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Symplectic and para-Kähler structures on Hom-Lie algebroids

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Abstract. The purpose of this paper is to study Hom-algebroids, among them left symmetric Hom-algebroids and symplectic Hom-algebroids by providing some characterizations and geometric interpretations. Therefore, we introduce and study para-Kähler Hom-Lie algebroids and show various properties and examples including these structures.

Introduction

Symplectic geometry is the mathematical device to express classical mechanics, geometrical optics, and thermodynamics problems. One gets a symplectic geometry by extracting the equation from the variational principle. Hamiltonian dynamics equations get more straightforward with symplectic geometry, as the ordinary geometry of linear spaces decreases the complexity of computations. This geometry has applications to the theory of elementary particles, oceanographic and atmospheric sciences, condensed matter, accelerator and plasma physics and other disciplines at the classical and quantum levels [1].

Hom-type algebras were motivated by σ-deformations of some algebras of vector fields like Witt and Virasoro algebras. The first instances appeared in papers by physicists, where it was noticed that the obtained structures satisfy modified Jacobi condition. The main feature of Hom-type algebras is that usual identities are twisted by a structure map (a homomorphism).

Hom-Lie algebroids were first studied by Laurent-Gengoux and Teles in [7], they mainly showed a one-to-one correspondence with Hom-Gerstenhaber algebras. Then Cai, Liu and Sheng, in [4], changed slightly the definition and introduced various related structures. They showed that there is a natural Hom-Lie algebroid structure on the pullback bundle of a Lie algebroid with respect to a diffeomorphism $\varphi : M \to M$. Moreover they introduced the notion of Hom-Poisson tensor on $C^\infty(M)$ and showed that there is a Hom-Lie algebroid structure on $\varphi^! T^*M$ associated to Hom-Poisson structure, providing an interesting geometric interpretation. They also consider dual structures and discuss Hom-Lie bialgebroids. See also [9, 10] for more recent results. In [13], Mandal and Kumar Mishra studied adjoint functors between the category of Hom-Gerstenhaber algebras and the category of Hom-Lie-Rinehart algebras, some geometric applications and cohomology.

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The aim of this paper is to make some geometric developments of these structures. We recall in Section 1 basic definitions. In Section 2, we introduce Hom-algebroids and left symmetric Hom-algebroids. Also, we deal with symplectic Hom-algebroids and show that there exists a Hom-connection, called left symmetric connections, which induces a left symmetric Hom-algebroid structure. Moreover we describe Hom-Levi-Cevita connections for which we provide some properties. The last section includes the main results of this paper. First, we introduce and provide examples of almost product, para-complex and para-Hermitian structures on Hom-Lie algebroids. Then consider para-Kähler Hom-Lie algebroids, discuss their properties and relationships with various structures.

1. Preliminaries

A Hom-algebra $(V, \cdot, \phi_V)$ consists of a linear space V, a bilinear map $\cdot : V \times V \to V$ and an algebra morphism $\phi_V : V \to V$. Also a Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a bilinear map $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and an algebra morphism $\phi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ such that

$$[u, v]_{\mathfrak{g}} = -[v, u]_{\mathfrak{g}}, \qquad \circlearrowleft_{u,v,w} [\phi_{\mathfrak{g}}(u), [v, w]] = 0,$$

for any $u, v, w \in \mathfrak{g}$. This algebra is called regular if $\phi_{\mathfrak{g}}$ is non-degenerate.

A *representation* of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ is a triple (V, A, ρ) where V is a vector space, $A \in gl(V)$ and $\rho : \mathfrak{g} \to gl(V)$ is a linear map such that for any $u, v \in \mathfrak{g}$, satisfying

$$\begin{cases} \rho(\alpha_{\mathfrak{g}}(u)) \circ A = A \circ \rho(u), \\ \rho([u, v]_{\mathfrak{g}}) \circ A = \rho(\alpha_{\mathfrak{g}}(u)) \circ \rho(v) - \rho(\alpha_{\mathfrak{g}}(v)) \circ \rho(u). \end{cases}$$

Let $A \to M$ be a vector bundle of rank n. Denote by $\Gamma(A)$ the $C^\infty(M)$-module of sections of $A \to M$. A *Hom-bundle* is a triple $(A \to M, \varphi, \phi_A)$ consisting of a vector bundle $A \to M$, a smooth map $\varphi : M \to M$ and a linear map $\phi_A : \Gamma(A) \to \Gamma(A)$ satisfying

$$\phi_A(fX) = \varphi^*(f)\phi_A(X),$$

for any $X \in \Gamma(A)$ and $f \in C^\infty(M)$ (in this case, ϕ_A is called a linear φ^*-function). If φ is a diffeomorphism and ϕ_A is an invertible map, then the Hom-bundle $(A \to M, \varphi, \phi_A)$ is called invertible. Considering $\varphi^! TM$ as a pullback bundle of φ over M and $Ad_{\varphi^*} : \Gamma(\varphi^! TM) \to \Gamma(\varphi^! TM)$ given by

$$Ad_{\varphi^*}(X) = \varphi^* \circ X \circ (\varphi^*)^{-1},$$

for any $X \in \Gamma(\varphi^! TM)$, then the triple $(\Gamma(\varphi^! TM), \varphi, Ad_{\varphi^*})$ is an example of Hom-bundles. Note that $\Gamma(\varphi^! TM)$ can be identified with $Der_{\varphi^*,\varphi^*}(C^\infty(M))$, i.e.

$$X(fg) = X(f)\varphi^*(g) + \varphi^*(f)X(g), \qquad \forall X \in \Gamma(\varphi^! TM), \forall f, g \in C^\infty(M).$$

As the linear map $\phi_A : \Gamma(A) \to \Gamma(A)$ can be extended to a linear map from $\Gamma(\wedge^k A)$ to $\Gamma(\wedge^k A)$ for which we use the same notation ϕ_A, i.e., $\phi_A(X) = \phi_A(X_1) \wedge \ldots \wedge \phi_A(X_k)$, for all $X = X_1 \wedge \ldots \wedge X_k \in \Gamma(\wedge^k A)$. Let A be an invertible Hom-bundle. We denote the inverses of φ and ϕ_A by φ^{-1} and ϕ_A^{-1}, respectively. In this case, it is easy to see that $(A \to M, \varphi^{-1}, \phi_A^{-1})$ is a Hom-bundle. Taking $\phi_A^\dagger : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$ defined by

$$(\phi_A^\dagger(\xi))(X) = \varphi^* \xi(\phi_A^{-1}(X)), \qquad \forall X \in \Gamma(\wedge^k A), \xi \in \Gamma(\wedge^k A^*),$$

we get the hom-bundle $(\wedge^k A^* \to M, \varphi, \phi_A^\dagger)$ [4].

A bundle map $\rho : A \to B$ between two Hom-bundles $(A \to M, \varphi, \phi_A)$ and $(B \to M, \varphi, \phi_B)$ is called a Hom-bundles morphism if the following condition holds

$$\rho \circ \phi_A = \phi_B \circ \rho.$$

Example 1.1. *[18] Let $(E \to M, \varphi, \phi_E)$ be an invertible Hom-bundle. The zero-order and first-order differential operators on E are defined, respectively by*

$$End(E) = \{D : \Gamma(E) \to \Gamma(E) | D(fX) = \varphi^*(f)D(X), \quad \forall X \in \Gamma(E), \forall f \in C^\infty(M)\},$$

and

$$Diff^1(E) = \{D : \Gamma(E) \to \Gamma(E) | [D, f] \in End(E), \quad \forall f \in C^\infty(M)\},$$

where

$$[D, f](X) = D(f\phi_E^{-1}(X)) - \varphi^*(f)D(\phi_E^{-1}X).$$

It is known that $(Diff^1(E), \varphi, Ad_{\phi_E})$ is a Hom-bundle, where $Ad_{\phi_E}(D) = \phi_E \circ D \circ \phi_E{}^{-1}$, for all $D \in Diff^1(E)$. Considering $\mathfrak{D}(E) = \sigma^{-1}(\mathcal{F}(\varphi^!TM))$, where $\sigma : Diff^1(E) \to Hom_{C^\infty(M)}((\varphi^!TM)^*, End(E))$ is a vector bundles morphism defined by

$$\sigma(D)(fdg)(X) = \varphi^*(f)[D, g]\phi_E(X), \quad \forall D \in Diff^1(E),\ \forall X \in \Gamma(\varphi^!TM),$$

and $\mathcal{F} : \varphi^!TM \to Hom_{C^\infty(M)}((\varphi^!TM)^*, End(E))$ given by

$$\mathcal{F}_X(\omega)(Y) = \omega(X)\phi_E(Y), \quad \forall \omega \in \Gamma(\varphi^!T^*M),\ \forall Y \in \Gamma(E),$$

$(\mathfrak{D}(E), \varphi, Ad_{\phi_E})$ is a Hom-bundle.

Definition 1.2. *[4] A Hom-Lie algebroid is a tuple $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ such that $(A \to M, \varphi, \phi_A)$ is a Hom-bundle, $(\Gamma(A), [\cdot,\cdot]_A, \phi_A)$ is a Hom-Lie algebra on the space of section $\Gamma(A)$, $a_A : A \to \varphi^!TM$ is a bundle map called the anchor map and if moreover, we have*

$$[X, fY]_A = \varphi^*(f)[X, Y]_A + a_A(\phi_A(X))(f)\phi_A(Y), \quad \forall X, Y \in \Gamma(A),\ \forall f \in C^\infty(M),$$

where $a_A : \Gamma(A) \to \Gamma(\varphi^!TM)$ is the representation of Hom-Lie algebra $(\Gamma(A), [\cdot,\cdot]_A, \phi_A)$ on $C^\infty(M)$ with respect to φ^* induced by the anchor map.

Example 1.3. *[4] Let M be a smooth manifold and $\varphi : M \to M$ be a diffeomorphism. $(\varphi^!TM, \varphi, Ad_{\varphi^*}, [\cdot,\cdot]_{\varphi^*}, Id)$ is a Hom-Lie algebroid, where $[\cdot,\cdot]_{\varphi^*}$ is given by*

$$[X, Y]_{\varphi^*} = \varphi^* \circ X \circ (\varphi^*)^{-1} \circ Y \circ (\varphi^*)^{-1} - \varphi^* \circ Y \circ (\varphi^*)^{-1} \circ X \circ (\varphi^*)^{-1},$$

for any $X, Y \in \Gamma(\varphi^!TM)$.

Example 1.4. *[18] Considr the Hom-bundle $(\mathfrak{D}(E), \varphi, Ad_{\phi_E})$ introduced in Example 1.1. Setting $a_{\mathfrak{D}(E)} = \sigma_{|\mathfrak{D}(E)}$, for every $D \in \mathfrak{D}(E)$ there exists $X = a_{\mathfrak{D}(E)}(D) \in \Gamma(\varphi^!TM)$ such that*

$$D(fY) = \varphi^*(f)D(Y) + a_{\mathfrak{D}(E)}(D)(f)\phi_E(Y), \quad \forall f \in C^\infty(M), Y \in \Gamma(E). \tag{1}$$

Then $(\Gamma(\mathfrak{D}(E)), Ad_{\phi_E}, [\cdot,\cdot]_{\mathfrak{D}(E)})$ is a Hom-Lie algebra where

$$[D_1, D_2]_{\mathfrak{D}(E)} = \phi_E \circ D_1 \circ \phi_E^{-1} \circ D_2 \circ \phi_E^{-1} - \phi_E \circ D_2 \circ \phi_E^{-1} \circ D_1 \circ \phi_E^{-1}, \tag{2}$$

for any $D_1, D_2 \in \mathfrak{D}(E)$. In addition, $(\mathfrak{D}(E), \varphi, Ad_{\phi_E}, [\cdot,\cdot]_{\mathfrak{D}(E)}, a_{\mathfrak{D}(E)} = \sigma)$ forms a Hom-Lie algebroid.

A subspace $B \subset A$ is a Hom-Lie subalgebroid of $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ if $\phi_A(B) \subset B$ and

$$[X, Y]_A \in \Gamma(B), \qquad \forall X, Y \in \Gamma(B).$$

In the sequel, we always assume that the Hom-bundle $(A \to M, \varphi, \phi_A)$ is invertible. The operator d^A : $\Gamma(\wedge^q A^*) \to \Gamma(\wedge^{q+1} A^*)$ defined by

$$d^A f(X) = a_A(X) f,$$
$$d^A \omega(X_1, \dots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} a_A(X_i) \omega(\phi_A^{-1}(X_1), \dots, \widehat{\phi_A^{-1}(X_i)}, \dots, \phi_A^{-1}(X_{q+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \phi_A^\dagger(\omega)([\phi_A^{-1}(X_i), \phi_A^{-1}(X_j)]_A, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}),$$

for any $X, X_1, \cdots, X_{q+1} \in \Gamma(A)$, is called *the exterior differentiation operator for the exterior differential algebra* of the Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$. Let $p, q > 0$. The operator

$$\Gamma(\wedge^p A) \times \Gamma(\wedge^q A) \xrightarrow{[\cdot,\cdot]_A} \Gamma(\wedge^{p+q-1} A) \ ,$$

given by

$$[X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q]_A = (-1)^{p+1} \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_A \wedge \phi_A(X_1) \wedge \dots \wedge$$
$$\wedge \widehat{\phi_A(X_i)} \wedge \dots \wedge \phi_A(X_p) \wedge \phi_A(Y_1) \wedge \dots \wedge \widehat{\phi_A(Y_j)} \wedge \dots \wedge \phi_A(Y_q), \quad (3)$$

is called Hom-Schouten bracket. The Hom-Schouten bracket satisfies the following conditions

1. $[X, Y]_A = -(-1)^{(p-1)(q-1)} [Y, X]_A$,
2. $[X, Y \wedge Z]_A = [X, Y]_A \wedge \phi_A(Z) + (-1)^{(p-1)q} \phi_A(Y) \wedge [X, Z]_A$,

where $X \in \Gamma(\wedge^p A)$, $Y \in \Gamma(\wedge^q A)$ and $Z \in \Gamma(\wedge^r A)$. For any $X \in \Gamma(A)$, the operator

$$L_X : \Gamma(\wedge^p A) \to \Gamma(\wedge^p A),$$

given by $L_X(Y) = [X, Y]_A$ is called the Lie derivative. Let $X \in L(A)$. The operator $L_X : \Gamma(\wedge A^*) \to \Gamma(\wedge A^*)$ defined by

$$L_X(f) = a_A(\phi_A(X))(f),$$
$$L_X \omega(X_1, \cdots, X_q) = a_A(\phi_A(X)) \omega(\phi_A^{-1}(X_1), \cdots, \phi_A^{-1}(X_q)) - \sum_{i=1}^{q} \phi_A^\dagger(\omega)(X_1, \cdots, [X, \phi_A^{-1}(X_i)]_A, \cdots, X_q),$$

for any $f \in C^\infty(M)$, $\omega \in \Gamma(\wedge^q A^*)$ and $X_1, \cdots, X_q \in \Gamma(A)$, is called the *covariant Lie derivative with respect to the element X*.

Let $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ be a finite-dimensional Hom-Lie algebroid and $\langle \cdot, \cdot \rangle$ be a bilinear symmetric non-degenerate form on A. We say that A admits a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ if the following equation is satisfied

$$\langle \phi_A(X), \phi_A(Y) \rangle = \varphi^* \langle X, Y \rangle, \quad \forall X, Y \in \Gamma(A). \quad (4)$$

In this case $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, \langle \cdot, \cdot \rangle)$ is called a pseudo-Riemannian Hom-Lie algebroid.

Assume that $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ is a Hom-Lie algebroid. An A-connection on a Hom-bundle $(E \to M, \varphi, \phi_E)$ is an operator

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E),$$

satisfying:
i) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$, iii) $\nabla_X (Z' + Z) = \nabla_X Z' + \nabla_X Z$,

ii) $\nabla_{fX}Z = \varphi^*(f)\nabla_X Z$, iv) $\nabla_X(fZ) = \varphi^*(f)\nabla_X Z + a_A(\phi_A(X))(f)\phi_E(Z)$,
for any $f \in C^\infty(M)$, $X, Y \in \Gamma(A)$, $Z', Z \in \Gamma(E)$. The map

$$\begin{array}{ccc} \Gamma(A)\times\Gamma(A) & \xrightarrow{T} & \Gamma(A) \\ (X,Y) & \longrightarrow & T(X,Y) \end{array},$$

defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]_A\,,$$

for any $X, Y \in \Gamma(A)$, is called the torsion associated to the A-connection ∇. If $T = 0$, then ∇ is said to be torsion-free. It is known that if ϕ_A is an isomorphism, then there exists a unique A-connection ∇ on the Hom-bundle $(A \to M, \varphi, \phi_A)$ such that:

$$[X,Y]_A = \nabla_X Y - \nabla_Y X, \tag{5}$$
$$a_A(\phi_A(X))\langle Y,Z\rangle = \langle \nabla_X Y, \phi_A(Z)\rangle + \langle \phi_A(Y), \nabla_X Z\rangle, \tag{6}$$

for any $X, Y, Z \in \Gamma(A)$. This connection is called the Hom-Levi-Civita connection [18], which is given by Koszul's formula

$$\begin{aligned} 2\langle \nabla_X Y, \phi_A(Z)\rangle =& a_A(\phi_A(X))\langle Y,Z\rangle + a_A(\phi_A(Y))\langle Z,X\rangle - a_A(\phi_A(Z))\langle X,Y\rangle \\ &+ \langle [X,Y]_A, \phi_A(Z)\rangle + \langle [Z,X]_A, \phi_A(Y)\rangle + \langle [Z,Y]_A, \phi_A(X)\rangle. \end{aligned} \tag{7}$$

Let $\pi \in \Gamma(\wedge^2\varphi^! TM)$ be bisection on a manifold M. If $[\pi,\pi]_{\varphi^! TM} = 0$ and $Ad_{\varphi^*}(\pi) = \pi$, the bisection π is called Hom-Poisson tensor. A manifold M equipped with a Hom-Poisson tensor π is said to be a Hom-Poisson manifold, which denotes by (M, φ, π) [4].

Theorem 1.5. *[4] Let (M, φ, π) be a Hom-Poisson manifold. Then $(\varphi^! TM, \varphi, Ad_{\varphi^*}, [\cdot,\cdot]_{\pi^\sharp}, \pi^\sharp)$ is a Hom-Lie algebroid, where $[\cdot,\cdot]_{\pi^\sharp}$ and the bundle map $\pi^\sharp : \varphi^! T^*M \to \varphi^! TM$ are given by*

$$[\alpha,\beta]_{\pi^\sharp} = L_{\pi^\sharp(\alpha)}\beta - L_{\pi^\sharp(\beta)}\omega - d^A(\pi(\alpha,\beta)),$$
$$\pi^\sharp(\alpha)(\beta) = \pi(\alpha,\beta),$$

for any $\alpha, \beta \in \Gamma(\varphi^! T^*M)$.

Example 1.6. *Consider a Hom-Poisson manifold (M, φ, π), we set $E := \varphi^! TM \oplus \varphi^! T^*M$, where $(\varphi^! TM, \varphi, Ad_{\varphi^*})$ and $(\varphi^! T^*M, \varphi, Ad^\dagger_{\varphi^*})$ are Hom-bundles. We define the bracket and linear map Φ on E as*

$$[(X,\alpha),(Y,\beta)] = ([X,Y]_{\varphi^*}, [\alpha,\beta]_{\pi^\sharp}),$$
$$\Phi(X,\alpha) = (Ad_{\varphi^*}(X), Ad^\dagger_{\varphi^*}(\alpha)),$$

where

$$[X,Y]_{\varphi^*} = \varphi^* \circ X \circ (\varphi^*)^{-1} \circ Y \circ (\varphi^*)^{-1} - \varphi^* \circ Y \circ (\varphi^*)^{-1} \circ X \circ (\varphi^*)^{-1},$$
$$[\alpha,\beta]_{\pi^\sharp} = L_{\pi^\sharp(\alpha)}\beta - L_{\pi^\sharp(\beta)}\omega - d^A(\pi(\alpha,\beta)),$$

for any $X, Y \in \Gamma(\varphi^! TM)$ and $\alpha, \beta \in \Gamma(\varphi^! T^*M)$. Easily we see that

$$\Phi[(X,\alpha),(Y,\beta)] = [\Phi(X,\alpha), \Phi(Y,\beta)],$$
$$[\Phi(X,\alpha), [(Y,\beta),(z,\gamma)]] + c.p. = 0,$$

where c.p. means the cyclic permutations of $(X,\alpha),(Y,\beta)$ *and* (z,γ). *Thus* $(E,[\cdot,\cdot],\Phi)$ *is a Hom-Lie algebra. Let* $a_E : E \to \varphi^! TM$ *be the bundle map defined by*

$$a_E(X,\alpha) = (X, \pi^\sharp(\alpha)).$$

Then $(E,\varphi,\Phi,[\cdot,\cdot],a_E)$ *is a Hom-Lie algebroid. If the metric* $\langle\cdot,\cdot\rangle$ *is determined as*

$$\langle (X,\alpha),(Y,\beta)\rangle = \alpha(Y) + \beta(X),$$

then we have

$$\langle \Phi(X,\alpha),\Phi(Y,\beta)\rangle = \varphi^*(\alpha(Y)) + \varphi^*(\beta(X)).$$

Hence $(E,\varphi,\Phi,[\cdot,\cdot],a_E,\langle\cdot,\cdot\rangle)$ *is a pseudo-Riemannian Hom-Lie algebroid.*

Definition 1.7. *Let* $(A,\varphi,\phi_A,[\cdot,\cdot]_A,a_A)$ *be a Hom-Lie algebroid. We define the curvature tensor* $\mathcal{R}$ *of* A *as*

$$\mathcal{R}(X,Y) = \nabla_{\phi_A(X)}\nabla_Y - \nabla_{\phi_A(Y)}\nabla_X - \nabla_{[X,Y]_A}\phi_A, \tag{8}$$

where $[X,Y]_A = \nabla_X Y - \nabla_Y X$, *for any* $X,Y \in \Gamma(A)$. *Moreover,* A *is called a flat Hom-Lie algebroid if* $\mathcal{R} = 0$.

Proposition 1.8. *In a Hom-Lie algebroid* $(A,\varphi,\phi_A,[\cdot,\cdot]_A,a_A)$, *we have*

$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0,$$

for any $X,Y,Z \in \Gamma(A)$. *This equation is called Hom-Bianchi first identity.*

Proof. If we consider $X,Y,Z \in \Gamma(A)$, then we have

$$0 = [\phi_A(X),[Y,Z]_A]_A + [\phi_A(Y),[Z,X]_A]_A + [\phi_A(Z),[X,Y]_A]_A,$$

which gives

$$\begin{aligned}
0 =& \nabla_{\phi_A(X)}[Y,Z]_A - \nabla_{[Y,Z]_A}\phi_A(X) + \nabla_{\phi_A(Y)}[Z,X]_A - \nabla_{[Z,X]_A}\phi_A(Y) + \nabla_{\phi_A(Z)}[X,Y]_A - \nabla_{[X,Y]_A}\phi_A(Z)\\
=& \nabla_{\phi_A(X)}\nabla_Y Z - \nabla_{\phi_A(X)}\nabla_Z Y - \nabla_{[Y,Z]_A}\phi_A(X) + \nabla_{\phi_A(Y)}\nabla_Z X - \nabla_{\phi_A(Y)}\nabla_X Z - \nabla_{[Z,X]_A}\phi_A(Y)\\
&+ \nabla_{\phi_A(Z)}\nabla_X Y - \nabla_{\phi_A(Z)}\nabla_Y X - \nabla_{[X,Y]_A}\phi_A(Z) = \mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y.
\end{aligned}$$

□

2. Hom-algebroids and left symmetric structures of Hom-algebroids

In this section, we introduce Hom-algebroids and left symmetric Hom-algebroids. Also, we deal with symplectic Hom-Lie algebroids and show that one may associate natural connections defined by the symplectic structure and that lead to left symmetric Hom-algebroids. They are called Hom-left symmetric connections.

Definition 2.1. *A Hom-algebroid structure on a Hom-bundle* $(A \to M,\varphi,\phi_A)$ *is a pair consisting of a Hom-algebra structure* $(\Gamma(A),\cdot_A,\phi_A)$ *on the space of sections* $\Gamma(A)$ *and a bundle morphism* $a_A : A \to \varphi^! TM$, *called the anchor, such that the following conditions are satisfied*

$$\begin{cases} X \cdot_A (fY) = \varphi^*(f)(X \cdot_A Y) + a_A(\phi_A(X))(f)\phi_A(Y),\\ (fX)\cdot_A Y = \varphi^*(f)(X \cdot_A Y),\\ \varphi^* \circ a_A(X) = a_A(\phi_A(X)) \circ \varphi^*, \end{cases} \tag{9}$$

for any $X,Y \in \Gamma(A)$ *and* $f \in C^\infty(M)$. *We denote a Hom-algebroid by* $(A,\varphi,\phi_A,\cdot_A,a_A)$.

Definition 2.2. *A Hom-algebroid $(A, \varphi, \phi_A, \cdot_A, a_A)$ is called left symmetric if*

$$ass_{\phi_A}(X, Y, Z) = ass_{\phi_A}(Y, X, Z),$$

where

$$ass_{\phi_A}(X, Y, Z) = (X \cdot_A Y) \cdot_A \phi_A(Z) - \phi_A(X) \cdot_A (Y \cdot_A Z),$$

for any $X, Y, Z \in \Gamma(A)$. In this case, the product $\cdot_A$ is called a Hom-left symmetric product on the Hom-bundle $(A \to M, \varphi, \phi_A)$.

Example 2.3. *A left symmetric Hom-algebroid over a one-point set with the zero anchor, is a Hom-left-symmetric algebra [15].*

Definition 2.4. *A symplectic Hom-Lie algebroid is a Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ endowed with a bilinear skew-symmetric non-degenerate form ω which is a 2-Hom-cocycle, i.e.*

$$\phi_A^{\dagger}\omega = \omega, \quad d^A\omega = 0. \tag{10}$$

In this case, ω is called symplectic structure on A and (A, ω) is called a symplectic Hom-Lie algebroid.

The conditions (10) are equivalent to

$$\begin{aligned} &a_A(\phi_A(X))\omega(Y, Z) - a_A(\phi_A(Y))\omega(X, Z) + a_A(\phi_A(Z))\omega(X, Y) \\ &- \omega([X, Y]_A, \phi_A(Z)) + \omega([X, Z]_A, \phi_A(Y)) - \omega([Y, Z]_A, \phi_A(X)) = 0, \quad \forall X, Y, Z \in \Gamma(A). \end{aligned} \tag{11}$$

Definition 2.5. *A representation of a Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ on a Hom-bundle (E, φ, ϕ_E) (shortly E) with respect to a linear φ^*-function map $\mu : \Gamma(E) \to \Gamma(E)$ is a Hom-bundle map $\rho : A \to \mathfrak{D}(E)$ such that for all $X, Y \in \Gamma(A)$, the following equalities are satisfied*

$$\begin{cases} a_{\mathfrak{D}(E)} \circ \rho = a_A \circ \phi_A, \\ \rho(\phi_A(X)) \circ \mu = \mu \circ \rho(X), \\ \rho([X, Y]_A) \circ \mu = \rho(\phi_A(X)) \circ \rho(Y) - \rho(\phi_A(Y)) \circ \rho(X). \end{cases} \tag{12}$$

We denote a representation of A by $(E; \mu, \rho)$ [17].

Let $(E^*, \varphi, \phi_E^{\dagger})$ be the dual Hom-bundle of the Hom-bundle (E, φ, ϕ_E), then the dual map of ρ is the map $\widetilde{\rho} : A \to \mathfrak{D}(E^*)$ given by

$$< \widetilde{\rho}(X)(\xi), Y >= a_A(\phi_A(X))(< \xi, \mu^{-1}(Y) >) - \varphi^* < \xi, \rho(\phi_A^{-1}(X))(\mu^{-2}(Y)) >, \tag{13}$$

for any $X \in \Gamma(A), Y \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$, we denote $(\rho(X)(\xi))(Y)$ by $< \widetilde{\rho}(X)(\xi), Y >$. Moreover, it is easy to see that $\widetilde{\rho}$ is a representation of $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ on the Hom-bundle $(E^*, \varphi, \phi_E^{\dagger})$ with respect to $\mu^{\dagger}$.

Corollary 2.6. *If $L = \widetilde{\rho}$, then $(A; \phi_A, L)$ and $(A^*; \phi_A^{\dagger}, \widetilde{L})$ are representations of A, where L and $\widetilde{L}$ are Lie derivative and covariant Lie derivative of A, respectively.*

Theorem 2.7. *Let (A, ω) be a symplectic Hom-Lie algebroid. Then the map $\nabla^{\mathbf{a}} : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ given by*

$$\omega(\nabla_X^{\mathbf{a}} Y, \phi_A(Z)) = a_A(\phi_A(X))\omega(Y, Z) - \omega(\phi_A(Y), [X, Z]_A) - \omega(\phi_A(Y), \nabla_Z X), \tag{14}$$

is an A-connection and satisfies

i) $\omega(\nabla^{\mathbf{a}}_X Y - \nabla^{\mathbf{a}}_Y X - [X,Y]_A, \phi_A(Z)) = -(\nabla_Z \omega)(X,Y)$,

ii) $$\begin{aligned}&\omega(\nabla^{\mathbf{a}}_{\phi_A(X)}\nabla^{\mathbf{a}}_Y Z - \nabla^{\mathbf{a}}_{\phi_A(Y)}\nabla^{\mathbf{a}}_X Z - \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_X Y}\phi_A(Z) + \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_Y X}\phi_A(Z), \phi^2_A(W))\\
&= a_A(\phi^2_A(X))a_A(\phi_A(Y))\omega(Z,W) + \omega(\phi^2_A(Z), \nabla_{[X,W]_A}\phi_A(Y)) + \omega(\phi^2_A(Z), \nabla_{\phi_A(Y)}\nabla_W Y)\\
&- a_A(\phi^2_A(Y))a_A(\phi_A(X))\omega(Z,W) - \omega(\phi^2_A(Z), \nabla_{[Y,W]_A}\phi_A(X)) - \omega(\phi^2_A(Z), \nabla_{\phi_A(X)}\nabla_W X)\\
&- a_A(\phi^2_A(W))(\nabla_Z\omega)(X,Y) + \omega(\phi^2_A(Z), \nabla_{\nabla^{\mathbf{a}}_X Y - \nabla^{\mathbf{a}}_Y X - [X,Y]_A}\phi_A(W)) + (\nabla_{\nabla^{\mathbf{a}}_W Z}\omega)(\phi_A(X), \phi_A(Y))\\
&- a_A(\phi_A(\nabla^{\mathbf{a}}_X Y))\omega(\phi_A(Z), \phi_A(W)) + \omega(\phi^2_A(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_X Y) + a_A(\phi_A(\nabla^{\mathbf{a}}_Y X))\omega(\phi_A(Z), \phi_A(W))\\
&- \omega(\phi^2_A(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_Y X),\end{aligned}$$

where ∇ *is an torsion-free connection on* A, *for any* $X, Y, Z \in \Gamma(A)$.

Proof. It is easy to check that

$$\begin{aligned}&\omega(\nabla^{\mathbf{a}}_{fX} Y, \phi_A(Z)) = \varphi^*(f)\omega(\nabla^{\mathbf{a}}_X Y, \phi_A(Z)),\\
&\omega(\nabla^{\mathbf{a}}_X fY, \phi_A(Z)) = \varphi^*(f)\omega(\nabla^{\mathbf{a}}_X Y, \phi_A(Z)) + a_A(\phi_A(X))(f)\varphi^*(\omega(Y,Z)).\end{aligned}$$

Thus $\nabla^{\mathbf{a}}$ is an A-connection. Using (11) and (14), we obtain

$$\begin{aligned}&\omega(\nabla^{\mathbf{a}}_X Y - \nabla^{\mathbf{a}}_Y X, \phi_A(Z)) = a_A(\phi_A(X))\omega(Y,Z) - a_A(\phi_A(Y))\omega(X,Z)\\
&+ \omega([X,Z]_A, \phi_A(Y)) - \omega([Y,Z]_A, \phi_A(X)) - \omega(\phi_A(Y), \nabla_Z X) + \omega(\phi_A(X), \nabla_Z Y)\\
&= -a_A(\phi_A(Z))\omega(X,Y) + \omega([X,Y]_A, \phi_A(Z)) - \omega(\phi_A(Y), \nabla_Z X) + \omega(\phi_A(X), \nabla_Z Y).\end{aligned}$$

On the other hand, we have

$$(\nabla_Z\omega)(X,Y) = a_A(\phi_A(Z))\omega(X,Y) - \omega(\nabla_Z X, \phi_A(Y)) - \omega(\phi_A(X), \nabla_Z Y).$$

The above equations give us (i). Using (14), we get

$$\begin{aligned}\omega(\nabla^{\mathbf{a}}_{\phi_A(X)}\nabla^{\mathbf{a}}_Y Z, \phi^2_A(W)) =& a_A(\phi^2_A(X))\omega(\nabla^{\mathbf{a}}_Y Z, \phi_A(W)) - \omega(\phi_A(\nabla^{\mathbf{a}}_Y Z), \phi_A[X,W]_A)\\
&- \omega(\phi_A(\nabla^{\mathbf{a}}_Y Z), \phi_A(\nabla_W X)),\end{aligned} \tag{15}$$

which gives

$$\begin{aligned}\omega(\nabla^{\mathbf{a}}_{\phi_A(X)}\nabla^{\mathbf{a}}_Y Z, \phi^2_A(W)) =& a_A(\phi^2_A(X))\{a_A(\phi_A(Y))\omega(Z,W) - \omega(\phi_A(Z), [Y,W]_A) - \omega(\phi_A(Z), \nabla_W Y)\}\\
&- a_A(\phi^2_A(Y))\omega(\phi_A(Z), [X,W]_A) + \omega(\phi^2_A(Z), [\phi_A(Y), [X,W]_A]_A\\
&+ \omega(\phi^2_A(Z), \nabla_{[X,W]_A}\phi_A(Y)) - a_A(\phi^2_A(Y))\omega(\phi_A(Z), \nabla_W X)\\
&+ \omega(\phi^2_A(Z), [\phi_A(Y), \nabla_W X]_A) + \omega(\phi^2_A(Z), \nabla_{\nabla_W X}\phi_A(Y)).\end{aligned} \tag{16}$$

Similarly

$$\begin{aligned}\omega(\nabla^{\mathbf{a}}_{\phi_A(Y)}\nabla^{\mathbf{a}}_X Z, \phi^2_A(W)) =& a_A(\phi^2_A(Y))\{a_A(\phi_A(X))\omega(Z,W) - \omega(\phi_A(Z), [X,W]_A) - \omega(\phi_A(Z), \nabla_W X)\}\\
&- a_A(\phi^2_A(X))\omega(\phi_A(Z), [Y,W]_A) + \omega(\phi^2_A(Z), [\phi_A(X), [Y,W]_A]_A\\
&+ \omega(\phi^2_A(Z), \nabla_{[Y,W]_A}, \phi_A(X)) - a_A(\phi^2_A(X))\omega(\phi_A(Z), \nabla_W Y)\\
&+ \omega(\phi^2_A(Z), [\phi_A(X), \nabla_W Y]_A) + \omega(\phi^2_A(Z), \nabla_{\nabla_W Y}\phi_A(X)),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\omega(\nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_Y X}\phi_A(Z) - \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_X Y}\phi_A(Z), \phi_A^2(W)) = &- \omega(\phi_A^2(Z), [\phi_A(W), [X,Y]_A]_A) - a_A(\phi_A^2(W))(\nabla_Z\omega)(X,Y) \\ &+ \omega(\phi_A^2(Z), \nabla_{\nabla^{\mathbf{a}}_X Y - \nabla^{\mathbf{a}}_Y X - [X,Y]_A}\phi_A(W)) + (\nabla_{\nabla^{\mathbf{a}}_W Z}\omega)(\phi_A(X), \phi_A(Y)) \\ &- a_A(\phi_A(\nabla^{\mathbf{a}}_X Y))\omega(\phi_A(Z), \phi_A(W)) + \omega(\phi_A^2(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_X Y) \\ &+ a_A(\phi_A(\nabla^{\mathbf{a}}_Y X))\omega(\phi_A(Z), \phi_A(W)) - \omega(\phi_A^2(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_Y X). \end{aligned} \tag{18}$$

On the other hand, we have

$$\omega(\phi_A^2(Z), [\phi_A(X), [Y,W]_A]_A + [\phi_A(Y), [W,X]_A]_A + [\phi_A(W), [X,Y]_A]_A) = 0. \tag{19}$$

From (16)-(19), we get (ii). □

Corollary 2.8. *The connection $\nabla^{\mathbf{a}}$ in the Theorem 2.7 induces a left symmetric structure on the symplectic Hom-Lie algebroid (A,ω) if and only if*

$$\begin{aligned}&a_A(\phi_A^2(X))a_A(\phi_A(Y))\omega(Z,W) + \omega(\phi_A^2(Z), \nabla_{[X,W]_A}\phi_A(Y)) + \omega(\phi_A^2(Z), \nabla_{\phi_A(Y)}\nabla_W Y) \\ &- a_A(\phi_A^2(Y))a_A(\phi_A(X))\omega(Z,W) - \omega(\phi_A^2(Z), \nabla_{[Y,W]_A}\phi_A(X)) - \omega(\phi_A^2(Z), \nabla_{\phi_A(X)}\nabla_W X) \\ &- a_A(\phi_A^2(W))(\nabla_Z\omega)(X,Y) + \omega(\phi_A^2(Z), \nabla_{\nabla^{\mathbf{a}}_X Y - \nabla^{\mathbf{a}}_Y X - [X,Y]_A}\phi_A(W)) + (\nabla_{\nabla^{\mathbf{a}}_W Z}\omega)(\phi_A(X), \phi_A(Y)) \\ &- a_A(\phi_A(\nabla^{\mathbf{a}}_X Y))\omega(\phi_A(Z), \phi_A(W)) + \omega(\phi_A^2(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_X Y) + a_A(\phi_A(\nabla^{\mathbf{a}}_Y X))\omega(\phi_A(Z), \phi_A(W)) \\ &- \omega(\phi_A^2(Z), \nabla_{\phi_A(W)}\nabla^{\mathbf{a}}_Y X) = 0.\end{aligned}$$

Proof. From (ii) of Theorem 2.7, we have the assertion. □

According to the above corollary, we consider $\nabla^{\mathbf{a}}$ as the Hom-left symmetric connection associated with (A,ω).

Applying Theorem 2.7 and Corollary 2.8, we obtain the following:

Corollary 2.9. *Let (A,ω) be a symplectic Hom-Lie algebroid. Then the following statements are equivalent:*

1. $\nabla\omega = 0$;
2. $\nabla^{\mathbf{a}} = \nabla$;

Moreover, if $\nabla^{\mathbf{a}}$ is a Hom-left symmetric connection, i.e.,

$$\nabla^{\mathbf{a}}_{\phi_A(X)}\nabla^{\mathbf{a}}_Y Z - \nabla^{\mathbf{a}}_{\phi_A(Y)}\nabla^{\mathbf{a}}_X Z - \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_X Y}\phi_A(Z) + \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_Y X}\phi_A(Z) = 0,$$

then A is flat.

3. Para-Kähler Hom-Lie algebroids

In this section, first we introduce almost product, para-complex and para-Hermitian structures on Hom-Lie algebroids. Then consider para-Kähler Hom-Lie algebroids and discuss their properties.

Definition 3.1. *An almost product structure on a Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$, is an invertible linear φ^*-function $K : \Gamma(A) \to \Gamma(A)$ such that*

$$(\phi_A \circ K)^2 = Id_{\text{ÄéÄé}A}, \quad \phi_A \circ K = K \circ \phi_A,$$

where $Id_A : \Gamma(A) \to \Gamma(A)$ is the identity map. We denote an almost product Hom-Lie algebroid by (A,K).

Remark 3.2. *From the condition* $(\phi_A \circ K)^2 = Id_A$ *we get* $(\phi_A \circ K)^2(fX) = fX$, *for all* $f \in C^\infty(M)$ *and* $X \in \Gamma(A)$. *This equation gives us* $(\varphi^*)^4(f)(\phi_A \circ K)^2(X) = fX$. *Considering* $(\phi_A \circ K)^2(X) = X$ *in the last equation we obtain* $(\varphi^*)^4(f)X = fX$ *which gives us* $(\varphi^*)^4(f) = f$, *for all* $f \in C^\infty(M)$. *So,* $(\varphi^*)^4 = Id_{C^\infty(M)}$ *is a necessary condition for the first equation of Definition 3.1. Indeed,* $(\varphi^*)^4 = Id_{C^\infty(M)}$ *is a necessary condition for the existence of an almost product structure on a Hom-Lie algebroid.*

Using Definition 3.1, one can write A as $A = A^1 \oplus A^{-1}$, such that

$$A^1 := ker(\phi_A \circ K - Id_A), \ \ A^{-1} := ker(\phi_A \circ K + Id_A).$$

Also K is called an *almost para-complex* structure on A, if A^1 and A^{-1} have the same dimension n (in this case the dimension of A is even). We define the *Nijenhuis torsion* of $\phi_A \circ K$ as follows

$$N_{\phi_A\circ K}(X,Y) = [(\phi_A \circ K)X, (\phi_A \circ K)Y]_A - (\phi_A \circ K)[(\phi_A \circ K)X, Y]_A - (\phi_A \circ K)[X, (\phi_A \circ K)Y]_A + [X,Y]_A, \quad (20)$$

for all $X, Y \in \Gamma(A)$. It easy to see that if $(\varphi^*)^2 = Id_{C^\infty(M)}$, then $N_{\phi_A\circ K}(fX,Y) = \varphi^*(f)N_{\phi_A\circ K}(X,Y)$, for any $f \in C^\infty(M)$, i.e., $N_{\phi_A\circ K}$ is a bilinear φ^*-function. In the sequel we assume that $(\varphi^*)^2 = Id_{C^\infty(M)}$. In addition, for simplicity, we often set $N := N_{\phi_A\circ K}$. If $N = 0$, then the almost product (almost para-complex) structure is called *product (para-complex)*.

Remark 3.3. *If* $\phi_A^2 = Id_A$, *then an almost product structure on a Hom-Lie algebroid* $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ *induces an almost product structure on the Hom-Lie algebra* $(\Gamma(A), \phi_A, [\cdot,\cdot]_A)$ *(see [14–16] for more details on almost product structures). But an almost product structure on the Home-Lie algebra* $(\Gamma(A), \phi_A, [\cdot,\cdot]_A)$ *does not necessarily induce an almost product structure on the Hom-Lie algebroid* $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ *(because a* $\mathbb{R}$*-linear function* $K : \Gamma(A) \to \Gamma(A)$ *is not necessarily a linear* φ^**-function). Therefore, an almost product structure on a Hom-Lie algebroid* $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ *can not be defined as an almost product structure on the Hom-Lie algebra* $(\Gamma(A), \phi_A, [\cdot,\cdot]_A)$. *Also, if* $\phi_A^2 = Id_A$, *the concepts of Nijenhuis torsion and (almost-)para-complex structure on a Hom-Lie algebroid* $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ *induce the same concepts given in [15] on the Hom-Lie algebra* $(\Gamma(A), \phi_A, [\cdot,\cdot]_A)$.

Example 3.4. *Let M be a* $2n$*-dimensional manifold endowed with an* n*-codimensional foliation F. Considering the Hom-bundle* $(\varphi^! TM, \varphi, Ad_{\varphi^*})$, *there is the decomposition* $\varphi^! TM = \varphi^! TF \oplus \varphi^! T^\perp F$ *where* $(\varphi^! TF, \varphi, Ad_{\varphi^*})$ *is the Hom-bundle (the pullback bundle of* φ *over the leaves of F) and* $(\varphi^! T^\perp F, \varphi, Ad_{\varphi^*})$ *is the transversal Hom-bundle of F. We define the bracket on* $\varphi^! TM$ *as follows*

$$\begin{aligned}
&[u,v]_{\varphi^*} = \varphi^* \circ u \circ (\varphi^*)^{-1} \circ v \circ (\varphi^*)^{-1} - \varphi^* \circ v \circ (\varphi^*)^{-1} \circ u \circ (\varphi^*)^{-1},\\
&[\bar{u},\bar{v}]_{\varphi^*} = \varphi^* \circ \bar{u} \circ (\varphi^*)^{-1} \circ \bar{v} \circ (\varphi^*)^{-1} - \varphi^* \circ \bar{v} \circ (\varphi^*)^{-1} \circ \bar{u} \circ (\varphi^*)^{-1},\\
&[u,\bar{v}]_{\varphi^*} = 0,
\end{aligned}$$

for any $u, v \in \Gamma(\varphi^! TF)$, $\bar{u}, \bar{v} \in \Gamma(\varphi^! T^\perp F)$. *Since*

$$\begin{aligned}
&Ad_{\varphi^*}[(u,\bar{u}),(v,\bar{v})]_{\varphi^*} = [Ad_{\varphi^*}(u,\bar{u}), Ad_{\varphi^*}(v,\bar{v})]_{\varphi^*},\\
&[Ad_{\varphi^*}(u,\bar{u}), [(v,\bar{v}),(z,\bar{z})]_{\varphi^*}]_{\varphi^*} + c.p. = 0,
\end{aligned}$$

one has $(\varphi^! TF \oplus \varphi^! T^\perp F, [\cdot,\cdot]_{\varphi^*}, Ad_{\varphi^*})$ *is a Hom-Lie algebra. Also we see that* $(\varphi^! TF \oplus \varphi^! T^\perp F, \varphi, Ad_{\varphi^*} \oplus Ad_{\varphi^*}, [\cdot,\cdot]_{\varphi^*}, Id)$ *is a Hom-Lie algebroid. If the isomorphism K is given by*

$$K(u) = Ad_{\varphi^*}^{-1}(u), \qquad K(\bar{u}) = -Ad_{\varphi^*}^{-1}(\bar{u}),$$

then using the above equations, we have

$$(K \circ Ad_{\varphi^*})(u) = u = (Ad_{\varphi^*} \circ K)(u), \quad (K \circ Ad_{\varphi^*})(\bar{u}) = -\bar{u} = (Ad_{\varphi^*} \circ K)(\bar{u}).$$

Also, $(K \circ Ad_{\varphi^*})^2 = Id$. *Therefore K is an almost product structure on* $\varphi^! TM$. *Since* $\varphi^! TF$ *and* $\varphi^! T^\perp F$ *have the same dimension n, we also deduce that K is an almost para-complex structure on* $\varphi^! TM$. *Moreover, K is a para-complex structure on* $\varphi^! TM$ *because*

$$N(u,v) = N(\bar{u},\bar{v}) = N(u,\bar{v}) = 0.$$

Definition 3.5. *An almost para-Hermitian structure on a Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ is a pair $(K, \langle\cdot,\cdot\rangle)$ consisting of an almost para-complex structure and a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$, such that*

$$\langle(\phi_A \circ K)X, (\phi_A \circ K)Y\rangle = -\langle X, Y\rangle, \quad \forall X, Y \in \Gamma(A). \tag{21}$$

Moreover, If $N = 0$, then the pair $(K, \langle\cdot,\cdot\rangle)$ is called a para-Hermitian structure. In this case, $(A, K, \langle\cdot,\cdot\rangle)$ is said to be a para-Hermitian Hom-Lie algebroid.

Definition 3.6. *A para-Kähler Hom-Lie algebroid is an almost para-Hermitian Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ such that $\phi_A \circ K$ is invariant with respect to the Hom-Levi-Civita connection ∇, i.e.,*

$$\nabla_X \phi_A(K(Y)) = \phi_A(K(\nabla_X Y)), \quad \forall X, Y \in \Gamma(A). \tag{22}$$

The above condition implies

$$\nabla_{\phi_A(K(X))}\phi_A(K(Y)) = \phi_A(K(\nabla_{\phi_A(K(X))}Y)), \quad \nabla_X Y = \phi_A(K(\nabla_X \phi_A(K(Y)))). \tag{23}$$

Example 3.7. *Assume that $(E, \varphi, \Phi, [\cdot,\cdot], a_E, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian Hom-Lie algebroid introduced in Example 1.6. If the invertible linear φ^*-function $K : \Gamma(E) \to \Gamma(E)$ is defined by*

$$K(X) = Ad_{\varphi^*}^{-1}(X), \quad K(\alpha) = -(Ad_{\varphi^*}^{\dagger})^{-1}(\alpha),$$

then we deduce that

$$\begin{aligned}
&(K \circ Ad_{\varphi^*})(X) = X = (Ad_{\varphi^*} \circ K)(X), \quad (K \circ Ad_{\varphi^*}^{\dagger})(\alpha) = -\alpha = (Ad_{\varphi^*}^{\dagger} \circ K)(\alpha),\\
&(K \circ Ad_{\varphi^*})^2(X) = X, \quad (K \circ Ad_{\varphi^*}^{\dagger})^2(\alpha) = \alpha.
\end{aligned}$$

Therefore K is an almost product structure on E. As $\varphi^! TM$ and $\varphi^! T^*M$ have the same dimension n, thus K is an almost para-complex structure on E. Also, we have

$$N(X, Y) = N(\alpha, \beta) = N(X, \alpha) = 0,$$

that is K is a para-complex structure on E. It is easy to check that

$$\begin{aligned}
&\langle(K \circ Ad_{\varphi^*})(X), (K \circ Ad_{\varphi^*})(Y)\rangle = 0 = \langle X, Y\rangle,\\
&\langle(K \circ Ad_{\varphi^*}^{\dagger})(\alpha), (K \circ Ad_{\varphi^*}^{\dagger})(\beta)\rangle = 0 = \langle\alpha, \beta\rangle,\\
&\langle(K \circ Ad_{\varphi^*})(X), (K \circ Ad_{\varphi^*}^{\dagger})(\alpha)\rangle = -\alpha(X) = -\langle X, \alpha\rangle,
\end{aligned}$$

and so $(E, \varphi, \Phi, [\cdot,\cdot], a_E, \langle\cdot,\cdot\rangle, K)$ is a para-Hermitian Hom-Lie algebroid. From Koszul's formula given by (7), we get

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\varphi^*}, \quad \nabla_\alpha \beta = \frac{1}{2}[\alpha, \beta]_{\pi^\sharp}, \quad \nabla_\alpha X = \nabla_X \alpha = 0.$$

It is easy to see that the Hom-Levi-Civita connection computed above satisfies in (22). Thus $(E, \varphi, \Phi, [\cdot,\cdot], a_E, \langle\cdot,\cdot\rangle, K)$ is a para-Kähler Hom-Lie algebroid.

Proposition 3.8. *Let $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ be a para-Kähler Hom-Lie algebroid. If we consider*

$$\Omega(X, Y) = \langle(\phi_A \circ K)X, Y\rangle, \quad \forall X, Y \in \Gamma(A), \tag{24}$$

then (A, Ω) is a symplectic Hom-Lie algebroid.

Proof. It turns out that (4) and (24) imply $\phi_A^\dagger\Omega = \Omega$, because

$$\begin{aligned}\Omega(\phi_A(X), \phi_A(Y)) =&\langle(\phi_A \circ K)\phi_A(X), \phi_A(Y)\rangle = \varphi^*\langle(K \circ \phi_A)X, Y\rangle = \varphi^*\langle(\phi_A \circ K)X, Y\rangle \\ =&\varphi^*\Omega(X, Y).\end{aligned}$$

Applying (11) and (24), we obtain

$$\begin{aligned}&\Omega([X, Y]_A, \phi_A(Z)) + \Omega([Y, Z]_A, \phi_A(X)) + \Omega([Z, X]_A, \phi_A(Y)) - a_A(\phi_A(X))\Omega(Y, Z) \\ &\quad + a_A(\phi_A(Y))\Omega(X, Z) - a_A(\phi_A(Z))\Omega(X, Y) \\ &= -\langle[X, Y]_A, (\phi_A \circ K)(\phi_A(Z))\rangle - \langle[Y, Z]_A, (\phi_A \circ K)(\phi_A(X))\rangle \\ &\quad - \langle[Z, X]_A, (\phi_A \circ K)(\phi_A(Y))\rangle - a_A(\phi_A(X))\Omega(Y, Z) + a_A(\phi_A(Y))\omega(X, Z) - a_A(\phi_A(Z))\Omega(X, Y) \\ &= -\langle\nabla_X Y, (\phi_A \circ K)(\phi_A(Z))\rangle + \langle\nabla_Y X, (\phi_A \circ K)(\phi_A(Z))\rangle - \langle\nabla_Y Z, (\phi_A \circ K)(\phi_A(X))\rangle \\ &\quad + \langle\nabla_Z Y, (\phi_A \circ K)(\phi_A(X))\rangle - \langle\nabla_Z X, (\phi_A \circ K)(\phi_A(Y))\rangle + \langle\nabla_X Z, (\phi_A \circ K)(\phi_A(Y))\rangle \\ &\quad - a_A(\phi_A(X))\Omega(Y, Z) + a_A(\phi_A(Y))\Omega(X, Z) - a_A(\phi_A(Z))\Omega(X, Y),\end{aligned} \tag{25}$$

for any $X, Y, Z \in \Gamma(A)$. Using (6) and (23), we conclude

$$\begin{aligned}&\langle\nabla_X Y, (\phi_A \circ K)(\phi_A(Z))\rangle = \langle(\phi_A \circ K)(\nabla_X(\phi_A \circ K)(Y)), (\phi_A \circ K)(\phi_A(Z))\rangle \\ &= -\langle\nabla_X(\phi_A \circ K)(Y), \phi_A(Z)\rangle = -a_A(\phi_A(X))\Omega(Y, Z) + \langle\nabla_X Z, \phi_A(\phi_A \circ K)(Y)\rangle.\end{aligned}$$

Setting the above equation in (25), we get the assertion i.e., $d^A\Omega = 0$. □

Proposition 3.9. *Let $(A = A^1 \oplus A^{-1}, \varphi, \phi_A, [\cdot,\cdot]_A, \langle\cdot,\cdot\rangle, K)$ be a para-Kähler Hom-Lie algebroid. Then*
i) Nijenhuis torsion N is zero,
ii) A^1 and A^{-1} are subalgebroids isotropic with respect to $\langle\cdot,\cdot\rangle$, and Lagrangian with respect to Ω,
iii) $\nabla_X\Gamma(A^1) \subset \Gamma(A^1)$ and $\nabla_X\Gamma(A^{-1}) \subset \Gamma(A^{-1})$, for any $X \in \Gamma(A)$ (∇ is the Hom-Levi-Civita connection),
iv) for any $X \in \Gamma(A^1)$, $\phi_A(X) \in \Gamma(A^1)$ and for any $\bar{X} \in \Gamma(A^{-1})$, $\phi_A(\bar{X}) \in \Gamma(A^{-1})$ i.e.,

$$\phi_A(X + \bar{X}) = \phi_{A^1}(X) + \phi_{A^{-1}}(\bar{X}),$$

v) $(A, K, \langle\cdot,\cdot\rangle)$ is a para-Hermitian Hom-Lie algebroid,
vi) $A^{-1} \simeq (A^1)^*$ ($(A^1)^*$ is the dual space of A^1) and the endomorphisms $\phi_{A^{-1}}$ and $(\phi_{A^1})^\dagger$ are the same.

Proof. Let $X, Y \in \Gamma(A)$. Then using (23), we obtain

$$\begin{aligned}N(X, Y) =&[(\phi_A \circ K)(X), (\phi_A \circ K)(Y)]_A - (\phi_A \circ K)[(\phi_A \circ K)(X), Y]_A \\ &- (\phi_A \circ K)[X, (\phi_A \circ K)(Y)]_A + [X, Y]_A \\ =&\nabla_{\phi_A(K(X))}\phi_A(K(Y)) - \nabla_{\phi_A(K(Y))}\phi_A(K(X)) - \nabla_{\phi_A(K(X))}\phi_A(K(Y)) \\ &+ \nabla_Y X - \nabla_X Y + \nabla_{\phi_A(K(Y))}\phi_A(K(X)) + \nabla_X Y - \nabla_Y X = 0,\end{aligned}$$

which gives (*i*). For any $X, Y \in \Gamma(A^1)$, we have $\langle(\phi_A \circ K)(X), (\phi_A \circ K)(Y) = \langle X, Y\rangle$. On the other hand, since $\phi_A \circ K$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$, we conclude that $\langle X, Y\rangle = 0$. Similarly, $\langle\bar{X}, \bar{Y}\rangle = 0$, for any $\bar{X}, \bar{Y} \in \Gamma(A^{-1})$. Hence, A^1 and A^{-1} are isotropic with respect to $\langle\cdot,\cdot\rangle$. Now we show that A^1 is a Lagrangian subspace of A with respect to Ω. Assume $X \in \Gamma(A^1)$. We have $\Omega(X, Y) = 0$, for any $Y \in \Gamma(A^1)$, which means that $X \in \Gamma(A^1)^\perp$ and thus $A^1 \subseteq (A^1)^\perp$. Let $0 \neq X + \bar{X} \in (A^1)^\perp$ such that $X \in \Gamma(A^1)$ and $\bar{X} \in \Gamma(A^{-1})$. Then, we have $0 = \Omega(X + \bar{X}, Y) = \Omega(X, Y) + \Omega(\bar{X}, Y)$, for all $Y \in \Gamma(A^1)$. As $\Omega(X, Y) = 0$, it yields $\Omega(\bar{X}, Y) = 0$. On the other hand, since $\Omega(\bar{X}, \bar{Y}) = 0$, for any $\bar{Y} \in \Gamma(A^{-1})$, we deduce $\Omega(\bar{X}, Y + \bar{Y}) = 0$, hence $\bar{X} = 0$. Therefore $(A^1)^\perp \subseteq A^1$ and consequently $(A^1)^\perp = A^1$. Similarly it follows that A^{-1} is a Lagrangian subspace of A with respect to Ω. Therefore we have (*ii*). Using (23) we have $(\phi_A \circ K)(\nabla_X Y) = \nabla_X\phi_A(KY) = \nabla_X Y$, for all $X \in \Gamma(A)$ and $Y \in \Gamma(A^1)$. The above equation implies $\nabla_X Y \in \Gamma(A^1)$. Similarly, $\nabla_X\bar{Y} \in \Gamma(A^{-1})$ for any $X \in \Gamma(A)$, which gives us (*iii*). To prove (*iv*), since $(K \circ \phi_A)(\phi_A(X)) = (\phi_A \circ K \circ \phi_A)(X) = \phi_A(X)$, for all $X \in \Gamma(A^1)$, we have

$\phi_A(X) \in \Gamma(A^1)$. Similarly, $\phi_A(\bar{X}) \in \Gamma(A^{-1})$ for any $\bar{X} \in \Gamma(A^{-1})$. To prove (*v*), using (*i*) and (*ii*) we have K is a para-complex structure on $(\Gamma(A), [\cdot,\cdot]_A, \phi_A)$. Since $\phi_A \circ K$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$, then we have the assertion. To prove (*vi*), let $\bar{X} \in \Gamma(A^{-1})$ and $\bar{X}^* \in (\Gamma(A^1)^*)$ such that $< \bar{X}^*, Y >= \langle \bar{X}, Y\rangle$, for all $Y \in \Gamma(A^1)$. Then we have map $A^{-1} \to (A^1)^*$, $\bar{X} \to \bar{X}^*$, which is an isomorphism. If we consider $\bar{X} \in \Gamma(A^{-1})$ and $\bar{X}^*$ be the corresponding element of it in $\Gamma((A^1)^*)$, then for any $Y \in \Gamma(A^1)$ we obtain

$$< (\phi_{A^1})^\dagger(\bar{X}^*), Y >= \varphi^* < \bar{X}^*, \phi_{A^1}^{-1}(Y) >= \varphi^*\langle \bar{X}, \phi_{A^1}^{-1}(Y)\rangle = \langle \phi_{A^{-1}}(\bar{X}), Y\rangle =< (\phi_{A^{-1}}(\bar{X}))^*, Y >,$$

which gives $(\phi_{A^1})^\dagger(\bar{X}^*) = (\phi_{A^{-1}}(\bar{X}))^*$, where $(\phi_{A^{-1}}(\bar{X}))^*$ is the corresponding element of $\phi_{A^{-1}}(\bar{X}) \in \Gamma(A^{-1})$ in $(A^1)^*$. □

Lemma 3.10. *Let $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ be a para-Kähler Hom-Lie algebroid. Then*

$$\nabla_X Y = \nabla^{\mathbf{a}}_X Y, \quad \forall X, Y \in \Gamma(A^1),$$
$$\nabla_{\bar{X}} \bar{Y} = \nabla^{\mathbf{a}}_{\bar{X}} \bar{Y}, \quad \forall \bar{X}, \bar{Y} \in \Gamma(A^{-1}),$$

where ∇ is the Hom-Levi-Civita connection and $\nabla^{\mathbf{a}}$ is given in Theorem 2.7.

Proof. Since A is a symplectic Hom-Lie algebroid, then using (14) and (5), we get

$$\begin{aligned} 0 =&\Omega(\nabla^{\mathbf{a}}_X Y, \phi_A(Z)) - a_A(\phi_A(X))\Omega(Y,Z) + \Omega(\phi_A(Y), [X,Z]_A) + \Omega(\phi_A(Y), \nabla_Z X) \\ =&\Omega(\nabla^{\mathbf{a}}_X Y, \phi_A(Z)) - a_A(\phi_A(X))\Omega(Y,Z) + \Omega(\phi_A(Y), \nabla_X Z), \end{aligned} \tag{26}$$

for any $Z \in \Gamma(A)$ and $X, Y \in \Gamma(A^1)$. Also, (4), (24) and parts (*iii*) and (*iv*) of Proposition 3.9 imply

$$\begin{aligned} &\Omega(\phi_A(Y), \nabla_X Z) = \langle (\phi_A \circ K)(\phi_A(Y)), \nabla_X Z\rangle = \langle \phi_A(Y), \nabla_X Z\rangle \\ &= a_A(\phi_A(X))\Omega(Y,Z) - \langle \phi_A(Z), \nabla_X Y\rangle = a_A(\phi_A(X))\Omega(Y,Z) - \Omega(\nabla_X Y, \phi_A(Z)). \end{aligned} \tag{27}$$

Setting (27) in (26) and using the non-degenerate property of Ω and ϕ_A, we have $\nabla_X Y = \nabla^{\mathbf{a}}_X Y$. Similarly, we get the second relation. □

Corollary 3.11. *If $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ is a para-Kähler Hom-Lie algebroid, then $(\Gamma(A^1), \nabla^{\mathbf{a}}, \phi_{A^1})$ and $(\Gamma(A^{-1}), \nabla^{\mathbf{a}}, \phi_{A^{-1}})$ are Hom-left symmetric algebras if and only if A^1 and A^{-1} are flat. Moreover, they induce Hom-Lie algebra structures on $\Gamma(A^1)$ and $\Gamma(A^{-1})$. Consequently, $(A^1, \varphi, \phi_{A^1}, [\cdot,\cdot]_{A^1}, a_{A^1})$ and $(A^{-1}, \varphi, \phi_{A^{-1}}, [\cdot,\cdot]_{A^{-1}}, a_{A^{-1}})$ are Hom-Lie algebroids.*

In the sequel, we let A^1 and A^{-1} be flat. As $A^{-1} \simeq (A^1)^*$, we denote the elements of A^{-1} by $\alpha, \beta, \ldots$. Also, to simplify we use $\phi_{(A^1)^*}$ instead of $(\phi_{A^1})^\dagger$. Considering $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ as a para-Kähler Hom-Lie algebroid, for any $X \in \Gamma(A^1), \alpha \in (\Gamma(A^1)^*)$, we let ∇_X and ∇_α be the Hom-Levi-Civita connection operators on X and α, respectively.

Proposition 3.12. *With the above notations, we have*
i) $(A^1; \phi_{A^1}, \nabla)$ is a representation of the Hom-Lie algebroid A^1 where $\nabla : A^1 \to \mathfrak{D}(A^1)$ with $X \to \nabla_X$,
ii) $((A^1)^*; \phi_{(A^1)^*}, \nabla)$ is a representation of the Hom-Lie algebroid $(A^1)^*$ where $\nabla : (A^1)^* \to \mathfrak{D}((A^1)^*)$ with $\alpha \to \nabla_\alpha$.

Proof. Since A^1 is the isotropic subspace and $\nabla_X Y - \nabla_Y X = [X,Y]_{A^1}$, then using Theorem 2.7 we have

$$\nabla^{\mathbf{a}}_{\phi_{A^1}(Y)} \nabla^{\mathbf{a}}_X Z - \nabla^{\mathbf{a}}_{\phi_{A^1}(X)} \nabla^{\mathbf{a}}_Y Z + \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_X Y} \phi_{A^1}(Z) - \nabla^{\mathbf{a}}_{\nabla^{\mathbf{a}}_Y X} \phi_A(Z) = 0, \quad \forall X, Y, Z \in \Gamma(A^1),$$

where $\nabla_X Y = \nabla^{\mathbf{a}}_X Y$. The above equation implies

$$\nabla_{[X,Y]_{A^1}} \circ \phi_{A^1} = \nabla_{\phi_{A^1}(X)} \circ \nabla_Y - \nabla_{\phi_{A^1} \ddot{A}\acute{e}(Y)} \circ \nabla_X.$$

Also, we have $\phi_{A^1}(\nabla_X Y) = \nabla_{\phi_{A^1}(X)} \phi_{A^1}(Y)$ and

$$a_{\mathfrak{D}(A^1)}(\nabla_X)(f) = \nabla_X(f) = a_{A^1}(\phi_{A^1}(X))(f).$$

So (*i*) holds. Similarly, we obtain (*ii*). □

Proposition 3.13. *Let $(A, \varphi, \phi_A, [\cdot,\cdot], a_A, K, \langle\cdot,\cdot\rangle)$ be a para-Kähler Hom-Lie algebroid. Then $(A^1 \oplus (A^1)^*, \varphi, \phi_A, [\cdot,\cdot], a)$ is a Hom-Lie algebroid, where*

$$\begin{cases} [X+\alpha, Y+\beta] = [X,Y]_{A^1} + \widetilde{\nabla}_X\beta - \widetilde{\nabla}_Y\alpha, \\ \phi_A(X+\alpha) = \phi_{A^1}(X) + \phi_{(A^1)^*}(\alpha), \\ a(X+\alpha) = a_A(X), \end{cases} \tag{28}$$

for any $X, Y \in \Gamma(A^1)$ and $\alpha, \beta \in \Gamma((A^1)^*)$.

Proof. Obviously, we see that

$$[X+\alpha, Y+\beta] = -[Y+\beta, X+\alpha].$$

As $((A^1)^*, \phi_{(A^1)^*}, \widetilde{\nabla})$ is a representation of Hom-Lie algebroid A^1, we have

$$\widetilde{\nabla}_{[Y,Z]_{A^1}} \circ \phi_{(A^1)^*} = \widetilde{\nabla}_{\phi_{A^1}(Y)} \circ \widetilde{\nabla}_Z - \widetilde{\nabla}_{\phi_{A^1}(Z)} \circ \widetilde{\nabla}_Y.$$

Using (28) and the above equation, we obtain

$$\begin{aligned} &[\phi_A(X+\alpha), [Y+\beta, Z+\gamma]] + c.p. = [\phi_{A^1}(X) + \phi_{(A^1)^*}(\alpha), [Y,Z]_{A^1} + \widetilde{\nabla}_Y\gamma - \widetilde{\nabla}_Z\beta] + c.p. \\ &= \Big([\phi_{A^1}(X), [Y,Z]_{A^1}]_{A^1} + \widetilde{\nabla}_{\phi_{A^1}(X)}(\widetilde{\nabla}_Y\gamma - \widetilde{\nabla}_Z\beta) - \widetilde{\nabla}_{[Y,Z]_{A^1}}\phi_{(A^1)^*}(\alpha)\Big) + c.p. \\ &= [\phi_{A^1}(X), [Y,Z]] + c.p. + \widetilde{\nabla}_{\phi_{A^1}(X)}(\widetilde{\nabla}_Y\gamma - \widetilde{\nabla}_Z\beta) + \widetilde{\nabla}_{\phi_{A^1}(Y)}(\widetilde{\nabla}_Z\alpha - \widetilde{\nabla}_X\gamma) + \widetilde{\nabla}_{\phi_{\mathfrak{g}^1}(Z)}(\widetilde{\nabla}_X\beta - \widetilde{\nabla}_Y\alpha) \\ &- \widetilde{\nabla}_{\phi_{A^1}(Y)}\widetilde{\nabla}_Z\alpha + \widetilde{\nabla}_{\phi_{A^1}(Z)}\widetilde{\nabla}_Y\alpha - \widetilde{\nabla}_{\phi_{A^1}(Z)}\widetilde{\nabla}_X\beta + \widetilde{\nabla}_{\phi_{A^1}(X)}\widetilde{\nabla}_Z\beta - \widetilde{\nabla}_{\phi_{A^1}(X)}\widetilde{\nabla}_Y\gamma + \widetilde{\nabla}_{\phi_{A^1}(Y)}\widetilde{\nabla}_X\gamma = 0. \end{aligned}$$

Therefore $(A^1 \oplus (A^1)^*, [\cdot,\cdot], \phi_A)$ is a Hom-Lie algebra. Since a_A is the representation of Hom-Lie algebroid $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$, we have

$$\varphi^* \circ a(X+\alpha) = \varphi^* \circ a_A(X) = a_A(\phi_A(X)) \circ \varphi^* = \varphi^* \circ a(\phi_A(X+\alpha)),$$

and

$$\begin{aligned} a([X+\alpha, Y+\beta]) \circ \varphi^* &= a([X,Y]_A + \widetilde{\nabla}_X\beta - \widetilde{\nabla}_Y\alpha) \circ \varphi^* \\ &= a_A([X,Y]_A \circ \varphi^* = a_A(\phi_A(X)) \circ a_A(Y) - a_A(\phi_A(Y)) \circ a_A(X) \\ &= a(\phi_A(X+\alpha)) \circ a(Y+\beta) - a(\phi_A(Y+\beta)) \circ a(X+\alpha). \end{aligned}$$

□

Definition 3.14. *Let $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A)$ be a Hom-Lie algebroid and A^* be the dual bundle of A. A phase space of A is defined as a Hom-Lie algebroid $(A \oplus A^*, \varphi, \phi_A \oplus \phi_{A^*}, [\cdot,\cdot]_{A\oplus A^*}, a)$ consisting of Hom-subalgebroids A, A^* and the natural skew-symmetry bilinear form ω on $A \oplus A^*$ given by*

$$\omega(X+\alpha, Y+\beta) =< \beta, X\text{Äé} > - < \alpha, Y\text{Äé} >, \qquad \forall X, Y \in \Gamma(A), \forall \alpha, \beta \in \Gamma(A^*), \tag{29}$$

which is a symplectic form, $\phi_{A^*} = \phi_A^\dagger$.

Lemma 3.15. *If $(A, \varphi, \phi_A, [\cdot,\cdot]_A, a_A, K, \langle\cdot,\cdot\rangle)$ is a para-Kähler Hom-Lie algebroid, then the Hom-Lie algebroid $A^1 \oplus (A^1)^*$ is a phase space of the Hom-Lie algebroid A^1.*

Proof. As the para-Kähler Hom-Lie algebroid A is a symplectic Hom-Lie algebroid, then we can write

$$\Omega(X+\alpha, Y+\beta) = \langle(\phi_A \circ K)(X+\alpha), Y+\beta\rangle = \langle X-\alpha, Y+\beta\rangle = - < \alpha, Y\text{Äé} > + < \beta, X\text{Äé} > .$$

for any $X, Y \in \Gamma(A^1)$ and $\alpha, \beta \in \Gamma((A^1)^*)$. Similar to the proof of Proposition 3.8, we can prove the following

$$\Omega(\phi_A(X+\alpha), \phi_A(Y+\beta)) = \varphi^*\Omega(X+\alpha, Y+\beta),$$

and

$$\Omega([X+\alpha, Y+\beta], \phi_A(Z+\gamma)) + c.p. = 0.$$

□

Example 3.16. *We consider the para-Kähler Hom-Lie algebroid* $(E, \varphi, \Phi, [\cdot,\cdot], a_E, \langle\cdot,\cdot\rangle, K)$ *in Example 3.7. Define a skew-symmetry bilinear form* Ω *on* E *as* $\Omega(\cdot,\cdot) = \langle(\Phi\circ K)\cdot,\cdot\rangle$, *we obtain* $\Omega((X,0),(0,\beta)) = \langle((\Phi\circ K)X,0),(0,\beta)\rangle = \langle(Ad_{\varphi^*}\circ Ad_{\varphi^*}^{-1})X,\beta\rangle =\prec \beta, XÄé \succ$. *Similarly, it follows*

$$\Omega((0,\alpha),(Y,0)) = -\prec \alpha, YÄé \succ, \quad \Omega((X,0),(Y,0)) = \Omega((0,\alpha),(0,\beta)) = 0.$$

Also, $\Phi^\dagger Äé\Omega = \Omega$, *because*

$$\Phi^\dagger Äé\Omega((X,\alpha),(Y,\beta)) = \varphi^*\Omega(\Phi^{-1}(X,\alpha),\Phi^{-1}(Y,\beta)) = \varphi^*\langle\Phi^{-1}(\Phi\circ K)(X,\alpha),\Phi^{-1}(Y,\beta)\rangle$$
$$= \varphi^*\langle(Ad_{\varphi^*}^{-1}X, -(Ad_{\varphi^*}^\dagger)^{-1}\alpha),(Ad_{\varphi^*}^{-1}Y,(Ad_{\varphi^*}^\dagger)^{-1}\beta)\rangle = -\prec\alpha, YÄé\succ + \prec\beta, XÄé\succ.$$

As $Ad_{\varphi^*}X \prec \beta, Z \succ = \prec \varphi^*\circ X\circ(\varphi^*)^{-1}\circ Z\circ(\varphi^*)^{-1}, Ad_{\varphi^*}^\dagger(\beta)\succ$, *thus*

$$d\Omega((X,0),(0,\beta),(Z,0)) = Ad_{\varphi^*}(X)\Omega((0,\beta),(Z,0)) + Ad_{\varphi^*}(Z)\Omega((X,0),(0,\beta))$$
$$+ \Omega([(X,0),(Z,0)], Ad_{\varphi^*}^\dagger(\beta)) - \Omega([(0,\beta),(Z,0)], Ad_{\varphi^*}(X)) = 0.$$

In the same way, we get

$$d\Omega((X,0),(Y,0),(Z,0)) = d\Omega((0,\alpha),(0,\beta),(\gamma,0)) = d\Omega((0,\alpha),(Y,0),(\gamma,0)) = 0,$$

i.e., Ω *is a symplectic form. Therefore* $(\varphi^! TM\oplus\varphi^! T^*M, \varphi, \Phi, [\cdot,\cdot], a_E)$ *is a phase space of the Hom-Lie algebroid* $\varphi^! TM$.

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