varphi(n)}\| \leq 1$. Thus, $y \in B(0, 1)$. Hence, $y \in \Pi^1$. Implies that the unit ball of the $\mathcal{N}(I - T)$ is weakly compact. Consequently, $\mathcal{N}(I - T)$ is reflexive. Now, we will show that $\mathcal{R}(I - T)$ is closed. To do this, let $(x_n)_n$ be a bounded sequence in $\mathcal{D}(T)$ such that

$$x_n - Tx_n \rightharpoonup x,\ x \in X.$$

Since T is weakly demicompact, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that

$$x_{\varphi(n)} \rightharpoonup y,\ y \in X.$$

We have $x_{\varphi(n)} \rightharpoonup y$ and $(I - T)x_{\varphi(n)} \rightharpoonup x$. Then, we obtain $x \in \mathcal{D}(T)$ and $x = (I - T)y$. Thus, $x \in \mathcal{R}(I - T)$. Hence, $\mathcal{R}(I - T)$ is closed. Therefore $(I - T) \in \Phi_{g+}(X)$.

Conversely, since X has the property (H_1) and $\mathcal{N}(I - T)$ is reflexive, then there exists a closed subspace X_0 of X such that

$$X = \mathcal{N}(I - T) \oplus X_0$$

and $\mathcal{D}(T) = \mathcal{N}(I - T) \oplus X_0 \cap \mathcal{D}(T)$. Taking into account that $\mathcal{R}(I - T) = \mathcal{R}(\widehat{I - T})$ is a closed subset of X, we deduce that the restriction $(\widehat{I - T})_{|(X_0 \cap \mathcal{D}(T))}$ of $I - T$ on $X_0 \cap \mathcal{D}(T)$ is an isomorphism between the Banach spaces $(X_0 \cap \mathcal{D}(T), \|.\|_{I-T})$ and $(\mathcal{R}(I - T), \|.\|)$. Hence

$$[\widehat{I - T}_{|(X_0 \cap \mathcal{D}(T), \|.\|_{I-T})}]^{-1} : \mathcal{R}(I - T) \longrightarrow (X_0 \cap \mathcal{D}(T), \|.\|_{I-T})$$

is bounded. Let $(x_n)_n$ be a bounded sequence in $\mathcal{D}(T)$ such that $x_n - Tx_n \rightharpoonup y,\ y \in X$, then $y \in \mathcal{R}(I - T)$. So we get, $[\widehat{I - T}_{|(X_0 \cap \mathcal{D}(T), \|.\|_{I-T})}]^{-1}(x_n - Tx_n) \rightharpoonup [\widehat{I - T}_{|(X_0 \cap \mathcal{D}(T), \|.\|_{I-T})}]^{-1}(y)$. Thus, $x_n \rightharpoonup [\widehat{I - T}_{|(X_0 \cap \mathcal{D}(T), \|.\|_{I-T})}]^{-1}(y) = z, z \in X$ and hence $T \in \mathcal{WDC}(X)$. $\square$

Remark 3.4. Let X be a Banach space having the property (H_1). If $T_1 : \mathcal{D}(T_1) \subset X \longrightarrow X$ is weakly demicompact and $T_2 : \mathcal{D}(T_2) \subset \mathcal{D}(T_1) \longrightarrow X$ is weakly compact. Then, $I - T_1 - T_2$ is upper generalized semi-Fredholm.

Theorem 3.5. Let X be a Banach space satisfying the property (H_1) and let $T \in \mathcal{L}(X)$. Then, for all $n \in \mathbb{N}\backslash\{0\}$, we have the following implication:

$$(I - T) \in \mathcal{WDC}(X) \text{ implies that } T^n \in \Phi_{g+}(X).$$

Proof. Since $(I - T) \in \mathcal{WDC}(X)$, then $(I - T^n) \in \mathcal{WDC}(X)$ and so by using Theorem 3.3 we infer that $T^n \in \Phi_{g+}(X)$. □

Next, we will give a characterization of weakly demicompact bounded projections on a Banach space satisfying the property (H_1) which may be seen as a generalization of [[23], Corollary 2.1]. Before that, we recall the following theorem.

Theorem 3.6. [34] Let X be a Banach space and let $T \in \mathcal{L}(X)$ with closed range. Then, T is weakly compact if and only if $\mathcal{R}(T)$ is reflexive.

Theorem 3.7. Let X be a Banach space having the property (H_1) and P be a bounded projection on X. Then the following statements are equivalent.
(*i*) P is weakly demicompact.
(*ii*) $P \in \mathcal{W}(X)$.
(*iii*) $I - P$ is generalized Fredholm operator.

Proof. (*i*) ⇒ (*ii*) Let P be a bounded projection on X such that $P \in \mathcal{WDC}(X)$. Then, by Theorem 3.3 we deduce that $I - P$ is an upper generalized semi-Fredholm operator. Since $\mathcal{R}(P) = \mathcal{N}(I - P)$ which is reflexive, then by using Theorem 3.6 we deduce that P is weakly compact on X.
(*ii*) ⇒ (*iii*) Let $P \in \mathcal{W}(X)$, then $P \in \mathcal{WDC}(X)$ and thus by applying Theorem 3.3 we show that

$$(I - P) \in \Phi_{g+}(X). \tag{4}$$

On the other hand, since I is co-tauberian and $P \in \mathcal{W}(X)$, then $(I - P) \in \mathcal{T}^d(X)$ and according to Eq. (4) we conclude that $\mathcal{R}(I - P)$ is closed. Hence, $I - P$ is a lower generalized semi-Fredholm operator and thus $(I - P) \in \Phi_g(X)$.
(*iii*) ⇒ (*i*) Suppose that $(I - P) \in \Phi_g(X)$, then $(I - P) \in \Phi_{g+}(X)$ and so by using Theorem 3.3 we obtain the desired result. □

We can now give an example of a bounded projection which verifies the above theorem. Before that, let us recall the following definition.

Definition 3.8. [2] The Rademacher functions $(r_k)_{k=1}^{\infty}$ are defined on $[0, 1]$ by:

$$r_k(t) := sgn(\sin 2^k \pi t),$$

satisfy the following properties:
- $(r_k)(t) = \pm 1$ a.e for all k,
- $\int_0^1 r_{k_1} r_{k_2} \dots r_{k_m}(t)dt = 0$, whenever $k_1 < k_2 < \dots k_m$.

We denote by L_p the space $L_p(0, 1)$ for $1 < p < \infty$. Let R_p be the closed subspace spanned in L_p, $1 < p < \infty$, by the Rademacher functions $(r_k)_{k=1}^{\infty}$ which is isomorphic to l_2 and complemented in L_p with $1 < p < \infty$. For more detail see [2].

Example 3.9. There exists a bounded projection P from L_p onto R_p with $1 < p < \infty$ such that the operator P is weakly demicompact.

Indeed, let $P : L_p \longrightarrow R_p$ such that $I - P : L_p \longrightarrow L_p$. We have that the space L_p with $1 < p < \infty$ satisfies the property (H_1) (see [[2], Proposition 6.4.2]) and R_p, with $1 < p < \infty$ is reflexive, then there exists a closed subspace X_0 of L_p such that $L_p = X_0 \oplus R_p$. Clearly, $\mathcal{N}(I - P) = \mathcal{R}(P)$ and $\mathcal{R}(I - P) = \mathcal{N}(P)$. Thus,

$$\mathcal{R}(I - P) \text{ is closed in } L_p. \tag{5}$$

While, the space L_p with $1 < p < \infty$ is reflexive and $\mathcal{N}(I - P)$ is closed, then

$$\mathcal{N}(I - P) \text{ is reflexive.} \tag{6}$$

Moreover, since L_p with $1 < p < \infty$ is reflexive and by using Eq. (5), then we deduce that $\mathcal{R}(I-P)$ is reflexive. This result combined with the fact that L_p with $1 < p < \infty$ has the property (H_1) allow us to infer that there exists a closed subspace X_1 of L_p such that $L_p = X_1 \oplus \mathcal{R}(I - P)$. Taking into account that $L_p/\mathcal{R}(I-P)$ is reflexive and X_1 is closed in L_p, then we deduce that

$$L_p/\mathcal{R}(I - P) \text{ is reflexive.} \tag{7}$$

Again, the use of Eqs. (5), (6) and (7) enables us to conclude that $(I-P) \in \Phi_g(L_p)$ and hence $P \in \mathcal{WDC}(L_p, R_p)$ by Theorem 3.7.

Now, let us recall the following definitions.

Definition 3.10. [19] Let X be a Banach space and let $T \in \mathcal{L}(X)$. We define the weak essential norm of T by:

$$\|T\|_w = \inf\{\|T - W\| : W \in \mathcal{W}(X)\}.$$

Definition 3.11. [22, 25, 31] Let X be a Banach space and let $T \in \mathcal{L}(X)$. The operator T is said to be:

(*i*) Quasi-compact if there exists $K \in \mathcal{K}(X)$ and a positive integer m such that $\|T^m - K\| < 1$.

(*ii*) Weakly quasi-compact if there exists $W \in \mathcal{W}(X)$ and a positive integer m such that $\|T^m - W\| < 1$.

We note $\mathcal{Q}P(X)$, (resp. $\mathcal{WQ}P(X)$) for the set of quasi-compact operators (resp. weakly quasi-compact operators) acting on a Banach space X.

The following theorem is an extension of [[23], Theorem 3.1]. Before that, let us present the following lemma. The proof can be found in [19].

Lemma 3.12. Let X, Y be two Banach spaces and Z be a linear subspace of X. For an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent.

(*i*) T is co-tauberian.

(*ii*) Every operator $S \in \mathcal{L}(Y, Z)$ is weakly compact whenever ST is weakly compact.

Theorem 3.13. Let X be a Banach space satisfying the property (H_1) and $T \in \mathcal{L}(X)$. Assume that $\|T\|_w < 1$, then

(*i*) T is weakly demicompact operator on X.

(*ii*) $I - T$ is a generalized Fredholm operator on X.

Proof. (*i*) Suppose that $\|T\|_w < 1$, then there exists $W \in \mathcal{W}(X)$, such that

$$\|T\|_w \leq \|T - W\| < 1.$$

Hence, $\overline{\omega}(T - W) < 1$ and thus by using assertions (*i*) and (*iv*) from Proposition 2.9, then we get

$$\overline{\omega}(T) < 1.$$

Let $(x_n)_n$ be a bounded sequence such that

$$x_n - Tx_n \rightharpoonup x,\ x \in X.$$

We have $\{x_n, n \in \mathbb{N}\} \subset \{x_n - Tx_n, n \in \mathbb{N}\} + \{Tx_n, n \in \mathbb{N}\}$. It follows that

$$\omega(\{x_n, n \in \mathbb{N}\}) \leq \omega(\{x_n - Tx_n, n \in \mathbb{N}\}) + \omega(\{Tx_n, n \in \mathbb{N}\}).$$

Accordingly, $(1 - \overline{\omega}(T))\omega(\{x_n, n \in \mathbb{N}\}) \leq 0$. Consequently, $\omega(\{x_n, n \in \mathbb{N}\}) = 0$. So there exists a weakly convergent subsequence. Hence, T is weakly demicompact.

(*ii*) We have from the above assertion (*i*) that T is weakly demicompact, then by Theorem 3.3, we deduce that

$$(I - T) \in \Phi_{g+}(X). \tag{8}$$

Now, we will show that $I - T$ is co-tauberian. To do this, let $A \in \mathcal{L}(X)$ such that $A(I - T)$ is weakly compact. By Lemma 3.12, it is enough to show that A is weakly compact.
We have

$$A = A(I - T) + AT.$$

Accordingly,

$$\overline{\omega}(A) \leq \overline{\omega}(A)\overline{\omega}(T).$$

Then, $(1 - \overline{\omega}(T))\overline{\omega}(A) \leq 0$. Hence, A is weakly compact, which implies that $I - T$ is co-tauberian. From Eq. (8), we obtain that $\mathcal{R}(I - T)$ is closed. Consequently,

$$(I - T) \in \Phi_{g-}(X). \tag{9}$$

Using Eqs. (8) and (9), we obtain the desired result. □

Theorem 3.14. Let X be a non-reflexive Banach space having the property (H_1) and $T \in \mathcal{L}(X)$. Then, the following statements hold.
(*i*) If $T \in \mathcal{WQ}P(X)$, then $(I - T) \in \Phi_g(X)$.
(*ii*) If $T \in \mathcal{Q}P(X)$, then $(I - T) \in \Phi_g(X)$.

Proof. (*i*) Let $T \in \mathcal{L}(X)$. If there exists a positive integer m and $W \in \mathcal{W}(X)$ such that $\|T^m - W\| < 1$, then

$$\overline{\omega}(T^m - W) \leq \|T^m - W\| < 1. \tag{10}$$

Now, let $(x_n)_n$ be a bounded sequence in X such that

$$x_n - (T^m - W)x_n \rightharpoonup x, \quad x \in X,$$

We set $y_n := x_n - (T^m - W)x_n \rightharpoonup x,\ x \in X$, then we have

$$x_n = y_n + (T^m - W)x_n.$$

Accordingly,

$$\omega(\{x_n, n \in \mathbb{N}\}) \leq \omega(\{y_n, n \in \mathbb{N}\}) + \overline{\omega}(T^m - W)\omega(\{x_n, n \in \mathbb{N}\}).$$

Which implies that

$$(1 - \overline{\omega}(T^m - W))\omega(\{x_n, n \in \mathbb{N}\}) \leq 0.$$

It follows from Eq. (10) that $\omega(\{x_n, n \in \mathbb{N}\}) = 0$ and consequently $T^m - W \in \mathcal{WDC}(X)$. Thus, the use of Theorem 3.3 leads to $I - T^m + W \in \Phi_{g+}(X)$. By using assertion (*iii*) from Theorem 2.7, we deduce that $I - T^m \in \Phi_{g+}(X)$. Taking into account that

$$I - T^m = (I + T + T^2 + \cdots + T^{m-1})(I - T),$$

then by using Theorem 2.6 (*i*), we get that

$$(I - T) \in \Phi_{g+}(X). \tag{11}$$

In order to finish the proof we only need to show that $I - T$ is co-tauberian. For this purpose let $S \in \mathcal{L}(X)$ such that $S(I-T)$ is weakly compact. From Lemma 3.12, we have to claim that S is weakly compact. $S(I-T^m)$ is weakly compact thanks to $S(I-T^m) = S(I-T)(I+T+\cdots+T^{m-1})$, where $(I+T+\cdots+T^{m-1})$ is bounded. Clearly,

$$\begin{aligned}\overline{\omega}(T^m) &\leq \overline{\omega}(T^m - W) \\ &\leq \|T^m - W\| < 1.\end{aligned}$$

From the expression $S = S - ST^m + ST^m = S(I-T^m) + ST^m$, we deduce that

$$\overline{\omega}(S) \leq \overline{\omega}(ST^m) \leq \overline{\omega}(S)\overline{\omega}(T^m)$$

and thus, $(1-\overline{\omega}(T^m))\overline{\omega}(S) \leq 0$, which implies that $\overline{\omega}(S) = 0$ and so S is weakly compact. Then, from Lemma 3.12, we infer that $I-T$ is co-tauberian and according to Eq. (3.4) we conclude that $\mathcal{R}(I-T)$ is closed. Hence,

$$(I-T) \in \Phi_{g-}(X). \tag{12}$$

From Eqs. (11) and (12), we infer that $(I-T) \in \Phi_g(X)$.

(*ii*) The proof of (*ii*) follows from the fact that $\mathcal{Q}P(X) \subset \mathcal{WQ}P(X)$ and by arguing similarly as for (*i*). □

4. Generalized Fredholm results and spectral properties for operator matrices involving weak demicompactness classes

In this section, we will characterize the generalized essential spectra of the operator matrix L, the closure of L_0, acting on the space $X \times X$, where X is a non-reflexive Banach space satisfying the properties (H_1) and (H_2) or one of them. For this, let us consider the operator L_0 in the product space $X \times X$ as follow

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{13}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(D)$) and acts on X.

Further, we suppose in the following that the entries of this matrix satisfy some conditions which were introduced in [32].

(P_1) A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.

(P_2) The operator B is a densely defined linear operator on X and for (hence all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A-\mu)^{-1}B$ is closable).

(P_3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(A)$, the operator $C(A-\mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A-\mu)^{-1}$ is bounded).

(P_4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X and for some (hence all) $\mu \in \rho(A)$, the operator $D - C(A-\mu)^{-1}B$ is closable. We will denote by $S(\mu)$ its closure.

Remark 4.1. (*i*) Under the assumptions (P_1) and (P_2), we infer that for each $\mu \in \rho(A)$ the operator $G(\mu) := (A-\mu)^{-1}B$ is bounded on X.

(*ii*) From the assumption (P_3), it follows that the operator: $F(\mu) := C(A-\mu)^{-1}$ is bounded on X.

In the next, we recall the following result which characterizes the operator L.

Theorem 4.2. [5] Let conditions (P_1)-(P_3) be satisfied and the lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ be dense in X. Then, the operator L_0 is closable and the closure L of L_0 is given by

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Remark 4.3. For $\lambda \in \mathbb{C}$, it yields from previous equation that

$$\lambda - L = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu)M(\mu),$$

where

$$M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}.$$

Now, let us present the next useful lemma which is related to the cartesian product of two Banach spaces possesses on the one hand the property (H_1) and on the other hand the property (H_2).

Lemma 4.4. Let X and Y be two Banach spaces.
(i) If X and Y have the property (H_1), then $X \times Y$ has the property (H_1).
(ii) If X and Y have the property (H_2), then $X \times Y$ has the property (H_2).

Proof. (i) Let F be a reflexive subspace of $X \times Y$ and let P and Q be the bounded projections of F onto X and Y respectively. Clearly $\mathcal{R}(P)$ is closed. Since F is reflexive, then $\mathcal{N}(P)$ is also reflexive and so P is tauberian operator. By applying [[19], Proposition 2.1.3] we get that P is weakly compact and then the use of Theorem 3.6 leads to $\mathcal{R}(P)$ is a reflexive subspace of X. This result combined with the fact that X has the property (H_1) show that there exists a closed subspace X_1 of X such that $X = X_1 \oplus \mathcal{R}(P)$. Similarly we can prove that $Y = Y_1 \oplus \mathcal{R}(Q)$, where Y_1 (resp. $\mathcal{R}(Q)$) is a closed (resp. reflexive) subspace of Y. Therefore,

$$X \times Y = (X_1 \times Y_1) \oplus (\mathcal{R}(P) \times \mathcal{R}(Q)). \tag{14}$$

To demonstrate this, it remains to check that for all $z \in X \times Y$ that $z = a + b$, where $a = (a_1, a_2) \in X_1 \times Y_1$ and $b = (b_1, b_2) \in \mathcal{R}(P) \times \mathcal{R}(Q)$ and that $(X_1 \times Y_1) \cap (\mathcal{R}(P) \times \mathcal{R}(Q)) = \{0\}$. In order to do so, let $z = (x, y) \in X \times Y$. Since $X = X_1 \oplus \mathcal{R}(P)$ and $Y = Y_1 \oplus \mathcal{R}(Q)$, then for $x \in X$ and $y \in Y$, there exist respectively unique $(a_1, b_1) \in X_1 \times \mathcal{R}(P)$ and $(a_2, b_2) \in Y_1 \times \mathcal{R}(Q)$ such that $x = a_1 + b_1$ and $y = a_2 + b_2$. Hence $z = (x, y) = (a_1, a_2) + (b_1, b_2)$. Furthermore,

$$(X_1 \times Y_1) \cap (\mathcal{R}(P) \times \mathcal{R}(Q)) = (X_1 \cap \mathcal{R}(P)) \times (Y_1 \cap \mathcal{R}(Q)) = \{0\}.$$

While $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are reflexive, then so is $\mathcal{R}(P) \times \mathcal{R}(Q)$ (see [[1], Proposition 1.8.3]), and thus by using Eq. (14) we conclude that $X \times Y$ satisfies the property (H_1).
(ii) Let E and F be two closed subspaces of X and Y respectively. Firstly, we will prove that

$$(X \times Y)/(E \times F) \cong (X/E) \times (Y/F). \tag{15}$$

Since E and F are closed subspaces of X and Y respectively, then X/E and Y/F are two Banach spaces. Hence, $(X \times Y)/(E \times F)$ is isomorphic to $(X/E) \times (Y/F)$.
Now, let M be a closed subspace with reflexive quotient of $X \times Y$ and let P_1 and Q_1 be the bounded projections of $(X \times Y)/M$ onto X and Y respectively. Obviously, $\mathcal{R}(P_1)$ is closed. While $(X \times Y)/M$ is reflexive, then P_1 is weakly compact and so by applying Theorem 3.6 we conclude that $\mathcal{R}(P_1)$ is a reflexive subspace of X. This combined with the fact that X has the property (H_2) allow us to deduce that there exists a closed subspace X_0 of X such that $X = X_0 \oplus \mathcal{N}(P_1)$. Similarly we can show that Y_0 and $\mathcal{N}(Q_1)$ are complemented in Y, where Y_0 (resp. $\mathcal{N}(Q_1)$) is a closed (resp. reflexive) subspace of Y. Moreover,

$$X \times Y = (\mathcal{N}(P_1) \times \mathcal{N}(Q_1)) \oplus (X_0 \times Y_0).$$

Since X and Y have the property (H_2) and by using Eq. (15) together with [[1], Proposition 1.8.3], we infer that $X \times Y$ satisfies the property (H_2). □

Next, we state some generalized Fredholm results by means of weak demicompactness classes. Before that, we give the following remark.

Remark 4.5. Let X and Y be two non-reflexives Banach spaces. Then $X\times Y$ is a non-reflexive Banach space. Indeed, we have from [[24], Page 164] that $(X\times Y)^* = X^*\times Y^*$. So $(X\times Y)^{**} = X^{**}\times Y^{**}$. Using the fact that X and Y are non-reflexives, then so is $X\times Y$.

If $T\in \mathcal{C}(X_1)$ (resp. $\mathcal{C}(X_2)$), then $X_{1,T} = (\mathcal{D}(T), \|.\|_{X_1})$ (resp. $X_{2,T} = (\mathcal{D}(T), \|.\|_{X_2})$) is a Banach space. From now, we assume that $X_{1,T}$ and $X_{2,T}$ are non-reflexive having the property (H).

Lemma 4.6. Let X_1 and X_2 be two non-reflexive Banach spaces having the the property (H) and let M_g be the operator matrix defined by:

$$M_g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A\in \mathcal{C}(X_1), B\in \mathcal{C}(X_2)$. Then, the following statements hold.

(*i*) If $M_g \in \Phi_{g+}(X_1\times X_2)$, then $A\in \Phi_{g+}(X_1)$ and $B\in \Phi_{g+}(X_2)$.

(*ii*) If $M_g \in \Phi_{g-}(X_1\times X_2)$, then $A\in \Phi_{g-}(X_1)$ and $B\in \Phi_{g-}(X_2)$.

(*iii*) If $A\in \Phi_{g+}(X_1)$ and $B\in \Phi_{g+}(X_2)$, then $M_g \in \Phi_{g+}(X_1\times X_2)$.

(*iv*) If $A\in \Phi_{g}(X_1)$ and $B\in \Phi_{g}(X_2)$, then $M_g \in \Phi_{g}(X_1\times X_2)$.

Proof. (*i*) Suppose that $M_g \in \Phi_{g+}(X_1\times X_2)$, then $\widehat{M_g} \in \Phi_{g+}(X_{1,T}\times X_{2,T}, X_1\times X_2)$. The operator $\widehat{M_g}$ can be written in the following from:

$$\widehat{M_g} = \begin{pmatrix} I & 0 \\ 0 & \widehat{B} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \widehat{A} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \widehat{A} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \widehat{B} \end{pmatrix}. \tag{16}$$

By applying Theorem 2.6 (*i*) to Eq. (16) we deduce that $\widehat{A}\in \Phi_{g+}(X_{1,T}, X_1)$ and hence by applying Remark 3.2 we infer that $A\in \Phi_{g+}(X_1)$.
The proof for the other operator will be similarly achieved and the proof of (*i*) is complete.

(*ii*) By using Lemma 2.4 (*ii*) and Lemma 4.4 (*ii*), the proof of (*ii*) follows by symmetry as (*i*).

(*iii*) The proof of (*iii*) follows immediately from Eq. (16).

(*iv*) The statement (*iv*) can be checked in the same way from the assertion (*iii*). □

Next, for $A\in \mathcal{C}(X_1), B\in \mathcal{C}(X_2)$ and $C\in \mathcal{C}(X_1, X_2)$, let us consider the 2×2 operator matrix M_C as

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}. \tag{17}$$

Lemma 4.7. Let X_1 and X_2 be two non-reflexive Banach spaces having the property (H) and let M_C be the matrix operator defined in Eq. (17), then

(*i*) If $A\in \Phi_{g}(X_1)$ and $B\in \Phi_{g}(X_2)$, then $M_C \in \Phi_{g}(X_1\times X_2)$.

(*ii*) If $A\in \Phi_{g+}(X_1)$ and $B\in \Phi_{g+}(X_2)$, then $M_C \in \Phi_{g+}(X_1\times X_2)$.

Proof. (*i*) Since $X_{1,T}$ and $X_{2,T}$ satisfy the property (H), then by Lemma 4.4 we deduce that $X_{1,T}\times X_{2,T}$ satisfies the property (H). Moreover, the operator M_C can be written in the following form

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = JKL, \quad J, K, L\in \mathcal{C}(X_1\times X_2).$$

Since $A\in \Phi_g(X_1)$ and $B\in \Phi_g(X_2)$, then from assertion (*iii*) of Lemma 4.6, it yields that J and L are both generalized Fredholm operators. Taking account that K is invertible, for every $C\in \mathcal{C}(X_2, X_1)$, then K is generalized Fredholm. So, $\widehat{J}, \widehat{K}$ and $\widehat{L}$ are both generalized Fredholm operators. By using Theorem 2.7 we infer that $\widehat{M_C}\in \Phi_g(X_{1,T}\times X_{2,T}, X_1\times X_2)$. Hence $M_C\in \Phi_g(X_1\times X_2)$.

(*ii*) The proof of (*ii*) is established similarly as that in (*i*) by using Lemma 2.4 (*ii*). □

Remark 4.8. Let X_1 and X_2 be two non-reflexive Banach spaces having the property (H). Then, by using a similar reasoning as in again Lemma 4.7, it is easy to check that:
If $A \in \Phi_g(X_1)$ and $B \in \Phi_g(X_2)$, then for every $D \in \mathcal{C}(X_1, X_2)$

$$M_D = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$$

is a generalized Fredholm operator on $X_1 \times X_2$.

Lemma 4.9. Let X_1 and X_2 be two non-reflexive Banach spaces having the property (H) and let M_C be the matrix operator defined in Eq. (17). Then,
(i) If $M_C \in \Phi_{g+}(X_1 \times X_2)$, then $A \in \Phi_{g+}(X_1)$.
(ii) If $M_C \in \Phi_{g-}(X_1 \times X_2)$, then $B \in \Phi_{g-}(X_2)$.
(iii) If $M_C \in \Phi_g(X_1 \times X_2)$, then $A \in \Phi_{g+}(X_1)$ and $B \in \Phi_{g-}(X_2)$.

Proof. (i) Since $X_{1,T}$ and $X_{2,T}$ have the property (H), then by Lemma 4.4 we infer that $X_{1,T} \times X_{2,T}$ satisfies the property (H). Since $M_C \in \Phi_{g+}(X_1 \times X_2)$, then $\widehat{M_C} \in \Phi_{g+}(X_{1,T} \times X_{2,T}, X_1 \times X_2)$. Moreover, we have the following decomposition

$$\widehat{M_C} = \begin{pmatrix} I & 0 \\ 0 & \widehat{B} \end{pmatrix}\begin{pmatrix} I & \widehat{C} \\ 0 & I \end{pmatrix}\begin{pmatrix} \widehat{A} & 0 \\ 0 & I \end{pmatrix}, \tag{18}$$

where $\widehat{C} \in \mathcal{C}(X_{2,T}, X_1)$. Using assertion (i) from Theorem 2.6, we get that $\begin{pmatrix} \widehat{A} & 0 \\ 0 & I \end{pmatrix} \in \Phi_{g+}(X_{1,T} \times X_{2,T}, X_1 \times X_2)$.
By using Lemma 4.6 we get $\widehat{A} \in \Phi_{g+}(X_{1,T}, X_1)$. Consequently, $A \in \Phi_{g+}(X_1)$.
(ii) Since $M_C \in \Phi_{g-}(X_1 \times X_2)$, the combination of the decomposition given in Eq. (18) and assertion (ii) from Lemma 4.6 shows that $B \in \Phi_{g-}(X_2)$.
(iii) The proof of (iii) is a direct consequence of (i) and (ii). □

Remark 4.10. Similarly as in Lemma 4.7, it can be shown for X_1 and X_2 are two non-reflexive Banach spaces having the property (H) that,

(i) if $A \in \Phi_{g+}(X_1)$ and $B \in \Phi_{g+}(X_2)$, then $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \in \Phi_{g+}(X_1 \times X_2)$ for every $D \in \mathcal{C}(X_1, X_2)$.

(ii) If $A \in \Phi_{g-}(X_1)$ and $B \in \Phi_{g-}(X_2)$, then $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \in \Phi_{g-}(X_1 \times X_2)$ for every $D \in \mathcal{C}(X_1, X_2)$.

The following lemma concerns the set of upper g-Riesz operators.

Lemma 4.11. Let X_1 and X_2 be two non-reflexive Banach spaces having the property (H_1) and let M_C be the matrix operator defined in Eq. (17). Then, A and B are weakly demicompact if and only if, $\lambda - M_C$ is an upper generalized semi-Fredholm, for all $\lambda \in \mathbb{C}\backslash(\{0\} \cap \sigma(B))$.

Proof. We have

$$\lambda - M_C = \begin{pmatrix} \lambda - A & C \\ 0 & \lambda - B \end{pmatrix}.$$

Since A and B are weakly demicompact and $X_1 \times X_2$ has the property (H_1) by Lemma 4.4, then the use of Theorem 3.3 leads to $\lambda - A$ and $\lambda - B$ are upper generalized semi-Fredholm operators for all $\lambda \in \mathbb{C}\backslash\{0\}$, by Lemma 4.7 (ii), $(\lambda - M_C) \in \Phi_{g+}(X_1 \times X_2)$. For the converse, let $\lambda \in \mathbb{C}\backslash\sigma(B)$, then we consider the following decomposition:

$$\lambda - M_C = \begin{pmatrix} \lambda - A & -C \\ 0 & \lambda - B \end{pmatrix} = \begin{pmatrix} I & -C(\lambda - B)^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix}.$$

From the facts that $(\lambda-\widehat{M_c}) \in \Phi_{g+}(X_{1,T}\times X_{2,T}, X_1\times X_2)$ and

$$\begin{pmatrix} \lambda-\widehat{A} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \lambda-\widehat{B} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda-\widehat{B} \end{pmatrix}\begin{pmatrix} \lambda-\widehat{A} & 0 \\ 0 & I \end{pmatrix},$$

then by using Theorem 2.6 and Lemma 4.6 (*i*), it follows that $\lambda - A$ and $\lambda - B$ are upper generalized semi-Fredholm. Again, the use of Theorem 3.3 leads to the desired result. □

In the following results, we assume that X_T is non-reflexive.

Proposition 4.12. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H_1) and let L_0 be the matrix operator defined in Eq. (13) satisfies (P_1)-(P_4). Suppose that there is $\mu \in \mathbb{C}\backslash\{0\}$ such that $\frac{1}{\mu} \in \rho(A)$. If the operator $\mu S(\frac{1}{\mu})$ is weakly demicompact, then $I-\mu L$ is an upper generalized semi-Fredholm operator.

Proof. The operator L can be written by the Frobenius-Schur factorization as follow:

$$\begin{aligned} L &= \frac{1}{\mu} - \begin{pmatrix} I & 0 \\ F(\frac{1}{\mu}) & I \end{pmatrix}\begin{pmatrix} \frac{1}{\mu}-A & 0 \\ 0 & \frac{1}{\mu}-S(\frac{1}{\mu}) \end{pmatrix}\begin{pmatrix} I & G(\frac{1}{\mu}) \\ 0 & I \end{pmatrix} \\ &= \frac{1}{\mu} - QRP. \end{aligned}$$

Clearly, Q and P are upper generalized semi-Fredholm operators on $X\times X$ (see Lemma 4.6 (*ii*)). Furthermore, we have the following decomposition

$$\mu L = I - \mu\begin{pmatrix} I & 0 \\ F(\frac{1}{\mu}) & I \end{pmatrix}\begin{pmatrix} \frac{1}{\mu}-A & 0 \\ 0 & \frac{1}{\mu}-S(\frac{1}{\mu}) \end{pmatrix}\begin{pmatrix} I & G(\frac{1}{\mu}) \\ 0 & I \end{pmatrix}.$$

Since $\frac{1}{\mu} \in \rho(A)$ for $\mu \neq 0$, then $(\frac{1}{\mu}-A) \in \Phi_g(X\times X)$ and so

$$(I-\mu A) \in \Phi_{g+}(X\times X). \tag{19}$$

Taking account that $\mu S(\frac{1}{\mu}) \in \mathcal{WDC}(X)$, then by Theorem 3.3 we deduce that

$$(I-\mu S(\frac{1}{\mu})) \in \Phi_{g+}(X\times X). \tag{20}$$

The use of Eqs. (19), (20) and Lemma 4.6 (*ii*) leads to $R \in \Phi_{g+}(X\times X)$. This result combined with the fact that Q and P are bounded with bounded inverses allow us to get that $\widehat{QRP}$ is an upper generalized semi-Fredholm operator and so is QRP. It yields from Theorem 3.3 that $\mu L = (\mu - QRP) \in \mathcal{WDC}(X\times X)$ for $\mu \in \mathbb{C}\backslash\{0\}$ and hence the result follows from Theorem 3.3. □

Now, we need the following lemma.

Lemma 4.13. Let X_1, X_2 be two non-reflexive Banach spaces satisfying the property (H_1) and let

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where $F_{ij} \in \mathcal{L}(X_j, X_i)$, with $i,j=1,2$. If $F_{ij} \in \mathcal{F}_g^l(X_j, X_i)$ then, $F \in \mathcal{F}_{g+}(X_1\times X_2)$.
Here, $\mathcal{F}_g^l$ denotes the set of perturbations of upper generalized semi-Fredholm with complemented range.

Proof. We have

$$\begin{aligned} F &= \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_{22} \end{pmatrix} \\ &= K + L + M + N, \end{aligned}$$

where

$$K := \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}, \ L := \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix}, \ M := \begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix} \text{ and } N =: \begin{pmatrix} 0 & 0 \\ 0 & F_{22} \end{pmatrix}.$$

First, we will prove that if $F_{11} \in \mathcal{F}_g^l(X_1)$ then $K \in \mathcal{F}_{g+}(X_1 \times X_2)$. To do this, let

$$S := \begin{pmatrix} A & C \\ D & B \end{pmatrix} \in \Phi_{g+}(X_1 \times X_2),$$

with complemented range. Since $X_1 \times X_2$ is non-reflexive and satisfies the property (H_1), it follows that there exist

$$S_0 := \begin{pmatrix} A_0 & C_0 \\ D_0 & B_0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2) \text{ and } W := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \in \mathcal{W}(X_1 \times X_2),$$

such that $S_0 S = I + W$ on $X_1 \times X_2$. Then,

$$S_0(S + K) = I + W + S_0 K = \begin{pmatrix} I + W_{11} + A_0 F_{11} & W_{12} \\ W_{21} + D_0 F_{11} & I + W_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}_g^l(X_1)$, then by using Theorem 2.7 we get $(I + W_{11} + A_0 F_{11}) \in \Phi_{g+}(X_1)$. This result combined with the fact that $(I + W_{22}) \in \Phi_{g+}(X_2)$, enables us to infer from Remark 4.10, that

$$\left[S_0(S + K) - \begin{pmatrix} 0 & W_{12} \\ 0 & 0 \end{pmatrix} \right] \in \Phi_{g+}(X_1 \times X_2).$$

Taking into account that $W_{12} \in \mathcal{W}(X_1, X_2)$ and $(S_0(S + K)) \in \Phi_{g+}(X_1 \times X_2)$, then by Theorem 2.6 we deduce that $(S + K) \in \Phi_{g+}(X_1 \times X_2)$. Consequently, $K \in \mathcal{F}_{g+}(X_1 \times X_2)$. The proofs for the other operators can be proved similarly. □

Theorem 4.14. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H_1). Let L_0 be the operator defined in Eq. (13) satisfies (P_1)-(P_4). Suppose that $\mu \in \rho(A)$, $G(\mu) \in \mathcal{W}(X)$ and $F(\mu) \in \mathcal{F}_g^l(X)$. If A and $S(\mu)$ are weakly demicompact, then $(\mu - L) \in \Phi_{g+}(X \times X)$.

Proof. For $\lambda = 1$, we have

$$\begin{aligned} I - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} I - A & 0 \\ 0 & I - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (1 - \mu) M(\mu) \\ &= UV(\mu)W - (1 - \mu) M(\mu). \end{aligned}$$

It is clear that U and W are upper generalized semi-Fredholm. Since A and $S(\mu)$ are weakly demicompact, then by Theorem 3.3 we deduce that $I - A$ and $I - S(\mu)$ are upper generalized semi-Fredholm operators on X. By using Lemma 4.6 we get $V(\mu)$ is upper generalized semi-Fredholm. It follows from the boundedness of the operators U and W and their inverses that $UV(\mu)W \in \Phi_{g+}(X \times X)$. We have

$$\begin{aligned} I - L = UV(\mu)W + (1 - \mu)M(\mu) &= UV(\mu)W + (1 - \mu) \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix} \\ &= Z + F_1, \end{aligned}$$

where,

$$Z := UV(\mu)W \ \text{ and } F_1 := (1 - \mu) \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}.$$

To prove that $(I-L) \in \Phi_{g+}(X \times X)$, it is enough to show that $F_1 \in \mathcal{F}_{g+}(X \times X)$. For this purpose, let $\mu \in \rho(A)$. Since $F(\mu) \in \mathcal{L}(X)$ and $G(\mu)$ is weakly compact, then the fact that $\mathcal{W}(X)$ is a two-sided ideal of $\mathcal{L}(X)$ shows that $F(\mu)G(\mu) \in \mathcal{W}(X) \subset \mathcal{F}_g^l(X)$. Moreover, taking into account that $F(\mu), G(\mu) \in \mathcal{F}_g^l(X)$, then according to Lemma 4.13 we conclude that $F_1 \in \mathcal{F}_{g+}(X \times X)$. Thus, $\left[UV(\mu)W - (1-\mu)M(\mu)\right] \in \Phi_{g+}(X \times X)$ and so the result follows. □

Finally, we will describe the generalized essential spectra of the matrix operator L. The following results are generalizations of some results found in [33].

Theorem 4.15. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H_1). Assume that L_0 satisfies (P_1)-(P_4). If $A \in \mathcal{WDC}(X)$, then for every $\mu \in \mathbb{C}\backslash\{0\}$

$$\sigma_{e_1,g}(L)\backslash\{0\} = \sigma_{e_1,g}(S(\mu))\backslash\{0\}.$$

Proof. We have

$$\mu - L = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Let us consider the following operators:

$$K := \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}, \; T := \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \text{ and } U := \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Again, let $\mu \notin \sigma_{e_1,g}(L)\backslash\{0\}$, then $(\mu - L) \in \Phi_{g+}(X \times X)$. Clearly, U and K are bounded and have bounded inverses. Then, it follows that $T \in \Phi_{g+}(X \times X)$. According to Remark 4.10 (*i*) we conclude that $(\mu - S(\mu)) \in \Phi_{g+}(X)$. Consequently, $\mu \notin \sigma_{e_1,g}(S(\mu))\backslash\{0\}$. To show the opposite inclusion, let $\mu \notin \sigma_{e_1,g}(S(\mu))\backslash\{0\}$, then $(S(\mu) - \mu) \in \Phi_{g+}(X)$. Since $A \in \mathcal{WDC}(X)$, then from Theorem 3.3 it follows that $(\mu - A) \in \Phi_{g+}(X)$. Applying Lemma 4.6 (*ii*) we infer that $T \in \Phi_{g+}(X \times X)$. The fact that U and K are bounded with bounded inverses allow us to conclude that $(\mu - L) \in \Phi_{g+}(X \times X)$, for all $\mu \in \mathbb{C}\backslash\{0\}$. Hence $\mu \notin \sigma_{e_1,g}(L)\backslash\{0\}$. □

Remark 4.16. In the rest of this paper, we impose the following assumptions (P'_1), (P'_2), (P'_3) and (P'_4) by replacing the operators A and C in (P_1), (P_2), (P_3) and (P_4), by D and B respectively. The closure of $A - B(D-\mu)^{-1}C$ is here denoted by $T(\mu)$.

(P'_5) The operator A satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\rho(A) \cap \rho(D) \neq \emptyset$ and $\mathcal{D}(B) \cap \mathcal{D}(D)$ is a core of D.

(P'_6) The operator D satisfies $\mathcal{D}(D) \subset \mathcal{D}(B)$, $\rho(A) \cap \rho(D) \neq \emptyset$ and $\mathcal{D}(A) \cap \mathcal{D}(C)$ is a core of A.

Theorem 4.17. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H_1). Let L_0 be the matrix operator defined in Eq. (13) satisfies the assumption (P'_1)-(P'_4). If $D \in \mathcal{WDC}(X)$, then for every $\mu \in \mathbb{C}\backslash\{0\}$

$$\sigma_{e_1,g}(L)\backslash\{0\} = \sigma_{e_1,g}(T(\mu))\backslash\{0\}.$$

Proof. Since assumption (P_5) is satisfied, then [[33], Theorem 2.2.18], shows that A is closable (resp. closed) if and only if $A - B(D-\mu)^{-1}C$ is closable (resp. closed) for some (and hence for all) $\mu \in \rho(D)$ and thus the operator $L - \mu$ can be written as follow

$$L - \mu = \begin{pmatrix} I & B(D-\mu)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} T(\mu) - \mu & 0 \\ 0 & D - \mu \end{pmatrix} \begin{pmatrix} I & 0 \\ \overline{(D-\mu)^{-1}C} & I \end{pmatrix}.$$

Now, similarly as in the proof of again Theorem 4.15 and for all $\mu \in \mathbb{C}\backslash\{0\}$, it can be shown that $(T(\mu) - \mu) \in \Phi_{g+}(X)$ if and only if $(L - \mu) \in \Phi_{g+}(X \times X)$, which achieves the result. □

The following theorems give a characterization of the generalized Wolf essential spectra of L. Before that, for an arbitrary fixed $\mu_0 \in \rho(A)$ (resp. $\mu_0 \in \rho(D)$) let us define the operator $\overline{L}_{1,\mu_0}$ (resp. $\overline{L}_{2,\mu_0}$) as:

$$\overline{L}_{1,\mu_0} := \begin{pmatrix} A & 0 \\ 0 & S(\mu_0)+\mu_0 \end{pmatrix} \quad \text{and} \quad \overline{L}_{2,\mu_0} := \begin{pmatrix} T(\mu_0)+\mu_0 & 0 \\ 0 & D \end{pmatrix}.$$

Theorem 4.18. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H) and let L_0 be the matrix operator defined in Eq. (13) satisfies the assumption (P_1)-(P_4) and (P_5'). If for some (and hence for all) $\mu \in \rho(A) \cap \rho(D)$:
(*i*) $(A-\mu)^{-1}B$ and $C(A-\mu)^{-1}B$ are bounded on $\mathcal{D}(B)$,
(*ii*) $(D-\mu)^{-1}C(A-\mu)^{-1}$ and $(A-\mu)^{-1}B(D-\mu)^{-1}$ are weakly compact,
then for every $\mu_0 \in \rho(A)$ with $\rho(L)\cap\rho(A)\cap\rho(D-\overline{C(A-\mu_0)^{-1}B}) \neq \emptyset$, the difference of the resolvents $(L-\lambda)^{-1}$ and $(\overline{L}_{1,\mu_0}-\lambda)^{-1}$ is weakly compact for $\lambda \in \rho(L)\cap\rho(A)\cap\rho(D-\overline{C(A-\mu_0)^{-1}B})$. In particular,

$$\sigma_{e_4,g}(L) = \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(D-\overline{C(A-\mu_0)^{-1}B}).$$

Proof. Let $\mu_0 \in \rho(A)$ and let $\lambda \in \mathbb{C}$ be so that $\rho(L)\cap\rho(A)\cap\rho(D-\overline{C(A-\mu_0)^{-1}B}) \neq \emptyset$, then we have

$$(L-\lambda)^{-1} - (\overline{L}_{1,\mu_0}-\lambda)^{-1} :=$$

$$\begin{pmatrix} \overline{(A-\lambda)^{-1}B}\,\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1} & -\overline{(A-\lambda)^{-1}B}\,\overline{S(\lambda)}^{-1} \\ -\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1} & \overline{S(\lambda)}^{-1} - (\overline{S(\mu_0)}+\mu_0-\lambda)^{-1} \end{pmatrix} \tag{21}$$

Furthermore,

$$\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1} = (D-\lambda)^{-1}C(A-\lambda)^{-1} + \overline{S(\lambda)}^{-1}\overline{C(A-\lambda)^{-1}B}(D-\lambda)^{-1}C(A-\lambda)^{-1},$$

and

$$\overline{S(\lambda)}^{-1} - (\overline{S(\mu_0)}+\mu_0-\lambda)^{-1} = (\lambda-\mu_0)\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1}\overline{(A-\mu_0)^{-1}B}(\overline{S(\mu_0)}+\mu_0-\lambda)^{-1}. \tag{22}$$

To prove that the operator $(L-\lambda)^{-1} - (\overline{L}_{1,\mu_0}-\lambda)^{-1}$ is weakly compact, it suffices to show that all its entries that defined in Eq. (21) are weakly compact. To do so, we have from the second condition in (*i*) and the first condition in (*ii*) that $\overline{(A-\lambda)^{-1}B}\,\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1}$ is weakly compact.

Moreover, by assertion (*i*) and the first condition from (*ii*) we infer that $\overline{S(\lambda)}^{-1}C(A-\lambda)^{-1}$ is weakly compact.
It suffices to use Eq. (22) to obtain that $\overline{S(\lambda)}^{-1} - (\overline{S(\mu_0)}+\mu_0-\lambda)^{-1}$ is weakly compact.
The proofs for the other operators will be similarly achieved.
Therefore, it follows from assertion (*iii*) of Lemma 4.9 and the weak compactness of $(L-\lambda)^{-1} - (\overline{L}_{1,\mu_0}-\lambda)^{-1}$ combined with the fact that

$$L-\lambda = (\lambda-\mu)(L-\mu)((\lambda-\mu)^{-1} - (L-\mu)^{-1})$$

that

$$\begin{aligned} \lambda \in \sigma_{e_4,g}(L) \quad &\Leftrightarrow \quad (\lambda-\mu)^{-1} \in \sigma_{e_4,g}((L-\mu)^{-1}) = \sigma_{e_4,g}((\overline{L}_{1,\mu_0}-\lambda)^{-1}) \\ &\Leftrightarrow \quad \lambda \in \sigma_{e_4,g}(\overline{L}_{1,\mu_0}). \end{aligned}$$

Accordingly,

$$\sigma_{e_4,g}(L) = \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(D-\overline{C(A-\mu_0)^{-1}B}). \quad \square$$

As a similar way as in theorem 4.18, we can easily obtain the following result.

Theorem 4.19. Let X be a non-reflexive Banach space. Assume that X and X_T satisfy the property (H) and let L_0 be the matrix operator defined in Eq. (13) satisfies the assumptions (P_1')-(P_4') and (P_6'). If for some (and hence for all) $\mu \in \rho(A) \cap \rho(D)$:

(i) $(D-\mu)^{-1}C$ and $B(D-\mu)^{-1}C$ are bounded on $\mathcal{D}(C)$,

(ii) $(A-\mu)^{-1}B(D-\mu)^{-1}$ and $(D-\mu)^{-1}C(A-\mu)^{-1}$ are weakly compact,

then for every $\mu_0 \in \rho(D)$ with $\rho(L) \cap \rho(D) \cap \rho(A - \overline{B(D-\mu_0)^{-1}C}) \neq \emptyset$, the difference of the resolvents $(L-\lambda)^{-1}$ and $(\overline{L}_{2,\mu_0} - \lambda)^{-1}$ is weakly compact for $\lambda \in \rho(L) \cap \rho(D) \cap \rho(A - \overline{B(D-\mu_0)^{-1}C})$. In particular,

$$\sigma_{e_4,g}(L) = \sigma_{e_4,g}(D) \cup \sigma_{e_4,g}(A - \overline{B(D-\mu_0)^{-1}C}).$$

Remark 4.20. In Theorems 4.18 and 4.19, if X and X_T satisfy the property (H_1) and the other conditions are satisfied, then

$$\sigma_{e_1,g}(L) = \sigma_{e_1,g}(A) \cup \sigma_{e_1,g}(D - \overline{C(A-\mu_0)^{-1}B}),$$

and

$$\sigma_{e_1,g}(L) = \sigma_{e_1,g}(D) \cup \sigma_{e_1,g}(A - \overline{B(D-\mu_0)^{-1}C}).$$

Now, we will finish with a simple example of weakly demicompact 2×2 matrix operator acting on the space $J(X_n) \times J(X_n)$.

Example 4.21. Let $J(X_n)$ be the James' quasi-reflexive Banach space with X_n denotes the subspace of l_1 generated by $\{e_1, \dots, e_n\}$. For more details see [3, 19].

For $k \in \mathbb{N}$, we define the operator $T_k : J(X_n) \to J(X_n)$ as follows:

$$T_k((x_n)) := (x_1, x_2, \dots, x_k, 0, 0, \dots); \;\; (x_n) \in J(X_n).$$

Clearly T_k is a projection onto a finite dimensional subspace. Then, T_k is compact.

Next, we define the following operators on $J(X_n)$ by:

$$\begin{aligned} B_k((x_n)) &= (x_1, 2x_2, x_3, 2x_4, \dots, x_k, 0, 0, \dots); \;\; (x_n) \in J(X_n), \\ C_k((x_n)) &= (x_1, 0, x_3, 0, x_5, \dots, x_k, 0, 0, \dots); \;\; (x_n) \in J(X_n), \\ S_k((x_n)) &= (0, x_2, 0, x_4, \dots, x_{k-1}, 0, 0, \dots); \;\; (x_n) \in J(X_n). \end{aligned}$$

Clearly, the operators B_k, C_k and S_k are weakly compact.

Now, we consider the matrix operator $\mathcal{M}_k$ defined on $J(X_n) \times J(X_n)$ as:

$$\mathcal{M}_k := \begin{pmatrix} T_k & B_k \\ C_k & S_k \end{pmatrix}.$$

Since all the entries of $\mathcal{M}_k$ are weakly compact then so is $\mathcal{M}_k$. Hence $\mathcal{M}_k \in \mathcal{WDC}(J(X_n) \times J(X_n))$. On the other hand, it yields from Lemma 4.4 and Remark 4.5 that $J(X_n) \times J(X_n)$ is non-reflexive and satisfies the property (H_1). This result combined with the use of Theorem 3.3, allow us to conclude that $(I - \mathcal{M}_k) \in \Phi_{g+}(J(X_n) \times J(X_n))$.

Finally, we present some additional interesting works:

(1) What about generalized Fredholm theory in Banach Lattice spaces.

(2) What about the current study (generalized Fredholm and weak demicompactness) in Banach Lattice spaces.

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