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On structures in the semi-tangent bundle

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Abstract. The main goal of this paper is to identify some important tensor structures in the semi-tangent bundle, to provide instances of these structures and to study lift problems in these structures. The definition of almost product, almost paracomplex, almost contact and almost paracontact structural geometry in a semi-tangent bundle is covered in this paper. Using the semi-tangent bundle theory, several properties of adapted frames are shown. Furthermore, the prolongations (horizontal lifts, complete lifts and vertical lifts) of affinor fields in the semi-tangent bundle are used in this study to build almost product, almost complex and Lorentzian almost paracontact structures.

1. Introduction

The tensor structures on smooth manifolds are remarkable geometric objects in popular differential geometry. Many authors have made important contributions to this field. In 1947, Weil noticed that there exist in a complex space a $(1,1)-$ tensor field (i.e., an affinor field) P whose square is minus unity [33]. Ehresmann and Libermann [5] researched and provided the prerequisites for a complex structure to generate an almost complex structure.

In 1955, A.G. Walker started the study of so-called an almost product spaces and showed that there exists a mixed tensor field P whose square is unity instead of being minus unity as in the case of an almost complex space [29]. In 1965, K. Yano tried to make as clear as possible the analogy between the almost complex and almost product structures in [32]. In reality, polynomial structures ($P-$structures) are $(1,1)-$tensor fields, which are roots of the algebraic equation

$$\varphi(P) = P^n + a_n P^{(n-1)} + ... + a_2 P + a_1 I = 0,$$

in which $a_1, a_2, ..., a_n \in$ R and $I = id_{M_n}$ is the identity tensor of type $(1,1)$. The polynomial structures on a manifold that we have discussed were defined as the following equations:

(i) If $\varphi(P) = P^2 + I = 0$, then P is referred to as an almost complex structure. That is to say, we have a smooth $(1,1)-$tensor field P such that $P^2 = -I$ when regarded as a vector bundle isomorphism $P : T(M_n) \rightarrow$

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$T(M_n)$ on the tangent bundle $T(M_n)$. Thus, we defined an almost-complex structure to be a linear bundle map $P : T(M_n) \to T(M_n)$ with $P^2 = -I$.

(*ii*) If $\varphi(P) = P^2 - I = 0$, then P is referred to as an almost product structure. That is to say, an almost-product structure on M_n is a field of endomorphisms of $T(M_n)$, i.e. an affinor field on M_n, so $P^2 = I$.

(*iii*) If $\varphi(P) = P^2 = 0$, then P is referred to as an almost tangent structure [4], [6].

Let B_m and M_n denote two differentiable manifolds of dimensions m and n respectively, let (M_n, π_1, B_m) be a differentiable bundle and let π_1 be the submersion (natural projection) $\pi_1 : M_n \to B_m$. We may consider $(x^i) = (x^a, x^\alpha)$, $i = 1, ..., n$; $a, b, ... = 1, ..., n-m$; $\alpha, \beta, ... = n-m+1, ..., n$ as local coordinates in a neighborhood $\pi_1^{-1}(U)$.

Let B_m be the base manifold and $T(B_m)$ be the tangent bundle over B_m and let $\widetilde{\pi} : T(B_m) \to B_m$ be the natural projection. Also, let $T_p(B_m)$ represent in for the tangent space at a p−point $(\widetilde{p} = (x^a, x^\alpha) \in M_n, p = \pi_1(\widetilde{p}))$ on the base manifold B_m. If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with regard to the natural base $\{\partial_\alpha\} = \{\frac{\partial}{\partial x^\alpha}\}$, then we have the set of all points $(x^a, x^\alpha, x^{\overline{\alpha}})$, $X^\alpha = x^{\overline{\alpha}} = y^\alpha$, $\overline{\alpha}, \overline{\beta}, ... = n+1, ..., n+m$ is by definition, the semi-tangent bundle $t(B_m)$ over the M_n manifold and the natural projection $\pi_2 : t(B_m) \to M_n$, $\dim t(B_m) = n + m$.

Specifically, assuming $n = m$, then the semi-tangent bundle [21] $t(B_m)$ becomes a tangent bundle $T(B_m)$. If given a tangent bundle $\widetilde{\pi} : T(B_m) \to B_m$ and a natural projection $\pi_1 : M_n \to B_m$, the pullback bundle (for example, see [7], [9], [14], [16], [17], [24], [27], [28]) is defined by $\pi_2 : t(B_m) \to M_n$ where

$$t(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}}\right) \in M_n \times T_x(B_m) \middle| \pi_1(x^a, x^\alpha) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\alpha}}\right) \right\}.$$

The induced coordinates $\left(x^{1'}, ..., x^{n-m'}, x^{1'}, ..., x^{m'}\right)$ with regard to $\pi^{-1}(U)$ will be given by

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), & a, b, ... = 1, ..., n-m \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), & \alpha, \beta, ... = n-m+1, ..., n, \end{cases} \tag{1}$$

if $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another coordinate chart on M_n.

The Jacobian matrice of (1) is given by [21]:

$$\left(A_j^{i'}\right) : \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} \frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta} \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta} \end{pmatrix},$$

where $i, j, ... = 1, ..., n$.

If (1) is the local coordinate system on M_n, then we have the induced fiber coordinates $(x^{a'}, x^{\alpha'}, x^{\overline{\alpha}'})$ on the semi-tangent bundle (change of coordinates):

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), & a, b, ... = 1, ..., n-m, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), & \alpha, \beta, ... = n-m+1, ..., n, \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta, & \overline{\alpha}, \overline{\beta}, ... = n+1, ..., n+m, \end{cases} \tag{2}$$

The Jacobian matrice for (2) is as follows [21]:

$$\bar{A} : \left(A_J^{I'}\right) = \begin{pmatrix} \frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta} & 0 \\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta} & 0 \\ 0 & y^\varepsilon \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon} & \frac{\partial x^{\alpha'}}{\partial x^\beta} \end{pmatrix}, \tag{3}$$

where $I, J, ... = 1, ..., n+m$.

Therefore, we obtain

$$\left(A_{J'}^{I}\right) = \begin{pmatrix} A_{b'}^a & A_{\beta'}^a & 0 \\ 0 & A_{\beta'}^\alpha & 0 \\ 0 & A_{\beta'\varepsilon'}^\alpha y^{\varepsilon'} & A_{\beta'}^\alpha \end{pmatrix}, \tag{4}$$

which is the Jacobian matrix of inverse (2).

There is numerous research on tangent bundle theory, which is a popular topic in engineering, physics and particularly differential geometry [8], [18].

The semi-tangent bundle considered in this work specifies a pull-back bundle and differs from the tangent bundle.

We note that almost paracontact structure and almost contact structure in the tangent bundles and their some features were studied in [1], [2], [13], [19], [22]. Numerous studies, including [21], [27], [28] and others, have studied the geometric properties of the semi-tangent bundle.

The study of projectable linear connections in the semi-tangent bundles and some of their properties is known to have occurred in [16], [27], [28].

This paper's main objectives are to find some significant tensor structures that have not yet been established in the semi-tangent bundle, to give examples of these structures and to research lift problems in these structures.

The definition of the geometry for almost product structures, almost paracomplex structures, almost contact structures and almost paracontact structures is the focus of this work. Some properties of adapted frames are presented by using the theory of the semi-tangent bundle $t(B_m)$. In addition, in this work, almost product, almost complex and Lorentzian almost paracontact structures are defined according to the prolongations (horizontal lifts, complete lifts and vertical lifts) of affinor fields in the semi-tangent bundle.

2. Basic formulas on the semi-tangent bundle

If f is a function on B_m, we write ${}^{vv}f$ for the function on the semi-tangent bundle $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^{v}f = f \circ \pi_1$, so that

$$ {}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. $$

Consequently,

$$ {}^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha) \tag{5} $$

is provided by the ${}^{vv}f$−vertical lift of the function $f \in \Im_0^0(B_m)$ to $t(B_m)$.

Consequently, we obtain from (5), the formula

$$ {}^{vv}(fg) = {}^{vv}f\,{}^{vv}g \tag{6} $$

for any $f, g \in \Im_0^0(B_m)$.

It should be observed that along every fiber of $\pi : t(B_m) \to B_m$, the value ${}^{vv}f$ stays constant. If $f = f(x^a, x^\alpha)$ is a function in M_n, then we write ${}^{cc}f$ for the function in $t(B_m)$ defined by

$$ {}^{cc}f = \imath(df) = x^{\overline{\beta}}\partial_\beta f = y^\beta \partial_\beta f \tag{7} $$

and name the complete lift of the function f [21].

${}^{HH}f = {}^{cc}f - \nabla_\gamma f$ determines the ${}^{HH}f$−horizontal lift of the function f to $t(B_m)$, where

$$ \nabla_\gamma f = \gamma \nabla f. $$

Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. From (3), on putting

$$ {}^{vv}X : \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{8} $$

we easily see that ${}^{vv}X' = \bar{A}({}^{vv}X)$. The vector field ${}^{vv}X$ is called the vertical lift of X to semi-tangent bundle [27].

Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$$^{vv}\omega : (0, \omega_\alpha, 0)\,, \tag{9}$$

from (3), we easily verify that $^{vv}\omega = \bar{A}^{vv}\omega'$. The vector field $^{vv}\omega$ is called the vertical lift of ω to $t(B_m)$ [27].

The complete lift $^{cc}\omega \in \Im_1^0(t(B_m))$ of $\omega \in \Im_1^0(B_m)$ with the components ω_α in B_m has the following components

$$^{cc}\omega : (0, y^\varepsilon \partial_\varepsilon \omega_\alpha, \omega_\alpha) \tag{10}$$

relative to the induced coordinates in the semi-tangent bundle [27]. Let ω be a covector field on B_m with an affine connection ∇.

Then the components of the $^{HH}\omega-$horizontal lift of ω have the form

$$^{HH}\omega = {}^{cc}\omega - \nabla_\gamma \omega$$

in $t(B_m)$, where $\nabla_\gamma \omega = \gamma \nabla \omega$. The horizontal lift $^{HH}\omega \in \Im_1^0(t(B_m))$ of ω has the following components

$$^{HH}\omega : (0, \Gamma_\alpha^\varepsilon \omega_\varepsilon, \omega_\alpha)$$

relative to the induced coordinates in $t(B_m)$.

Now, consider that there is given a $(p,q)-$tensor field S whose local expression is

$$S = S^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

in the base manifold B_m with $\nabla-$affine connection and a $\nabla_\gamma S-$tensor field defined by

$$\nabla_\gamma S = y^\varepsilon \nabla_\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ in $\pi^{-1}(U)$ in the semi-tangent bundle. Additionally, we define a $\nabla_X S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla_X S = \left(X^\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\varepsilon \beta_1 \ldots \beta_q} \right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

and a $\gamma S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla S = \left(y^\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\varepsilon \beta_1 \ldots \beta_q} \right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$, U being an arbitrary coordinate neighborhood in B_m. Next, we obtain

$$\nabla_X S = {}^{vv}(S_X)$$

for any $X \in \Im_0^1(B_m)$ and $S \in \Im_s^0(B_m)$ or $S \in \Im_s^1(B_m)$, where $S_X \in \Im_{s-1}^0(B_m)$ or $\Im_{s-1}^1(B_m)$.

The $^{HH}S-$horizontal lift of $(p,q)-$tensor field S in the base manifold B_m to $t(B_m)$ has the following equation:

$$^{HH}S = {}^{cc}S - \nabla_\gamma S.$$

Assuming $P, Q \in t(B_m)$, we get,

$$\begin{aligned} \nabla_\gamma (P \otimes Q) &= {}^{vv}P \otimes \left(\nabla_\gamma Q\right) + \left(\nabla_\gamma P\right) \otimes {}^{vv}Q, \\ ^{HH}(P \otimes Q) &= {}^{HH}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{HH}Q. \end{aligned}$$

Assume $\widetilde{X} \in \Im_0^1(M_n)$ is a projectable $(1,0)-$tensor field with projection $X = X^\alpha(x^\alpha)\partial_\alpha$, i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$.

Now, take into account $\widetilde{X} \in \Im_0^1(M_n)$, in that case complete lift ${}^{cc}\widetilde{X}$ has components of the form [21]:

$$ {}^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{11} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on the semi-tangent bundle $t(B_m)$.

For an arbitrary affinor field $F \in \Im_1^1(B_m)$, if (3) is taken into consideration, we may demonstrate that $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a $(1,0)-$tensor field defined by [16]:

$$ \gamma F : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{12} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

For each projectable $(1,0)-$tensor field $\widetilde{X} \in \Im_0^1(M_n)$ [28], we well-know that the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ is given by ${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X})$ (see [16]). In the above situation, a differentiable manifold B_m has a projectable symmetric linear connection denoted by ∇. We recall that $\gamma(\nabla\widetilde{X})-$ vector field has components [16]:

$$ \gamma(\nabla\widetilde{X}) : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$ (\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_{\beta\alpha}^\varepsilon. $$

Consequently, for the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$, we obtain the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ with the following components [16]:

$$ {}^{HH}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma_\beta^\alpha X^\beta \end{pmatrix} \tag{13} $$

where

$$ \Gamma_\beta^\alpha = y^\varepsilon \Gamma_{\varepsilon\beta}^\alpha. \tag{14} $$

Let $\left\{U, x^\beta\right\}$ be coordinate neighborhood of B_m. Then if we put in $\pi^{-1}(U)$

$$ \widehat{X}_{(a)} = {}^{HH}\left(\frac{\partial}{\partial x^a}\right), \quad \widehat{X}_{(\alpha)} = {}^{HH}\left(\frac{\partial}{\partial x^\alpha}\right), \quad \widehat{X}_{(\overline{\alpha})} = {}^{vv}\left(\frac{\partial}{\partial x^\alpha}\right), \tag{15} $$

we have in $\pi^{-1}(U)$ $n+m-$frame $\left\{\widehat{X}_{(A)}\right\} = \left\{\widehat{X}_{(a)}, \widehat{X}_{(\alpha)}, \widehat{X}_{(\overline{\alpha})}\right\}$, which is called the frame adapted to the non-linear connection. Thus the $(1,0)-$tensor field $\overline{\overline{X}}$, the horizontal lift ${}^{HH}\widetilde{X}$ and the vertical lift ${}^{vv}X$ of an element $\widetilde{X}$ of $\Im_0^1(M_n)$ have respectively components of the form

$$ \overline{\overline{X}} : \begin{pmatrix} X^a \\ 0 \\ 0 \end{pmatrix}, \quad {}^{HH}\widetilde{X} : \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, \quad {}^{vv}X : \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{16} $$

with regard to the adapted frame $\left\{\widehat{X}_{(B)}\right\} = \left\{\widehat{X}_{(b)}, \widehat{X}_{(\beta)}, \widehat{X}_{(\overline{\beta})}\right\}$ in each $\pi^{-1}(U)$, where X^α are local components of X in U. There for the complete lift ${}^{cc}\widetilde{X}$ of $\widetilde{X}$ has the components

$$ {}^{cc}\widetilde{X} : \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \widehat{\nabla}_\varepsilon X^\alpha \end{pmatrix} \tag{17} $$

with regard to the adapted frame $\left\{\widehat{X}_{(B)}\right\}$, where we have put

$$ \widehat{\nabla}_\varepsilon X^\alpha = \partial_\varepsilon X^\alpha + \Gamma^\alpha_{\beta\varepsilon} X^\beta, \quad \Gamma^\alpha_{\beta\varepsilon} = \frac{\partial}{\partial y^\beta}\Gamma^\alpha_\varepsilon. \tag{18} $$

The coframe $\left\{\widehat{\theta}^{(A)}\right\} = \left\{\widehat{\theta}^{(a)}, \widehat{\theta}^{(\alpha)}, \widehat{\theta}^{(\overline{\alpha})}\right\}$ is dual to the adapted frame $\left\{\widehat{X}_{(B)}\right\}$ in $\pi^{-1}(U)$, where $\widehat{\theta}^a = dx^a$, $\widehat{\theta}^\alpha = dx^\alpha$ and $\widehat{\theta}^{\overline{\alpha}} = \Gamma^\alpha_\varepsilon dx^\varepsilon + dy^\alpha$ with regard to the induced coordinates.

Vertical lifts are given by the following relations:

$$ {}^{vv}(P \otimes Q) = {}^{vv}P \otimes {}^{vv}Q, \quad {}^{vv}(P + R) = {}^{vv}P + {}^{vv}R, \tag{19} $$

to an algebraic isomorphism (unique) of the $\Im(B_m)$-tensor algebra into the $\Im(t(B_m))$-tensor algebra with regard to constant coefficients.

Where P, Q and R being arbitrary elements of $t(B_m)$. For an arbitrary affinor field $F \in \Im^1_1(B_m)$, if (3) is taken into consideration, we may demonstrate that ${}^{vv}F^I_J = A^I_{I'} A^{J'}_J ({}^{vv}F^{I'}_{J'})$, where ${}^{vv}F$ is a $(1,1)-$tensor field defined by [27]:

$$ {}^{vv}F : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & F^\alpha_\beta & 0 \end{pmatrix} \tag{20} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. The $(1,1)-$tensor field (20) is called the vertical lift of affinor field F to semi-tangent bundle $t(B_m)$ [27].

Complete lifts are given by the following relations:

$$ {}^{cc}(P + R) = {}^{cc}P + {}^{cc}R, \quad {}^{cc}(P \otimes Q) = {}^{cc}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{cc}Q, \tag{21} $$

to an algebraic isomorphism (unique) of the $\Im(B_m)$-tensor algebra into the $\Im(t(B_m))$-tensor algebra with regard to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$.

For an arbitrary projectable affinor field $\widetilde{F} \in \Im^1_1(M_n)$ [28] with projection $F = F^\alpha_\beta(x^\alpha)\,\partial_\alpha \otimes dx^\beta$ i.e. $\widetilde{F}$ has components

$$ \widetilde{F} : \begin{pmatrix} \widetilde{F}^a_b(x^a, x^\alpha) & \widetilde{F}^a_\beta(x^a, x^\alpha) \\ 0 & \widetilde{F}^\alpha_\beta(x^\alpha) \end{pmatrix} $$

relative to the coordinates (x^a, x^α). If (3) is taken into consideration, we may demonstrate that ${}^{cc}\widetilde{F}^I_J = A^I_{I'} A^{J'}_J ({}^{cc}\widetilde{F}^{I'}_{J'})$, where ${}^{cc}\widetilde{F}$ is a $(1,1)-$tensor field defined by [27]:

$$ {}^{cc}\widetilde{F} : \begin{pmatrix} \widetilde{F}^a_b & \widetilde{F}^a_\beta & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F^\alpha_\beta & F^\alpha_\beta \end{pmatrix}, \tag{22} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. The $(1,1)-$tensor field (22) is called the complete lift of affinor field F to semi-tangent bundle $t(B_m)$ [27].

We will now give below some important equations that we will use.

Lemma 2.1. *Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and $(1,1)$ −tensor fields on M_n with projections X, Y and F on the base manifold B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$ and $I = id_{M_n}$, then [27], [28]:*

$(i)\ {}^{vv}I^{cc}\widetilde{X} = {}^{vv}X,$	$(xi)\ \left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right] = {}^{cc}\left[\widetilde{X,Y}\right],$
$(ii)\ {}^{cc}\widetilde{X}^{vv}f = {}^{vv}(Xf),$	$(xii)\ {}^{cc}\widetilde{F}^{vv}X = {}^{vv}(FX),$
$(iii)\ {}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{vv}(\omega(X)),$	$(xiii)\ {}^{cc}\widetilde{X}^{cc}f = {}^{cc}(Xf),$
$(iv)\ {}^{vv}F^{cc}\widetilde{X} = {}^{vv}(FX),$	$(xiv)\ {}^{cc}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}(\omega X),$
$(v)\ {}^{vv}X^{cc}f = {}^{vv}(Xf),$	$(xv)\ {}^{cc}\left(\widetilde{FX}\right) = {}^{cc}\widetilde{F}^{cc}\widetilde{X},$
$(vi)\ {}^{cc}\left(\widetilde{fX}\right) = {}^{cc}f^{vv}X + {}^{vv}f^{cc}\widetilde{X},$	$(xvi)\ {}^{vv}(fX) = {}^{vv}f^{vv}X,$
$(vii)\ {}^{vv}I^{vv}X = 0,$	$(xvii)\ {}^{vv}\omega^{vv}X = 0,$
$(viii)\ \left[{}^{vv}X, {}^{cc}\widetilde{Y}\right] = {}^{vv}[X,Y],$	$(xviii)\ {}^{vv}(f\omega) = {}^{vv}f^{vv}\omega,$
$(ix)\ {}^{cc}I = \widetilde{I},$	$(xix)\ {}^{vv}F^{vv}X = 0,$
$(x)\ {}^{cc}\omega({}^{vv}X) = {}^{vv}(\omega(X)),$	$(xx)\ {}^{vv}X^{vv}f = 0.$

3. Main results

3.1. Almost product structure

Definition 3.1. An almost product structure on a smooth manifold B_m is a tensor field $\widehat{F}$ of type (1,1) on B_m, such that: $\widehat{F}^2 = I$ ($\widehat{F} \neq I$), where I is the identity tensor field of type (1,1) on B_m. The pair $\left(B_m, \widehat{F}\right)$ is called an almost product manifold.

We shall define a tensor field $\widehat{F} \in \Im_1^1(t(B_m))$ by

$$\widehat{F} : \left(\widehat{F}_B^A\right) = \begin{pmatrix} \delta_b^a & 0 & 0 \\ 0 & -\delta_\beta^\alpha & 0 \\ 0 & 0 & \delta_{\overline{\beta}}^{\overline{\alpha}} \end{pmatrix} \tag{23}$$

or $\widehat{F} = \widehat{X}_{(b)} \otimes \widehat{\theta}^{(a)} - \widehat{X}_{(\beta)} \otimes \widehat{\theta}^{(\alpha)} + \widehat{X}_{(\overline{\beta})} \otimes \widehat{\theta}^{(\overline{\alpha})}$ with regard to the adapted frame $\left\{\widehat{X}_{(B)}\right\} = \left\{\widehat{X}_{(b)}, \widehat{X}_{(\beta)}, \widehat{X}_{(\overline{\beta})}\right\}$ in each $\pi^{-1}(U)$. The matrix $\widehat{F}$ in (23) has the inverse since it is not singular. We have this inverse represented by

$$\left(\widehat{F}\right)^{-1} : \left(\widehat{F}_C^B\right)^{-1} = \begin{pmatrix} \delta_c^b & 0 & 0 \\ 0 & -\delta_\theta^\beta & 0 \\ 0 & 0 & \delta_{\overline{\theta}}^{\overline{\beta}} \end{pmatrix}, \tag{24}$$

where $\widehat{F}\left(\widehat{F}\right)^{-1} = (\widehat{F}_B^A)\left(\widehat{F}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = (a, \alpha, \overline{\alpha})$, $B = \left(b, \beta, \overline{\beta}\right)$, $C = \left(c, \theta, \overline{\theta}\right)$.

Theorem 3.2. *Given the above Definition 3.1, we say that $\widehat{F}$ gives an almost product structure and we call the manifold B_m an almost product manifold on the semi-tangent bundle $t(B_m)$ (see, for example [3], [11]).*

Theorem 3.3. *Suppose that $\widehat{F}$ has a condition that $tr\widehat{F} = 0$ is an almost product structure. As such, $\widehat{F}$ is an almost paracomplex structure on the semi-tangent bundle $t(B_m)$.*

Let B_m denote a smooth manifold with a given affine connection ∇. In each coordinate neighborhood $\{U, x^\alpha\}$ of B_m, we put

$$X_{(a)} = \frac{\partial}{\partial x^a}, \quad X_{(\alpha)} = \frac{\partial}{\partial x^\alpha}.$$

Then $n+m$ local vector fields ${}^{HH}\widetilde{X}_{(a)}$, ${}^{HH}\widetilde{X}_{(\alpha)}$ and ${}^{vv}X_{(\alpha)}$ have respectively components of the form

$$ {}^{HH}\widetilde{X}_{(a)}:\begin{pmatrix}\delta^a_b\\0\\0\end{pmatrix},\quad {}^{HH}\widetilde{X}_{(\alpha)}:\begin{pmatrix}\delta^a_\beta\\\delta^\alpha_\beta\\-\Gamma^\alpha_\beta\end{pmatrix},\quad {}^{vv}X_{(\alpha)}:\begin{pmatrix}0\\0\\\delta^\alpha_\beta\end{pmatrix}, \tag{25}$$

with regard to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $\pi^{-1}(U)$. We call the set

$$\left\{{}^{HH}\widetilde{X}_{(a)},\ {}^{HH}\widetilde{X}_{(\alpha)},\ {}^{vv}X_{(\alpha)}\right\}$$

the frame adapted to the affine connection ∇ in $\pi^{-1}(U)$. When we set

$$\widehat{e_{(a)}} = {}^{HH}\widetilde{X}_{(a)},\quad \widehat{e_{(\alpha)}} = {}^{HH}\widetilde{X}_{(\alpha)},\quad \widehat{e_{(\overline{\alpha})}} = {}^{vv}X_{(\alpha)}, \tag{26}$$

we can write the adapted frame as

$$\left\{\widehat{e_{(B)}}\right\} = \left\{\widehat{e_{(a)}}, \widehat{e_{(\alpha)}}, \widehat{e_{(\overline{\alpha})}}\right\}. \tag{27}$$

The adapted frame $\left\{\widehat{e_{(B)}}\right\} = \left\{\widehat{e_{(a)}}, \widehat{e_{(\alpha)}}, \widehat{e_{(\overline{\alpha})}}\right\}$ is expressed as

$$\widehat{A} : \left(\widehat{A}^A_B\right) = \begin{pmatrix}\delta^a_b & \delta^a_\beta & 0\\ 0 & \delta^\alpha_\beta & 0\\ 0 & -\Gamma^\alpha_\beta & \delta^\alpha_\beta\end{pmatrix}, \tag{28}$$

where $\Gamma^\beta_\alpha = y^\varepsilon\Gamma^\beta_{\varepsilon\alpha}$ then the matrix $\widehat{A}$ in (28) has the inverse since it is not singular. Using $\left(\widehat{A}\right)^{-1}$ to denote this inverse, we obtain

$$\left(\widehat{A}\right)^{-1} : \left(\widehat{A}^B_C\right)^{-1} = \begin{pmatrix}\delta^b_c & -\delta^b_\theta & 0\\ 0 & \delta^\beta_\theta & 0\\ 0 & \Gamma^\beta_\theta & \delta^\beta_\theta\end{pmatrix}, \tag{29}$$

where $\widehat{A}\left(\widehat{A}\right)^{-1} = (\widehat{A}^A_B)\left(\widehat{A}^B_C\right)^{-1} = \delta^A_C = \widetilde{I}$, where $A = (a, \alpha, \overline{\alpha})$, $B = \left(b, \beta, \overline{\beta}\right)$, $C = \left(c, \theta, \overline{\theta}\right)$.

Proof. In fact, from (28) and (29), we easily see that

$$\begin{aligned}\widehat{A}\left(\widehat{A}\right)^{-1} &= \begin{pmatrix}\delta^a_b & \delta^a_\beta & 0\\ 0 & \delta^\alpha_\beta & 0\\ 0 & -\Gamma^\alpha_\beta & \delta^\alpha_\beta\end{pmatrix}\begin{pmatrix}\delta^b_c & -\delta^b_\theta & 0\\ 0 & \delta^\beta_\theta & 0\\ 0 & \Gamma^\beta_\theta & \delta^\beta_\theta\end{pmatrix}\\ &= \begin{pmatrix}\delta^a_c & -\delta^a_\theta+\delta^a_\theta & 0\\ 0 & \delta^\alpha_\theta & 0\\ 0 & \Gamma^\alpha_\theta-\Gamma^\alpha_\theta & \delta^\alpha_\theta\end{pmatrix} = \begin{pmatrix}\delta^a_c & 0 & 0\\ 0 & \delta^\alpha_\theta & 0\\ 0 & 0 & \delta^\alpha_\theta\end{pmatrix}\\ &= \delta^A_C = \widehat{I}.\end{aligned}$$

□

Proposition 3.4. *Let $\widetilde{S}$ and $\widetilde{T}$ be two tensor fields of type (r,s) in $t(B_m)$ such that*

$$\widetilde{S}(\widetilde{X}_s, ..., \widetilde{X}_1) = \widetilde{T}(\widetilde{X}_s, ..., \widetilde{X}_1)$$

for all vector fields $\widetilde{X}_t$ $(t = 1, 2, ..., s)$ which are of the form ${}^{HH}\widetilde{X}_{(a)}$, ${}^{vv}X_{(\alpha)}$ or ${}^{HH}\widetilde{X}_{(\alpha)}$, where $X \in \Im^1_0(M_n)$. Then $\widetilde{S} = \widetilde{T}$.

Proof. The local vector fields ${}^{HH}\widetilde{X}_{(a)}$, ${}^{vv}X_{(\alpha)}$ and ${}^{HH}\widetilde{X}_{(\alpha)}$ span the module of vector fields in $\pi^{-1}(U)$. Therefore, each tensor field is determined in $\pi^{-1}(U)$ by its action of ${}^{HH}\widetilde{X}_{(a)}$, ${}^{vv}X_{(\alpha)}$ and ${}^{HH}\widetilde{X}_{(\alpha)}$. Thus, we can conclude our proposition. □

Let B_m be an $m-$dimensional differentiable manifold ($m = 2k+1,\ k \geq 0$) endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im^1_1(M_n)$ [28] with projection $\varphi = \varphi^\alpha_\beta(x^\alpha)\,\partial_\alpha \otimes dx^\beta$ i.e., and let $\widetilde{\xi} \in \Im^1_0(M_n)$ be a projectable $(1,0)-$tensor field with projection $\xi = \xi^\alpha(x^\alpha)\,\partial_\alpha$ i.e. $\widetilde{\xi} = \widetilde{\xi^a}(x^\alpha, x^\alpha)\partial_a + \xi^\alpha(x^\alpha)\,\partial_\alpha$ [28], and let η be a 1-form, and let $I = id_{M_n}$ be an idendity and let them also satisfy

$$\widetilde{\varphi}^2 = -I + \eta \otimes \widetilde{\xi}, \quad \widetilde{\varphi}\left(\widetilde{\xi}\right) = 0, \quad \eta \circ \widetilde{\varphi} = 0, \quad \eta\left(\widetilde{\xi}\right) = 1. \tag{30}$$

Then $\left(\widetilde{\varphi}, \widetilde{\xi}, \eta\right)$ define almost contact structure on B_m (see, for example [12], [19], [20], [25], [31]), where $\widetilde{\varphi}^2 = -I + \eta \otimes \widetilde{\xi}$ means

$$\widetilde{\varphi}\left(\widetilde{\varphi}\widetilde{X}\right) = \widetilde{\varphi}^2\widetilde{X} = -\widetilde{X} + \eta\left(\widetilde{X}\right)\cdot\widetilde{\xi}$$

for all $\widetilde{X} \in \Im^1_0(M_n)$.

An almost contact manifold is a differentiable odd-dimensional manifold with an almost contact structure (see, for example [11]).

It is to be noted that the conditions (30) imply the following

$$rank\widetilde{\varphi} = 2k \tag{31}$$

on B_m everywhere. In fact, $\widetilde{\varphi}_p$ and η_p being the values of $\widetilde{\varphi}$ and η at $p \in B_m$, the linear map $\widetilde{\varphi}_p$ leaves invariant the subspace $V_p = \eta_p^{-1}(0)$ of the tangent space of B_m at p. Moreover, the restriction $\widetilde{\varphi}'_p$ of $\widetilde{\varphi}_p$ to V_p satisfies $\widetilde{\varphi}'_p \circ \widetilde{\varphi}'_p = -1$. Hence, $rank\widetilde{\varphi}'_p = 2k$. On the other hand, the equations in (30) imply that $rank\widetilde{\varphi}_p \leq 2k$, whence $rank\widetilde{\varphi}_p = 2k$.

3.2. *Almost complex structures in the semi-tangent bundle*

Let now there be given a non-linear connection Γ in B_m. We shall define a projectable tensor field $\widehat{F}$ of type $(1,1)$ in M_n by

$$\widehat{F}^{HH}\widetilde{X} = -\overline{\overline{X}}, \quad \widehat{F}\overline{\overline{X}} = -{}^{HH}\widetilde{X}, \quad \widehat{F}^{vv}X = -{}^{HH}\widetilde{X} \tag{32}$$

for any projectable vector field $\widetilde{X} \in \Im^1_0(M_n)$. Then $\widehat{F}$ has components of the form $(n, m \in Z^+; n \geq m)$

$$\underbrace{\left(\begin{array}{cccccc|cccc}
0 & \cdots & \cdots & \cdots & 0 & \delta^a_\beta & 0 & \cdots & \cdots & 0 \\
\vdots & \iddots & \iddots & \iddots & \iddots & 0 & \vdots & \iddots & \iddots & \vdots \\
\vdots & \iddots & \iddots & \delta^\alpha_\beta & \iddots & \vdots & \vdots & \iddots & \iddots & \vdots \\
\vdots & \iddots & -\delta^\alpha_\beta & \iddots & \iddots & \vdots & \vdots & \iddots & \iddots & \vdots \\
0 & \delta^\alpha_\beta & \iddots & \iddots & \iddots & \vdots & \vdots & \iddots & \iddots & \vdots \\
-\delta^\alpha_b & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\hline
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & -\delta^\alpha_\beta \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots & \iddots & \iddots & 0 \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & 0 & -\delta^\alpha_\beta & \iddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \delta^\alpha_\beta & 0 & \cdots & 0
\end{array}\right)}_{n-m=2f \qquad\qquad 2m}
\begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\end{array}}\right\} n-m=2f \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} 2m \end{array} \tag{33}$$

with regard to the adapted frame $\left\{\widehat{\theta}^{(A)}\right\} = \left\{\widehat{\theta}^{(a)}, \widehat{\theta}^{(\alpha)}, \widehat{\theta}^{(\overline{\alpha})}\right\}$ in each $\pi^{-1}(U)$. Thus, we obtain $\widehat{F}^2 = -I$, which implies

Theorem 3.5. *If there is given a non-linear connection in B_m, then there exists an almost complex structure $\widehat{F}$ defined by (32) in $t(B_m)$.*

N being the Nijenhuis tensor of F, we define a tensor field $S \in \Im^1_2(B_m)$ by

$$S(X,Y) = N(X,Y) + (X(\omega(Y)) - (Y(\omega(X)) - \omega([X,Y])\, U$$

for any $X, Y \in \Im^1_0(B_m)$,

The almost contact structure $(\widetilde{F}, \widetilde{U}, \omega)$ is said to be normal if and only if $S = 0$ (see, for example [23]).

Taking account of (21) and Lemma 2.1, we find from (32)

$$({}^{cc}\widetilde{F})^2 = -I + ({}^{vv}U \otimes {}^{cc}\omega + {}^{cc}\widetilde{U} \otimes {}^{vv}\omega),$$

and

$$\begin{array}{llll} (i) & {}^{cc}\widetilde{F}\,{}^{vv}U = 0, & (ii) & {}^{cc}\widetilde{F}\,{}^{cc}\widetilde{U} = 0, \\ (iii) & {}^{vv}\omega \circ {}^{cc}\widetilde{F} = 0, & (iv) & {}^{cc}\omega \circ {}^{cc}\widetilde{F} = 0, \\ (v) & {}^{vv}\omega({}^{vv}U) = 0, & (vi) & {}^{vv}\omega({}^{cc}\widetilde{U}) = 1, \\ (vii) & {}^{cc}\omega({}^{vv}U) = 1, & (viii) & {}^{cc}\omega({}^{cc}\widetilde{U}) = 0. \end{array} \tag{34}$$

Now, if we define an element

$$\tilde{J} = {}^{cc}\widetilde{F} + ({}^{vv}U \otimes {}^{vv}\omega - {}^{cc}\widetilde{U} \otimes {}^{cc}\omega), \tag{35}$$

then we obtain by (34) and (35) that $\tilde{J}^2$ is equal to $-I$. As a result, $\tilde{J}$ is an almost complex structure in the semi-tangent bundle. The almost complex structure $\tilde{J}$ has components

$$\tilde{J} : \begin{pmatrix} \widetilde{F}^b_a & \widetilde{F}^b_\alpha - (\partial\omega_\alpha)\,\widetilde{U}^b & -\omega_\alpha \widetilde{U}^b \\ 0 & F^\beta_\alpha - (\partial\omega_\alpha)\,U^\beta & -\omega_\alpha U^\beta \\ 0 & \partial F^\beta_\alpha - (\partial\omega_\alpha)\left(\partial U^\beta\right) + \omega_\alpha U^\beta & F^\beta_\alpha - \omega_\alpha \partial U^\beta \end{pmatrix}$$

with regard to the local coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t(B_m)$, where F^β_α, ω_α and U^β are respectively the local components of $\widetilde{F} \in \Im^1_1(M_n)$, $\omega \in \Im^0_1(B_m)$ and $\widetilde{U} \in \Im^1_0(M_n)$ (see, for example [30], [31]). Thus we have:

Theorem 3.6. *There exists in $t(B_m)$ an almost complex structure $\tilde{J}$ defined by (35), if there is given in B_m an almost contact structure $(\widetilde{F}, \widetilde{U}, \omega)$.*

We get from (35)

$$\begin{array}{ll} (i) & \tilde{J}\,{}^{vv}X = {}^{vv}(FX) - {}^{vv}(\omega(X))\,{}^{cc}\widetilde{U}, \\ (ii) & \tilde{J}\,{}^{cc}\widetilde{X} = {}^{cc}\left(\widetilde{FX}\right) + {}^{vv}(\omega(X))\,{}^{vv}U - {}^{cc}(\omega(X))\,{}^{cc}\widetilde{U}, \end{array} \tag{36}$$

for any $\widetilde{X} \in \Im^1_0(M_n)$.

In particular, we have

$$\begin{array}{ll} (i)\ \tilde{J}\,{}^{vv}X = {}^{vv}(FX), & (ii)\ \tilde{J}\,{}^{cc}\widetilde{X} = {}^{cc}(\widetilde{FX}), \\ (iii)\ \tilde{J}\,{}^{vv}U = -{}^{cc}\widetilde{U}, & (iv)\ \tilde{J}\,{}^{cc}\widetilde{U} = {}^{cc}\widetilde{U}, \end{array}$$

X being an arbitrary projectable vector field in M_n such that $\omega(X) = 0$.

The required result now follows from (36).

Theorem 3.7. *The almost complex structure $\tilde{J}$ in $t(B_m)$ defined by (36) is complex analytic if and only if the almost contact structure $(\widetilde{F}, \widetilde{U}, \omega)$ given in B_m is normal.*

Let B_m be an $m-$dimensional differentiable manifold $(m = 2k+1,\ k \geq 0)$ endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im_1^1(M_n)$ [28] with projection $\varphi = \varphi_\beta^\alpha (x^\alpha)\, \partial_\alpha \otimes dx^\beta$ i.e., and let $\widetilde{\xi} \in \Im_0^1(M_n)$ be a projectable $(1,0)-$tensor field with projection $\xi = \xi^\alpha (x^\alpha)\, \partial_\alpha$ i.e. $\widetilde{\xi} = \widetilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha (x^\alpha)\, \partial_\alpha$ [28], and let η be a 1-form , and let $I = id_{M_n}$ be an idendity and let them also satisfy

$$\widetilde{\varphi}^2 = I - \eta \otimes \widetilde{\xi}, \quad \widetilde{\varphi}\left(\widetilde{\xi}\right) = 0, \quad \eta \circ \widetilde{\varphi} = 0, \quad \eta\left(\widetilde{\xi}\right) = 1. \tag{37}$$

Then $\left(\widetilde{\varphi}, \widetilde{\xi}, \eta\right)$ define almost paracontact structure on B_m (see, for example [12], [20], [25], [31]), where $\widetilde{\varphi}^2 = I - \eta \otimes \widetilde{\xi}$ means

$$\widetilde{\varphi}\left(\widetilde{\varphi}\widetilde{X}\right) = \widetilde{\varphi}^2\widetilde{X} = \widetilde{X} - \eta\left(\widetilde{X}\right) \cdot \widetilde{\xi}$$

for all $\widetilde{X} \in \Im_0^1(M_n)$.

A differentiable manifold of odd dimension with an almost paracontact structure is called an almost paracontact manifold (see, for example [15]).

It is to be noted that the conditions (37) imply the following:

$$rank\widetilde{\varphi} = m - 1 \tag{38}$$

on B_m everywhere.

3.3. Complete lifts of Lorentzian almost paracontact structure

Let $\overline{B}_m$ be an $m-$dimensional differentiable manifold $(m = 2k+1,\ k \geq 0)$ endowed with a projectable $(1,1)-$tensor field $\widetilde{F} \in \Im_1^1(M_n)$ [28] with projection $F = F_\beta^\alpha (x^\alpha)\, \partial_\alpha \otimes dx^\beta$ i.e., and let $\widetilde{U} \in \Im_0^1(M_n)$ be a projectable $(1,0)-$tensor field with projection $U = U^\alpha (x^\alpha)\, \partial_\alpha$ i.e. $\widetilde{U} = \widetilde{U}^a(x^\alpha, x^\alpha)\partial_a + U^\alpha (x^\alpha)\, \partial_\alpha$ [28], and let ω be a 1-form and let them also satisfy

$$\widetilde{F}^2 = I + \widetilde{U} \otimes \omega, \quad \omega \circ \widetilde{F} = 0, \quad \widetilde{F}\widetilde{U} = 0, \quad \omega(\widetilde{U}) = 1. \tag{39}$$

Then $(\widetilde{F}, \widetilde{U}, \omega)$ define Lorentzian almost paracontact structure on $\overline{B}_m$ (see, for example [10], [26]).

Theorem 3.8. *Let $\overline{B}$ be a differentiable manifold endowed with Lorentzian almost paracontact structure $(\widetilde{F}, \widetilde{U}, \omega)$. Prove that*

$$\tilde{J} = {}^{cc}\widetilde{F} + ({}^{vv}U \otimes {}^{vv}\omega - {}^{cc}\widetilde{U} \otimes {}^{cc}\omega)$$

is almost product structure on $t(\overline{B})$.

Proof. According to (39), we find

$$({}^{cc}\widetilde{F})^2 = I + ({}^{vv}U \otimes {}^{cc}\omega - {}^{cc}\widetilde{U} \otimes {}^{vv}\omega), \tag{40}$$

and

$$\begin{array}{lll} (i)\ {}^{cc}\widetilde{F}{}^{vv}U = 0, & (ii)\ {}^{cc}\widetilde{F}{}^{cc}\widetilde{U} = 0, & (iii)\ {}^{vv}\omega \circ {}^{cc}\widetilde{F} = 0, \\ (iv)\ {}^{cc}\omega \circ {}^{vv}F = 0, & (v)\ {}^{cc}\omega \circ {}^{cc}\widetilde{F} = 0, & (vi)\ {}^{vv}\omega({}^{vv}U) = 0, \\ (vii)\ {}^{vv}\omega({}^{cc}\widetilde{U}) = 1, & (viii)\ {}^{cc}\omega({}^{vv}U) = 1, & (ix)\ {}^{cc}\omega({}^{cc}\widetilde{U}) = 0. \end{array} \tag{41}$$

Let us define an element $\tilde{J}$ of $J(t(\overline{B}))$ by

$$\tilde{J} = {}^{cc}\widetilde{F} + ({}^{vv}U \otimes {}^{vv}\omega - {}^{cc}\widetilde{U} \otimes {}^{cc}\omega) \tag{42}$$

then we easily obtain by (40), (41) and (42),

$$\tilde{J}^2 = I$$

thus $\tilde{J}$ is an almost product structure in $t(\overline{B})$. □

In view of equation (42), we have

$$\begin{array}{ll} (i) & \tilde{J}^{vv}X = -{}^{vv}(FX) + {}^{vv}(\omega(X))^{cc}\widetilde{U}, \\ (ii) & \tilde{J}^{cc}\widetilde{X} = -{}^{cc}(\widetilde{FX}) - {}^{vv}(\omega(X))^{cc}\widetilde{U} - {}^{cc}(\omega(X))^{cc}\widetilde{U}. \end{array} \tag{43}$$

In particular, we have

$$\begin{array}{llll} (i) & \tilde{J}^{vv}X = -{}^{vv}(FX), & (ii) & \tilde{J}^{cc}\widetilde{X} = -{}^{cc}(\widetilde{FX}), \\ (iii) & \tilde{J}^{vv}U = {}^{cc}\widetilde{U}, & (iv) & \tilde{J}^{cc}\widetilde{U} = {}^{cc}\widetilde{U}. \end{array} \tag{44}$$

X being an arbitrary projectable vector field in $\overline{B}$ such that $\omega(X) = 0$.

Theorem 3.9. *Let the semi-tangent bundle $t(\overline{B})$ of the manifold $\overline{B}$ admits $\tilde{J}$ defined in (42), then for projectable vector fields X, Y such that $\omega(Y) = 0$, we obtain*

$$\begin{array}{rcl} (i)\ (L_{{}^{vv}X}\tilde{J})^{cc}\widetilde{Y} & = & -{}^{vv}(L_XF)Y) +{}^{vv} (L_X\omega)Y)^{cc}\widetilde{U} \\ (ii)\ (L_{{}^{vv}X}\tilde{J})^{cc}\widetilde{U} & = & -{}^{vv}((L_XF)U) +{}^{vv} (L_X\omega)U)^{cc}\widetilde{U} \\ (iii)\ (L_{{}^{vv}X}\tilde{J})^{vv}U & = & {}^{vv}(L_XU) \\ (iv)\ (L_{{}^{vv}X}\tilde{J})^{vv}Y & = & 0 \end{array}$$

By similar devices, we have also

$$\begin{array}{rcl} (i)\ (L_{{}^{cc}\widetilde{X}}\tilde{J})^{cc}\widetilde{Y} & = & -{}^{vv}((L_X\omega)Y)^{cc}\widetilde{U} + (L_X\omega))^{cc}\widetilde{Y}^{cc}\widetilde{U}, \\ (ii)\ (L_{{}^{cc}\widetilde{X}}\tilde{J})^{vv}Y & = & -{}^{vv}((L_XF)Y) +{}^{vv} (L_X\omega)Y)^{cc}\widetilde{U}, \\ (iii)\ (L_{{}^{cc}\widetilde{X}}\tilde{J})^{vv}U & = & {}^{cc}((\widetilde{L_XF})U) +{}^{cc} \left[\widetilde{X,U}\right] +{}^{vv} ((L_X\omega)U)^{cc}\widetilde{U}, \\ (iv)\ (L_{{}^{cc}\widetilde{X}}\tilde{J})^{cc}\widetilde{U} & = & {}^{cc}((\widetilde{L_XF})U) -{}^{vv} ((L_X\omega)U)^{cc}\widetilde{U} -{}^{cc} \left[\widetilde{X,U}\right] +{}^{cc} ((L_X\omega)U)^{cc}\widetilde{U}. \end{array}$$

Proof. Using (41), (43) and (44), the proofs can be easily done. □

3.4. Horizontal lifts of Lorentzian almost paracontact structure

Let now $(\widetilde{F}, \widetilde{U}, \omega)$ be Lorentzian almost paracontact structure in $\overline{B}$ with an affine connection ∇. Then by (39), we obtain

$$\begin{array}{rcl} (i)\ ({}^{HH}\widetilde{F})^2 & = & I +{}^{HH} (\widetilde{U \otimes \omega}) \\ (ii)\ ({}^{HH}\widetilde{F})^2 & = & {}^{HH}(I + \widetilde{U \otimes} \omega) \\ (iii)\ ({}^{HH}\widetilde{F})^2 & = & I +{}^{HH} \widetilde{U} \otimes^{vv} \omega +{}^{vv} U \otimes^{HH} \omega \end{array}$$

also,

$$\begin{array}{rcl} (i)\ {}^{HH}\widetilde{F}\left({}^{HH}\widetilde{U}\right) & = & 0, {}^{HH}\widetilde{F}({}^{vv}U) = 0 \\ (ii)\ \ {}^{HH}\omega({}^{HH}\widetilde{U}) & = & 0, {}^{HH}\omega({}^{vv}U) = 1, {}^{vv}\omega({}^{HH}\widetilde{U}) = 1 \\ (iii)\ \ {}^{HH}\omega \circ {}^{HH}\widetilde{F} & = & 0, {}^{vv}\omega \circ {}^{HH}\widetilde{F} = 0. \end{array}$$

Let us define a projectable tensor field $\tilde{J}^{*}$ of type (1,1) in $t(\overline{B})$ by

$$\tilde{J}^{*} = {}^{HH}\widetilde{F} + \left({}^{vv}U \otimes {}^{vv}\omega - {}^{HH}\widetilde{U} \otimes {}^{HH}\omega\right),$$

then it is easy to show that

$$\tilde{J}^{*2} = I$$

consequently, $\tilde{J}^{*}$ is an almost product structure in $t(\overline{B})$.

Moreover, a direct result shows that:

Theorem 3.10. *Let $(\widetilde{F}, \widetilde{U}, \omega)$ be Lorentzian almost paracontact structure in $\overline{B}$ with an affine connection ∇. Then $\tilde{J}^{*}$ is almost product structure in the semi-tangent bundle $t(\overline{B})$.*

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