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On the lattice of z°-ideals (resp., z-ideals) and its applications

Maryam Ahmadi[a], **Ali Taherifar**[a,*]

[a]*Department of Mathematics, Yasouj University, Yasouj, Iran*

Abstract. An ideal I of a commutative ring R is a z°-ideal (resp., z-ideal) if, for each $a \in I$, the intersection of all minimal prime ideals (resp., maximal ideals) containing a is contained in I. A ring R is termed a WSA-ring if, for any two ideals I, J of R, where $I \cap J = 0$, we have $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_\circ = R$. It is observed that for a reduced ring R, the lattice of z°-ideals of R ($Z^\circ Id(R)$) is a co-normal lattice if and only if R is a WSA-ring. This concept is then applied to characterize spaces X for which $C(X)$ is a WSA-ring. In this context, a space X is termed a WED-space if every two disjoint open sets can be separated by two disjoint Z-zero-sets (i.e., the interior of a zero-set). The class of WED-spaces contains the class of extremally disconnected spaces and the class of perfectly normal spaces. It has been proven that $C(X)$ is a WSA-ring if and only if X is a WED-space, and also if and only if $C^*(X)$ is a WSA-ring. Moreover, it has been demonstrated that the lattice of z-ideals of a commutative ring R ($ZId(R)$) is a co-normal lattice if and only if R is an SA-ring, and also if and only if the lattice of radical ideals of R ($RId(R)$) is a co-normal lattice.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity. Almost all our rings are reduced, which are the rings with no non-zero nilpotent elements. Let $ZId(R) = \{I : I \text{ is a } z\text{-ideal of } R\}$. Additionally, for an ideal K of R, we use K_z (resp., $K_\circ$) to denote the smallest z-ideal (resp., z°-ideal) containing K. The lattice $(ZId(R), \subseteq)$ equipped with the operations $I \vee J = (I+J)_z$ and $I \wedge J = I \cap J$ forms a fundamental structure. Similarly, the set $Z^\circ Id(R) = \{I : I \text{ is a } z^\circ\text{-ideal of } R\}$ partially ordered by inclusion, also forms a lattice under the operations $I \vee J = (I + J)_\circ$ and $I \wedge J = I \cap J$. The concept of z-ideal (resp., z°-ideal) was originally introduced by Khols [16] in the study of rings of continuous functions. After that, Mason in [18] and [19] generalized these concepts in any commutative ring. Martınez and Zenk in [17] started the study of the lattice of z-ideals. They proved that the lattice of z-ideals of the ring $C(X)$ is a frame. They actually proved that it is a coherently normal Yosida frame. Ighedo [14] extended the results of Martinez and Zenk to the lattices of z-ideals of the ring $\mathcal{R}L$ of continuous real-valued functions on a completely regular frame. This was further extended by Dube [10] to the lattices of z-ideals of an f-ring with bounded inversion. Recently, Ighedo and McGovern [13] investigated many properties of this lattice in any commutative ring. Actually, they characterize when the lattice $ZId(R)$ is a Yosida frame.

In the present paper, we recall in section 2 the necessary background, and we fix notation. Section 3 is devoted to the lattice of z°-ideals. Whenever R is a reduced ring, the lattice $Z^\circ Id(R)$ is the one that was

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* Corresponding author: Ali Taherifar

Email addresses: ahmadymryam66@gmail.com (Maryam Ahmadi), ataherifar@yu.ir, ataherifar54@gmail.com (Ali Taherifar)

presented by Dube [9] as the lattice $DId(R)$. We prove that for a z°-ideal J of a reduced ring R and a family $\{I_\alpha : \alpha \in S\}$ of ideals of R, $J \cap (\sum_{\alpha\in S} I_\alpha)_\circ = (\sum_{\alpha\in S}(J \cap I_\alpha))_\circ$ (Lemma 3.2), where for an ideal I of R, $I_\circ$ is the smallest z°-ideal containing it. This leads to the conclusion that $Z^\circ Id(R)$ forms a frame when R is a reduced ring. To study the lattice properties of $Z^\circ Id(R)$, it suffices to assume R is reduced, as the lattices $Z^\circ Id(R)$ and $Z^\circ Id(R/N(R))$ are isomorphic. We define a ring R a WSA-ring if for any two ideals I, J of R with $I \cap J = 0$, $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_\circ = R$. Additionally, we designate a completely regular space X as a WED-space if every pair of disjoint open sets can be separated by two disjoint Z-zero-sets (i.e., the interior of a zero-set). We prove that a space X is a WED-space if and only if βX is a WED-space (Theorem 3.7). Using this result, we establish that the ring $C(X)$ is a WSA-ring if and only if the space X is WED which is also equivalent to the $C^*(X)$ is a WSA-ring (Theorem 3.8). Furthermore, we demonstrate that a reduced ring R is WSA if and only if $Z^\circ Id(R)$ is a co-normal lattice (Proposition 3.9). For a reduced ring R with property A, we prove that R is a WSA-ring if and only if for each two ideals I, J of R, $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_\circ = \mathrm{Ann}(I \cap J)$ (Proposition 3.11). In Section 4, we extend the results to z-ideals, proving that for a z-ideal J of a ring R and a set $\{I_\alpha : \alpha \in S\}$ of ideals of R, $J \cap (\sum_{\alpha\in S} I_\alpha)_z = (\sum_{\alpha\in S}(J \cap I_\alpha))_z$ (Lemma 4.2). Using this result, we reaffirm that the lattice $ZId(R)$ forms a frame. Additionally, for a semiprimitive ring R, we demonstrate that the lattice $ZId(R)$ is a co-normal lattice if and only if R is an SA-ring, which is also equivalent to the lattice $RId(R)$ forming a co-normal lattice (Theorem 4.4).

2. Background and notation

2.1. Rings

Let S be a subset of a ring R. We write $\mathrm{Ann}(S)$ for the annihilator of S in R. The ideal generated by S in R is denoted by $< S >$. The *radical* of an ideal I of R is the ideal

$$\sqrt{I} = \{x \in R : x^n \in I \quad for \quad some \quad n \in \mathbb{N}\}.$$

Whenever $I = \sqrt{I}$, we say I is a *radical ideal*. It is well-known that

$$I \quad is \quad a \quad radical \quad ideal \Leftrightarrow a^2 \in I \quad implies \quad a \in I.$$

The *Jacobson radical* of a ring R is denoted by $J(R)$. It is well-know that $J(R)$ is the intersection of all maximal ideals of R. For each element a in a ring R, the intersection of all maximal ideals in R containing a is denoted by M_a, an ideal I of R is a z-ideal if $M_a \subseteq I$ for each $a \in I$, see [11, 7A]. Maximal ideals, minimal prime ideals (in reduced rings) and annihilator ideals (in semiprimitive rings in which the intersection of all maximal ideals is zero) and most of familiar ideals are z-ideals. Intersections of z-ideals are z-ideals. Hence the smallest z-ideal containing an ideal I of R always exists and it is denoted by I_z. We refer the reader to Mason [18] for more details and characterizations of ideal I_z in commutative rings and in $C(X)$, the ring of all real valued continuous functions on a completely regular Hausdorff space X.

The following lemma is well-known and is needed in the sequel.

Lemma 2.1. *The following statements hold.*

(1) *If P is minimal in the class of prime ideals containing a z-ideal I, then P is a z-ideal.*

(2) *If I, J are two ideals in R, then $(I \cap J)_z = I_z \cap J_z$.*

Proof. (1) See [18, Theorem 1.1] for Part (1).

(2) Trivially $(I \cap J)_z \subseteq I_z \cap J_z$. To see the reverse inclusion, let $a \in I_z \cap J_z$. Since $(I \cap J)_z$ is a z-ideal, so it is an intersection of minimal prime ideals over it, each of which is a z-ideal. Let P be a prime ideal contains $(I \cap J)_z$. Then $P \supseteq I$ or $P \supseteq J$. Thus $P \supseteq I_z$ or $P \supseteq J_z$. This shows that $a \in P$. So we are done. $\square$

An ideal I of R is called a z^o-ideal if for each $a \in I$, $P_a \subseteq I$, where P_a is the intersection of all minimal prime ideals of R containing a. Important z°-ideals in any ring are minimal prime ideals. An intersection of z°-ideals is a z°-ideal. Hence the nilradical of R (i.e, $N(R)$) which is the intersection of all minimal prime ideals of R, is a z°-ideal. The smallest z°-ideal containing a proper ideal I is denoted by $I_\circ$. It is well-known that whenever I is an ideal of $C(X)$, (see [4]),

$$I_\circ = \{f \in C(X) : \exists g \in I \quad with \quad \operatorname{int} Z(g) \subseteq \operatorname{int} Z(f)\}.$$

It is important to mention that for a proper ideal I of a ring R we may have $I_\circ = R$. For example, consider an ideal I of $C(X)$ containing some f such that $\operatorname{int} Z(f) = \emptyset$ (i.e., f is a non-zero-divisors). Then $I_\circ = C(X)$.

The following lemma also is needed in the sequel.

Lemma 2.2. *Let R be a reduced ring.*

(1) *If I is a z°-ideal in R, then every prime ideal, minimal over I is a prime z°-ideal.*

(2) *A proper ideal I of R is a z°-ideal if and only if it is an intersection of prime z°-ideals.*

(3) *If I, J are two ideals of R, then $(I \cap J)_\circ = I_\circ \cap J_\circ$.*

Proof. (1) See [4, Theorem 1.1.16].

(2) See [4, Corolarry 1.18]

(3) Always $(I \cap J)_\circ \subseteq I_\circ \cap J_\circ$. If $(I \cap J)_\circ = R$, then $I_\circ = R$ and $J_\circ = R$, hence the equality holds. Now let $(I \cap J)_\circ$ be a proper ideal of R. Then it is an intersection of prime z°-ideals containing it. Let $a \in I_\circ \cap J_\circ$ and P be a prime z°-ideal containing $(I \cap J)_\circ$. Then P containing I or P containing J. Thus $a \in P$. So we are done. □

In this paper, we use $\operatorname{Max}(R)$ (resp., $\operatorname{Min}(R)$) for the spaces of maximal ideals (resp., minimal prime ideals) of R with the *hull-kernel* topology.

2.2. Rings of continuous functions and topological concepts

In this paper, $C(X)$ ($C^*(X)$) is the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space X. In fact, for every topological space X there exists a completely regular Hausdorff space Y such that $C(X)$ and $C(Y)$ are isomorphic as two rings. So, whenever we speak about $C(X)$, X is a completely regular and Hasdorff space.

In studying relations between topological properties of a space X and algebraic properties of $C(X)$, it is natural to look at the subsets of X of the form $f^{-1}\{0\}$, for each $f \in C(X)$. The set $f^{-1}\{0\}$ is called the zero-set of f and denoted by $Z(f)$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X. Thus, Z is a mapping from the ring $C(X)$ onto the set of all zero-sets in X. A $\operatorname{coz} f$ is the set $X \setminus Z(f)$ which is called the cozero-set of f. The set of all zero-sets in X is denoted by $Z[X]$ and for each ideal I in $C(X)$, $Z[I]$ is the set of all zero-sets of the form $Z(f)$, where $f \in I$. The space βX is known as the *Stone-Čech compactification* of X. It is characterized as that compactification of X in which X is C^*-embedded as a dense subspace. The space υX is the *real-compactification* of X, and X is C-embedded in this space as a dense subspace. For a completely regular Hausdorff space X, we have $X \subseteq \upsilon X \subseteq \beta X$. For each $Z(f) \in Z[X]$, $Z^\beta = Z(f^\beta)$, where f^β is the unique continuous extension of f on βX.

2.3. Basic facts and definitions of lattices

Recall from [6], [8] and [22] that a lattice $< L, \wedge, \vee, 0, 1 >$ is called a normal lattice whenever it is a distributive lattice and for all $a, b \in L$ with $a \wedge b = 0$ there exist $x, y \in L$ such that $x \vee y = 1$ and $x \wedge a = y \wedge b = 0$. We prefer to call these classes of lattices co-normal lattices, since in the frame literature the adjective normal refers to the dual property. Trivially, every Boolean algebra is a co-normal lattice. To see more details about lattices the reader is referred to [21].

A frame is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b : a \in A\},$$

for any subset A of L and any $b \in L$.Our reference for frames and their homomorphisms is [20].

3. On the lattice $Z^{\circ}Id(R)$

The set $Z^{\circ}Id(R)$, partially ordered by inclusion forms a lattice with

$$I \wedge J = I \cap J \quad \text{and} \quad I \vee J = (I + J)_{\circ}, \quad \text{for} \quad I, J \in Z^{\circ}Id(R).$$

An ideal I of R is a d-ideal if $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$ and $a \in I$, then $b \in I$. Since every z°-ideal is a d-ideal, for an ideal I of a ring R, we have $I \subseteq I_d \subseteq I_{\circ}$, where I_d is the smallest d-ideal containing I. When R is a reduced ring, the class of d-ideals and the class of z°-ideals coincide, as shown in [21, Proposition 2.8]. Consequently, for each ideal I of a reduced ring R, $I_{\circ} = I_d$. Thus, when R is a reduced ring, the lattices $Z^{\circ}Id(R)$ and $DId(R)$ are the same.

In [9], Dube extensively investigated the properties of the lattice $DId(R)$ and demonstrated that it forms a frame. In this section, we aim to provide alternative characterizations of this lattice and utilize them to introduce novel classes of topological spaces.

Rings in which the sum of two z°-ideals is a z°-ideal are also important. We now turn to characterizing them. We recall that the set of all basic z°-ideals of R is $\{P_a : a \in R\}$.

Proposition 3.1. *The following statements are equivalent.*

(1) *The sum of two z°-ideals in R is a z°-ideal.*

(2) *For each two ideals I and J of R, $(I_{\circ} + J_{\circ})_{\circ} = I_{\circ} + J_{\circ}$.*

(3) *The lattice $Z^{\circ}Id(R)$ is a sublattice of the lattice of ideals of R.*

(4) *For each two families $\{P_a : a \in S\}$ and $\{P_b : b \in K\}$ of basic z°-ideals,*

$$\Big(\sum_{a\in S} P_a + \sum_{b\in K} P_b\Big)_{\circ} = \Big(\sum_{a\in S} P_a\Big)_{\circ} + \Big(\sum_{b\in K} P_b\Big)_{\circ}.$$

Proof. (1) ⇒ (2) Let I, J be two ideals of R. By hypothesis, $I_{\circ} + J_{\circ}$ is a z°-ideal of R and hence $(I_{\circ} + J_{\circ})_{\circ} = I_{\circ} + J_{\circ}$.

(2) ⇒ (3) Consider two z°-ideals I, J of R. Then by (2), $I + J = I_{\circ} + J_{\circ} = (I_{\circ} + J_{\circ})_{\circ}$. This shows $I + J$ is a z°-ideal. So we are done.

(3) ⇒ (4) By hypothesis, $(\sum_{a\in S} P_a)_{\circ} + (\sum_{b\in K} P_b)_{\circ} \in Z^{\circ}Id(R)$ and it contains $\sum_{a\in S} P_a + \sum_{b\in K} P_b$. Also, it is clear that $(\sum_{a\in S} P_a)_{\circ} + (\sum_{b\in K} P_b)_{\circ} \in Z^{\circ}Id(R)$ is contained in $(\sum_{a\in S} P_a + \sum_{b\in K} P_b)_{\circ}$. So we are done.

(4) ⇒ (1) Let I, J be two z°-ideals of R. Then $I = \sum_{a\in I} P_a$, $J = \sum_{b\in J} P_b$ and,

$$I + J = \sum_{a\in I} P_a + \sum_{b\in J} P_b = \Big(\sum_{a\in I} P_a\Big)_{\circ} + \Big(\sum_{b\in J} P_b\Big)_{\circ} = \Big(\sum_{a\in I} P_a + \sum_{b\in J} P_b\Big)_{\circ}.$$

□

Lemma 3.2. *For a z°-ideal J of a reduced ring R and a set $\{I_\alpha : \alpha \in S\}$ of ideals of R we have,*

$$J \cap \Big(\sum_{\alpha\in S} I_\alpha\Big)_{\circ} = \Big(\sum_{\alpha\in S} (J \cap I_\alpha)\Big)_{\circ}.$$

Proof. We have $J \cap (\sum_{\alpha\in S} I_\alpha)_{\circ}$ is a z°-ideal containing $\sum_{\alpha\in S} I_\alpha \cap J$. As $(\sum_{\alpha\in S}(I_\alpha \cap J))_{\circ}$ is the smallest z°-ideal containing $\sum_{\alpha\in S} I_\alpha \cap J$,

$$\Big(\sum_{\alpha\in S} (I_\alpha \cap J)\Big)_{\circ} \subseteq J \cap \Big(\sum_{\alpha\in S} I_\alpha\Big)_{\circ}.$$

We can assume $(\sum_{\alpha\in S}(I_\alpha \cap J))_{\circ}$ is a proper ideal of R. Since $(\sum_{\alpha\in S}(I_\alpha \cap J))_{\circ}$ is a z°-ideal, so it is an intersection of minimal prime ideals over it, each of which is a z°-ideal. Now, let $a \in (\sum_{\alpha\in S} I_\alpha)_{\circ} \cap J$ and P be a minimal prime ideal over $(\sum_{\alpha\in S}(I_\alpha \cap J))_{\circ}$. Then $a \in J$, $a \in (\sum_{\alpha\in S} I_\alpha)_{\circ}$ and P contains $I_\alpha \cap J$ for each $\alpha \in S$. If $P \not\supseteq J$, then $P \supseteq I_\alpha$, for all $\alpha \in S$, hence $P \supseteq \sum_{\alpha\in S} I_\alpha$. But P is a z°-ideal, so $P \supseteq (\sum_{\alpha\in S} I_\alpha)_{\circ}$. This implies $a \in P$. Hence $a \in (\sum_{\alpha\in S}(J \cap I_\alpha))_{\circ}$. □

Lemma 3.2 implies Theorem 2.2 in [9] (i.e., $Z^\circ Id(R)$ is a frame). In fact, if J and the family $\{I_\alpha : \alpha \in S\}$ are z°-ideals of R, then we have,

$$J \wedge (\bigvee_{\alpha\in S} I_\alpha) = J \cap (\sum_{\alpha\in S} I_\alpha)_\circ = (\sum_{\alpha\in S}(J \cap I_\alpha))_\circ = \bigvee_{\alpha\in S}(J \wedge I_\alpha).$$

It is well-known that $Z^\circ Id(R/H) \cong \uparrow H$ for every radical ideal H, hence $Z^\circ Id(R/N(R)) \cong Z^\circ Id(R)$, because $N(R)$ is the bottom element of $Z^\circ Id(R)$. Although this is well-known we provide a direct proof for this result.

Lemma 3.3. *For a ring R, the two lattices $Z^\circ Id(R)$ and $Z^\circ Id(R/N(R))$ are isomorphic.*

Proof. (1) First we denote join and meet in $R/N(R)$ by $\vee'$ and $\wedge'$, respectively. It is easy to see that for two z°-ideals I_1, I_2 of R, we have

$$I_1/N(R) \vee' I_2/N(R) = (I_1 \vee I_2)/N(R) = (I_1 + I_2)_\circ/N(R) \quad \text{and}$$

$$I_1/N(R) \wedge' I_2/N(R) = (I_1 \cap I_2)/N(R).$$

These show two operations $(\vee', \wedge')$ on the lattice $(Z^\circ Id(R/N(R)), \subseteq)$. Next, define $\phi : Z^\circ Id(R) \to Z^\circ Id(R/N(R))$ by $\phi(I) = I/N(R)$. By the fact that for two z°-ideals I_1, I_2 of R, we have $I_1 = I_2$ if and only if $I_1/N(R) = I_2/N(R)$, the map ϕ is well-defined and injective from $Z^\circ Id(R)$ onto $Z^\circ Id(R/N(R))$. We also have,

$$\phi(I_1 \vee I_2) = \phi((I_1 + I_2)_\circ) = (I_1 + I_2)_\circ/N(R) = I_1/N(R) \vee' I_2/N(R) = \phi(I_1) \vee' \phi(I_2),$$

$$\phi(I_1 \wedge I_2) = \phi(I_1 \cap I_2) = (I_1 \cap I_2)/N(R) = I_1/N(R) \wedge' I_2/N(R) = \phi(I_1) \wedge' \phi(I_2).$$

Thus ϕ is a lattice isomorphism. □

The above result tells us that for the investigation of the lattice properties of $Z^\circ Id(R)$, we can assume R to be a reduced ring.

Definition 3.4. A ring R is called *WSA* if for each two ideals I and J of R where $I \cap J = 0$, we have

$$(\mathrm{Ann}(I) + \mathrm{Ann}(J))_\circ = R.$$

Recall from [7] that a ring R is an *SA*-ring if the sum of two annihilator ideals is an annihilator ideal. According to Theorem 4.4 in [7], every reduced *SA*-ring is *WSA*. However, we will demonstrate that the reverse is not necessarily true. To explore this further, we need to introduce the following topological concept in the sequel.

Definition 3.5. A completely regular space X is called *W. Extremally disconnected* (briefly, *WED*-space) if every two disjoint open sets can be separated by two disjoint Z-zero-sets (i.e., the interior of a zero-set).

Example 3.6. (1) Every extremally disconnected space is a *WED*-space. This follows from [11, 1H.2], where it is established that in an extremally disconnected space, any two disjoint open sets are completely separated, and hence, they are separated by two disjoint Z-zero-sets.

(2) Every perfectly normal space, such as a metric space X, is a *WED*-space. To see it, consider two disjoint open sets A and B in X. Let $\mathrm{cl}\, A$ and $\mathrm{cl}\, B$ be the closures of A and B, respectively, which are two zero-sets in X. We claim that $\mathrm{int}\,\mathrm{cl}\, A \cap \mathrm{int}\,\mathrm{cl}\, B = \emptyset$. Assume, to the contrary, that $x \in \mathrm{int}\,\mathrm{cl}\, A \cap \mathrm{int}\,\mathrm{cl}\, B$. Then, there exist open sets U and V in X such that $x \in U \subseteq \mathrm{cl}\, A$ and $x \in V \subseteq \mathrm{cl}\, B$. This implies $V \cap A \neq \emptyset$, which leads to a contradiction. Therefore, $\mathrm{int}\,\mathrm{cl}\, A \cap \mathrm{int}\,\mathrm{cl}\, B$ is empty, and A and B are contained in two disjoint Z-zero-sets.

(3) If we consider $\mathbb{R}$ with usual topology, it serves as an example of a *WED*-space that is not an extremally disconnected space, as shown in Part (2).

Theorem 3.7. *Let X be a completely regular Hausdorff space. Then X is a WED-space if and only if βX is a WED-space.*

Proof. $\Rightarrow$ Assume X is a *WED*-space. Let U, V be two disjoint open sets in βX. Then $U \cap X \cap V \cap X = \emptyset$. By hypothesis, there exist two disjoint Z-zero-sets $\operatorname{int}_X Z_1$ and $\operatorname{int}_X Z_2$ in X such that $U \cap X \subseteq \operatorname{int}_X Z_1$ and $V \cap X \subseteq \operatorname{int}_X Z_2$. These imply $U \subseteq \operatorname{cl}_{\beta X} U = \operatorname{cl}_{\beta X}(U \cap X) \subseteq \operatorname{cl}_{\beta X} Z_1$ and $V \subseteq \operatorname{cl}_{\beta X} V = \operatorname{cl}_{\beta X}(V \cap X) \subseteq \operatorname{cl}_{\beta X} Z_2$. Thus $U \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z_1 = \operatorname{int}_{\beta X} Z_1^\beta$ and $V \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z_2 = \operatorname{int}_{\beta X} Z_2^\beta$. On the other hand, $\operatorname{int}_X Z_1 \cap \operatorname{int}_X Z_2 = \emptyset$ and X is dense in βX, hence $\operatorname{int}_X Z_1^\beta \cap \operatorname{int}_X Z_2^\beta = \emptyset$. Therefore, $U \subseteq \operatorname{int}_{\beta X} Z_1^\beta$ and $V \subseteq \operatorname{int}_{\beta X} Z_2^\beta$ in βX , proving the forward direction.

$\Leftarrow$ Assume βX is a *WED*-space. Let U, V be two disjoint open sets in X. Then there are two open sets U_1, V_1 in βX such that $U = U_1 \cap X$ and $V = V_1 \cap X$. Since U_1 and V_1 are disjoint in βX (as U, V are disjoint in X and X is dense in βX), by the hypothesis, there exist two disjoint Z-zero-sets $\operatorname{int}_{\beta X} Z_1^\beta$ and $\operatorname{int}_{\beta X} Z_2^\beta$ in βX such that $U_1 \subseteq \operatorname{int}_{\beta X} Z_1^\beta$ and $V_1 \subseteq \operatorname{int}_{\beta X} Z_2^\beta$. Thus, $U = U_1 \cap X \subseteq \operatorname{int}_{\beta X} Z_1^\beta \cap X = \operatorname{int}_X Z_1$ and $V = V_1 \cap X \subseteq \operatorname{int}_{\beta X} Z_2^\beta \cap X = \operatorname{int}_X Z_2$. Furthermore,

$$\operatorname{int}_X Z_1 \cap \operatorname{int}_X Z_2 = \operatorname{int}_{\beta X} Z_1^\beta \cap \operatorname{int}_{\beta X} Z_2^\beta \cap X = \emptyset.$$

This completes the proof. □

The next result shows that $C(\mathbb{R})$ is a *WSA*-ring which is not an *SA*-ring.

Theorem 3.8. *Let X be a completely regular Hausdorff space. The following statements are equivalent.*

(1) *$C(X)$ is a WSA-ring.*

(1) *The space X is a WED-space.*

(1) *$C^*(X)$ is a WSA-ring.*

Proof. (1) $\Rightarrow$ (2) Let A, B be two disjoint open sets in X. As X is a completely regular space, there are two subsets S, H of $C(X)$ such that

$$A = \bigcup_{f \in S}(X \setminus Z(f)) \quad \text{and} \quad B = \bigcup_{g \in H}(X \setminus Z(g)).$$

Consider the two ideals I and J, where $I =< S >$ and $J =< H >$, respectively. Then $A \cap B = \emptyset$ implies $I \cap J = 0$. Since $f \in I \cap J$ follows $X \setminus Z(f) \subseteq A \cap B$, i.e., $f = 0$. By the hypothesis, $(\operatorname{Ann}(I) + \operatorname{Ann}(J))_\circ = C(X)$. This shows that there exists a non-zero-divisor element $f \in \operatorname{Ann}(I) + \operatorname{Ann}(J)$. Hence there are $h \in \operatorname{Ann}(I)$ and $k \in \operatorname{Ann}(J)$ such that $f = h + k$ and $\operatorname{int} Z(h) \cap \operatorname{int} Z(k) = \emptyset$. $h \in \operatorname{Ann}(I)$ and $k \in \operatorname{Ann}(J)$ imply $A \subseteq Z(h)$ and $B \subseteq Z(k)$, respectively. So we are done.

(2) $\Rightarrow$ (1) Let I and J be two ideals of $C(X)$ with $I \cap J = 0$. Put

$$A = \bigcup_{f \in I}(X \setminus Z(f)) \quad \text{and} \quad B = \bigcup_{g \in J}(X \setminus Z(g)).$$

The equality $I \cap J = 0$ implies $A \cap B = \emptyset$. By the hypothesis, there are two zero-sets $Z(f), Z(g) \in Z[X]$ such that

$$A \subseteq \operatorname{int} Z(f), \quad B \subseteq \operatorname{int} Z(g) \quad \text{and} \quad \operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset.$$

$A \subseteq \operatorname{int} Z(f)$ implies $f \in \operatorname{Ann}(I)$ and $B \subseteq \operatorname{int} Z(g)$ implies $g \in \operatorname{Ann}(J)$. On the other hand, $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$ implies $f^2 + g^2$ is a non-zero-divisor element in $\operatorname{Ann}(I) + \operatorname{Ann}(J)$. Therefore $(\operatorname{Ann}(I) + \operatorname{Ann}(J))_\circ = C(X)$.

(3) $\Leftrightarrow$ (4) As $C^*(X)$ is isomorphic to $C(\beta X)$, this follows from Theorem 3.7 and (1)$\Leftrightarrow$(2). □

Now we want to characterize the co-normality of the lattice $Z^\circ Id(R)$ $(DId(R))$ in the class of reduced rings.

Proposition 3.9. *Let R be a reduced ring. Then the lattice $Z^{\circ}Id(R)$ is co-normal if and only if R is a WSA-ring.*

Proof. $\Rightarrow$ Let I, J be two ideals of R with $I \cap J = 0$. Then $I_{\circ} \cap J_{\circ} = (I \cap J)_{\circ} = 0_{\circ} = 0$, since R is a reduced ring. By the hypothesis, there are two z°-ideals I_1, J_1 such that $I_1 \vee J_1 = R$, $I_{\circ} \cap I_1 = 0$ and $J_{\circ} \cap J_1 = 0$. The first equality shows that $(I_1 + J_1)_{\circ} = R$ and the others show

$$I_1 \subseteq \mathrm{Ann}(I_{\circ}) \subseteq \mathrm{Ann}(I) \quad \text{and} \quad J_1 \subseteq \mathrm{Ann}(J_{\circ}) \subseteq \mathrm{Ann}(J).$$

This implies that

$$(I_1 + J_1)_{\circ} \subseteq (\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ}.$$

Thus $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ} = R$.

$\Leftarrow$ Consider two z°-ideals I, J of R with $I \cap J = 0$. Then by the hypothesis,

$$(\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ} = R.$$

Put $I_1 = \mathrm{Ann}(I)$ and $J_1 = \mathrm{Ann}(J)$. Then I_1, J_1 are two z°-ideals of R, $I \cap I_1 = 0$, $J \cap J_1 = 0$ and $I_1 \vee J_1 = (I_1 + J_1)_{\circ} = R$. This shows $Z^{\circ}Id(R)$ is a co-normal lattice. □

From Theorem 3.8 and Proposition 3.9, we have the next result.

Corollary 3.10. *Let X be a completely regular Hausdorff space. Then $Z^{\circ}Id(C(X))$ is co-normal if and only if X is a WED-space.*

Recall from [12], a ring R satisfies property A if each f.g. ideal of R consisting of zero divisors has a nonzero annihilator. Noetherian rings, $C(X)$, Zero-dimensional rings (each prime ideal is maximal), the polynomial ring $R[x]$ and rings whose classical ring of quotients are regular are examples of rings with the property A.

Proposition 3.11. *Let R be a reduced ring with property A. Then R is a WSA-ring if and only if for each pair of ideals I, J of R, $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ} = \mathrm{Ann}(I \cap J)$.*

Proof. The necessity is obvious. Now, let R be a WSA-ring and I, J be two ideals of R. Trivially, we have $(\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ} \subseteq \mathrm{Ann}(I \cap J)$. Let $x \in \mathrm{Ann}(I \cap J) = \mathrm{Ann}(IJ)$. Then $xIJ = 0$. This shows that $xI \cap J = 0$. By the hypothesis,

$$(\mathrm{Ann}(xI) + \mathrm{Ann}(J))_{\circ} = R.$$

According to [3, Theorem 1.21], there exists a non-zero-divisor element $a + b$ in $\mathrm{Ann}(xI) + \mathrm{Ann}(J)$, where $a \in \mathrm{Ann}(xI)$ and $b \in \mathrm{Ann}(J)$. This implies

$$ax + bx \in \mathrm{Ann}(I) + \mathrm{Ann}(J) \subseteq (\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ}.$$

Since $\mathrm{Ann}(a + b) = 0$, we have $\mathrm{Ann}(ax + bx) = \mathrm{Ann}(x)$. Thus, $x \in (\mathrm{Ann}(I) + \mathrm{Ann}(J))_{\circ}$. Therefore, the proof is complete. □

4. On the lattice of z-ideals in a commutative ring

Rings in which the sum of two z-ideals is a z-ideal are important (e.g., $C(X)$). We now turn to characterizing them. We recall that the set of all basic z-ideals of R is $\{M_a : a \in R\}$.

Proposition 4.1. *The following statements are equivalent.*

(1) *The sum of two z-ideals in R is a z-ideal.*

(2) *For each* $I, J \trianglelefteq R$, $(I_z + J_z)_z = I_z + J_z$.

(3) *The lattice* $ZId(R)$ *is a sublattice of the lattice of ideals of* R.

(4) *For every two families* $\{M_a : a \in S\}$ *and* $\{M_b : b \in K\}$ *of basic z-ideals,*

$$(\sum_{a\in S} M_a + \sum_{b\in K} M_b)_z = (\sum_{a\in S} M_a)_z + (\sum_{b\in K} M_b)_z.$$

Proof. The proof is similar to the proof of Proposition 3.1. □

The lattice $RId(R)$ of radical ideals of R, ordered by inclusion, constitutes a coherent frame (refer to [5]). In this frame, the meet operation corresponds to intersection, and the join operation is defined as the radical of the sum.

In [13], Ighedo and McGovern provided a characterization of various properties of the lattice of z-ideals using the tools of frame and locale theory. In this context, we offer direct proofs for some of these properties. Additionally, we establish results for the lattice $RId(R)$. To support these results, the following lemma is required.

Lemma 4.2. *For a ring* R *the following statements hold.*

(1) *For a z-ideal* J *of* R *and a set* $\{I_\alpha : \alpha \in S\}$ *of ideals of* R,

$$J \cap (\sum_{\alpha\in S} I_\alpha)_z = (\sum_{\alpha\in S}(J \cap I_\alpha))_z.$$

(2) *For a radical ideal* J *of* R *and a set* $\{I_\alpha : \alpha \in S\}$ *of ideals of* R,

$$J \cap \sqrt{\sum_{\alpha\in S} I_\alpha} = \sqrt{\sum_{\alpha\in S}(J \cap I_\alpha)}.$$

Proof. (1) We have $J \cap (\sum_{\alpha\in S} I_\alpha)_z$ is a z-ideal containing $\sum_{\alpha\in S} I_\alpha \cap J$. As $(\sum_{\alpha\in S}(I_\alpha \cap J))_z$ is the smallest z-ideal containing $\sum_{\alpha\in S} I_\alpha \cap J$,

$$(\sum_{\alpha\in S}(I_\alpha \cap J))_z \subseteq J \cap (\sum_{\alpha\in S} I_\alpha)_z.$$

We can assume $(\sum_{\alpha\in S}(I_\alpha \cap J))_z$ is a proper ideal. Since $(\sum_{\alpha\in S}(I_\alpha \cap J))_z$ is a z-ideal, so it is an intersection of minimal prime ideals over it, each of which is a z-ideal. Now, let $a \in (\sum_{\alpha\in S} I_\alpha)_z \cap J$ and P be a minimal prime ideal over $(\sum_{\alpha\in S}(I_\alpha \cap J))_z$. Then $a \in J$, $a \in (\sum_{\alpha\in S} I_\alpha)_z$ and P contains $I_\alpha \cap J$ for each $\alpha \in S$. If $P \not\supseteq J$, then $P \supseteq I_\alpha$, for all $\alpha \in S$, hence $P \supseteq \sum_{\alpha\in S} I_\alpha$. But P is a z-ideal, so $P \supseteq (\sum_{\alpha\in S} I_\alpha)_z$. This implies $a \in P$. Hence $a \in (\sum_{\alpha\in S}(J \cap I_\alpha))_z$.

(2) The proof is similar to the proof of Part (1). □

Lemma 4.2 implies Theorem 3.1 in [13] (i.e., $ZId(R)$ is a frame). In fact, if J and the family $\{I_\alpha : \alpha \in S\}$ are z-ideals of R, then we have,

$$J \wedge (\bigvee_{\alpha\in S} I_\alpha) = J \cap (\sum_{\alpha\in S} I_\alpha)_z = (\sum_{\alpha\in S}(J \cap I_\alpha))_z = \bigvee_{\alpha\in S}(J \wedge I_\alpha).$$

Similarly, we can apply Part 2 of Lemma 4.2 to show that $RId(R)$ is a frame, see also [5].

It is well-known that $ZId(R/H) \cong \uparrow H$ for every radical ideal H, hence

$$ZId(R/J(R)) \cong ZId(R).$$

Because $J(R)$ is the bottom element of $ZId(R)$. Although this is well-known, we provide a direct proof for this result. Hence, to investigate the lattice properties of $ZId(R)$ (resp., $RId(R)$), we can consider R to be a semiprimitive ring.

Lemma 4.3. *For a ring R the following statements hold.*

(1) *The two lattices $ZId(R)$ and $ZId(R/J(R))$ are isomorphic.*

(2) *The two lattices $RId(R)$ and $RId(R/N(R))$ are isomorphic.*

Proof. (1) First we denote join and meet in $R/J(R)$ by $\vee'$ and $\wedge'$, respectively. It is easy to see that for two z-ideals I_1, I_2 of R, we have

$$I_1/J(R) \vee' I_2/J(R) = (I_1 \vee I_2)/J(R) = (I_1 + I_2)_z/J(R) \quad \text{and}$$

$$I_1/J(R) \wedge' I_2/J(R) = (I_1 \cap I_2)/J(R).$$

These show two operations $(\vee', \wedge')$ on the lattice $(ZId(R/J(R)), \subseteq)$. Next, we define $\phi : ZId(R) \to ZId(R/J(R))$ by $\phi(I) = I/J(R)$. By the fact that for two z-ideals I_1, I_2 of R, we have $I_1 = I_2$ if and only if $I_1/J(R) = I_2/J(R)$, the map ϕ is well-defined and injective from $ZId(R)$ onto $ZId(R/J(R))$. We also have,

$$\phi(I_1 \vee I_2) = \phi((I_1 + I_2)_z) = (I_1 + I_2)_z/J(R) = I_1/J(R) \vee' I_2/J(R) = \phi(I_1) \vee' \phi(I_2),$$

$$\phi(I_1 \wedge I_2) = \phi(I_1 \cap I_2) = (I_1 \cap I_2)/J(R) = I_1/J(R) \wedge' I_2/J(R) = \phi(I_1) \wedge' \phi(I_2).$$

Thus ϕ is a lattice isomorphism.

(2) Similar to the proof of (1), for two radical ideals I_1, I_2 of R, we have

$$I_1/N(R) \vee' I_2/N(R) = (I_1 \vee I_2)/N(R) = \sqrt{(I_1 + I_2)}/N(R) \quad \text{and}$$

$$I_1/N(R) \wedge' I_2/N(R) = (I_1 \cap I_2)/N(R).$$

These show two operations $(\vee', \wedge')$ on the lattice $(RId(R/N(R)), \subseteq)$. Now, define $\psi : RId(R) \to RId(R/N(R))$ by $\psi(I) = I/N(R)$. We can see that I is a radical ideal of R if and only if $I/N(R)$ is a radical ideal of $R/N(R)$. It also is easy to see that for two radical ideals I_1, I_2 of R, we have $I_1 = I_2$ if and only if $I_1/N(R) = I_2/N(R)$. Thus ψ is injective and surjective from $RId(R)$ onto $RId(R/N(R))$. We also have,

$$\psi(I_1 \vee I_2) = \psi(\sqrt{I_1 + I_2}) = \sqrt{I_1 + I_2}/N(R) = I_1/N(R) \vee' I_2/N(R) = \psi(I_1) \vee' \phi(I_2),$$

$$\psi(I_1 \wedge I_2) = \psi(I_1 \cap I_2) = (I_1 \cap I_2)/N(R) = I_1/N(R) \wedge' I_2/N(R) = \psi(I_1) \wedge' \psi(I_2).$$

Thus ψ is a lattice isomorphism. □

We now come to the characterization of the co-normality of the lattice $ZId(R)$ (resp., $RId(R)$). Whenever R is a reduced ring, it is proved in [7, Corollary 4.5] that R is an SA-ring if and only if R is a Baer ring. We remind the reader that a lattice $< L, \wedge, \vee, 0, 1 >$ is co-normal whenever it is distributive and for all $a, b \in L$ with $a \wedge b = 0$ there exist $x, y \in L$ such that $x \vee y = 1$ and $a \wedge x = b \wedge y = 0$.

Theorem 4.4. *For a semiprimitive ring R the following statements are equivalent.*

(1) *The lattice $ZId(R)$ is a co-normal lattice.*

(2) *R is an SA-ring.*

(3) *The lattice $RId(R)$ is a co-normal lattice.*

Proof. (1) ⇒ (2) Let I and J be two annihilator ideals of R and $I \cap J = 0$. Notably, since R is a semiprimitive ring, I and J are two z-ideals of R. This can be shown as follows:

$$I = \mathrm{Ann}(\mathrm{Ann}(I)) = \bigcap_{M \in \mathrm{Max}(R)} \;\; \bigcap_{\mathrm{Ann}(I) \not\subseteq M} M.$$

This establishes that I is a z-ideal. Similarly, J is a z-ideal. According to the hypothesis, there exist z-ideals I_1 and J_1 in R with $(I_1 + J_1)_z = I_1 \vee J_1 = R$ and $I \cap I_1 = 0$ and $J \cap J_1 = 0$. The first equality implies $I_1 + J_1 = R$ and the other conditions imply $I_1 \subseteq \text{Ann}(I)$ and $J_1 \subseteq \text{Ann}(J)$, respectively. Hence $\text{Ann}(I) + \text{Ann}(J) = R$. Now, by Corollary 4.9 in [7], R is an SA-ring.

(2) ⇒ (3) Let I and J be two radical ideals with $I \cap J = 0$. According to the hypothesis and Theorem 2.14 in [1], $\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(I \cap J) = R$. Set $I_1 = \text{Ann}(I)$ and $J_1 = \text{Ann}(J)$. Since R is a semiprimitive ring, I_1 and J_1 are two radical ideals, $I \cap I_1 = 0$, $J \cap J_1 = 0$ and $I_1 \vee J_1 = \sqrt{\text{Ann}(I) + \text{Ann}(J)} = R$. Therefore, $RId(R)$ is a co-normal lattice.

(3) ⇒ (1) Assume that I and J are two z-ideals and $I \cap J = 0$. As I, J are two radical ideals, there are two radical ideals I_1 and J_1 with $I_1 + J_1 = R$, $I \cap I_1 = 0$ and $J \cap J_1 = 0$, by the hypothesis. Thus $(I_1)_z + (J_1)_z = R$, $I \cap (I_1)_z = (I \cap I_1)_z = 0_z = 0$ and $J \cap (J_1)_z = (J \cap J_1)_z = 0_z = 0$. So we are done. □

It is well known fact, as established by [2, Theorem 3.5] and [23, Theorem 3.12], that $C(X)$ is a Baer ring if and only if X is an extremally disconnected space (i.e., the closure of every open set is open). This, combined with Theorem 3.4, leads to the following result.

Corollary 4.5. *The following statements are equivalent.*

(1) *The lattice $ZId(C(X))$ is a co-normal lattice.*

(2) *The space X is extremally disconnected.*

(3) *The lattice $RId(C(X))$ is a co-normal lattice.*

Lemma 4.6. *Let R, S be two rings and $\phi : R \to S$ be a ring isomorphism. The following statements hold.*

(1) *If $a \in R$, then $\phi(M_a) = M_{\phi(a)}$.*

(2) *If I is a z-ideal of R, then $\phi(I)$ is a z-ideal of S.*

(3) *If J is a z-ideal of S, then $\phi^{-1}(J)$ is a z-ideal of R.*

(4) *If I is an ideal of R, then $\phi(I_z) = (\phi(I))_z$.*

Proof. (1) Let $\phi(x) \in \phi(M_a)$, where $x \in M_a$. Consider a maximal ideal M in S, where $\phi(a) \in M$. Then $a \in \phi^{-1}(M)$. Since $\phi^{-1}(M)$ is a maximal ideal in R, $x \in \phi^{-1}(M)$, i.e., $\phi(x) \in M$. This shows $\phi(M_a) \subseteq M_{\phi(a)}$. To show other inclusion, let $y = \phi(x) \in M_{\phi(a)}$. We must show that $x \in M_a$. Assume that M is a maximal ideal in R containing a. Then $\phi(a) \in \phi(M)$ and $\phi(M)$ is a maximal ideal in S. Hence $\phi(x) \in \phi(M)$. This implies $x \in \phi^{-1}(\phi(M) = M$. Thus $x \in M_a$.

(2) Let $\phi(a) \in \phi(I)$, where $a \in I$. By the hypothesis, $M_a \subseteq I$. By Part (1), $M_{\phi(a)} = \phi(M_a) \subseteq \phi(I)$.

(3) Suppose that $x \in \phi^{-1}(J)$. Then $\phi(x) \in J$. Thus $\phi(M_x) = M_{\phi(x)} \subseteq J$. This implies $M_x \subseteq \phi^{-1}(J)$.

(4) By Part (1) and the fact that ϕ is a ring isomorphism,

$$\phi(I_z) = \phi(\sum_{x\in I} M_x) = \sum_{x\in I} \phi(M_x) = \sum_{\phi(x)\in\phi(I)} M_{\phi(x)} = (\phi(I))_z.$$

□

The following result immediately follows from Proposition 6.3 of [13], since their functor ZId clearly sends a ring isomorphism to an isomorphism in the category $CohFrm$. However, we provide a direct proof.

Theorem 4.7. *Let R and S be two isomorphic rings. Then the two lattices $ZId(R)$ and $ZId(S)$ are isomorphic.*

Proof. Let $\phi : R \to S$ be a ring isomorphism. Define

$$\varphi : ZId(R) \to ZId(S), \quad \text{by} \quad \varphi(I) = \phi(I) \quad \text{where} \quad I \in ZId(R).$$

By Lemma 4.6, φ is a well-defined and injective map. Now, let $J \in ZId(S)$. Then $\phi^{-1}(J) \in ZId(R)$, by Lemma 4.6. We have $\varphi(\phi^{-1}(J)) = J$. Demonstrating that φ is a surjective map. Consider two ideals $I, J \in ZId(R)$. Then we have the following equalities:

$$\varphi(I \vee J) = \varphi((I+J)_z) = \phi((I+J)_z) = (\phi(I+J))_z, \quad \text{by} \quad \text{Lemma} \quad 4.6$$

$$= (\phi(I) + \phi(J))_z = \phi(I) \vee \phi(J) = \varphi(I) \vee \varphi(J).$$

$$\varphi(I \wedge J) = \varphi(I \cap J) = \phi(I \cap J) = \phi(I) \cap \phi(J) = \varphi(I) \wedge \varphi(J).$$

So φ is a lattice isomorphism. □

It is easy to see that if R is a semiprimitive ring, then

$$Id(R) = \{Re : e \text{ is an idempotent of } R\}$$

partially ordered by inclusion is a lattice and for two idempotents e and f of R, we have $eR \vee' fR = (e+f-ef)R$ and $eR \wedge' fR = efR$.

Proposition 4.8. *For a semiprimitive ring R the following statements are equivalent.*

(1) *The lattice $ZId(R)$ is a Boolean algebra.*

(2) *Two lattices $< ZId(R), \vee, \wedge >$ and $< Id(R), \vee', \wedge' >$ coincide.*

(3) *Every maximal ideal of R is generated by an idempotent.*

(4) *R is a semisimple ring.*

Proof. (1) ⇒ (2) Initially, we demonstrate that two sets $ZId(R)$ and $Id(R)$ are equal. By the hypothesis, each element of $Id(R)$ is a z-ideal. Let I be a z-ideal. By Part (1), there is a z-ideal J such that $I \cap J = 0$ and $I + J = R$. Thus $I = eR$ for some idempotent e of R. Therefore the two sets coincide. Now, let I and J be two z-ideals of R. Then $I = eR$ and $J = fR$ for some idempotents $e, f \in R$. Thus $I \vee' J = (e + f - ef)R$. It is easy to see that $(e + f - ef)R$ is the smallest z-ideal containing $eR + fR$. Hence

$$I \vee' J = (I + J)_z = I \vee J.$$

We also have

$$I \wedge' J = eR \wedge' fR = efR = eR \cap fR = I \cap J = I \wedge J.$$

Suppose that eR and fR are two elements of $Id(R)$. Hence

$$eR \vee fR = (eR + fR)_z = (e + f - ef)R = eR \vee' fR.$$

And

$$eR \wedge fR = eR \cap fR = efR = eR \wedge' fR.$$

Thus, we have shown that they are equal as two lattices.

(2)⇒(3) Trivial.

(3) ⇒ (4) By Theorem in [15], R is a finite direct sum of simple rings. Since R is commutative, every simple ring is a field, implying R is a finite direct sum of fields, i.e., R is a semisimple ring.

(4) ⇒ (1) By the hypothesis, there exist finitely number fields $F_1, F_2, ..., F_n$ such that R is isomorphic to $F_1 \times F_2 \times ... \times F_n$. Theorem 4.7 implies two lattices $ZId(R)$ and $ZId(F_1 \times F_2 \times ... \times F_n)$ are isomorphic. Trivially, every ideal of $F_1 \times F_2 \times ... \times F_n$ is a z-ideal. On the othe hand, it is easy to calculate that every ideal of $F_1 \times F_2 \times ... \times F_n$ has a complement. Thus, $ZId(R)$ is a Boolean algebra. □

We apply Theorem 4.7 for the ring of continuous function in the next result.

Corollary 4.9. *Let X, Y be two completely regular Hausdorff spaces.*

(1) *If X and Y are two homeomorphic spaces, then $ZId(C(X))$ and $ZId(C(Y))$ are two isomorphic lattices.*

(2) *The two lattices $ZId(C(X))$ and $ZId(C(\upsilon X))$ are isomorphic.*

(3) *The two lattices $ZId(C^*(X))$ and $ZId(C(\beta X))$ are isomorphic.*

Proof. (1) If X and Y are two homeomorphic spaces, then $C(X)$ and $C(Y)$ are isomorphic rings and hence $ZId(C(X))$ and $ZId(C(Y))$ are two isomorphic lattices, by Theorem 4.7.

(2) Since $C(X)$ and $C(\upsilon X)$ are two isomorphic rings, it follows from Theorem 4.7.

(3) The two rings $C^*(X)$ and $C(\beta X)$ are isomorphic, so $ZId(C^*(X))$ and $ZId(C(\beta X))$ are two isomorphic lattices, by Theorem4.7. □

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